VARIATIONAL ANALYSIS FOR FIXED POINT PROBLEMS AND CONVERGENCE THEOREMS FOR NONLINEAR MAPPINGS

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หัวข้อวิทยานิพนธ์	การวิเคราะห์การแปรผันสำหรับปัญหาจุดตรึง และทฤษฎีบทการลู่เข้าสำหรับการส่งแบบไม่เชิงเส้น
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บทคัดย่อ

จุดประสงค์ในงานวิจัยฉบับนี้แบ่งออกได้ดังนี้ อันดับแรกผู้วิจัยได้เสนอวิธีการแก้ปัญหาอสมการการแปรผัน และปัญหาเชิงดุลยภาพ และได้ยกตัวอย่างการคำนวณเพื่อสนับสนุนทฤษฎีบทหลัก อันดับที่สอง ผู้วิจัยได้เสนอวิธีการแก้ปัญหาเชิงดุลยภาพ และปัญหาจุดตรึงของวงศ์จำกัดของการส่งแบบไม่ขยายและการส่งแบบหดเทียมโดยแท้ อันดับที่สาม ผู้วิจัยได้ประยุกต์ทฤษฎีบทหลักเพื่อพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการหาสมาชิกร่วมของเซตคำตอบของปัญหาจุดตรึงของการส่งแบบหดเทียมโดยแท้ และทฤษฎีบทการลู่เข้าที่เกี่ยวข้องกับปัญหาค่าน้อยที่สุด

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Abstract

The purpose of this research is divided into the following; First, we present a method for solving the variational inequality problems and equilibrium problems and we utilize our main theorem for the numerical example. Secondly, we present a method for solving an equilibrium problems and fixed point problems of finite families of nonexpansive mappings and strictly pseudo-contractive mappings. Thirdly, we apply our main result to prove a strong convergence theorem for finding a common element of the set of fixed point problems of strictly pseudocontractive mappings and a convergence theorem involving minimization problems.

Keywords : Equilibrium problem, Optimization problems, κ-Strict pseudo contraction mapping, Variational inequality

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Introduction

1.1 Research Motivation

The fixed point theorem based on the contraction principle is studied the existence and uniqueness of the solutions. In the past few decades, many mathematicians have created the convergence theorem to find the solution of the fixed point problem. This makes the fixed point theory developed extensively and apply in various fields as well such as mathematics, economics, physics, engineering, optimizations, and computer sciences, etc. For an interesting example, equilibrium problem or variational inequality problem are fundamental concepts in the field economics, engineering, physics, which can be transformed in terms of fixed points.

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A mapping T is said to be *κ-strictly pseudocontractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Note that the class of κ-strictly pseudocontractions strictly included the class of nonexpansive mappings. It is well-know that (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\kappa}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T).$$

Remark 1.1.

i) It is well-know that $I - T$ is $\frac{1-\kappa}{2}$-inverse strongly monotone mapping.

ii) If $T : C \to C$ be κ-strictly pseudocontractive with $F(T) \neq \emptyset$, then $F(T) = VI(C, I - T)$, See more detail in [8]

Throughout this paper we denote $F(T)$ is the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$).

A mapping A of C into H is called *α-inverse strongly monotone*(see [10]), if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If $\langle x - y, Ax - Ay\rangle \geq 0$, a mapping A is called monotone operator.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problems of F is to find $x \in C$, such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of the equilibrium problems is denoted by $EP(F)$. Many problems in physics, optimization, and economics are seeking some elements of $EP(F)$, see more detail in [6, 5]. Over decades ago, there are many researches studied the equilibrium problems, see, for instance [7, 8, 12]

Let $B : C \to H$. The variational inequality problems is to find a point $u \in C$ such that

$$\langle Bu, v-u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted by $VI(C, B)$.
Numerous problems in physics, optimization, minimax problems , game theory, the Nash equilibrium problems in noncooperative games reduce to find element of (1.3), see more detail in [11, 12].

By modification of (1.3), Kangtunyakarn introduce the combination of variational inequality problems which is to find a point $x^* \in C$ such that

$$\langle y - x^*, (aA + (1-a)B)x^*\rangle \geq 0, \quad \forall y \in C, \ a \in (0,1). \tag{1.4}$$

The set of all solution of (1.4) is denoted by $VI(C, aA + (1-a)B)$.
If $A = B$, $VI(C, aA + (1-a)B)$ reduce to $VI(C, B)$.

So, He proved a strong convergence theorem for finding a common element of the set of fixed point problems of infinite family of strictly pseudocontractive mappings and the set of equilibrium problems and two set of variational inequality problems as follows;

Theorem 1.2. Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$, let $A, B : C \to H$ be α and β-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strict pseudocontractive mappings of C into itself with $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S-mapping generated by $T_n, \ldots, T_1$ and $\rho_n, \rho_{n-1}, \ldots, \rho_1$ and $T_n, T_{n-1}, \ldots,$ and $\rho_n, \rho_{n-1}, \ldots,$ respectively. Assume that $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \bigcap VI(C, A) \bigcap VI(C, B) \neq \emptyset$ and let $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) S_n P_C\Big(I - \gamma(aA + (1-a)B)\Big)u_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\alpha_n, a \in (0,1), 0 < \gamma < \min\{2\alpha, 2\beta\}$ and $\{r_n\} \subset [b, c] \subset (0, \min\{2\alpha, 2\beta\})$, satisfy the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \ \sum\limits_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$

(iii) $\sum_{n=1}^{\infty} \alpha_1^n < \infty$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathcal{F}$ where $z = P_{\mathcal{F}}u$.

The purpose of this research is divided into the following; First, we present a method for solving the variational inequality problems and equilibrium problems and we utilize our main theorem for the numerical example. Secondly, we present a method for solving an equilibrium problems and fixed point problems of finite families of nonexpansive mappings and strictly pseudo-contractive mappings. Thirdly, we apply our main result to prove a strong convergence theorem for finding a common element of the set of fixed point problems of strictly pseudocontractive mappings and a convergence theorem involving minimization problems.

1.2 Objectives of the study

1) To prove a strong convergence theorem for finding a common element of the set of solutions of the variational inequality problems, the set of solutions of an equilibrium problems.

2) To prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problems and fixed point problems of finite family of nonexpansive mappings and strictly pseudo-contractive mappings.

3) For apply our main result to obtain a strong convergence theorem for finding a common element of the set of fixed point problems of strictly pseudocontractive mappings and a convergence theorem involving minimization problems.

4) To give numerical example for our results to compare converge of them.

1.3 Scopes of the study

1) Study equilibrium problems in real Hilbert space.

2) Study variational inequality problems in real Hilbert space.

3) Study the fixed point problems of strictly pseudocontractive mappings and a convergence theorem involving minimization problems.

4) All strong convergence theorems are considered and proved in a real Hilbert space.

1.4 Benefits of the study

1) To obtain new tools for fixed point problems on real Hilbert space.

2) To obtain a strong convergence theorem for finding a common element of the set of solutions of the variational inequality problems, the set of solutions of an equilibrium problems.
3) To obtain a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problems and fixed point problems of finite family of nonexpansive mappings and strictly pseudo-contractive mappings.
4) To obtain a strong convergence theorem for finding a common element of the set of fixed point problems of strictly pseudocontractive mappings and a convergence theorem involving minimization problems.

1.5 Research methodology

1) Study advanced topics in fixed point theory for a strictly pseudocontractive mappings.
2) Study background in a real Hilbert space.
3) Collect and study research papers and textbooks concerning fixed point theorem.
4) Determine the objectives and scope of the research.
5) Produce tools for a strong convergence theorem of fixed point problem.
6) Prove a strong convergence theorem for fixed point problems in a real Hilbert space.
7) Provide examples and applications.
8) Conclude the results, make suggestions for further works and write the thesis.

Chapter 2

Preliminaries and Literature Reviews

The purpose of this chapter is to collect lemma, definition, theorem and terminology for used throughout of the thesis.

2.1 Fundamental properties in Hilbert spaces

Definition 2.1 (Strong convergence [13]). A sequence $\{x_n\}$ of vectors in an inner product space K is called *strongly convergent* to a vector x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.2 (Weak convergence [13]). A sequence $\{x_n\}$ of vectors in an inner product space K is called *weakly convergent* to a vector x in K if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty \text{ for every } y \in K.$$

Theorem 2.1 ([13]). A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.2 ([14]). If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Theorem 2.3 ([14]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Definition 2.3 ([14]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called *convex* if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Similarly, f is said to be *concave* if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.4 ([14]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Lemma 2.5 ([15]). Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Theorem 2.6 ([14]). Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Remark 2.7 ([14]). Let H be an inner product space. Then we know that the following (i) and (ii) are equivalent:

1) (i) H is complete,

2) (ii) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 2.4 (Metric projection [14]). The (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

Lemma 2.8 ([1]). Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.9 ([16]). For a given $z \in H$ and $u \in C$,

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \forall v \in C.$$

It is well-known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H.$$

Lemma 2.10 ([16]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.11 ([3]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.12 ([3]). Let $\{s_n\}$ be a sequence of nonnegative real number satisfying

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

(1) $\{\alpha_n\} \subset [0,1], \displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\displaystyle\limsup_{n\to\infty} \beta_n \leq 0 \text{ or } \sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\displaystyle\lim_{n\to\infty} s_n = 0$.

Lemma 2.13 ([7]). Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C,A)\bigcap VI(C,B) \neq \emptyset$. Then

$$VI(C, aA + (1-a)B) = VI(C,A)\bigcap VI(C,B), \quad \forall a \in (0,1).$$

Furthermore if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is a nonexpansive mapping.

Definition 2.5. Let C be a nonempty convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strict pseudo-contractions of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
&U_0 = I\\
&U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\\
&U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\\
&U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\\
&\vdots\\
&U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\\
&S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called S-mapping generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 2.14. (See [19]) Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ-strict pseudo-contractive mapping of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N \in [\kappa, 1)$, $\alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^{N} F(T_i)$ and S is a nonexpansive mapping.

Lemma 2.15. (See [14]) The following inequality holds in an inner product space X:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \qquad \forall x, y \in X.$$

2.2 Fixed Point Theorems

Let X be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and $F(T)$ represents the set of all fixed points of T.

Example 2.16 ([17]). 1) If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$;

2) If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $F(T) = \{0, 2\}$;

3) If $X = \mathbb{R}$ and $T(x) = x + 2$, then $F(T) = \emptyset$;

4) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$;

Theorem 2.17 ([14]). Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.18 ([14]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Lemma 2.19 (Demiclosedness principle [18]). Assume that T is a nonexpansive self-mapping of closed convex C subset of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y it follows that $(I - T)x = y$. Here, I is the identity mapping of H.

2.3 Equilibrium problems and generalized equilibrium problem in Hilbert spaces

The equilibrium problem provides us a natural to study problems arising in economics, finance, minimization problems, Nash equilibria in noncooperative games and certain fixed point problems.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. *The classical equilibrium problem* for F is to find $u \in C$ satisfying the following inequality

$$F(u, v) \geq 0, \forall v \in C. \tag{2.1}$$

We use $EP(F)$ to represent the set of solution of (2.1).
Let the bifunction F satisfy the following conditions for solving equilibrium problem.
(A1) $F(u, u) = 0$ for all $u \in C$;
(A2) F is monotone, i.e., $F(u, v) + F(v, u) \leq 0$ for all $u, v \in C$;
(A3) For each $u, v, w \in C$,

$$\lim_{t \to 0^+} F\left(tw + (1 - t)u, v\right) \leq F(u, v);$$

(A4) For each $u \in C, v \mapsto F(u, v)$ is convex and lower semicontinuous.

