EXPLICIT DETERMINATION OF ADDITIVE FUNCTIONS WITH TWO PARAMETERS

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE (APPLIED MATHEMATICS)

DEPARTMENT OF MATHEMATICS

FACULTY OF SCIENCE

KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

2018

KMITL-2018-SC-D-001-022

การหารูปชัดแจ้งของฟังก์ชันแยกบวกเทียบกับสองตัวแปรเสริม

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พ.ศ. 2561
KMITL-2018-SC-D-001-022

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Degree	Master of Science (Applied Mathematics)
Department	Mathematics
Year	2018
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Abstract

For an integer $g \geq 2$, a function φ defined over the nonnegative integers is said to be g-additive if

$$\varphi(n) = \sum_{r \geq 0} \varphi(a_r(n)g^r),$$

where $n = \sum_{r \geq 0} a_r(n)g^r, a_r(n) \in \{0, 1, \ldots, g-1\}$, is the base g-representation of n.

Let V be the set of all simultaneously g-additive and h-additive functions with $g \nmid h$ and $h \nmid g$. In 2003, Puchta and Spilker proved that V is a complex vector space, with precise dimension, each of whose elements can be uniquely written as a linear combination of step-functions and certain periodic functions. Here, explicit shapes of the elements and a basis of V based on the prime factorization of the parameters g and h are determined.

Keywords : q-additive function, complex vector space.

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บทคัดย่อ

สำหรับจำนวนเต็ม $g \geq 2$, ฟังก์ชัน φ นิยามบนจำนวนเต็มไม่เป็นลบเรียกว่า g-แยกบวก ถ้า

$$\varphi(n) = \sum_{r\geq 0} \varphi(a_r(n)g^r),$$

เมื่อ $n = \sum_{r\geq 0} a_r(n)g^r, a_r(n) \in \{0, 1, \ldots, g-1\}$ เป็นการเรียง n ในรูปฐาน g

ให้ V เป็นเซตของฟังก์ชันที่เป็น g-แยกบวก และ h-แยกบวก พร้อมกันซึ่ง $g \nmid h$ และ $h \nmid g$

ในปี ค.ศ. 2003 Puchta และ Spilker พิสูจน์ว่า V เป็นปริภูมิเวกเตอร์เชิงซ้อนที่มีมิติจำกัดและแต่ละสมาชิกของ V สามารถเขียนเป็นผลรวมเชิงเส้นของฟังก์ชันขั้นบันไดและฟังก์ชันคาบในรูปแบบเดียว ในงานนี้แสดงการบวกรูปแบบชัดแจ้งของสมาชิกและฐานของ V ในรูปของแยกตัวประกอบเฉพาะของตัวแปร g และ h

คำสำคัญ : q-ฟังก์ชันแยกบวก, ปริภูมิเวกเตอร์เชิงซ้อน

Acknowledgements

Immeasurable appreciation and deepest gratitude for the advices, help and support are extended to the following persons who have contributed to making this thesis possible.

Firstly, I would like to express my sincere gratitude to my advisors, Asst. Prof. Dr. Sukrawan Mavecha and Prof. Dr. Vichian Laohakosol, from Kasetsart University for the continuous support of my M. Sc. study, for their patience, motivation, and immense knowledge. Their guidance helped me in all the time of study and writing of this thesis. I could not have imagined having a better advisors and mentors for my M. Sc. study.

Besides my advisor, I would like to thank the rest of my thesis committee: Asst. Prof. Dr. Boonrod Yuttanan, Assoc. Prof. Praiboon Pantaragphong, and Asst. Prof. Dr. Thawatchai Khumprapussorn, for their insightful comments.

Finally, I must express my very profound gratitude to my family for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

Nonpawit Seekam

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Chapter 1

Introduction

1.1 Research Motivation

Let g be an arbitrary fixed natural number ≥ 2. Then every $n \in \mathbb{N}_0$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, can be uniquely represented through base g-representation as

$$n = \sum_{r=0}^{\infty} a_r(n) g^r, \quad a_r(n) \in \{0, 1, \ldots, g-1\}.$$

An arithmetic function $\varphi : \mathbb{N}_0 \to \mathbb{C}$ is said to be g-**additive** if

$$\varphi\left(\sum_{r\geq 0} a_r(n)\, g^r\right) = \sum_{r\geq 0} \varphi\left(a_r(n)\, g^r\right)$$

holds for all $n \in \mathbb{N}_0$. These functions were introduced by Gelfond in [2] and Delange in [1]. A function $\varphi(n)$ is said to be **strongly** g-**additive** if

$$\varphi(ag^r) = \varphi(a)$$

for any $a \in \{0, 1, \ldots, g-1\}$ and $r \in \mathbb{N}_0$.

In 1983, Toshimitsu [6] proved that, for fixed $g_1, g_2 \in \mathbb{N}$; $g_1, g_2 \geq 2$, if $\varphi(n)$ is both strongly g_1-additive and strongly g_2-additive and if $\frac{\log g_1}{\log g_2}$ is not a rational number, then $\varphi(n)$ must be identically zero. In 1999, Uchida [7] showed that for g_1 and g_2 as in the work of Toshimitsu, if $\varphi(n)$ is a g_1-additive and g_2-additive, then there exist $\ell, m \in \mathbb{N}$ with $g = \gcd(g_1^\ell, g_2^m)$ such that $\varphi(ng) = n\varphi(g)$ for each $n \in \mathbb{N}$. Moreover, if $g \geq 2$ then $\varphi(n)$ is g-additive. In 2003, Puchta and Spilker [4] characterized those functions φ that are simultaneously g-additive and h-additive. This research recapitulate the work of Puchta and Spilker [4].

1.2 Objectives of the study

Let g, h be integers ≥ 2. Let V be the set of all simultaneously g-additive and h-additive with $g \nmid h$ and $h \nmid g$. In this work fine explicit formulas of simultaneously g-additive and h-additive functions in V and a basis of V are determined.

1.3 Scope(s) of the study

We consider the element of the set of all simultaneously g-additive and h-additive functions with the conditions $g \nmid h$ and $h \nmid g$ for fixed integers $g, h \geq 2$.

1.4 Benefits of the study

As in the work of Puchta and Spilker [4], we analyse how to find the formula of the element in V. Then find explicit formulas of the element and a basis of V.

1.5 Research methodology

1) Study background in additive function, periodic function and constant function.

2) Study the research paper of Puchta nad Spilker [4].

3) Find explicit formula of simultaneously g-additive and h-additive function in V.

4) Find a basis of V.

5) Summarize obtained results and write the thesis.

Table 1.1: The research schedule

Activity	Time frame					
	2016		2017		2018	
	Jul.-Sep.	Oct.-Dec.	Jan.-Jun.	Jul.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	←→					
Step 2		←	→			
Step 3			←	—	→	
Step 4			←	—	→	
Step 5						←→

Chapter 2

Preliminaries

Definition 2.1. *[3, Definition 1] A complex vector space is a collection V of vectors and two operators $+, \cdot$, such that the following hold for $u, v, w \in V$ and $c, d \in \mathbb{C}$:*

1. *For any vectors $u, v \in V$, $u + v \in V$.*
2. *For any vectors $u, v \in V$, $u + v = v + u$.*
3. *For any vectors $u, v, w \in V$, $u + (v + w) = (u + v) + w$.*
4. *For any vector $u \in V$, $0 + u = u$ for $0 \in V$.*
5. *For any vector $u \in V$ there is vector in V, denote by $-u$, for which $u + (-u) = 0$.*
6. *For any scalar $c \in \mathbb{C}$ and any vector $u \in V$, $cu \in V$.*
7. *For any scalar $c \in \mathbb{C}$ and any vectors $u, v \in V$, $c(u + v) = cu + cv$.*
8. *For any scalars $c, d \in \mathbb{C}$ and any vector $u \in V$, $(c + d)u = cu + du$.*
9. *For any scalars $c, d \in \mathbb{C}$ and any vector $u \in V$, $c(du) = (cd)u$.*
10. *For any vector $u \in V$, $1u = u$.*

Theorem 2.1. *[3, Theorem 4.2] Let W ve a subset of a complex vector space V. W is called a* **subspace** *of V if and only if*

1. *W is nonempty.*
2. *W is closed under vector addition: $v, w \in W$ implies $v + w \in W$.*
3. *W is closed under scalar multiplication: $v \in W$ implies $kv \in W$ for every $k \in \mathbb{C}$.*

Definition 2.2. *[3] Let V be a vector space over a field $\mathbb{C}$ and let $v_1, v_2, \ldots, v_m \in V$. Any vector in V of the form*

$$a_1 v_1 + a_2 v_2 + \ldots + a_m v_m$$

where $a_i \in \mathbb{C}$, is called a **linear combination** *of $v_1, v_2, \ldots, v_m$.*

Theorem 2.2. *[3, Theorem 4.5] Let S be a nonempty subset of vector space V. The set of all linear combination of vectors in S, denoted by $L(S)$, is a subspace of V containing S. Furthermore, if W is any other subspace of V containing S, then $L(S) \subset W$.*

In other words, $L(S)$ is the smallest subspace of V containing S; hence it is called the subspace **spanned** or **generated** by S. For convenience, we define $L(\emptyset) = \{0\}$.