Example 2.20. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x, y) = x^2 + 2xy - 3y^2, \forall x, y \in \mathbb{R}.$$

Then a bifunction F satisfies the condition (A1)-(A4) and $0 \in EP(F)$.
Solution. Let $x, y, z \in \mathbb{R}$. Since

$$F(x, x) = x^2 + 2x^2 - 3x^2 = 0,$$

thus we obtain (A1) holds. Next, observe that

$$F(x,y)+F(y,x) = (x^2 + 2xy - 3y^2) + (y^2 + 2xy - 3x^2) = -2x^2+4xy-2y^2 = -2(x-y)^2 \leq 0.$$

This implies that F satisfies (A2). Let $t \in [0, 1]$. Consider

$$\lim_{t \to 0^+} F(tz+(1-t)x, y) = \lim_{t \to 0^+} [(tz+(1-t)x)^2+2(tz+(1-t)x)y-3y^2] = x^2+2xy-3y^2 = F(x, y).$$

Therefore, (A3) is true. To show (A4), first let $\alpha \in (0, 1)$. Then we derive that

$$\begin{aligned}
& F(x, \alpha z + (1 - \alpha)y) \\
&= x^2 + 2x(\alpha z + (1 - \alpha)y) - 3(\alpha z + (1 - \alpha)y)^2 \\
&= x^2 + 2\alpha xz + (1 - \alpha)2xy - 3(\alpha^2 z^2 + 2\alpha(1 - \alpha)zy + (1 - \alpha)^2 y^2) \\
&\leq x^2 + 2\alpha xz + (1 - \alpha)2xy - 3(\alpha^2 z^2 + \alpha(1 - \alpha)(z^2 + y^2) + (1 - \alpha)^2 y^2) \\
&= \alpha\left(x^2 + 2xz - 3(\alpha z^2 + (1 - \alpha)z^2)\right) + (1 - \alpha)\left(x^2 + 2xy - 3(\alpha y^2 + (1 - \alpha)y^2)\right) \\
&= \alpha\left(x^2 + 2xz - 3z^2\right) + (1 - \alpha)\left(x^2 + 2xy - 3y^2\right) \\
&= \alpha F(x, z) + (1 - \alpha)F(x, y).
\end{aligned}$$

Hence F is a convex function. Let $\{y_n\} \subset \mathbb{R}$ with $y_n \to y$ as $n \to \infty$. Thus we get

$$\lim_{n \to \infty} F(x, y_n) = \lim_{n \to \infty} x^2 + 2xy_n - 3y_n^2 = x^2 + 2xy - 3y^2. \tag{2.2}$$

This yields that F is lower semicontinuous and (A4) holds.

In 1994, Blum and Oettli [5] proved the following existence result:

Lemma 2.21 ([5]). Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.22 ([6]). Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

1) T_r is single-valued;

2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle ;$$

3) $F(T_r) = EP(F)$;

4) $EP(F)$ is closed and convex.

Chapter 3

A Theorem for Solving Equilibrium and Variational Inequality Problems

In this chapter, we prove our main result and utilize Theorem 3.1 for the numerical example.

Now,we prove a strong convergence theorem for finding a common element of the set of solutions of the variational inequality problems, the set of solutions of an equilibrium problems.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $A, B : C \to H$ be α and β-inverse strongly monotone respectively with $\mathbb{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(aA + (1-a)B))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, where $\eta = min\{\alpha, \beta\}$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

Proof. We will divide our prove into 5 steps.

Step 1. We will show that the sequence $\{x_n\}$ is bounded. Since

$$F_i(u_n^i, v) + \frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N.$$

By Lemma 2.22, we have $u_n^i = T_{r_n}^i(x_n)$ and $EP(F_i) = F(T_{r_n}^i)$, for all $i = 1, 2, ..., N$.

Let $z \in \mathbb{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^{N} EP(F_i)$.

By Lemma 2.13 and Lemma 2.10, we have

$$z \in VI\Big(C, aA + (1-a)B\Big) = F\Big(P_C(I - \lambda(aA + (1-a)B))\Big).$$

From Lemma 2.13 and nonexpansiveness of $T^i_{r_n}$, we have

$$\begin{aligned}
\|x_{n+1} - z\| =& \|\alpha_n u + \beta_n P_C(I - \lambda(aA + (1-a)B))x_n + \gamma_n \sum_{i=1}^{N} a_i u^i_n - z\| \\
\leq& \alpha_n\|u - z\| + \beta_n\|P_C(I - \lambda(aA + (1-a)B))x_n - z\| + \gamma_n\|\sum_{i=1}^{N} a_i(u^i_n - z)\| \\
\leq& \alpha_n\|u - z\| + \beta_n\|P_C(I - \lambda(aA + (1-a)B))x_n - z\| + \gamma_n \sum_{i=1}^{N} a_i\|T^i_{r_n}(x_n) - z\| \\
\leq& \alpha_n\|u - z\| + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\| \\
=& \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \qquad (3.2)
\end{aligned}$$

Putting $M = max\{\|u - z\|, \|x_1 - z\|\}$. From (3.2), we can show by induction that $\|x_n - z\| \leq M, \ \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u^i_n\}$ for all $i = 1, 2, ..., N$.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Putting $D = aA + (1-a)B, \ \forall a \in (0,1)$. From difinition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| =& \|\alpha_n u + \beta_n P_C(I - \lambda D)x_n + \gamma_n \sum_{i=1}^{N} a_i u^i_n - \alpha_{n-1}u - \beta_{n-1}P_C(I - \lambda D)x_{n-1} \\
&- \gamma_{n-1}\sum_{i=1}^{N} a_i u^i_{n-1}\| \\
\leq& |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}|\|\sum_{i=1}^{N} a_i u^i_{n-1}\| + \gamma_n \sum_{i=1}^{N} a_i\|u^i_n - u^i_{n-1}\|. \qquad (3.3)
\end{aligned}$$

Since $u^i_n = T^i_{r_n}x_n$. By difinition of $T^i_{r_n}$, we have

$$F_i(T^i_{r_n}x_n, v) + \frac{1}{r_n}\langle v - T^i_{r_n}x_n, T^i_{r_n}x_n - x_n\rangle \ \geq \ 0, \ \forall v \in C. \qquad (3.4)$$

Similarly

$$F_i(T^i_{r_{n+1}}x_{n+1}, v) + \frac{1}{r_{n+1}}\langle v - T^i_{r_{n+1}}x_{n+1}, T^i_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ \geq \ 0, \ \forall v \in C. \qquad (3.5)$$

From (3.4) and (3.5), we obtain

$$F_i(T^i_{r_n}x_n, T^i_{r_{n+1}}x_{n+1}) + \frac{1}{r_n}\langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_n}x_n - x_n\rangle \ \geq \ 0 \qquad (3.6)$$

and

$$F_i(T^i_{r_{n+1}}x_{n+1}, T^i_{r_n}x_n) + \frac{1}{r_{n+1}}\langle T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1}, T^i_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ \geq \ 0. \qquad (3.7)$$

By (3.6) and (3.7), we have

$$\frac{1}{r_n}\langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_n}x_n - x_n\rangle + \frac{1}{r_{n+1}}\langle T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1}, T^i_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ \geq \ 0$$

it follows that

$$\langle T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1}, \frac{T^i_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T^i_{r_n}x_n - x_n}{r_n}\rangle \ \geq \ 0.$$

This implies that

$$\langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1} + T^i_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle \geq 0. \quad (3.8)$$

It follows that

$$\begin{aligned}
\|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\|^2 \leq & \langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle \\
= & \langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle \\
\leq & \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\|\|x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\| \\
\leq & \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
& \times \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right) \\
= & \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
& \times \left(\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right) \\
\leq & \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
& \times \left(\|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right). \qquad (3.9)
\end{aligned}$$

It follows that

$$\|u^i_{n+1} - u^i_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|u^i_{n+1} - x_{n+1}\|. \qquad (3.10)$$

Substituting (3.10) into (3.3), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq & |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
& + |\gamma_n - \gamma_{n-1}|\Big\|\sum_{i=1}^N a_i u^i_{n-1}\Big\| + \gamma_n \sum_{i=1}^N a_i\|u^i_n - u^i_{n-1}\| \\
\leq & |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
& + |\gamma_n - \gamma_{n-1}|\Big\|\sum_{i=1}^N a_i u^i_{n-1}\Big\| \\
& + \gamma_n \sum_{i=1}^N a_i\left(\|x_n - x_{n-1}\| + \frac{1}{a}|r_n - r_{n-1}|\|u^i_n - x_n\|\right) \\
= & |\alpha_n - \alpha_{n-1}|\|u\| + (1 - \alpha_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
& + |\gamma_n - \gamma_{n-1}|\Big\|\sum_{i=1}^N a_i u^i_{n-1}\Big\| + \frac{\gamma_n}{a}|r_n - r_{n-1}|\sum_{i=1}^N a_i\|u^i_n - x_n\| \\
\leq & |\alpha_n - \alpha_{n-1}|M_1 + (1 - \alpha_n)\|x_n - x_{n-1}\| \\
& + |\beta_n - \beta_{n-1}|M_1 + |\gamma_n - \gamma_{n-1}|M_1 + \frac{\gamma_n}{a}|r_n - r_{n-1}|M_1, \qquad (3.11)
\end{aligned}$$

where $M_1 = \max_{n\in\mathbb{N}}\{\|u\|, \|x_n\|, \Big\|\sum_{i=1}^{N} a_i u_n^i\Big\|, \sum_{i=1}^{N} a_i\|u_n^i - x_n\|\}$.
By (3.11), Lemma 2.11, and conditions (i)-(vi), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Step 3. We will show that $\lim_{n\to\infty} \|u_n^i - x_n\| = 0$ and $\lim_{n\to\infty} \|P_C(I - \lambda D)x_n - x_n\| = 0$, where $D = aA + (1-a)B$, $\forall a \in (0,1)$. Since $u_n^i = T_{r_n}^i x_n$ and $T_{r_n}^i$ is a firmly nonexpansive mapping, we have

$$\begin{aligned}\|z - T_{r_n}^i x_n\|^2 &= \|T_{r_n}^i z - T_{r_n}^i x_n\|^2\\ &\le \langle T_{r_n}^i z - T_{r_n}^i x_n, z - x_n\rangle\\ &= \frac{1}{2}(\|T_{r_n}^i x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n}^i x_n - x_n\|^2).\end{aligned}$$

Hence

$$\|u_n^i - z\|^2 \le \|x_n - z\|^2 - \|u_n^i - x_n\|^2. \tag{3.13}$$

From Lemma 2.13, (3.13) and definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 =& \Big\|\alpha_n(u - z) + \beta_n\Big(P_C(I - \lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2\\ \le& \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big\|\sum_{i=1}^{N} a_i u_n^i - z\Big\|^2\\ \le& \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\sum_{i=1}^{N} a_i\|u_n^i - z\|^2\\ \le& \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\sum_{i=1}^{N} a_i(\|x_n - z\|^2 - \|u_n^i - x_n\|^2)\\ \le& \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \gamma_n\sum_{i=1}^{N} a_i\|u_n^i - x_n\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\gamma_n\sum_{i=1}^{N} a_i\|u_n^i - x_n\|^2 \le& \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2\\ \le& \alpha_n\|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)(\|x_n - z\| + \|x_{n+1} - z\|)\\ \le& \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|). \qquad (3.14)\end{aligned}$$

By (3.12) and condition (i), we have

$$\lim_{n\to\infty} \|u_n^i - x_n\| = 0, \text{ for all } i = 1, 2, ..., N. \tag{3.15}$$

From nonexpansiveness of P_C, we have

$$\begin{aligned}\|P_C(I - \lambda D)x_n - z\|^2 =& \|P_C(I - \lambda D)x_n - P_C(I - \lambda D)z\|^2\\ \le& \|(I - \lambda D)x_n - (I - \lambda D)z\|^2\\ =& \|x_n - z - \lambda(Dx_n - Dz)\|^2\\ \le& \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2. \qquad (3.16)\end{aligned}$$

For every $x, y \in C$, we have

$$\begin{aligned}
\langle Dx - Dy, x - y\rangle &= \langle aAx + (1-a)Bx - aAy - (1-a)By, x - y\rangle \\
&= \langle a(Ax - Ay) + (1-a)(Bx - By), x - y\rangle \\
&= a\langle Ax - Ay, x - y\rangle + (1-a)\langle Bx - By, x - y\rangle \\
&\geq a\alpha\|Ax - Ay\|^2 + (1-a)\beta\|Bx - By\|^2 \\
&\geq \eta(a\|Ax - Ay\|^2 + (1-a)\|Bx - By\|^2) \\
&\geq \eta\|Dx - Dy\|^2. \qquad (3.17)
\end{aligned}$$

Then D is η-inverse strongly monotone.
From (3.16) and (3.17), we have

$$\begin{aligned}
\|P_C(I - \lambda D)x_n - z\|^2 &\leq \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2 \\
&\leq \|x_n - z\|^2 - 2\lambda\eta\|Dx_n - Dz\|^2 + \lambda^2\|Dx_n - Dz\|^2 \\
&= \|x_n - z\|^2 - \lambda(2\eta - \lambda)\|Dx_n - Dz\|^2. \qquad (3.18)
\end{aligned}$$

By definition of x_n and (3.18), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \Big\|\alpha_n(u - z) + \beta_n\Big(P_C(I - \lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2 \\
&\leq \alpha_n\|u - z\|^2 + \beta_n\Big[\|x_n - z\|^2 - \lambda(2\eta - \lambda)\|Dx_n - Dz\|^2\Big] + \gamma_n\|x_n - z\|^2 \\
&\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \lambda\beta_n(2\eta - \lambda)\|Dx_n - Dz\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\lambda\beta_n(2\eta - \lambda)\|Dx_n - Dz\|^2 &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|). \qquad (3.19)
\end{aligned}$$

From (3.12), (3.19) and condition (i)

$$\lim_{n\to\infty}\|Dx_n - Dz\|^2 = 0. \qquad (3.20)$$

From definition of $P_C(I-\lambda D)$ and Lemma 2.13, we have

$$\begin{aligned}
\|P_C(I-\lambda D)x_n - z\|^2 =& \|P_C(I-\lambda D)x_n - P_C(I-\lambda D)z\|^2\\
\le& \langle (I-\lambda D)x_n - (I-\lambda D)z, P_C(I-\lambda D)x_n - z\rangle\\
=& \frac{1}{2}\Big(\|(I-\lambda D)x_n - (I-\lambda D)z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2\\
&- \|(I-\lambda D)x_n - (I-\lambda D)z - (P_C(I-\lambda D)x_n - z)\|^2\Big)\\
\le& \frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2\\
&- \|x_n - P_C(I-\lambda D)x_n - \lambda(Dx_n - Dz)\|^2\Big)\\
\le& \frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2\\
&- \|x_n - P_C(I-\lambda D)x_n\|^2 - \|\lambda(Dx_n - Dz)\|^2\\
&+ 2\lambda\langle x_n - P_C(I-\lambda D)x_n, Dx_n - Dz\rangle\Big)\\
\le& \frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2\\
&- \|x_n - P_C(I-\lambda D)x_n\|^2 + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|P_C(I-\lambda D)x_n - z\|^2 \le& \|x_n - z\|^2 - \|x_n - P_C(I-\lambda D)x_n\|^2\\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|. \qquad (3.21)
\end{aligned}$$

By definition of x_n, (3.21) and nonexpansiveness of $T^i_{r_n}$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 =& \Big\|\alpha_n(u-z) + \beta_n\Big(P_C(I-\lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2\\
\le& \alpha_n\|u-z\|^2 + \beta_n\Big\|P_C\Big(I-\lambda D\Big)x_n - z\Big\|^2 + \gamma_n\|x_n - z\|^2\\
\le& \alpha_n\|u-z\|^2 + \beta_n\Big[\|x_n - z\|^2 - \|x_n - P_C(I-\lambda D)x_n\|^2\\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|\Big] + \gamma_n\|x_n - z\|^2\\
\le& \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 - \beta_n\|x_n - P_C(I-\lambda D)x_n\|^2\\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\| + \gamma_n\|x_n - z\|^2\\
\le& \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \beta_n\|x_n - P_C(I-\lambda D)x_n\|^2\\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\beta_n\|x_n - P_C(I-\lambda D)x_n\|^2 \le& \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2\\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|\\
\le& \alpha_n\|u-z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|\\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|. \qquad (3.22)
\end{aligned}$$

From condition (i), (3.22), (3.12) and (3.20), we have

$$\lim_{n\to\infty} \|x_n - P_C(I-\lambda D)x_n\| = 0. \tag{3.23}$$

Step 4. We will show that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_{\mathbb{F}}u$.
To show this inequality, take a subsequence$\{x_{n_k}\}$ of $\{x_n\}$, such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle. \tag{3.24}$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$ where $\omega\in C$.
From (3.15), we have $u^i_{n_k} \rightharpoonup \omega$ as $k\to\infty$, for all $i = 1, 2, ..., N$.
Assume that $\omega \neq P_C(I-\lambda D)\omega$, where $D = aA + (1-a)B$.
By nonexpansiveness of $P_C(I-\lambda D)$, (3.23) and Opial's property, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k} - \omega\| &< \liminf_{k\to\infty}\|x_{n_k} - P_C(I-\lambda D)\omega\|\\ &\le \liminf_{k\to\infty}\Big(\|x_{n_k} - P_C(I-\lambda D)x_{n_k}\|\\ &\quad + \|P_C(I-\lambda D)x_{n_k} - P_C(I-\lambda D)\omega\|\Big)\\ &\le \liminf_{k\to\infty}\|x_{n_k} - \omega\|.\end{aligned}$$

This is a contradiction, then we have

$$\omega \in F\Big(P_C(I-\lambda D)\Big) = F\left(P_C\Big(I - \lambda(aA + (1-a)B)\Big)\right). \tag{3.25}$$

From Lemma 2.10 and Lemma 2.13, we have

$$F\left(P_C\Big(I - \lambda(aA + (1-a)B)\Big)\right) = VI(C,A)\bigcap VI(C,B). \tag{3.26}$$

From (3.25) and (3.26), we have

$$\omega \in VI(C,A)\bigcap VI(C,B).$$

Since

$$F_i(u^i_n, v) + \frac{1}{r_n}\langle v - u^i_n, u^i_n - x_n\rangle \ge 0,$$

for all $v\in C$ and $i = 1, 2, ..., N$.
By (A2), we have

$$\frac{1}{r_n}\langle v - u^i_n, u^i_n - x_n\rangle \ge F_i(v, u^i_n), \qquad \forall v\in C.$$

In particular

$$\left\langle v - u^i_{n_k}, \frac{1}{r_{n_k}}(u^i_{n_k} - x_{n_k})\right\rangle \ge F_i(v, u^i_{n_k}),$$

for all $v\in C$ and $i = 1, 2, ..., N$.
From (A4) and (3.15), we have

$$F_i(v,\omega) \le 0, \tag{3.27}$$

for all $v \in C$ and $i = 1, 2, ..., N$.

Let $u_t := tv + (1-t)\omega,\ \forall t \in (0,1]$, we have $u_t \in C$ and from (A1), (A4) and (3.27), we obtain

$$0 = F_i(u_t, u_t) \leq tF_i(u_t, v) + (1-t)F_i(u_t, \omega) \leq tF_i(u_t, v),$$

for all $i = 1, 2, ..., N$.

Hence $F_i(tv + (1-t)\omega, v) \geq 0,\ \forall t \in (0,1]$ and $\forall v \in C$.

Letting $t \to 0^+$ and using assumption (A3), we can conclude that

$$F_i(\omega, v) \geq 0, \qquad \forall v \in C \text{ and } i = 1, 2, ..., N.$$

Therefore, $\omega \in \bigcap_{i=1}^{N} EP(F_i)$. Hence $\omega \in \mathbb{F}$.

Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathbb{F}$, we have

$$\limsup_{n\to\infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k\to\infty} \langle u - z_0, x_{n_k} - z_0 \rangle = \langle u - z_0, \omega - z_0 \rangle \leq 0. \tag{3.28}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$ where $z_0 = P_{\mathbb{F}}u$.

By nonexpansive of $P_C\Big(I - \lambda(aA + (1-a)B)\Big)$, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 =& \Bigg\| \alpha_n(u - z_0) + \beta_n\bigg(P_C\Big(I - \lambda(aA + (1-a)B)\Big)x_n - z_0\bigg) \\
& + \gamma_n\bigg(\sum_{i=1}^{N} a_i u_n^i - z_0\bigg)\Bigg\|^2 \\
\leq& \Bigg\| \beta_n\bigg(P_C\Big(I - \lambda(aA + (1-a)B)\Big)x_n - z_0\bigg) + \gamma_n\bigg(\sum_{i=1}^{N} a_i u_n^i - z_0\bigg)\Bigg\|^2 \\
& + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq& \beta_n\Big\|P_C\Big(I - \lambda(aA + (1-a)B)\Big)x_n - z_0\Big\|^2 + \gamma_n\bigg\|\sum_{i=1}^{N} a_i u_n^i - z_0\bigg\|^2 \\
& + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq& \beta_n\|x_n - z_0\|^2 + \gamma_n\sum_{i=1}^{N} a_i\|u_n^i - z_0\|^2 \\
& + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq& \beta_n\|x_n - z_0\|^2 + \gamma_n\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
=& (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.
\end{aligned} \tag{3.29}$$

From (3.28) and Lemma 2.12, we obtain that $\{x_n\}$ converge strongly to $z_0 = P_{\mathbb{F}}u$.

This completes the proof of Theorem 3.1. □

Next, we give the numerical example to support Theorem 3.1.

Example 3.2. Let $\mathbb{R}$ be the set of real numbers. For every $i = 1, 2, ..., N$, let $F_i : [0, 100] \times [0, 100] \to \mathbb{R}$ defined by $F_i(x, y) = i(2x^2 + xy + y^2),\ \forall x, y \in \mathbb{R}$. Let $A, B : [0, 100] \to [0, 100]$

defined by $Ax = 3x$ and $Bx = \frac{6x}{7},\ \forall x \in \mathbb{R}$. Then $VI(C,A) \cap VI(C,B) \cap \bigcap_{i=1}^{N} EP(F_i) = \{0\}$. Let $u \in C$ and $\{x_n\}$,$\{u_n^i\}$ be the sequences generated by ((1)), for all $i = 1, 2, ..., N$. By the definition of F_i, we have

$$0 \leq F_i(u_n^i, v) + \frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle,$$

for all $i = 1, 2, ..., N$ and $n \in \mathbb{N}$.

Choose $r_n = 1$,

$$\begin{aligned}
0 &\leq F_i(u_n^i, v) + \langle v - u_n^i, u_n^i - x_n\rangle \\
&= i(-2{u_n^i}^2 + u_n^i v + v^2) + (v - u_n^i)(u_n^i - x_n) \\
&= i(-2{u_n^i}^2 + u_n^i v + v^2) + vu_n^i - vx_n - {u_n^i}^2 + u_n^i x_n \\
&= (-2i-1){u_n^i}^2 + (i+1)u_n^i v + iv^2 - vx_n + u_n^i x_n \\
&= iv^2 + ((i+1)u_n^i - x_n)v + (-2i-1){u_n^i}^2 + u_n^i x_n.
\end{aligned}$$

Let $G(v) = iv^2 + ((i+1)u_n^i - x_n)v + (-2i-1){u_n^i}^2 + u_n^i x_n$.

$G(v)$ is a quadratic function of u with coefficient $a = i,\ b = (i+1)u_n^i - x_n$, $c = (-2i-1){u_n^i}^2 + u_n^i x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= ((i+1)u_n^i - x_n)^2 - 4(i)((-2i-1){u_n^i}^2 + u_n^i x_n) \\
&= (i+1)^2{u_n^i}^2 - 2(i+1)u_n^i x_n + {x_n}^2 - 4i(-2i-1){u_n^i}^2 - 4iu_n^i x_n \\
&= ((i+1)^2 - 4i(-2i-1)){u_n^i}^2 - 2(3i+1)u_n^i x_n + {x_n}^2 \\
&= (9i^2 + 6i + 1){u_n^i}^2 - 2(3i+1)u_n^i x_n + {x_n}^2 \\
&= ((3i+1)u_n^i - x_n)^2,
\end{aligned}$$

we know that $G(v) \geq 0,\ \forall v \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we obtain

$$u_n^i = \frac{x_n}{3i+1}, \tag{3.30}$$

for all $i = 1, 2, ..., N$.