Definition 2.3. *[3, Definition 5.1] Let V be a vector space over a field $\mathbb{C}$. The vectors $v_1, \ldots, v_m \in V$ are said to be* ***linearly independent*** *over $\mathbb{C}$, if there exist scalars $a_1, \ldots, a_m \in \mathbb{C}$, such that*

$$a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0$$

only if $a_1 = a_2 = \ldots = a_m = 0$.

Definition 2.4. *[3, Definition 5.2] A vector space V is said to be of* ***finite dimension*** *n or to be n-**dimensional**, written $\dim V = n$, if there exist linearly independent vectors $e_1, e_2, \ldots, e_n$ which span V. The sequence $\{e_1, e_2, \ldots, e_n\}$ is then called a* ***basis of*** *V.*

Throughout this thesis, we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Example 2.3. Consider the complex vector space $V := \{f : \mathbb{N}_0 \to \mathbb{C} : f(n) = M, M \in \mathbb{C}\}$. Since $f(n) = M \cdot 1 = M \cdot e(n)$, for all $n \in \mathbb{N}_0$, where $e(n)$ is constant function in V. It is easy to see that $\{e(n)\}$ is a basis of V and $\dim V = 1$.

Definition 2.5. *[5, Definition 1.1] We say that a nonzero integer a* ***divides*** *an integer b, if there exists an integer c such that*

$$ac = b.$$

If a divides b, we write $a \mid b$. If a does not divide b, we write $a \nmid b$.

Theorem 2.4. *[5, Theorem 1.2]*

1. *If a is a nonzero integer, then $a \mid 0$.*
2. *If a is an integer, then $1 \mid a$ and, if $a \neq 0$, then $a \mid a$.*
3. *If $a \mid b$ and $b \mid c$, then $a \mid c$.*
4. *If $a \mid b$ and c is a nonzero integer, then $ac \mid bc$ and $a \mid bc$.*
5. *If $a \mid b$ and $a \mid c$, then for all integers m and n we have $a \mid (mb + nc)$.*
6. *If $a \mid b$ and $b \mid a$, then $a = \pm b$.*
7. *If $a \mid b$ and a and b are positive integers, then $a < b$.*

Theorem 2.5. *[5, Theorem 1.3] Let a and b be integers with a positive. Then there exist unique integers q and r such that*

$$b = qa + r,$$

where $0 \leq r < a$. If $a \nmid b$, then we have $0 < r < a$.

Theorem 2.6. *[5, Theorem 1.4] Let a and g be positive integers with $g > 1$. Then a can be uniquely represented in the form*

$$a = c_0 + c_1g + c_2g^2 + \ldots + c_ng^n,$$

where $0 \leq c_m < g$, $m = 0, 1, \ldots, n$.

Definition 2.6. *[5, Definition 1.2] We define the* ***greatest integer in*** x*, denoted by* $\lfloor x \rfloor$*, to be that unique integer* n *satisfying*

$$n \leq x < n + 1.$$

Definition 2.7. *[5, Definition 1.3] An integer* g *is said to be a* ***common divisor*** *of the integers* a *and* b *if* $g \mid a$ *and* $a \mid b$*. The largest one of these common divisors is call the* ***greatest common divisor*** *and is denoted by* $\gcd(a, b)$*. We say that* a *and* b *are* ***relatively prime*** *if* $\gcd(a, b) = 1$*.*

Definition 2.8. *[5, Definition 1.4] A positive integer* p*, greater than one is called a* ***prime number****, if, whenever* $p = ab$*, have* $p = a$ *or* $p = b$*. An integer greater than one that is not a prime number is called a* ***composite number****.*

Theorem 2.7. *[5, Theorem 1.11] Every positive integer greater than one can be written as a product of prime numbers.*

Theorem 2.8. *[5, Theorem 1.12] If* n *is composite, then there exists a prime* $p \leq \sqrt{n}$ *such that* $p \mid n$*.*

Theorem 2.9. *[5, Theorem 1.19] (Unique Factorization, The Fundamental Theorem of Arithmetic). Let* $n > 1$ *be an integer. Then the factoring of* n *into prime factors is unique apart from the order of the prime factors.*

Corollary 2.10. *[5, Corollary 1.19.1] Let*

$$a = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \text{ and } b = p_1^{\beta_1} \cdots p_r^{\beta_r},$$

where $\alpha_i, \beta_i \geq 0, 1 \leq i \leq r$*. Then*

1. $a \mid b$ *if and only if* $\alpha_i \leq \beta_i, 1 \leq i \leq r$*;*
2. $\gcd(a, b) = 1$ *if and only if* $\alpha_i > 0$ *implies* $\beta_i = 0$ *and* $\beta_i > 0$ *implies Unique Factorization*

$$\alpha_i = 0, 1 \leq i \leq r.$$

Theorem 2.11. *[5, Theorem 1.20] Let* $a_1, \ldots, a_n$ *be positive integers and suppose that, for* $1 \leq i \leq n$*,*

$$a_i = p_1^{\alpha_{1,j}} \cdots p_r^{\alpha_{r,j}},$$

where $\alpha_{i,j} \geq 0$ *for* $1 \leq i \leq r, 1 \leq j \leq n$*. Then*

$$[a_1, \ldots, a_n] = p_1^{\max(\alpha_{1,1}, \ldots, \alpha_{1,n})} \cdots p_r^{\max(\alpha_{r,1}, \ldots, \alpha_{r,n})}$$

Corollary 2.12. *[5, Corollary 1.20.2] If* m *is a positive integer, then*

$$[ma_1, \ldots, ma_n] = m[a_1, \ldots, a_n].$$

Corollary 2.13. *[5, Corollary 1.20.3] If a and b are nonzero integers, then*

$$[a, b]\gcd(a, b) = |ab|.$$

Definition 2.9. *[5, Definition 2.2] Let a and b be integers and let m be a positive integer. We say a is* **congruent** *to b modulo m, written*

$$a \equiv b(\text{mod} m),$$

if and only if $m \mid (a-b)$. If $m \nmid (a-b)$, then a and b are said to be **incongruent** *modulo m and we denote this by*

$$a \not\equiv b(\text{mod} m).$$

Theorem 2.14. *[5, Theorem 2.1] Let m be a positive integer.*

1. *For all integers a we have*
$$a \equiv a(\text{mod} m).$$

2. *For all integers a and b we have*
$$a \equiv b(\text{mod} m) \text{ if and only if } b \equiv a(\text{mod} m).$$

3. *For all integers a, b and c we have that*
$$\text{If } a \equiv b(\text{mod} m), b \equiv c(\text{mod} m), \text{ then } a \equiv c(\text{mod} m).$$

4. *If a is any integer, then*
$$m \mid a \text{ if and only if } a \equiv 0(\text{mod} m).$$

5. *If $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}$ and $\{k_1, \ldots, k_n\}$ are any sets of integers such that $a_i \equiv b_i(\text{mod} m), 1 \leq i \leq n$, then*
$$\sum_{i=1}^{n} k_i a_i \equiv \sum_{i=1}^{n} k_i b_i(\text{mod} m).$$

6. *If a, b, c and d are any integers, then*
$$a \equiv b(\text{mod} m) \text{ and } c \equiv d(\text{mod} m) \text{ implies } ac \equiv bd(\text{mod} m).$$

7. *If $a \equiv b(\text{mod} m)$ and n is a natural number, then*
$$a^n \equiv b^n(\text{mod} m)$$

Definition 2.10. *A function $\varphi : \mathbb{N}_0 \to \mathbb{C}$ is called p-***periodic** *$(p \in \mathbb{N})$ if $\varphi(n+p) = \varphi(n)$ for all $n \in \mathbb{N}_0$.*

Definition 2.11. *A function $\varphi : \mathbb{N}_0 \to \mathbb{C}$ is called q-***constant** *$(q \in \mathbb{N})$ if $\varphi(aq+b) = \varphi(aq)$ for all $a, b \in \mathbb{N}_0$, $b < q$.*

Remark 2.15. Every function is a 1-constant.