Put $\alpha_n = \frac{1}{3n}, \beta_n = \frac{2n+1}{9n}, \gamma_n = \frac{14n-8}{18n}, \lambda = \frac{1}{3}, a = \frac{1}{2}$.

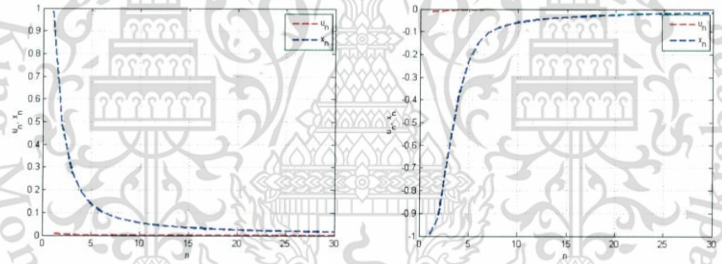
From (3.30) we rewrite ((1)) as follows:

$$x_{n+1} = \frac{1}{3n}u + \frac{2n+1}{9n}P_{[0,100]}\Big(I - \frac{1}{3}(\frac{1}{2}A + (1-\frac{1}{2})B)\Big)x_n + \frac{14n-8}{18n}\sum_{i=1}^{N}\Big(\frac{1}{3^i} + \frac{1}{N3^N}\Big)\frac{x_n}{3i+1}. \tag{3.31}$$

It is clear that the sequences $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ satisfy all the conditions of theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{u_n^i\}$ converge strongly to 0. The table 3.1 shows the values of sequences $\{u_n^i\}$ and $\{x_n\}$, where $u = x_1 = 1$ and $u = x_1 = -1$.

Table 3.1: The values of $\{u_n^i\}$ and $\{x_n\}$ with $n = N = 30$

n	$u = x_1 = 1$		$u = x_1 = -1$	
	u_n^i	x_n	u_n^i	x_n
1	0.010989	1.000000	-0.010989	-1.000000
2	0.005353	0.487120	-0.010063	-0.915692
3	0.003163	0.287809	-0.006855	-0.623774
4	0.002097	0.190808	-0.004161	-0.378611
5	0.001521	0.138441	-0.002498	-0.227329
⋮	⋮	⋮	⋮	⋮
15	0.000388	0.035350	-0.000396	-0.036052
⋮	⋮	⋮	⋮	⋮
26	0.000214	0.019487	-0.000216	-0.019670
27	0.000206	0.018724	-0.000208	-0.018891
28	0.000198	0.018018	-0.000200	-0.018172
29	0.000191	0.017364	-0.000192	-0.017506
30	0.000184	0.016755	-0.000186	-0.016887



(a) $u = x_1 = 1$. (b) $u = x_1 = -1$.

Figure 3.1: The convergence comparison with different initial values u and x_1

Next, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problems and fixed point problems of finite family of nonexpansive mappings and strictly pseudo-contractive mappings.

Theorem 3.3. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings and let $\rho_i = (\alpha_1^i, \alpha_2^i, \alpha_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^i + \alpha_2^i + \alpha_3^i = 1$, $\alpha_1^i + \alpha_2^i \leq e < 1$, and $\alpha_1^i, \alpha_2^i, \alpha_3^i \in (0, 1)$ for all $i = 1, 2, ..., N$ and let $\{P_i\}_{i=1}^N$ be κ_i-strict pseudo-contractive mappings of C into itself with $\kappa = \sup_{i=1,2,...,N} \kappa_i$ and let $\overline{\rho}_i = (\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i + \overline{\alpha}_3^i = 1$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i \leq f < 1$, and $\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i \in (\kappa, 1)$ for all $i = 1, 2, ..., N$. Let S_A be S-mapping generated by $T_N, ..., T_1$ and $\rho_N, \rho_{N-1}, ..., \rho_1$ and let S_B be S-mapping generated by $P_N, ..., P_1$ and $\overline{\rho}_N, \overline{\rho}_{N-1}, ..., \overline{\rho}_1$.

Assume that $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(P_i) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$ and let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a(I - S_A) + (1-a)(I - S_B)))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases} \tag{3.32}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 1)$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

Proof. Since S_A and S_B are nonexpansive mapping, then $I - S_A$, $I - S_B$ are $\frac{1}{2}$-inverse strongly monotone.

First, put $D = a(I - S_A) + (1 - a)(I - S_B), \ \forall a \in (0, 1)$. we show that D is $\frac{1}{2}$-inverse strongly monotone. Let $x, y \in C$, we have

$$\begin{aligned} \langle Dx - Dy, x - y\rangle &= \langle a(I - S_A)x + (1 - a)(I - S_B)x - a(I - S_A)y - (1 - a)(I - S_B)y, x - y\rangle \\ &= \langle a((I - S_A)x - (I - S_A)y) + (1 - a)((I - S_B)x - (I - S_B)y), x - y\rangle \\ &= a\langle (I - S_A)x - (I - S_A)y, x - y\rangle + (1 - a)\langle (I - S_B)x - (I - S_B)y, x - y\rangle \\ &\geq a\frac{1}{2}\|(I - S_A)x - (I - S_A)y\|^2 + (1 - a)\frac{1}{2}\|(I - S_B)x - (I - S_B)y\|^2 \\ &\geq \frac{1}{2}(a\|(I - S_A)x - (I - S_A)y\|^2 + (1 - a)\|(I - S_B)x - (I - S_B)y\|^2) \\ &\geq \frac{1}{2}\|Dx - Dy\|^2. \end{aligned} \tag{3.33}$$

Then D is $\frac{1}{2}$-inverse strongly monotone.

Let $x, y \in C$. Since D is $\frac{1}{2}$-inverse strongly monotone, we have

$$\begin{aligned}\|(I-\lambda D)x-(I-\lambda D)y\|^2 &= \|x-y-\lambda(Dx-Dy)\|^2\\ &= \|x-y\|^2-2\lambda\langle x-y, Dx-Dy\rangle+\lambda^2\|Dx-Dy\|^2\\ &\le \|x-y\|^2-\lambda\|Dx-Dy\|^2+\lambda^2\|Dx-Dy\|^2\\ &= \|x-y\|^2+\lambda(\lambda-1)\|Dx-Dy\|^2\\ &\le \|x-y]\|^2. \end{aligned} \tag{3.34}$$

Thus, $(I-\lambda D)$ is nonexpansive. We will divide our prove into 5 steps.

Step 1. We will show that the sequence $\{x_n\}$ is bounded. Since

$$F_i(u_n^i, v)+\frac{1}{r_n}\langle v-u_n^i, u_n^i-x_n\rangle \ge 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N.$$

By Lemma 2.22, we have $u_n^i = T_{r_n}^i(x_n)$ and $EP(F_i) = F(T_{r_n}^i)$, for all $i = 1, 2, ..., N$.

Let $z \in \mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(P_i) \cap \bigcap_{i=1}^{N} EP(F_i)$.

Since P_C is a nonexpansive mapping, we have $P_C(I-\lambda D)$ is a nonexpansive mapping.

By Lemma 2.13 and Lemma 2.10, we have

$$\begin{aligned} z \in VI\Big(C, a(I-S_A)+(1-a)(I-S_B)\Big) &= VI(C, I-S_A) \cap VI(C, I-S_B)\\ &= F(S_A) \cap F(S_B)\\ &= \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(P_i). \end{aligned}$$

From Lemma 2.13 and nonexpansiveness of $T_{r_n}^i$, we have

$$\begin{aligned} \|x_{n+1}-z\| =& \|\alpha_n u+\beta_n P_C(I-\lambda(a(I-S_A)+(1-a)(I-S_B)))x_n+\gamma_n\sum_{i=1}^{N} a_i u_n^i - z\|\\ \le& \alpha_n\|u-z\|+\beta_n\|P_C(I-\lambda(a(I-S_A)+(1-a)(I-S_B)))x_n-z\|\\ &+\gamma_n\|\sum_{i=1}^{N} a_i(u_n^i-z)\|\\ \le& \alpha_n\|u-z\|+\beta_n\|P_C(I-\lambda(a(I-S_A)+(1-a)(I-S_B)))x_n-z\|\\ &+\gamma_n\sum_{i=1}^{N} a_i\|T_{r_n}^i(x_n)-z\|\\ \le& \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\gamma_n\|x_n-z\|\\ =& \alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|. \end{aligned} \tag{3.35}$$

Putting $M = max\{\|u-z\|, \|x_1-z\|\}$. From (3.35), we can show by induction that $\|x_n-z\| \le M, \ \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n^i\}$ for all $i = 1, 2, ..., N$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

From difinition of x_n, we have

$$\begin{aligned}\|x_{n+1}-x_n\| =& \|\alpha_n u+\beta_n P_C(I-\lambda D)x_n+\gamma_n\sum_{i=1}^{N}a_iu_n^i-\alpha_{n-1}u-\beta_{n-1}P_C(I-\lambda D)x_{n-1}\\ &-\gamma_{n-1}\sum_{i=1}^{N}a_iu_{n-1}^i\|\\ \le& |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)x_{n-1}\|\\ &+|\gamma_n-\gamma_{n-1}|\|\sum_{i=1}^{N}a_iu_{n-1}^i\|+\gamma_n\sum_{i=1}^{N}a_i\|u_n^i-u_{n-1}^i\|. \qquad (3.36)\end{aligned}$$

Since $u_n^i=T_{r_n}^ix_n$ and difinition of $T_{r_n}^i$, we have

$$F_i(T_{r_n}^ix_n,v)+\frac{1}{r_n}\langle v-T_{r_n}^ix_n,T_{r_n}^ix_n-x_n\rangle \ \ge\ 0,\ \forall v\in C, \qquad (3.37)$$

Similarly

$$F_i(T_{r_{n+1}}^ix_{n+1},v)+\frac{1}{r_{n+1}}\langle v-T_{r_{n+1}}^ix_{n+1},T_{r_{n+1}}^ix_{n+1}-x_{n+1}\rangle \ \ge\ 0,\ \forall v\in C. \qquad (3.38)$$

From (3.37) and (3.38), we obtain

$$F_i(T_{r_n}^ix_n,T_{r_{n+1}}^ix_{n+1})+\frac{1}{r_n}\langle T_{r_{n+1}}^ix_{n+1}-T_{r_n}^ix_n,T_{r_n}^ix_n-x_n\rangle \ \ge\ 0 \qquad (3.39)$$

and

$$F_i(T_{r_{n+1}}^ix_{n+1},T_{r_n}^ix_n)+\frac{1}{r_{n+1}}\langle T_{r_n}^ix_n-T_{r_{n+1}}^ix_{n+1},T_{r_{n+1}}^ix_{n+1}-x_{n+1}\rangle \ \ge\ 0. \qquad (3.40)$$

By (3.39) and (3.40), we have

$$\frac{1}{r_n}\langle T_{r_{n+1}}^ix_{n+1}-T_{r_n}^ix_n,T_{r_n}^ix_n-x_n\rangle+\frac{1}{r_{n+1}}\langle T_{r_n}^ix_n-T_{r_{n+1}}^ix_{n+1},T_{r_{n+1}}^ix_{n+1}-x_{n+1}\rangle \ \ge\ 0.$$

It follows that

$$\langle T_{r_n}^ix_n-T_{r_{n+1}}^ix_{n+1},\frac{T_{r_{n+1}}^ix_{n+1}-x_{n+1}}{r_{n+1}}-\frac{T_{r_n}^ix_n-x_n}{r_n}\rangle \ \ge\ 0.$$