Example 2.16. Let $\varphi(n) = \sin(\frac{n\pi}{2})$. We see That

$$\begin{aligned}\varphi(n+4) &= \sin(\frac{(n+4)\pi}{2})\\ &= \sin(\frac{n\pi}{2} + 2\pi)\\ &= \sin(\frac{n\pi}{2})\cos(2\pi) + \cos(\frac{n\pi}{2})\sin(2\pi)\\ &= \sin(\frac{n\pi}{2})\\ &= \varphi(n).\end{aligned}$$

Therefore φ is 4-periodic function.

Example 2.17. Let $x \in \mathbb{N}_0$ and define function $\varphi(n) = 2i + x$, for $n \equiv x(\text{mod}3)$. Thus $x \in \{0, 1, 2\}$. Since $n + 3 \equiv n(\text{mod}3)$, then $\varphi(n+3) = \varphi(n)$.
Therefore φ is 3-periodic function.

Example 2.18. Define $\varphi(n) = \sum_{m=0}^{3} e^{2\pi imn/4}$.
Since $\varphi(n)$ is the sum terms in a geometric progression

$$\varphi(n) = \sum_{m=0}^{3} x^m,$$

where $x = e^{2\pi in/4}$, we have

$$\varphi(n) = \begin{cases} 0 & \text{if } 4 \nmid n, \\ 4 & \text{if } 4 \mid n. \end{cases}$$

Hence

$$\varphi(n+4) = \varphi(n).$$

Therefore φ is 4-periodic function.

Example 2.19. Define $\varphi(n) = \lfloor \frac{n+1}{2} \rfloor + n - 1$.
From the Definition 2.11, we get $q = 2$ and $b \in \mathbb{N}_0, b < 2$, we distinguish two cases:

Case 1: if $b = 1$, we have

$$\begin{aligned}\varphi(2a+1) &= \left\lfloor \frac{2a+1+1}{2} \right\rfloor + 2a + 1 - 1\\ &= \left\lfloor \frac{2a}{2} \right\rfloor + 1 + a\\ &= \left\lfloor \frac{2a+1}{2} \right\rfloor + 1 + a = \varphi(2a).\end{aligned}$$

Case 2: if $b = 0$, we have

$$\varphi(2a+0) = \left\lfloor \frac{2a+1}{2} \right\rfloor + 2a - 1 = \varphi(2a).$$

Therefore $\varphi(n)$ is 2-constant function.

2.1 The work of J. -C. Puchta and J. Spilker ([4])

In 2003, Puchta and Spilker [4] characterized the function φ that are simultaneously g-additive and h-additive.
For $g, h, k \in \mathbb{N}$ with $g, h \geq 2$, define the step-function ℓ_k by

$$\ell_k(n) := [g,h]^{k-1} \left\lfloor \frac{n}{[g,h]^{k-1}} \right\rfloor \quad (n \in \mathbb{N}_0).$$

where $[g, h]$ denotes the least common multiple of g, h and $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n.
Note that $\ell_1(n) = n$.

Proposition 2.20. *[4, Proposition 1] For $g, h \in \mathbb{N}$ with $g, h \geq 2$ and for $k \in \mathbb{N}$, if $[g,h]^{k-1} | \gcd(g,h)^k$, then ℓ_k is simultaneously g-additive and h-additive; if $\gcd(g,h) > 1$, then ℓ_k is $\gcd(g,h)$-additive.*

For $k, g, h \in \mathbb{N}$, define a $\mathbb{C}$-vector space

$$V_k := V_k(g;h) := \left\{\varphi_k : \mathbb{N}_0 \to \mathbb{C} : \varphi_k(0) = 0, \ \varphi_k \text{ is } \gcd(g,h)^k\text{-periodic and } [g,h]^{k-1}\text{-constant}\right\}.$$

Proposition 2.21. *[4, Proposition 4] If $g, h, k \in \mathbb{N}$ and $\varphi_k \in V_k(g;h)$, then*

(i) φ_k is simultaneously g-additive and h-additive;

(ii) if $\gcd(g,h) > 1$, then φ_k is $\gcd(g,h)$-additive.

Denote by V be the complex vector space of all simultaneously g-additive and h-additive functions. The main result in [4] is

Theorem 2.22. *[4, Theorem] Let $g, h \in \mathbb{Z}$ with $g, h \geq 2$ and assume $g \nmid h$, $h \nmid g$. Then*

(i) every function $\varphi \in V$, can be uniquely represented in the form

$$\varphi(n) = \sum_k^* c_k \ell_k(n) + \sum_k^* \varphi_k(n).$$

where $c_k \in \mathbb{C}$, $\varphi_k \in V_k(g;h)$, and the star indicates that the sum extends over $k \in \mathbb{N}$ with the property that $[g,h]^{k-1} | \gcd(g,h)^k$;

(ii) the vector space V has dimension $\dim V = \sum_{k=1}^* \frac{\gcd(g,h)^k}{[g,h]^{k-1}}$;

(iii) every function $\varphi \in V$ is already $\gcd(g,h)$-additive, if $\gcd(g,h) > 1$.

Based upon Theorem 2.22, it is natural to ask for explicit shapes of elements and bases of the vector space V. Following the analysis of small cases, i.e., those with small values of g and h, we are able to obtain general forms of the above mentioned quantities.

Example 2.23. Let $g = 12$ and $h = 18$. We have gcd$(g.h) = 6$. $[g,h] = 36$, and so $[g,h]^{k-1} |$ gcd$(g,h)^k$, for $k = 1, 2$.
By Theorem 2.22(i), we have

$$\varphi(n) = \sum_{k=1}^{2} c_k l_k(n) + \sum_{k=1}^{2} \varphi_k(n)$$

where $c_k \in \mathbb{C}$ and $\varphi_k \in V_k$ such that

$$V_k = \left\{\varphi_k : \mathbb{N}_0 \to \mathbb{C} : \varphi_k(0) = 0,\ \varphi_k \text{ is } 6^k\text{-periodic and } 36^{k-1}\text{-constant}\right\}.\ k = 1, 2.$$

By Theorem 2.22(ii), dim$V = \sum_{k=1}^{2} \frac{\text{gcd}(g,h)^k}{[g,h]^{k-1}} = \frac{6}{1} + \frac{36}{36} = 7$.
By Theorem 2.22(iii), since gcd$(g,h) = 6 > 1$, thus every function $\varphi \in V$ is already gcd(g,h)-additive.

Chapter 3

Explicit shapes of elements and bases of V

Based upon Theorem 2.22, it is natural to ask for explicit shapes of elements and bases of the vector space V. Following the analysis of small cases, i.e., those with small values of g and h, we are able to obtain general forms of the above mentioned quantities.

Our main result is:

Theorem 3.1. *Let g, h be integer ≥ 2. Assume $g \nmid h$ and $h \nmid g$ so that they can be written as*

$$g = p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_j^{b_j} w_1^{d_1} \cdots w_f^{d_f}, \quad h = p_1^{a_1} \cdots p_s^{a_s} q_1^{c_1} \cdots q_j^{c_j} w_1^{e_1} \cdots w_f^{e_f}$$

where, for $1 \leq i \leq s$, $1 \leq r \leq j$, $1 \leq \ell \leq f$, we set

- p_i, q_r, w_ℓ *distinct prime numbers*
- $b_r, e_\ell \in \mathbb{N}$
- $c_r, d_\ell, a_i \in \mathbb{N}_0$ *with* $b_r > c_r$, $d_\ell < e_\ell$.