This implies that

$$0 \ \le\ \langle T_{r_{n+1}}^ix_{n+1}-T_{r_n}^ix_n,T_{r_n}^ix_n-T_{r_{n+1}}^ix_{n+1}+T_{r_{n+1}}^ix_{n+1}-x_n-\frac{r_n}{r_{n+1}}(T_{r_{n+1}}^ix_{n+1}-x_{n+1})\rangle. \qquad (3.41)$$

It follows that

$$
\begin{aligned}
\|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\|^2 \leq& \langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle \\
=& \langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, x_{n+1} - x_n + \Big(1 - \frac{r_n}{r_{n+1}}\Big)(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle \\
\leq& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\|\|x_{n+1} - x_n + \Big(1 - \frac{r_n}{r_{n+1}}\Big)(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\| \\
\leq& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
&\times \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right) \\
=& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
&\times \left(\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right) \\
\leq& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
&\times \left(\|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right). \qquad (3.42)
\end{aligned}
$$

It implies that

$$
\|u^i_{n+1} - u^i_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|u^i_{n+1} - x_{n+1}\|. \qquad (3.43)
$$

Substituting (3.43) into (3.36), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq& |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}|\Big\|\sum_{i=1}^N a_i u^i_{n-1}\Big\| + \gamma_n\sum_{i=1}^N a_i\|u^i_n - u^i_{n-1}\| \\
\leq& |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}|\Big\|\sum_{i=1}^N a_i u^i_{n-1}\Big\| \\
&+ \gamma_n\sum_{i=1}^N a_i\left(\|x_n - x_{n-1}\| + \frac{1}{a}|r_n - r_{n-1}|\|u^i_n - x_n\|\right) \\
=& |\alpha_n - \alpha_{n-1}|\|u\| + (1 - \alpha_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}|\Big\|\sum_{i=1}^N a_i u^i_{n-1}\Big\| \\
&+ \frac{\gamma_n}{a}|r_n - r_{n-1}|\sum_{i=1}^N a_i\|u^i_n - x_n\| \\
\leq& |\alpha_n - \alpha_{n-1}|M_1 + (1 - \alpha_n)\|x_n - x_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}|M_1 + |\gamma_n - \gamma_{n-1}|M_1 + \frac{\gamma_n}{a}|r_n - r_{n-1}|M_1, \qquad (3.44)
\end{aligned}
$$

where $M_1 = \max_{n\in\mathbb{N}}\{\|u\|, \|x_n\|, \Big\|\sum_{i=1}^N a_i u^i_n\Big\|, \sum_{i=1}^N a_i\|u^i_n - x_n\|\}$.

By (3.44), Lemma 2.11, and conditions (i)-(vi), we obtain

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (3.45)
$$

Step 3. We will show that $\lim_{n\to\infty}\|u_n^i - x_n\| = 0$ and $\lim_{n\to\infty}\|P_C(I-\lambda D)x_n - x_n\| = 0$, where $D = a(I-S_A)+(1-a)(I-S_B)$, $\forall a\in(0,1)$. Since $u_n^i = T_{r_n}^i x_n$ and $T_{r_n}^i$ is a firmly nonexpansive mapping, we have

$$\begin{aligned}\|z - T_{r_n}^i x_n\|^2 &= \|T_{r_n}^i z - T_{r_n}^i x_n\|^2\\ &\le \langle T_{r_n}^i z - T_{r_n}^i x_n, z - x_n\rangle\\ &= \frac{1}{2}(\|T_{r_n}^i x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n}^i x_n - x_n\|^2).\end{aligned}$$

Hence

$$\|u_n^i - z\|^2 \le \|x_n - z\|^2 - \|u_n^i - x_n\|^2. \tag{3.46}$$

From Lemma 2.13, (3.46) and definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \Big\|\alpha_n(u-z) + \beta_n\Big(P_C(I-\lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^N a_i u_n^i - z\Big)\Big\|^2\\ &\le \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big\|\sum_{i=1}^N a_i u_n^i - z\Big\|^2\\ &\le \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\sum_{i=1}^N a_i\|u_n^i - z\|^2\\ &\le \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\sum_{i=1}^N a_i(\|x_n - z\|^2 - \|u_n^i - x_n\|^2)\\ &\le \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \gamma_n\sum_{i=1}^N a_i\|u_n^i - x_n\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\gamma_n\sum_{i=1}^N a_i\|u_n^i - x_n\|^2 &\le \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2\\ &\le \alpha_n\|u-z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)(\|x_n - z\| + \|x_{n+1} - z\|)\\ &\le \alpha_n\|u-z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|).\end{aligned} \tag{3.47}$$

By (3.45) and condition (i), we have

$$\lim_{n\to\infty}\|u_n^i - x_n\| = 0, \text{ for all } i = 1,2,...,N. \tag{3.48}$$

From nonexpansiveness of P_C, we have

$$\begin{aligned}\|P_C(I-\lambda D)x_n - z\|^2 &= \|P_C(I-\lambda D)x_n - P_C(I-\lambda D)z\|^2\\ &\le \|(I-\lambda D)x_n - (I-\lambda D)z\|^2\\ &= \|x_n - z - \lambda(Dx_n - Dz)\|^2\\ &\le \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2.\end{aligned} \tag{3.49}$$

From (3.49) and (3.33), we have

$$\begin{aligned}\|P_C(I-\lambda D)x_n - z\|^2 &\le \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2\\ &\le \|x_n - z\|^2 - \lambda\|Dx_n - Dz\|^2 + \lambda^2\|Dx_n - Dz\|^2\\ &= \|x_n - z\|^2 - \lambda(1-\lambda)\|Dx_n - Dz\|^2.\end{aligned} \tag{3.50}$$

By definition of x_n and (3.50), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \Big\|\alpha_n(u-z)+\beta_n\Big(P_C(I-\lambda D)x_n - z\Big)+\gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2\\ &\le \alpha_n\|u-z\|^2+\beta_n\Big[\|x_n-z\|^2-\lambda(1-\lambda)\|Dx_n-Dz\|^2\Big]+\gamma_n\|x_n-z\|^2\\ &\le \alpha_n\|u-z\|^2+\|x_n-z\|^2-\lambda\beta_n(1-\lambda)\|Dx_n-Dz\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}\lambda\beta_n(1-\lambda)\|Dx_n-Dz\|^2 &\le \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\ &\le \alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|). \qquad (3.51)\end{aligned}$$

From (3.45), (3.51) and condition (i)

$$\lim_{n\to\infty}\|Dx_n-Dz\|^2=0. \qquad (3.52)$$

From definition of $P_C(I-\lambda D)$ and Lemma 2.13, we have

$$\begin{aligned}\|P_C(I-\lambda D)x_n-z\|^2 =&\|P_C(I-\lambda D)x_n-P_C(I-\lambda D)z\|^2\\ \le&\langle (I-\lambda D)x_n-(I-\lambda D)z, P_C(I-\lambda D)x_n-z\rangle\\ =&\frac{1}{2}\Big(\|(I-\lambda D)x_n-(I-\lambda D)z\|^2+\|P_C(I-\lambda D)x_n-z\|^2\\ &-\|(I-\lambda D)x_n-(I-\lambda D)z-(P_C(I-\lambda D)x_n-z)\|^2\Big)\\ \le&\frac{1}{2}\Big(\|x_n-z\|^2+\|P_C(I-\lambda D)x_n-z\|^2\\ &-\|x_n-P_C(I-\lambda D)x_n-\lambda(Dx_n-Dz)\|^2\Big)\\ \le&\frac{1}{2}\Big(\|x_n-z\|^2+\|P_C(I-\lambda D)x_n-z\|^2\\ &-\|x_n-P_C(I-\lambda D)x_n\|^2-\|\lambda(Dx_n-Dz)\|^2\\ &+2\lambda\langle x_n-P_C(I-\lambda D)x_n, Dx_n-Dz\rangle\Big)\\ \le&\frac{1}{2}\Big(\|x_n-z\|^2+\|P_C(I-\lambda D)x_n-z\|^2\\ &-\|x_n-P_C(I-\lambda D)x_n\|^2+2\lambda\|x_n-P_C(I-\lambda D)x_n\|\|Dx_n-Dz\|\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\|P_C(I-\lambda D)x_n-z\|^2 \le& \|x_n-z\|^2-\|x_n-P_C(I-\lambda D)x_n\|^2\\ &+2\lambda\|x_n-P_C(I-\lambda D)x_n\|\|Dx_n-Dz\|. \qquad (3.53)\end{aligned}$$

By definition of x_n, (3.53) and nonexpansiveness of $T^i_{r_n}$, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 =& \Big\|\alpha_n(u-z)+\beta_n\Big(P_C(I-\lambda D)x_n-z\Big)+\gamma_n\Big(\sum_{i=1}^{N}a_iu_n^i-z\Big)\Big\|^2\\ \le& \alpha_n\|u-z\|^2+\beta_n\Big\|P_C\Big(I-\lambda D\Big)x_n-z\Big\|^2+\gamma_n\|x_n-z\|^2\\ \le& \alpha_n\|u-z\|^2+\beta_n\Big[\|x_n-z\|^2-\|x_n-P_C(I-\lambda D)x_n\|^2\\ &+2\lambda\|x_n-P_C(I-\lambda D)x_n\|\|Dx_n-Dz\|\Big]+\gamma_n\|x_n-z\|^2\\ \le& \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2-\beta_n\|x_n-P_C(I-\lambda D)x_n\|^2\\ &+2\lambda\|x_n-P_C(I-\lambda D)x_n\|\|Dx_n-Dz\|+\gamma_n\|x_n-z\|^2\\ \le& \alpha_n\|u-z\|^2+\|x_n-z\|^2-\beta_n\|x_n-P_C(I-\lambda D)x_n\|^2\\ &+2\lambda\|x_n-P_C(I-\lambda D)x_n\|\|Dx_n-Dz\|.\end{aligned}$$

It follows that

$$\begin{aligned}\beta_n\|x_n-P_C(I-\lambda D)x_n\|^2 \le& \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\ &+2\lambda\|x_n-P_C(I-\lambda D)x_n\|\|Dx_n-Dz\|\\ \le& \alpha_n\|u-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|\\ &+2\lambda\|x_n-P_C(I-\lambda D)x_n\|\|Dx_n-Dz\|. \qquad (3.54)\end{aligned}$$

From condition (i), (3.54), (3.45) and (3.52), we have

$$\lim_{n\to\infty}\|x_n-P_C(I-\lambda D)x_n\|=0. \qquad (3.55)$$

Step 4. We will show that $\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle\le 0$, where $z_0=P_{\mathbb{F}}u$.
To show this inequality, take a subsequence$\{x_{n_k}\}$ of $\{x_n\}$, such that

$$\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle=\lim_{k\to\infty}\langle u-z_0,x_{n_k}-z_0\rangle. \qquad (3.56)$$

Without loss of generality, we may assume that $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$ where $\omega\in C$.
From (3.48), we have $u^i_{n_k}\rightharpoonup\omega$ as $k\to\infty$, for all $i=1,2,...,N$.
Assume that $\omega\ne P_C(I-\lambda D)\omega$, where $D=a(I-S_A)+(1-a)(I-S_B)$.
By nonexpansiveness of $P_C(I-\lambda D)$, (3.55) and Opial's property, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-\omega\| <& \liminf_{k\to\infty}\|x_{n_k}-P_C(I-\lambda D)\omega\|\\ \le& \liminf_{k\to\infty}\Big(\|x_{n_k}-P_C(I-\lambda D)x_{n_k}\|+\|P_C(I-\lambda D)x_{n_k}-P_C(I-\lambda D)\omega\|\Big)\\ \le& \liminf_{k\to\infty}\|x_{n_k}-\omega\|.\end{aligned}$$

This is a contradiction, then we have

$$\omega\in F\Big(P_C(I-\lambda D)\Big)=F\Bigg(P_C\Big(I-\lambda(a(I-S_A)+(1-a)(I-S_B))\Big)\Bigg). \qquad (3.57)$$

From Lemma 2.10 and Lemma 2.13, we have

$$F\left(P_C\Big(I-\lambda(a(I-S_A)+(1-a)(I-S_B))\Big)\right) = VI(C,(I-S_A))\bigcap VI(C,(I-S_B))$$
$$= \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(P_i). \qquad (3.58)$$

From (3.57) and (3.58), we have

$$\omega \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(P_i).$$

Since

$$F_i(u_n^i, v) + \frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0,$$

for all $v \in C$ and $i = 1, 2, ..., N$.