Then (i) every function $\varphi \in V$ can be uniquely represented in the form

$$\varphi(n) = \sum_{k=1}^{x} c_k \ell_k(n) + \sum_{k=1}^{x} \sum_{i=0}^{\frac{\text{gcd}(g,h)^k}{[g,h]^{k-1}} - 1} N_i^k B_i^k(n)$$

where $c_k, N_i^k \in \mathbb{C}$,

$$x = \min\left(\left\lfloor \frac{b_1}{b_1 - c_1} \right\rfloor, \ldots, \left\lfloor \frac{b_j}{b_j - c_j} \right\rfloor, \left\lfloor \frac{e_1}{e_1 - d_1} \right\rfloor, \ldots, \left\lfloor \frac{e_f}{e_f - d_f} \right\rfloor \right),$$

and

$$B_i^k(n) = \begin{cases} 1 & \text{if } n \equiv i[g,h]^{k-1}, i[g,h]^{k-1} + 1, \ldots, (i+1)[g,h]^{k-1} - 1 (\text{mod gcd}(g,h)^k) \\ 0 & \text{otherwise,} \end{cases}$$

for $i \in \{0, 1, \ldots, \frac{\text{gcd}(g,h)^k}{[g,h]^{k-1}} - 1\}$;
(ii) a basis of the complex vector-space V is given by

$$\left\{ \ell_1(n), \ldots, \ell_x(n), B_1^1(n), \ldots, B_{\text{gcd}(g,h)-1}^1(n), \ldots, B_1^x(n), \ldots, B_{\frac{\text{gcd}(g,h)^x}{[g,h]^{x-1}}-1}^x(n) \right\}.$$

Proof. From the definition of V_k, since $\varphi_k \in V_k$ is $[g,h]^{k-1}$-constant and $\varphi_k(0) = 0$, we have

$$\varphi_k(0) = \varphi_k(1) = \ldots = \varphi_k([g,h]^{k-1} - 1) = 0,$$
$$\varphi_k([g,h]^{k-1}) = \varphi_k([g,h]^{k-1} + 1) = \ldots = \varphi_k(2[g,h]^{k-1} - 1),$$
$$\varphi_k(2[g,h]^{k-1}) = \varphi_k(2[g,h]^{k-1} + 1) = \ldots = \varphi_k(3[g,h]^{k-1} - 1),$$
$$\vdots$$
$$\varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1)[g,h]^{k-1}) = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1)[g,h]^{k-1} + 1) = \ldots = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}})[g,h]^{k-1} - 1),$$
$$\vdots$$

From the definition of V_k, since $\varphi_k \in V_k$ is $\gcd(g,h)^k$-periodic, we have

$$\begin{aligned}
\varphi_k(0) &= \varphi_k(\gcd(g,h)^k) = \varphi_k(2\gcd(g,h)^k) = \ldots = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1)\gcd(g,h)^k) \\
&= \ldots = 0, \\
\varphi_k(1) &= \varphi_k(\gcd(g,h)^k + 1) = \varphi_k(2\gcd(g,h)^k + 1) = \ldots = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1)\gcd(g,h)^k + 1) \\
&= \ldots, \\
\varphi_k(2) &= \varphi_k(\gcd(g,h)^k + 2) = \varphi_k(2\gcd(g,h)^k + 2) = \ldots = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1)\gcd(g,h)^k + 2) \\
&= \ldots, \\
&\vdots \\
\varphi_k([g,h]^{k-1} - 1) &= \varphi_k(\gcd(g,h)^k + ([g,h]^{k-1} - 1)) = \varphi_k(2\gcd(g,h)^k + ([g,h]^{k-1} - 1)) = \ldots \\
&= \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1)\gcd(g,h)^k + ([g,h]^{k-1} - 1)) = \ldots.
\end{aligned}$$

Since $[g,h]^{k-1} | \gcd(g,h)^k$, then $\gcd(g,h)^k = A[g,h]^{k-1}$ for some positive integer A. It follow that

$$\begin{aligned}
\varphi_k(0) &= \varphi_k(1) = \ldots = \varphi_k([g,h]^{k-1} - 1) = \varphi_k(\gcd(g,h)^k) = \varphi_k(\gcd(g,h)^k + 1) \\
&= \ldots = \varphi_k(\gcd(g,h)^k + ([g,h]^{k-1} - 1)) = \varphi_k(2\gcd(g,h)^k) = \ldots \\
&= \varphi_k(2\gcd(g,h)^k + ([g,h]^{k-1} - 1)) = \ldots = N_0^k = 0, \\
\varphi_k([g,h]^{k-1}) &= \varphi_k([g,h]^{k-1} + 1) = \ldots = \varphi_k(2[g,h]^{k-1} - 1) = \varphi_k([g,h]^{k-1} + \gcd(g,h)^k) \\
&= \varphi_k(([g,h]^{k-1} + 1) + \gcd(g,h)^k) = \ldots = \varphi_k((2[g,h]^{k-1} - 1) + \gcd(g,h)^k) \\
&= \varphi_k([g,h]^{k-1} + 2\gcd(g,h)^k) = \ldots = N_1^k \in \mathbb{C}, \\
\varphi_k(2[g,h]^{k-1}) &= \varphi_k(2[g,h]^{k-1} + 1) = \ldots = \varphi_k(3[g,h]^{k-1} - 1) = \varphi_k(2[g,h]^{k-1} + \gcd(g,h)^k) \\
&= \varphi_k((2[g,h]^{k-1} + 1) + \gcd(g,h)^k) = \ldots = \varphi_k((3[g,h]^{k-1} - 1) + \gcd(g,h)^k) \\
&= \varphi_k(2[g,h]^{k-1} + 2\gcd(g,h)^k) = \ldots = N_2^k \in \mathbb{C}, \\
&\vdots
\end{aligned}$$

$$\varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1)[g,h]^{k-1}) = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1)[g,h]^{k-1}+1) = \ldots = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}})[g,h]^{k-1}-1)$$
$$= \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1)[g,h]^{k-1}+\gcd(g,h)^k)$$
$$= \varphi_k(((\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1)[g,h]^{k-1}+1)+\gcd(g,h)^k)$$
$$= \ldots = \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}}[g,h]^{k-1}-1)+\gcd(g,h)^k)$$
$$= \varphi_k((\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1)[g,h]^{k-1}+2\gcd(g,h)^k)$$
$$= \ldots = N^k_{\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1} \in \mathbb{C}.$$

We see that the function $\varphi_k \in V_k$ $(k \geq 1)$ satisfies

$$\varphi_k(n) = N_i^k \in \mathbb{C}$$

whenever

$$n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1}-1 \pmod{\gcd(g,h)^k}$$
$$\left(i \in \left\{0, 1, \ldots, \frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1\right\}\right)$$

and $N_0^k = 0$ (because of $\varphi_k(0) = 0$).

Consider k such that $[g,h]^{k-1} \mid \gcd(g,h)^k$. Since $\gcd(g,h) = p_1^{a_1}\cdots p_s^{a_s} q_1^{c_1} \cdots q_j^{c_j} w_1^{d_1} \cdots w_f^{d_f}$ and $[g,h] = p_1^{a_1}\cdots p_s^{a_s} q_1^{b_1} \cdots q_j^{b_j} w_1^{e_1} \cdots w_f^{e_f}$.

If $k = 1$, then $\gcd(g,h)$ is divisible by $[g,h]^{k-1} = 1$.

For $k \geq 2$, we have

$$\frac{\gcd(g,h)^k}{[g,h]^{k-1}} = \frac{\left(p_1^{a_1}\cdots p_s^{a_s} q_1^{c_1} \cdots q_j^{c_j} w_1^{d_1} \cdots w_f^{d_f}\right)^k}{\left(p_1^{a_1}\cdots p_s^{a_s} q_1^{b_1} \cdots q_j^{b_j} w_1^{e_1} \cdots w_f^{e_f}\right)^{k-1}} = \frac{p_1^{a_1}\cdots p_s^{a_s} q_1^{c_1} \cdots q_j^{c_j} w_1^{d_1} \cdots w_f^{d_f}}{\left(q_1^{b_1-c_1} \cdots q_j^{b_j-c_j} w_1^{e_1-d_1} \cdots w_f^{e_f-d_f}\right)^{k-1}}.$$

The divisibility condition requires that $(b_r - c_r)(k-1) \leq c_r$ and $(e_\ell - d_\ell)(k-1) \leq d_\ell$, and so

$$k \leq \frac{b_r}{b_r - c_r}, \; k \leq \frac{e_l}{e_l - d_l}$$

for all $r = 1, 2, \ldots, j$; $\ell = 1, 2, \ldots f$. Thus,

$$k \leq \min\left(\left\lfloor \frac{b_1}{b_1 - c_1} \right\rfloor, \ldots, \left\lfloor \frac{b_j}{b_j - c_j} \right\rfloor, \left\lfloor \frac{e_1}{e_1 - d_1} \right\rfloor, \ldots, \left\lfloor \frac{e_f}{e_f - d_f} \right\rfloor\right) = x.$$

From Theorem 2.22 (i), every function $\varphi \in V$ can be uniquely represented in the form

$$\varphi(n) = \sum_{k=1}^{x} c_k \ell_k(n) + \sum_{k=1}^{x} \varphi_k(n)$$

where $c_k \in \mathbb{C}$ and $\varphi_k \in V_k$. Using the definition of V_k, we get

$$\varphi(n) = \sum_{k=1}^{x} c_k \ell_k(n) + \sum_{k=1}^{x} \sum_{i=0}^{\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1} N_i^k B_i^k(n)$$

where, for $i \in \{0, 1, \ldots, \frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1\}$, we have

$$n \equiv i[g,h]^{k-1}, i[g,h]^{k-1} + 1, \ldots, (i+1)[g,h]^{k-1} - 1 \pmod{\gcd(g,h)^k}$$

with $N_i^k \in \mathbb{C}$ and

$$B_i^k(n) = \begin{cases} 1 & \text{if } n \equiv i[g,h]^{k-1}, i[g,h]^{k-1} + 1, \ldots, (i+1)[g,h]^{k-1} - 1 \pmod{\gcd(g,h)^k} \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.20, we know that $\ell_k(n) \in V$. Since $B_i^k(n) \in V_k(g;h)$, by Proposition 2.21(i), we get $B_i^k(n) \in V$ for all i.

We next show that the set

$$S = \left\{ \ell_1(n), \ldots, \ell_x(n), B_1^1(n), \ldots, B_{\gcd(g,h)-1}^1(n), \ldots, B_1^x(n), \ldots, B^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1}(n) \right\},$$

is $\mathbb{C}$-linearly independent. If there is a $\mathbb{C}$-linear relation

$$\begin{aligned} &A_1^0 \ell_1(n) + \ldots + A_x^0 \ell_x(n) + A_1^1 B_1^1(n) + \ldots + A^1_{\gcd(g,h)-1} B^1_{\gcd(g,h)-1}(n) + A_1^2 B_1^2(n) + \cdots \\ &+ A^2_{\frac{\gcd(g,h)^2}{[g,h]}-1} B^2_{\frac{\gcd(g,h)^2}{[g,h]}-1}(n) + \cdots + A_1^x B_1^x(n) + \cdots + A^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1} B^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1}(n) = 0, \end{aligned}$$

then upon substituting n by $1, 2, \ldots, \gcd(g,h)$ we get the following system of equations

$$\begin{aligned} A_1^0 \ell_1(1) + A_1^1 &= 0 \\ A_1^0 \ell_1(2) + A_2^1 &= 0 \\ &\vdots \\ A_1^0 \ell_1(\gcd(g,h) - 1) + A^1_{\gcd(g,h)-1} &= 0 \\ A_1^0 \ell_1(\gcd(g,h)) &= 0. \end{aligned}$$

Solving this system back to forth, we get

$$A_1^0 = A_1^1 = A_2^1 = \ldots = A^1_{\gcd(g,h)-1} = 0.$$

Substituting n by $[g,h], 2[g,h], \ldots, \left(\frac{\gcd(g,h)^2}{[g,h]} - 1\right)[g,h], \gcd(g,h)^2$, we get the following system of equations

$$\begin{aligned} A_2^0 \ell_2([g,h]) + A_1^2 &= 0 \\ A_2^0 \ell_2(2[g,h]) + A_2^2 &= 0 \\ &\vdots \\ A_2^0 \ell_2\left(\left(\frac{\gcd(g,h)^2}{[g,h]} - 1\right)[g,h]\right) + A^2_{\frac{\gcd(g,h)^2}{[g,h]}-1} &= 0 \\ A_2^0 \ell_2\left(\gcd(g,h)^2\right) &= 0. \end{aligned}$$

Again solving the system back to forth, we get

$$A_2^0 = A_1^2 = A_2^2 = \ldots = A^2_{\frac{\gcd(g,h)^2}{[g,h]}-1} = 0.$$

Substituting n by $[g,h]^2, 2[g,h]^2, \ldots, \left(\frac{\gcd(g,h)^3}{[g,h]^2}-1\right)[g,h]^2, \gcd(g,h)^3$, we get the following system of equations

$$\begin{aligned} A_3^0\ell_3\left([g,h]^2\right) + A_1^3 &= 0 \\ A_3^0\ell_3\left(2[g,h]^2\right) + A_2^3 &= 0 \\ &\vdots \\ A_3^0\ell_3\left(\left(\frac{\gcd(g,h)^3}{[g,h]^2}-1\right)[g,h]^2\right) + A^3_{\frac{\gcd(g,h)^3}{[g,h]^2}-1} &= 0 \\ A_3^0\ell_3\left(\gcd(g,h)^3\right) &= 0. \end{aligned}$$

Again solving the system back to forth, we get

$$A_3^0 = A_1^3 = A_2^3 = \ldots = A^3_{\frac{\gcd(g,h)^3}{[g,h]^2}-1} = 0.$$

Continuing in the same manner, until finally, substituting n by $[g,h]^{x-1}, 2[g,h]^{x-1}, \ldots, \left(\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1\right)[g,h]^{x-1}, \gcd(g,h)^x$, we get the following system of equations

$$\begin{aligned} A_x^0\ell_x\left([g,h]^{x-1}\right) + A_1^x &= 0 \\ A_x^0\ell_x\left(2[g,h]^{x-1}\right) + A_2^x &= 0 \\ &\vdots \\ A_x^0\ell_x\left(\left(\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1\right)[g,h]^{x-1}\right) + A^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1} &= 0 \\ A_x^0\ell_x\left(\gcd(g,h)^x\right) &= 0. \end{aligned}$$

As before, the solution of this last system is

$$A_x^0 = A_1^x = A_2^x = \ldots = A^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1} = 0,$$

showing that the set S is $\mathbb{C}$-linearly independent and is thus a basis of V, yielding

$$\dim V = \sum_{k=1}^{x} \frac{\gcd(g,h)^k}{[g,h]^{k-1}},$$

which concludes the proof of the main theorem. □

Example 3.2. We consider in the case of $\gcd(g,h) = 1$. By the conditions $g \nmid h$ and $h \nmid g$, we can write

$$g = q_1^{b_1}\cdots q_j^{b_j}, \quad h = w_1^{e_1}\cdots w_f^{e_f},$$

where $q_1, q_2, \ldots, q_j, w_1, w_2, \ldots, w_f$ are distinct prime numbers and $b_i, e_i \in \mathbb{N}$. Thus $x = \min\left(\left\lfloor\frac{b_1}{b_1-0}\right\rfloor, \ldots, \left\lfloor\frac{b_j}{b_j-0}\right\rfloor, \left\lfloor\frac{e_1}{e_1-0}\right\rfloor, \ldots, \left\lfloor\frac{e_f}{e_f-0}\right\rfloor\right) = 1$. By Theorem 3.1(i), we have

$$\varphi(n) = \sum_{k=1}^{1} c_k\ell_k(n) + \sum_{k=1}^{1}\sum_{i=0}^{\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1} N_i^k B_i^k(n) = c_1 n + \sum_{i=0}^{0} N_i^1 B_i^1(n) = c_1 n,$$

where $c_1 \in \mathbb{C}$.

By Theorem 3.1(ii), a basis of V is $\{\ell_1(n)\}$ and $\dim V = 1$.