By (A2), we have

$$\frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq F_i(v, u_n^i), \qquad \forall v \in C.$$

In particular

$$\left\langle v - u_{n_k}^i, \frac{1}{r_{n_k}}(u_{n_k}^i - x_{n_k})\right\rangle \geq F_i(v, u_{n_k}^i),$$

for all $v \in C$ and $i = 1, 2, ..., N$.

From (A4) and (3.48), we have

$$F_i(v, \omega) \leq 0, \qquad (3.59)$$

for all $u \in C$ and $i = 1, 2, ..., N$.

Let $u_t := tv + (1-t)\omega,\ \forall t \in (0, 1]$, we have $u_t \in C$ and from (A1),(A4) and (3.27), we obtain

$$0 = F_i(u_t, u_t) \leq tF_i(u_t, v) + (1-t)F_i(u_t, \omega) \leq tF_i(u_t, v),$$

for all $i = 1, 2, ..., N$.

Hence $F_i(tv + (1-t)\omega, v) \geq 0, \forall t \in (0, 1]$ and $\forall v \in C$.

Letting $t \to 0^+$ and using assumption (A3), we can conclude that

$$F_i(\omega, v) \geq 0, \qquad \forall v \in C \text{ and } i = 1, 2, ..., N.$$

Therefore, $\omega \in \bigcap_{i=1}^{N} EP(F_i)$. Hence $\omega \in \mathbb{F}$.

Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathbb{F}$, we have

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \qquad (3.60)$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$ where $z_0 = P_{\mathbb{F}}u$.

By nonexpansive of $P_C\Big(I-\lambda(a(I-S_A)+(1-a)(I-S_B))\Big)$, we have

$$\begin{aligned}\|x_{n+1}-z_0\|^2 =&\left\|\alpha_n(u-z_0)+\beta_n\left(P_C\Big(I-\lambda(a(I-S_A)+(1-a)(I-S_B))\Big)x_n-z_0\right)\right.\\&\left.+\gamma_n\left(\sum_{i=1}^{N}a_iu_n^i-z_0\right)\right\|^2\\ \leq&\left\|\beta_n\left(P_C\Big(I-\lambda(a(I-S_A)+(1-a)(I-S_B))\Big)x_n-z_0\right)\right.\\&\left.+\gamma_n\left(\sum_{i=1}^{N}a_iu_n^i-z_0\right)\right\|^2+2\alpha_n\langle u-z_0,x_{n+1}-z_0\rangle\\ \leq&\beta_n\left\|P_C\Big(I-\lambda(a(I-S_A)+(1-a)(I-S_B))\Big)x_n-z_0\right\|^2\\&+\gamma_n\left\|\sum_{i=1}^{N}a_iu_n^i-z_0\right\|^2+2\alpha_n\langle u-z_0,x_{n+1}-z_0\rangle\\ \leq&(1-\alpha_n)\|x_n-z_0\|^2+2\alpha_n\langle u-z_0,x_{n+1}-z_0\rangle. \end{aligned} \tag{3.61}$$

From(3.60) and Lemma 2.12, we have $\{x_n\}$ converge strongly to $z_0=P_{\mathbb{F}}u$.
This completes the proof of Theorem 3.3. ☐

Next, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problems and the set of fixed point problems of nonexpansive mappings.

Corollary 3.4. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i=1,2,...,n$, let $F_i:C\times C\to\mathbb{R}$ be a bifunction satisfying $(A_1)-(A_4)$. Let T,S be nonexpansive mapping of C into itself, with $\mathbb{F}=F(T)\cap F(S)\cap\bigcap_{i=1}^{N}EP(F_i)\neq\emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u,x_1\in C$ and

$$\begin{cases}F_i(u_n^i,v)+\dfrac{1}{r_n}\langle v-u_n^i,u_n^i-x_n\rangle\geq 0, \text{ for all } v\in C \text{ and } i=1,2,...,N,\\ x_{n+1}=\alpha_nu+\beta_nP_C(I-\lambda(a(I-T)+(1-a)(I-S)))x_n+\gamma_n\displaystyle\sum_{i=1}^{N}a_iu_n^i, \text{ for all } n\geq 1,\end{cases} \tag{3.62}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}\subseteq[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$ for all $n\in\mathbb{N}$ and $a\in(0,1)$.
Suppose that the following conditions hold :

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $\beta_n,\gamma_n\in[c,d]\subset(0,1),\ \forall n\in\mathbb{N}$,

(iii) $\sum_{i=1}^{N}a_i=1$, where $a_i>0$ for all $i=1,2,...,N$,

(iv) $0<a<r_n<b$ for all $n\in\mathbb{N}$,

(v) $\lambda\in(0,1)$

(vi) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

Proof. We will divide our prove into 5 steps.

Step 1. We will show that the sequence $\{x_n\}$ is bounded. Since

$$F_i(u_n^i, v) + \frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N.$$

By Lemma 2.22, we have $u_n^i = T_{r_n}^i(x_n)$ and $EP(F_i) = F(T_{r_n}^i)$, for all $i = 1, 2, ..., N$.

Let $z \in \mathbb{F} = F(T) \cap F(S) \cap \bigcap_{i=1}^{N} EP(F_i)$.

By Lemma 2.13 and Lemma 2.10, we have

$$z \in VI\Big(C, a(I-T) + (1-a)(I-S)\Big) = F\Big(P_C(I - \lambda(a(I-T) + (1-a)(I-S)))\Big).$$

From Lemma 2.13 and nonexpansiveness of $T_{r_n}^i$, we have

$$\begin{aligned}
\|x_{n+1} - z\| =& \|\alpha_n u + \beta_n P_C(I - \lambda(a(I-T) + (1-a)(I-S)))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i - z\| \\
\leq& \alpha_n\|u - z\| + \beta_n\|P_C(I - \lambda(a(I-T) + (1-a)(I-S)))x_n - z\| \\
&+ \gamma_n\|\sum_{i=1}^{N} a_i(u_n^i - z)\| \\
\leq& \alpha_n\|u - z\| + \beta_n\|P_C(I - \lambda(a(I-T) + (1-a)(I-S)))x_n - z\| \\
&+ \gamma_n\sum_{i=1}^{N} a_i\|T_{r_n}^i(x_n) - z\| \\
\leq& \alpha_n\|u - z\| + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\| \\
=& \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \qquad (3.63)
\end{aligned}$$

Putting $M = max\{\|u - z\|, \|x_1 - z\|\}$. From (3.63), we can show by induction that $\|x_n - z\| \leq M, \ \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n^i\}$ for all $i = 1, 2, ..., N$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Putting $D = a(I-T) + (1-a)(I-S), \ \forall a \in (0,1)$. From difinition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| =& \|\alpha_n u + \beta_n P_C(I - \lambda D)x_n + \gamma_n\sum_{i=1}^{N} a_i u_n^i - \alpha_{n-1}u - \beta_{n-1}P_C(I - \lambda D)x_{n-1} \\
&- \gamma_{n-1}\sum_{i=1}^{N} a_i u_{n-1}^i\| \\
\leq& |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda D)x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}|\|\sum_{i=1}^{N} a_i u_{n-1}^i\| + \gamma_n\sum_{i=1}^{N} a_i\|u_n^i - u_{n-1}^i\|. \qquad (3.64)
\end{aligned}$$

Since $u_n^i = T_{r_n}^i x_n$. and difinition of $T_{r_n}^i$, we have

$$F_i(T_{r_n}^i x_n, v) + \frac{1}{r_n}\langle v - T_{r_n}^i x_n, T_{r_n}^i x_n - x_n\rangle \ \geq \ 0, \ \forall v \in C \text{ and } i = 1, 2, ..., N. \qquad (3.65)$$

Similarly

$$F_i(T^i_{r_{n+1}}x_{n+1}, v) + \frac{1}{r_{n+1}}\langle v - T^i_{r_{n+1}}x_{n+1}, T^i_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ \geq\ 0,\ \forall v \in C \text{ and } i = 1, 2, ..., N. \tag{3.66}$$

From (3.65) and (3.66), we obtain

$$F_i(T^i_{r_n}x_n, T^i_{r_{n+1}}x_{n+1}) + \frac{1}{r_n}\langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_n}x_n - x_n\rangle \ \geq\ 0 \tag{3.67}$$

and

$$F_i(T^i_{r_{n+1}}x_{n+1}, T^i_{r_n}x_n) + \frac{1}{r_{n+1}}\langle T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1}, T^i_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ \geq\ 0. \tag{3.68}$$

By (3.67) and (3.68), we have

$$\frac{1}{r_n}\langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_n}x_n - x_n\rangle + \frac{1}{r_{n+1}}\langle T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1}, T^i_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ \geq\ 0.$$

It follows that

$$\langle T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1}, \frac{T^i_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T^i_{r_n}x_n - x_n}{r_n}\rangle \ \geq\ 0.$$

This implies that

$$0 \ \leq\ \langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_n}x_n - T^i_{r_{n+1}}x_{n+1} + T^i_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle. \tag{3.69}$$

It follows that

$$\begin{aligned}
\|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\|^2 \leq& \langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, T^i_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle \\
=& \langle T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\rangle \\
\leq& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\|\|x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T^i_{r_{n+1}}x_{n+1} - x_{n+1})\| \\
\leq& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
& \times \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right) \\
=& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
& \times \left(\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right) \\
\leq& \|T^i_{r_{n+1}}x_{n+1} - T^i_{r_n}x_n\| \\
& \times \left(\|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|T^i_{r_{n+1}}x_{n+1} - x_{n+1}\|\right).
\end{aligned} \tag{3.70}$$

It follows that

$$\|u^i_{n+1} - u^i_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|u^i_{n+1} - x_{n+1}\|. \tag{3.71}$$

Substituting (3.71) into (3.64), we have

$$\begin{aligned}
\|x_{n+1-x_n}\| \leq& |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)x_{n-1}\| \\
&+|\gamma_n-\gamma_{n-1}|\Big\|\sum_{i=1}^{N}a_iu_{n-1}^i\Big\|+\gamma_n\sum_{i=1}^{N}a_i\|u_n^i-u_{n-1}^i\| \\
\leq& |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)x_{n-1}\| \\
&+|\gamma_n-\gamma_{n-1}|\Big\|\sum_{i=1}^{N}a_iu_{n-1}^i\Big\|+\gamma_n\sum_{i=1}^{N}a_i\left(\|x_n-x_{n-1}\|+\frac{1}{a}|r_n-r_{n-1}|\|u_n^i-x_n\|\right) \\
=& |\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)x_{n-1}\| \\
&+|\gamma_n-\gamma_{n-1}|\Big\|\sum_{i=1}^{N}a_iu_{n-1}^i\Big\|+\frac{\gamma_n}{a}|r_n-r_{n-1}|\sum_{i=1}^{N}a_i\|u_n^i-x_n\| \\
\leq& |\alpha_n-\alpha_{n-1}|M_1+(1-\alpha_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|M_1 \\
&+|\gamma_n-\gamma_{n-1}|M_1+\frac{\gamma_n}{a}|r_n-r_{n-1}|M_1,
\end{aligned} \tag{3.72}$$

where $M_1=\max_{n\in\mathbb{N}}\{\|u\|,\|x_n\|,\Big\|\sum_{i=1}^{N}a_iu_n^i\Big\|,\sum_{i=1}^{N}a_i\|u_n^i-x_n\|\}$.
By (3.72), Lemma 2.11, and conditions (i)-(vi), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.73}$$