Table 3.1: Sample of explicit forms for simultaneously g-additive and h-additive function φ of given g and h

g	h	$\gcd(g,h)$	$[g,h]$	x	$\varphi(n)$	basis	dim V
2	$2m+1$, $m \in \mathbb{N}$	1	$4m+2$	1	$c_1\ell_1(n)$	$\{\ell_1(n)\}$	1
2^2	$2m+1$, $m \in \mathbb{N}$	1	$8m+4$	1	$c_1\ell_1(n)$	$\{\ell_1(n)\}$	1
2^2	$2(2m+1)$, $m \in \mathbb{N}$	2	$8m+4$	1	$c_1\ell_1(n) + N_i^1 B_i^1(n)$	$\{\ell_1(n), B_1^1(n)\}$	2
2^3	$(2m+1)$, $m \in \mathbb{N}$	1	$16m+8$	1	$c_1\ell_1(n)$	$\{\ell_1(n)\}$	1
2^3	$2(2m+1)$, $m \in \mathbb{N}$	2	$16m+8$	1	$c_1\ell_1(n) + N_i^1 B_i^1(n)$	$\{\ell_1(n), B_1^1(n)\}$	2
2^3	$2^2(2m+1)$, $m \in \mathbb{N}$	4	$16m+8$	1	$c_1\ell_1(n)$ $+\sum_{i=1}^{3} N_i^1 B_i^1(n)$	$\{\ell_1(n), B_1^1(n),$ $B_2^1(n), B_3^1(n)\}$	4
2^a, $a \in \mathbb{N}$	$2^b(2m+1)$ $m \in \mathbb{N}$, $b \in \{0,1,$ $\ldots, a-1\}$	2^b	$2^{a+1}m$ $+2^a$	1	$c_1\ell_1(n)$ $+\sum_{i=1}^{2^b-1} N_i^1 B_i^1(n)$	$\{\ell_1(n), B_1^1(n),$ $B_2^1(n), \ldots,$ $B_{2^b-1}^1(n)\}$	2^b
$2^2 3^3 7^4$	$2^2 3^2 13^3$	36	569699676	1	$c_1\ell_1(n)$ $+\sum_{i=1}^{35} N_i^1 B_i^1(n)$	$\{\ell_1(n), B_1^1(n),$ $B_2^1(n), \ldots,$ $B_{35}^1(n)\}$	36
$2^2 3^7 7^2$	$2^2 3^6 7^4$	142884	21003948	2	$c_1\ell_1(n) + c_2\ell_2(n)$ $+\sum_{i=1}^{142883} N_i^1 B_i^1(n)$ $+\sum_{i=1}^{971} N_i^2 B_i^2(n)$	$\{\ell_1(n), B_1^1(n),$ $B_2^1(n), \ldots,$ $B_{142883}^1(n), B_1^2(n),$ $B_2^2(n), \ldots,$ $B_{971}^2(n)\}$	143857

Chapter 4

Conclusions

In our work, we consider additive function $\varphi : \mathbb{N}_0 := \mathbb{N} \cup \{0\} \to \mathbb{C}$ to form

$$\varphi\left(\sum_{r\geq 0} a_r(n)\, g^r\right) = \sum_{r\geq 0} \varphi\left(a_r(n)\, g^r\right),$$

where g an arbitrary fixed natural number ≥ 2, and $a_r(n) \in \{0, 1, \ldots, g-1\}$.
For $g, h, k \in \mathbb{N}$ with $g, h \geq 2$, define the step-function ℓ_k by

$$\ell_k(n) := [g,h]^{k-1} \left\lfloor \frac{n}{[g,h]^{k-1}} \right\rfloor \quad (n \in \mathbb{N}_0),$$

For $k, g, h \in \mathbb{N}$, we define complex vector space V_k by

$$V_k := V_k(g;h) := \left\{\varphi_k : \mathbb{N}_0 \to \mathbb{C} : \varphi_k(0) = 0,\ \varphi_k \text{ is } \gcd(g,h)^k\text{-periodic and } [g,h]^{k-1}\text{-constant}\right\},$$

and denote by V be the complex vector space of all simultaneously g-additive and h-additive functions.

We determines explicit shapes of the elements and a basis of V based on the prime factorization of the parameters g and h. The main results are summarized as follows : Let g, h be integer ≥ 2. Assume $g \nmid h$ and $h \nmid g$ so that they can be written as

$$g = p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_j^{b_j} w_1^{d_1} \cdots w_f^{d_f}, \quad h = p_1^{a_1} \cdots p_s^{a_s} q_1^{c_1} \cdots q_j^{c_j} w_1^{e_1} \cdots w_f^{e_f}$$

where, for $1 \leq i \leq s$, $1 \leq r \leq j$, $1 \leq \ell \leq f$, we set

- $p_i,\ q_r,\ w_\ell$ distinct prime numbers
- $b_r, e_\ell \in \mathbb{N}$
- $c_r, d_\ell, a_i \in \mathbb{N}_0$ with $b_r > c_r$, $d_\ell < e_\ell$.

Then 1) every function $\varphi \in V$ can be uniquely represented in the form

$$\varphi(n) = \sum_{k=1}^{x} c_k \ell_k(n) + \sum_{k=1}^{x} \sum_{i=0}^{\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1} N_i^k B_i^k(n)$$

where $c_k, N_i^k \in \mathbb{C}$,

$$x = \min\left(\left\lfloor \frac{b_1}{b_1 - c_1} \right\rfloor, \ldots, \left\lfloor \frac{b_j}{b_j - c_j} \right\rfloor, \left\lfloor \frac{e_1}{e_1 - d_1} \right\rfloor, \ldots, \left\lfloor \frac{e_f}{e_f - d_f} \right\rfloor\right),$$

and

$$B_i^k(n) = \begin{cases} 1 & \text{if } n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1} - 1 \pmod{\gcd(g,h)^k} \\ 0 & \text{otherwise.} \end{cases}$$

for $i \in \{0, 1, \ldots, \frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1\}$;

2) a basis of the complex vector-space V is given by

$$\left\{\ell_1(n), \ldots, \ell_x(n), B_1^1(n), \ldots, B_{(g,h)-1}^1(n), \ldots, B_1^x(n), \ldots, B_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1}^x(n)\right\}.$$

References

[1] Delange, H., Sur les fonctions q-additives ou q-multiplicative. *Acta Arith.,* **21**(1972) 285–298.

[2] Gelfond, A., Sur les nombres qui ont des propriétés additives et multiplicatives domées. *Acta Arith.,* **13**(1968) 259–265.

[3] Lipschutz, S., Theory and problems of linear algebra, Mcgraw-hill book company, New York, 1968.

[4] Puchta, J. and Spilker, J., Function that are both g- and h-additive. *Acta Arith.,* **80** (2003) 264–270.

[5] Redmond, D., Number Theory An Introduction, Marcel Dekker, New York, 1996.

[6] Toshimitsu, T., Strongly q-additive functions and algebraic independence. *Tokyo J.Math.,* **21**(1998) 107–113.

[7] Uchida, Y., On p- and q-additive functions. *Tokyo J. Math.,* **22**(1999) 83–87.

Appendix/Appendices

The research paper

Explicit Determination of Functions which Are Additive with Respect to Two Parameters

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Abstract

For an integer $g \geq 2$, a function φ defined over the nonnegative integers is said to be g-additive if

$$\varphi\left(\sum_{r\geq 0} a_r g^r\right) = \sum_{r\geq 0} \varphi(a_r g^r),$$

where $n = \sum_{r\geq 0} a_r g^r$ is the base g-representation of n.

Let V be the set of all simultaneously g-additive and h-additive functions with $g \nmid h$ and $h \nmid g$. In 2003, Puchta and Spilker proved that V is a complex vector space, with a precise dimension, each of whose elements can be uniquely written as a linear combination of step-functions and certain periodic functions. Here, we determine explicit shapes of the elements and a basis of V based on the prime factorization of the parameters g and h.

Mathematics Subject Classification: 11A25, 11K65
Keywords: g-additive function, Complex vector space

1 Introduction

Let g be an arbitary fixed natural number ≥ 2. Then every $n \in \mathbb{N}$ can be uniquely represented through base g-representation as

$$n = \sum_{r=0}^{\infty} a_r(n) g^r, \quad a_r(n) \in \{0, 1, \ldots, g-1\}.$$

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An arithmetic function $\varphi : \mathbb{N}_0 := \mathbb{N} \cup \{0\} \to \mathbb{C}$ is said to be g-**additive** if

$$\varphi\left(\sum_{r\geq 0} a_r(n)\, g^r\right) = \sum_{r\geq 0} \varphi\left(a_r(n)\, g^r\right)$$

holds for all $n \in \mathbb{N}_0$. These functions were introduced by Gelfond in [2] and Delange in [1]. A function $\varphi(n)$ is said to be **strongly** g-**additive** if

$$\varphi\left(ag^r\right) = \varphi(a)$$

for any $a \in \{0, 1, \ldots, g-1\}$ and $r \in \mathbb{N}_0$.

In 1983, Toshimitsu [4] proved that, for fixed $g_1, g_2 \in \mathbb{N}$; $g_1, g_2 \geq 2$, if $\varphi(n)$ is both strongly g_1-additive and strongly g_2-additive and if $\frac{\log g_1}{\log g_2}$ is not a rational number, then $\varphi(n)$ must be identically zero. In 1999, Uchida [5] showed that for g_1 and g_2 as in the work of Toshimitsu, if $\varphi(n)$ is a g_1-additive and g_2-additive, then there exist $\ell, m \in \mathbb{N}$ with $g = \gcd\left(g_1^\ell, g_2^m\right)$ such that $\varphi(ng) = n\varphi(g)$ for each $n \in \mathbb{N}$. Moreover, if $g \geq 2$ then $\varphi(n)$ is g-additive. In 2003, Puchta and Spilker [3] characterized those functions φ that are simultaneously g-additive and h-additive. In this work, we interest in g-additive for $g \in \mathbb{N}$ with $g \geq 2$ because we recapitulate work of Puchta and Spilker [3]. For $g, h, k \in \mathbb{N}$ with $g, h \geq 2$, define the step-function ℓ_k by

$$\ell_k(n) := [g,h]^{k-1} \left\lfloor \frac{n}{[g,h]^{k-1}} \right\rfloor \quad (n \in \mathbb{N}_0),$$

where $[g, h]$ denotes the least common multiple of g, h.

Proposition 1.1. [3, Proposition 1] *For $g, h \in \mathbb{N}$ with $g, h \geq 2$ and for $k \in \mathbb{N}$, if $[g,h]^{k-1} | \gcd(g,h)^k$, then ℓ_k is simultaneously g-additive and h-additive; if $\gcd(g,h) > 1$, then ℓ_k is $\gcd(g,h)$-additive.*

A function $\varphi : \mathbb{N}_0 \to \mathbb{C}$ is called

- p-periodic $(p \in \mathbb{N})$ if $\varphi(n+p) = \varphi(n)$ for all $n \in \mathbb{N}_0$,
- q-constant $(q \in \mathbb{N})$ if $\varphi(aq+b) = \varphi(aq)$ for all $a, b \in \mathbb{N}_0,\ b < q$.

For $k, g, h \in \mathbb{N}$, define

$$V_k := V_k(g;h) := \left\{\varphi_k : \mathbb{N}_0 \to \mathbb{C} : \varphi_k(0) = 0,\ \varphi_k \text{ is } \gcd(g,h)^k\text{-periodic and } [g,h]^{k-1}\text{-constant}\right\}.$$

It is easily checked that V_k is a $\mathbb{C}$-vector space.

Proposition 1.2. [3, Proposition 4] *If* $g, h, k \in \mathbb{N}$ *and* $\varphi_k \in V_k(g; h)$, *then*

(i) φ_k *is simultaneously* g*-additive and* h*-additive;*

(ii) *if* $\gcd(g, h) > 1$, *then* φ_k *is* $\gcd(g, h)$*-additive* .

Denote by V be the complex vector space of all simultaneously g-additive and h-additive functions. The main result in [3] is

Theorem 1.3. [3, Theorem] *Let* $g, h \in \mathbb{Z}$ *with* $g, h \geq 2$ *and assume* $g \nmid h$, $h \nmid g$. *Then*

(i) *every function* $\varphi \in V$, *can be uniquely represented in the form*

$$\varphi(n) = \sum_k^* c_k \ell_k(n) + \sum_k^* \varphi_k(n),$$

where $c_k \in \mathbb{C}$, $\varphi_k \in V_k(g; h)$, *and the star indicates that the sum extends over* $k \in \mathbb{N}$ *with the property that* $[g, h]^{k-1} | \gcd(g, h)^k$;

(ii) *the vector space* V *has dimension* $\dim_{\mathbb{C}} V = \sum_{k=1}^* \frac{\gcd(g,h)^k}{[g,h]^{k-1}}$;

(iii) *every function* $\varphi \in V$ *is already* $\gcd(g, h)$*-additive, if* $\gcd(g, h) > 1$.

Based upon Theorem 1.3, it is natural to ask for explicit shapes of elements and bases of the vector space V. Following the analysis of small cases, i.e., those with small values of g and h, we are able to obtain general forms of the above mentioned quantities.

Our main result is:

Theorem 1.4. *Let* g, h *be integer* ≥ 2. *Assume* $g \nmid h$ *and* $h \nmid g$ *so that they can be written as*

$$g = p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_j^{b_j} w_1^{d_1} \cdots w_f^{d_f}, \quad h = p_1^{a_1} \cdots p_s^{a_s} q_1^{c_1} \cdots q_j^{c_j} w_1^{c_1} \cdots w_f^{e_f}$$

where, for $1 \leq i \leq s$, $1 \leq r \leq j$, $1 \leq \ell \leq f$, *we set*

- p_i, q_r, w_ℓ *distinct prime numbers*
- $b_r, e_\ell \in \mathbb{N}$
- $c_r, d_\ell, a_i \in \mathbb{N}_0$ *with* $b_r > c_r$, $d_\ell < e_\ell$.

Then (i) every function $\varphi \in V$ can be uniquely represented in the form

$$\varphi(n) = \sum_{k=1}^{x} c_k \ell_k(n) + \sum_{k=1}^{x} \sum_{i=1}^{\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1} N_i^k B_i^k(n)$$

where $c_k, N_i^k \in \mathbb{C}$,

$$x = \min\left(\left\lfloor \frac{b_1}{b_1 - c_1} \right\rfloor, \ldots, \left\lfloor \frac{b_j}{b_j - c_j} \right\rfloor, \left\lfloor \frac{e_1}{e_1 - d_1} \right\rfloor, \ldots, \left\lfloor \frac{e_f}{e_f - d_f} \right\rfloor \right),$$

and

$$B_i^k(n) = \begin{cases} 1 & \text{if } n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1} - 1 (\operatorname{mod} \gcd(g,h)^k) \\ 0 & \text{otherwise,} \end{cases}$$

for $i \in \{0, 1, \ldots, \frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1\}$;
(ii) a basis of the complex vector-space V is given by

$$\left\{ \ell_1(n), \ldots, \ell_x(n), B_1^1(n), \ldots, B_{(g,h)-1}^1(n), \ldots, B_1^x(n), \ldots, B_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1}^x \right\}.$$

2 Proof of the main theorem

From the definition of V_k, we see that the function $\varphi_k \in V_k$ $(k \geq 1)$ satisfies

$$\varphi_k(n) = N_i^k \in \mathbb{C}$$

whenever

$$n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1} - 1 \ (\operatorname{mod} \gcd(g,h)^k)$$
$$\left(i \in \left\{ 0, 1, \ldots, \frac{(g,h)^k}{[g,h]^{k-1}} - 1 \right\} \right)$$

and $N_0^k = 0$ (because of $\varphi_k(0) = 0$).

Consider k such that $[g,h]^{k-1} |\ \gcd(g,h)^k$. Note that $\gcd(g,h) = p_1^{a_1} \cdots p_s^{a_s} q_1^{c_1} \cdots q_j^{c_j} w_1^{d_1} \cdots w_f^{d_f}$ and $[g,h] = p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_j^{b_j} w_1^{e_1} \cdots w_f^{e_f}$.

If $k = 1$, then $\gcd(g,h)$ is divisible by $[g,h]^{k-1} = 1$.