Step 3. We will show that $\lim_{n\to\infty}\|u_n^i-x_n\|=0$ and $\lim_{n\to\infty}\|P_C(I-\lambda D)x_n-x_n\|=0$, where $D=a(I-T)+(1-a)(I-S)$, $\forall a\in(0,1)$. Since $u_n^i=T_{r_n}^ix_n$ and $T_{r_n}^i$ is a firmly nonexpansive mapping, we have

$$\begin{aligned}
\|z-T_{r_n}^ix_n\|^2 &= \|T_{r_n}^iz-T_{r_n}^ix_n\|^2 \\
&\leq \langle T_{r_n}^iz-T_{r_n}^ix_n, z-x_n\rangle \\
&= \frac{1}{2}(\|T_{r_n}^ix_n-z\|^2+\|x_n-z\|^2-\|T_{r_n}^ix_n-x_n\|^2).
\end{aligned}$$

Hence

$$\|u_n^i-z\|^2\leq\|x_n-z\|^2-\|u_n^i-x_n\|^2. \tag{3.74}$$

From Lemma 2.13, (3.74) and definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \Big\|\alpha_n(u-z)+\beta_n\Big(P_C(I-\lambda D)x_n-z\Big)+\gamma_n\Big(\sum_{i=1}^{N}a_iu_n^i-z\Big)\Big\|^2 \\
&\leq \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\Big\|\sum_{i=1}^{N}a_iu_n^i-z\Big\|^2 \\
&\leq \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\sum_{i=1}^{N}a_i\|u_n^i-z\|^2 \\
&\leq \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\sum_{i=1}^{N}a_i(\|x_n-z\|^2-\|u_n^i-x_n\|^2) \\
&\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2-\gamma_n\sum_{i=1}^{N}a_i\|u_n^i-x_n\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}\gamma_n \sum_{i=1}^{N} a_i \|u_n^i - x_n\|^2 &\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n \|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)(\|x_n - z\| + \|x_{n+1} - z\|) \\ &\leq \alpha_n \|u - z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|). \qquad (3.75)\end{aligned}$$

By (3.73) and condition (i), we have

$$\lim_{n\to\infty} \|u_n^i - x_n\| = 0, \text{ for all } i = 1, 2, ..., N. \qquad (3.76)$$

From nonexpansiveness of P_C, we have

$$\begin{aligned}\|P_C(I - \lambda D)x_n - z\|^2 &= \|P_C(I - \lambda D)x_n - P_C(I - \lambda D)z\|^2 \\ &\leq \|(I - \lambda D)x_n - (I - \lambda D)z\|^2 \\ &= \|x_n - z - \lambda(Dx_n - Dz)\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2. \qquad (3.77)\end{aligned}$$

For every $x, y \in C$, we have

$$\begin{aligned}\langle Dx - Dy, x - y\rangle &= \langle a(I - T)x + (1 - a)(I - S)x - a(I - T)y - (1 - a)(I - S)y, x - y\rangle \\ &= \langle a((I - T)x - (I - T)y) + (1 - a)((I - S)x - (I - S)y), x - y\rangle \\ &= a\langle (I - T)x - (I - T)y, x - y\rangle + (1 - a)\langle (I - S)x - (I - S)y, x - y\rangle \\ &\geq a\alpha\|(I - T)x - (I - T)y\|^2 + (1 - a)\beta\|(I - S)x - (I - S)y\|^2 \\ &\geq \frac{1}{2}(a\|(I - T)x - (I - T)y\|^2 + (1 - a)\|(I - S)x - (I - S)y\|^2) \\ &\geq \frac{1}{2}\|Dx - Dy\|^2. \qquad (3.78)\end{aligned}$$

Then D is $\frac{1}{2}$-inverse strongly monotone.
From (3.77) and (3.78), we have

$$\begin{aligned}\|P_C(I - \lambda D)x_n - z\|^2 &\leq \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2 \\ &\leq \|x_n - z\|^2 - \lambda\|Dx_n - Dz\|^2 + \lambda^2\|Dx_n - Dz\|^2 \\ &= \|x_n - z\|^2 - \lambda(1 - \lambda)\|Dx_n - Dz\|^2. \qquad (3.79)\end{aligned}$$

By definition of x_n and (3.79), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \Big\|\alpha_n(u - z) + \beta_n\Big(P_C(I - \lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\Big[\|x_n - z\|^2 - \lambda(1 - \lambda)\|Dx_n - Dz\|^2\Big] + \gamma_n\|x_n - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \lambda\beta_n(1 - \lambda)\|Dx_n - Dz\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}\lambda\beta_n(1 - \lambda)\|Dx_n - Dz\|^2 &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|). \qquad (3.80)\end{aligned}$$

From (3.73), (3.80) and condition (i)

$$\lim_{n\to\infty} \|Dx_n - Dz\|^2 = 0. \tag{3.81}$$

From definition of $P_C(I-\lambda D)$ and Lemma 2.13, we have

$$\begin{aligned}
\|P_C(I-\lambda D)x_n - z\|^2 =& \|P_C(I-\lambda D)x_n - P_C(I-\lambda D)z\|^2 \\
\leq& \langle (I-\lambda D)x_n - (I-\lambda D)z, P_C(I-\lambda D)x_n - z\rangle \\
=& \frac{1}{2}\Big(\|(I-\lambda D)x_n - (I-\lambda D)z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2 \\
& - \|(I-\lambda D)x_n - (I-\lambda D)z - (P_C(I-\lambda D)x_n - z)\|^2\Big) \\
\leq& \frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2 \\
& - \|x_n - P_C(I-\lambda D)x_n - \lambda(Dx_n - Dz)\|^2\Big) \\
\leq& \frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2 \\
& - \|x_n - P_C(I-\lambda D)x_n\|^2 - \|\lambda(Dx_n - Dz)\|^2 \\
& + 2\lambda\langle x_n - P_C(I-\lambda D)x_n, Dx_n - Dz\rangle\Big) \\
\leq& \frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2 \\
& - \|x_n - P_C(I-\lambda D)x_n\|^2 + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|P_C(I-\lambda D)x_n - z\|^2 \leq& \|x_n - z\|^2 - \|x_n - P_C(I-\lambda D)x_n\|^2 \\
& + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|.
\end{aligned} \tag{3.82}$$

By definition of x_n, (3.82) and nonexpansiveness of $T^i_{r_n}$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 =& \Big\|\alpha_n(u - z) + \beta_n\Big(P_C(I-\lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2 \\
\leq& \alpha_n\|u - z\|^2 + \beta_n\Big\|P_C\Big(I-\lambda D\Big)x_n - z\Big\|^2 + \gamma_n\|x_n - z\|^2 \\
\leq& \alpha_n\|u - z\|^2 + \beta_n\Big[\|x_n - z\|^2 - \|x_n - P_C(I-\lambda D)x_n\|^2 \\
& + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|\Big] + \gamma_n\|x_n - z\|^2 \\
\leq& \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 - \beta_n\|x_n - P_C(I-\lambda D)x_n\|^2 \\
& + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\| + \gamma_n\|x_n - z\|^2 \\
\leq& \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \beta_n\|x_n - P_C(I-\lambda D)x_n\|^2 \\
& + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|.
\end{aligned}$$

It follows that

$$\begin{aligned}\beta_n\|x_n - P_C(I-\lambda D)x_n\|^2 \leq& \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1}-z\|^2 \\ &+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\| \\ \leq& \alpha_n\|u-z\|^2 + (\|x_n - z\| + \|x_{n+1}-z\|)\|x_{n+1}-x_n\| \\ &+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|. \end{aligned} \tag{3.83}$$

From condition (i), (3.83), (3.73) and (3.81), we have

$$\lim_{n\to\infty}\|x_n - P_C(I-\lambda D)x_n\| = 0. \tag{3.84}$$

Step 4. We will show that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \leq 0$, where $z_0 = P_{\mathbb{F}}u$.
To show this inequality, take a subsequence$\{x_{n_k}\}$ of $\{x_n\}$, such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle. \tag{3.85}$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$ where $\omega\in C$.
From (3.76), we have $u^i_{n_k} \rightharpoonup \omega$ as $k\to\infty$, for all $i = 1,2,...,N$.
Assume that $\omega \neq P_C(I-\lambda D)\omega$, where $D = a(I-T) + (1-a)(I-S)$.
By nonexpansiveness of $P_C(I-\lambda D)$, (3.84) and Opial's property, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k} - \omega\| &< \liminf_{k\to\infty}\|x_{n_k} - P_C(I-\lambda D)\omega\| \\ &\leq \liminf_{k\to\infty}\Big(\|x_{n_k} - P_C(I-\lambda D)x_{n_k}\| + \|P_C(I-\lambda D)x_{n_k} - P_C(I-\lambda D)\omega\|\Big) \\ &\leq \liminf_{k\to\infty}\|x_{n_k} - \omega\|.\end{aligned}$$

This is a contradiction, then we have

$$\omega \in F\Big(P_C(I-\lambda D)\Big) = F\Bigg(P_C\Big(I - \lambda(a(I-T) + (1-a)(I-S))\Big)\Bigg). \tag{3.86}$$

From Lemma 2.10 and Lemma 2.13, we have

$$\begin{aligned}F\Bigg(P_C\Big(I-\lambda(a(I-T)+(1-a)(I-S))\Big)\Bigg) &= VI(C, I-T)\bigcap VI(C, I-S) \\ &= F(T)\bigcap F(S).\end{aligned} \tag{3.87}$$

From (3.86) and (3.87), we have

$$\omega \in F(T)\bigcap F(S).$$

Since

$$F_i(u^i_n, v) + \frac{1}{r_n}\langle v - u^i_n, u^i_n - x_n\rangle \geq 0,$$

for all $v\in C$ and $i = 1,2,...,N$.
By (A2), we have

$$\frac{1}{r_n}\langle v - u^i_n, u^i_n - x_n\rangle \geq F_i(v, u^i_n), \qquad \forall v\in C.$$

In particular

$$\left\langle v - u^i_{n_k}, \frac{1}{r_{n_k}}(u^i_{n_k} - x_{n_k})\right\rangle \geq F_i(v, u^i_{n_k}),$$

for all $v \in C$ and $i = 1, 2, ..., N$.

From (A4) and (3.76), we have

$$F_i(v, \omega) \leq 0, \tag{3.88}$$

for all $v \in C$ and $i = 1, 2, ..., N$.

Let $u_t := tu + (1-t)\omega,\ \forall t \in (0, 1]$, we have $u_t \in C$ and from (A1), (A4) and (3.88), we obtain

$$0 = F_i(u_t, u_t) \leq tF_i(u_t, v) + (1-t)F_i(u_t, \omega) \leq tF_i(u_t, v),$$

for all $i = 1, 2, ..., N$.

Hence $F_i(tu + (1-t)\omega, v) \geq 0, \forall t \in (0, 1]$ and $\forall v \in C$.

Letting $t \to 0^+$ and using assumption (A3), we can conclude that

$$F_i(\omega, v) \geq 0, \qquad \forall v \in C \text{ and } i = 1, 2, ..., N.$$

Therefore, $\omega \in \bigcap_{i=1}^{N} EP(F_i)$. Hence $\omega \in \mathbb{F}$.

Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathbb{F}$, we have

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \tag{3.89}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$ where $z_0 = P_{\mathbb{F}}u$.

By nonexpansive of $P_C\Big(I - \lambda(a(I-T) + (1-a)(I-S))\Big)$, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 = &\Bigg\|\alpha_n(u - z_0) + \beta_n\bigg(P_C\Big(I - \lambda(a(I-T) + (1-a)(I-S))\Big)x_n - z_0\bigg) \\
&+ \gamma_n\bigg(\sum_{i=1}^{N} a_i u^i_n - z_0\bigg)\Bigg\|^2 \\
\leq &\Bigg\|\beta_n\bigg(P_C\Big(I - \lambda(a(I-T) + (1-a)(I-S))\Big)x_n - z_0\bigg) \\
&+ \gamma_n\bigg(\sum_{i=1}^{N} a_i u^i_n - z_0\bigg)\Bigg\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq &\beta_n\Big\|P_C\Big(I - \lambda(a(I-T) + (1-a)(I-S))\Big)x_n - z_0\Big\|^2 \\
&+ \gamma_n\Bigg\|\sum_{i=1}^{N} a_i u^i_n - z_0\Bigg\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq &\beta_n\|x_n - z_0\|^2 + \gamma_n\sum_{i=1}^{N} a_i\|u^i_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq &\beta_n\|x_n - z_0\|^2 + \gamma_n\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
= &(1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.
\end{aligned} \tag{3.90}$$

From(3.89) and Lemma 2.12, we obtain that $\{x_n\}$ converge strongly to $z_0 = P_{\mathbb{F}}u$.

This completes the proof of Corollary 3.4. ☐

Chapter 4

Applications

4.1 Fixed point problems of strictly pseudo-contractive mapping

In this section, we prove a strong convergence theorem involving fixed point problems of κ-strict pseudocontractive mapping.

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let T, S be κ and $\overline{\kappa}$-strict pseudocontractive mapping of C into itself, with $\mathbb{F} = F(T) \cap F(S) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a(I-T) + (1-a)(I-S)))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, where $\eta = min\{\frac{1-\kappa}{2}, \frac{1-\overline{\kappa}}{2}\}$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

Proof. The conclusion of Theorem 4.1 can be obtained from Theorem 3.1 and Remark 1.1. ☐

4.2 Constrained convex optimization problems

Let $f : C \to \mathbb{R}$ be a convex, Fréchet differentiable function, where C is a closed convex subset of a real Hilbert space of H. The constrain convex optimization problem is to find $x^* \in C$, such that

$$f(x^*) = \min_{x\in C} f(x), \tag{4.2}$$

we use the symbol Ω_f is the set of all solution of (4.2).
Before prove Theorem 4.3 and Theorem 4.4, we need the following Lemma 4.2.

Lemma 4.2. (See [9]) (Optimality condition) A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (4.2) is that x^* solves the variational inequality

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C. \tag{4.3}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(x^* - \lambda \nabla f(x^*)),$$

for every constant $\lambda > 0$. If, in addition, f is convex, then the optimality condition (4.3) is also sufficient.

Theorem 4.3. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $f, g : C \to \mathbb{R}$ be convex functions with gradient ∇f is $\frac{1}{L_f}$-inverse strongly monotone and continuous function for all $L_f > 0$ and ∇g is $\frac{1}{L_g}$-inverse strongly monotone and continuous function for all $L_g > 0$ with $\mathbb{F} = \Omega_f \cap \Omega_g \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n \rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a\nabla f + (1-a)\nabla g))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases} \tag{4.4}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$.
Suppose that the following conditions hold :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum\limits_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, where $\eta = min\{\frac{1}{L_f}, \frac{1}{L_g}\}$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

Proof. The conclusion of Theorem 4.3 can be obtained from Theorem 3.1 and Lemma 4.2. ☐

Theorem 4.4. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$.

Let $f_i : C \to \mathbb{R}$ be convex functions for all $i = 1, 2, ..., N$ with gradient $\bigtriangledown f_i$ is $\frac{1}{L_{f_i}}$-inverse strongly monotone and continuous function for all $L_{f_i} > 0$ and let $\{P_i\}_{i=1}^{N}$ be κ_i-strict pseudo-contractive mappings of C into itself with $\kappa = \sup_{i=1,2,\dots,N} \kappa_i$ and let $\overline{\rho}_i = (\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i + \overline{\alpha}_3^i = 1$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i \le f < 1$, and $\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i \in (\kappa, 1)$ for all $i = 1, 2, ..., N$. Let S_A be S-mapping generated by $P_C(I - \delta_1 \bigtriangledown f_1), P_C(I - \delta_2 \bigtriangledown f_2), ..., P_C(I - \delta_N \bigtriangledown f_N)$ and $\rho_N, \rho_{N-1}, ..., \rho_1$ where $0 < \delta_i < 2(\frac{1}{L_{f_i}})$ for all $i = 1, 2, ..., N$. Let S_B be S-mapping generated by $P_N, ..., P_1$ and $\overline{\rho}_N, \overline{\rho}_{N-1}, ..., \overline{\rho}_1$. Assume that $\mathbb{F} = \bigcap_{i=1}^{N} \Omega_{f_i} \cap \bigcap_{i=1}^{N} F(P_i) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$ and let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \ge 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(aS_A + (1-a)(I - S_B)))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \ge 1, \end{cases} \tag{4.5}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$.
Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1)$, $\forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 1)$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

Proof. The conclusion of Theorem 4.4 can be obtained from Theorem 3.3 and Lemma 4.2. $\square$

Chapter 5

Conclusions

In this chapter, we summarize all main theorems and applications obtained in this thesis.

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $A, B : C \to H$ be α and β-inverse strongly monotone respectively with $\mathbb{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(aA + (1-a)B))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1)$, $\forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, where $\eta = min\{\alpha, \beta\}$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings and let $\rho_i = (\alpha_1^i, \alpha_2^i, \alpha_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^i + \alpha_2^i + \alpha_3^i = 1$, $\alpha_1^i + \alpha_2^i \leq e < 1$, and $\alpha_1^i, \alpha_2^i, \alpha_3^i \in (0, 1)$ for all $i = 1, 2, ..., N$ and let $\{P_i\}_{i=1}^{N}$ be κ_i-strict pseudo-contractive mappings of C into itself with $\kappa = \sup_{i=1,2,...,N} \kappa_i$ and let $\overline{\rho}_i = (\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i + \overline{\alpha}_3^i = 1$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i \leq f < 1$, and $\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i \in (\kappa, 1)$ for all $i = 1, 2, ..., N$. Let S_A be S-mapping generated by $T_N, ..., T_1$ and $\rho_N, \rho_{N-1}, ..., \rho_1$ and let S_B be S-mapping generated by $P_N, ..., P_1$ and $\overline{\rho}_N, \overline{\rho}_{N-1}, ..., \overline{\rho}_1$. Assume that $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(P_i) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$

and let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a(I - S_A) + (1-a)(I - S_B)))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0,1)$. Suppose that the following conditions hold :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c,d] \subset (0,1), \ \forall n \in \mathbb{N}$,

(iii) $\sum\limits_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0,1)$,

(vi) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum\limits_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

(3) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let T, S be nonexpansive mapping of C into itself, with $\mathbb{F} = F(T) \cap F(S) \cap \bigcap\limits_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a(I - T) + (1-a)(I - S)))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0,1)$. Suppose that the following conditions hold :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c,d] \subset (0,1), \ \forall n \in \mathbb{N}$,

(iii) $\sum\limits_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0,1)$

(vi) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum\limits_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

(4) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let T, S be κ and $\overline{\kappa}$-strict pseudocontractive mapping of C into itself, with $\mathbb{F} = F(T) \cap F(S) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a(I - T) + (1 - a)(I - S)))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, where $\eta = min\{\frac{1-\kappa}{2}, \frac{1-\overline{\kappa}}{2}\}$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

(5) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $f, g : C \to \mathbb{R}$ be convex functions with gradient ∇f is $\frac{1}{L_f}$-inverse strongly monotone and continuous function for all $L_f > 0$ and ∇g is $\frac{1}{L_g}$-inverse strongly monotone and continuous function for all $\overline{L_g} > 0$ with $\mathbb{F} = \Omega_f \cap \Omega_g \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a\nabla f + (1 - a)\nabla g))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, where $\eta = min\{\frac{1}{L_f}, \frac{1}{L_g}\}$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

(6) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $f_i : C \to \mathbb{R}$ be convex functions for all $i = 1, 2, ..., N$ with gradient $\bigtriangledown f_i$ is $\frac{1}{L_{f_i}}$-inverse strongly monotone and continuous function for all $L_{f_i} > 0$ and let $\{P_i\}_{i=1}^{N}$ be κ_i-strict pseudo-contractive mappings of C into itself with $\kappa = \sup_{i=1,2,...,N} \kappa_i$ and let $\overline{\rho}_i = (\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i + \overline{\alpha}_3^i = 1$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i \leq f < 1$, and $\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i \in (\kappa, 1)$ for all $i = 1, 2, ..., N$. Let S_A be S-mapping generated by $P_C(I - \delta_1 \bigtriangledown f_1), P_C(I - \delta_2 \bigtriangledown f_2), ..., P_C(I - \delta_N \bigtriangledown f_N)$ and $\rho_N, \rho_{N-1}, ..., \rho_1$ where $0 < \delta_i < 2(\frac{1}{L_{f_i}})$ for all $i = 1, 2, ..., N$. Let S_B be S-mapping generated by $P_N, ..., P_1$ and $\overline{\rho}_N, \overline{\rho}_{N-1}, ..., \overline{\rho}_1$. Assume that $\mathbb{F} = \bigcap_{i=1}^{N} \Omega_{f_i} \cap \bigcap_{i=1}^{N} F(P_i) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$ and let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(aS_A + (1 - a)(I - S_B)))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \text{ for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1)$, $\forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ for all $n \in \mathbb{N}$,

(v) $\lambda \in (0, 1)$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

References

[1] Takahashi, W. 2000. **Functional Analysis.** Yokohama: Yokohama Publishers.

[2] Xu, H.K. 2003. “An iterative approach to quadratic optimization.” *J. Optim. Theory Appl.* 116(3): 659-678.

[3] Xu, H.K. 2003. “An iterative approach to quadratic optimization.” *J. Optim. Theory Appl.* 116: 659-678.

[4] Marino, G., Xu, H.K. 2007. “Weak and strong convergence theorem for strict pseudo-contractions in Hilbert spaces.” *J. Math. Anal. Appl.* 329: 336-346.

[5] Blum, E., Oettli, W. 1994. “From optimization and variational inequilities to equilibrium problems.” *Math. Student* 63(14): 123-145.

[6] Combettes, P.L., Hirstoaga, S.A. 2005. “Equilibrium programming in Hilbert spaces.” *J. Nonlinear Convex Anal.* 6(1): 117-136.

[7] Kangtunyakarn, A. 2013. “A new iterative scheme for fixed point problems of infinite family of κ_i-pseudo contractive mappings, equilibrium problem, variational inequality problems.” *J. Glob. Optim.* 56: 1543-1562.

[8] Kangtunyakarn, A. 2012. “Convergence theorem of κ-strictly pseudo-contractive mapping and a modification of generalized equilibrium problems.” *Fixed Point Theory and Applications.* 89.

[9] Su, M, Xu, HK. 2010. “Remarks on the gradient-projection algorithm.” *J. Nonlinear Anal. Optim.* 1: 35-43.

[10] Iiduka, H., Takahashi, W. 2006. “Weak convergence theorem by Ces’aro means for nonexpansive mappings and inverse-strongly monotone mappings.” *J. Nonlinear Convex Anal.*7105-113.

[11] Lions, J.L., Stampacchia, G. 1967. “Variational inequalities.” *Comm. Pure Appl. Math.* 20: 493-517.

[12] Kangtunyakarn, A. 2011. “A new iterative algorithm for the set of fixed-point problems of nonexpansive mappings and the set of Equilibrium problem and variational inequalitie problem.” *Abstract and Applied Analysis* Article ID 562689, p. 24. doi:10.1155/2011/562689.

[13] Debnath, L., Mikusinski, P. 2005. **Introduction to Hilbert Spaces with Applications.** London: PSW-KENT.

[14] Takahashi, W. 2009. **Introduction to Nonlinear and Convex Analysis.** Japan: Yokohama Publishers.

[15] Opial, Z. 1967. "Weak convergence of the sequence of successive approximation of nonexpansive mappings." *Bull. Amer. Math. Soc.* 73: 591-597.

[16] Takahashi, W. 2000. **Nonlinear Functional Analysis.** Yokohama: Yokohama Publishers.

[17] Berinde, V. 2002. **Iterative approximation of fixed points.** Romania: Editura Efemeride.

[18] Browder, F.E. 1967. "Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach space." *Arch Ration Mech Anal.* 24: 82-89.

[19] Kangtunyakarn, A, Suantai, S: Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions. Comput. Math. Appl. **60**, 680-694 (2010)