For $k \geq 2$, we have

$$\frac{\gcd(g,h)^k}{[g,h]^{k-1}} = \frac{\left(p_1^{a_1}\cdots p_s^{a_s} q_1^{c_1}\cdots q_j^{c_j} w_1^{d_1}\cdots w_f^{d_f}\right)^k}{\left(p_1^{a_1}\cdots p_s^{a_s} q_1^{b_1}\cdots q_j^{b_j} w_1^{e_1}\cdots w_f^{e_f}\right)^{k-1}} = \frac{\left(p_1^{a_1}\cdots p_s^{a_s} q_1^{c_1}\cdots q_j^{c_j} w_1^{d_1}\cdots w_f^{d_f}\right)}{\left(q_1^{b_1-c_1}\cdots q_j^{b_j-c_j} w_1^{e_1-d_1}\cdots w_f^{e_f-d_f}\right)^{k-1}}.$$

The divisibility condition requires that $(b_r - c_r)(k-1) \leq c_r$ and $(e_\ell - d_\ell)(k-1) \leq d_\ell$, and so

$$k \leq \frac{b_r}{b_r - c_r},\ k \leq \frac{e_l}{e_l - d_l}$$

for all $r = 1, 2, \ldots, j$; $\ell = 1, 2, \ldots, f$. Thus,

$$k \leq \min\left(\left\lfloor \frac{b_1}{b_1 - c_1}\right\rfloor, \ldots, \left\lfloor \frac{b_j}{b_j - c_j}\right\rfloor, \left\lfloor \frac{e_1}{e_1 - d_1}\right\rfloor, \ldots, \left\lfloor \frac{e_f}{e_f - d_f}\right\rfloor\right) =: x.$$

From Theorem 1.3 (i), every function $\varphi \in V$ can be uniquely represented in the form

$$\varphi(n) = \sum_{k=1}^{x} c_k \ell_k(n) + \sum_{k=1}^{x} \varphi_k(n)$$

where $c_k \in \mathbb{C}$ and $\varphi_k \in V_k$. Using the definition of V_k, we get

$$\varphi(n) = \sum_{k=1}^{x} c_k \ell_k(n) + \sum_{k=1}^{x} \sum_{i=1}^{\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1} N_i^k B_i^k(n)$$

where, for $i \in \{0, 1, \ldots, \frac{\gcd(g,h)^k}{[g,h]^{k-1}} - 1\}$, we have

$$n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1} - 1 (\bmod \gcd(g,h)^k)$$

with $N_i^k \in \mathbb{C}$ and

$$B_i^k(n) = \begin{cases} 1 & \text{if } n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1} - 1 (\bmod \gcd(g,h)^k) \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 1.1, we know that $\ell_k(n) \in V$. Since $B_i^k(n) \in V_1(g;h)$, by Proposition 1.2(**i**), we get $B_i^k(n) \in V$ for all i.

We next show that the set

$$S = \left\{\ell_1(n), \ldots, \ell_x(n), B_1^1(n), \ldots, B_{\gcd(g,h)-1}^1(n), \ldots, B_1^x(n), \ldots, B_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1}^x\right\},$$

is $\mathbb{C}$-linearly independent. If there is a $\mathbb{C}$-linear relation

$$A_1^0\ell_1(n) + \ldots + A_x^0\ell_x(n) + A_1^1B_1^1(n) + \ldots + A^1_{\gcd(g,h)-1}B^1_{\gcd(g,h)-1}(n) + A_1^2B_1^2(n) + \cdots$$
$$+ A^2_{\frac{\gcd(g,h)^2}{[g,h]}-1}B^2_{\frac{\gcd(g,h)^2}{[g,h]}-1}(n) + \cdots + A_1^xB_1^x(n) + \cdots + A^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1}B^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1}(n) = 0,$$

then upon substituting n by $1, 2, \ldots, \gcd(g,h)$ we get the following system of equations

$$\begin{aligned} A_1^0\ell_1(1) + A_1^1 &= 0 \\ A_1^0\ell_1(2) + A_2^1 &= 0 \\ &\vdots \\ A_1^0\ell_1(\gcd(g,h)-1) + A^1_{\gcd(g,h)-1} &= 0 \\ A_1^0\ell_1(\gcd(g,h)) &= 0. \end{aligned}$$

Solving this system back and forth, we get

$$A_1^0 = A_1^1 = A_2^1 = \ldots = A^1_{\gcd(g,h)-1} = 0.$$

Substituting n by $[g,h], 2[g,h], \ldots, \left(\frac{\gcd(g,h)^2}{[g,h]} - 1\right)[g,h], \gcd(g,h)^2$, we get the following system of equations

$$\begin{aligned} A_2^0\ell_2([g,h]) + A_1^2 &= 0 \\ A_2^0\ell_2(2[g,h]) + A_2^2 &= 0 \\ &\vdots \\ A_2^0\ell_2\left(\left(\frac{\gcd(g,h)^2}{[g,h]} - 1\right)[g,h]\right) + A^2_{\frac{\gcd(g,h)^2}{[g,h]}-1} &= 0 \\ A_2^0\ell_2\left(\gcd(g,h)^2\right) &= 0. \end{aligned}$$

Again solving the system back and forth, we get

$$A_2^0 = A_1^2 = A_2^2 = \ldots = A^2_{\frac{\gcd(g,h)^2}{[g,h]}-1} = 0.$$

Substituting n by $[g,h]^2, 2[g,h]^2, \ldots, \left(\frac{\gcd(g,h)^3}{[g,h]^2} - 1\right)[g,h]^2, \gcd(g,h)^3$, we get the following system of equations

$$\begin{aligned} A_3^0 \ell_3\left([g,h]^2\right) + A_1^3 &= 0 \\ A_3^0 \ell_3\left(2\,[g,h]^2\right) + A_2^3 &= 0 \\ &\vdots \\ A_3^0 \ell_3\left(\left(\frac{\gcd(g,h)^3}{[g,h]^2} - 1\right)[g,h]^2\right) + A^3_{\frac{\gcd(g,h)^3}{[g,h]^2}-1} &= 0 \\ A_3^0 \ell_3\left(\gcd(g,h)^3\right) &= 0. \end{aligned}$$

Again solving the system back and forth, we get

$$A_3^0 = A_1^3 = A_2^3 = \ldots = A^3_{\frac{\gcd(g,h)^3}{[g,h]^2}-1} = 0.$$

Continuing in the same manner, until finally, substituting n by $[g,h]^{x-1}, 2[g,h]^{x-1}, \ldots, \left(\frac{\gcd(g,h)^x}{[g,h]^{x-1}} - 1\right)[g,h]^{x-1}, \gcd(g,h)^{x+1}$, we get the following system of equations

$$\begin{aligned} A_x^0 \ell_x\left([g,h]^{x-1}\right) + A_1^x &= 0 \\ A_x^0 \ell_x\left(2[g,h]^{x-1}\right) + A_2^x &= 0 \\ &\vdots \\ A_x^0 \ell_x\left(\left(\frac{\gcd(g,h)^x}{[g,h]^{x-1}} - 1\right)[g,h]^{x-1}\right) + A^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1} &= 0 \\ A_x^0 \ell_x\left(\gcd(g,h)^x\right) &= 0. \end{aligned}$$

As before, the solution of this last system is

$$A_x^0 = A_1^x = A_2^x = \ldots = A^x_{\frac{\gcd(g,h)^x}{[g,h]^{x-1}}-1} = 0,$$

showing that the set S is $\mathbb{C}$-linearly independent and is thus a basis of V, yielding

$$\dim V = \sum_{k=1}^{x} \frac{\gcd(g,h)^k}{[g,h]^{k-1}},$$

which concludes the proof of the main theorem.

We end this paper with two examples.

Example 2.1. Let $g = 2^2 3^3 5^3$ and $h = 2^2 3^2 7^2$. Adopting the notation of Theorem 1.4, we have

$$p_1 = 2, q_1 = 3, q_2 = 5, w_1 = 7, a_1 = 2, b_1 = 3, b_2 = 3, c_1 = 2, c_2 = 0, d_1 = 0, e_1 = 2.$$

Thus, $\gcd(g,h) = 2^2 3^2 = 36$, $[g,h] = 2^2 3^3 5^3 7^2 = 66150$, and so
$x = \min\left(\left\lfloor \frac{3}{3-2} \right\rfloor, \left\lfloor \frac{3}{3-0} \right\rfloor, \left\lfloor \frac{2}{2-0} \right\rfloor \right) = 1$. By Theorem 1.4(i), we have

$$\varphi(n) = \sum_{k=1}^{1} c_k \ell_k(n) + \sum_{k=1}^{1} \sum_{i=1}^{\frac{\gcd(g,h)^k}{g,h^{k-1}}-1} N_i^k B_i^k(n) = cn + \sum_{i=1}^{35} N_i^1 B_i^1(n),$$

where $n \equiv i(\mathrm{mod}36)$, $c, N_i^1 \in \mathbb{C}$ and $B_i^1(n) = 1$ for $n \equiv i(\mathrm{mod}36)$ and $B_i^1(n) = 0$ otherwise. By Theorem 1.4(ii), a basis of V is

$$\{n, B_1^1(n), B_2^1(n), \ldots, B_{35}^1(n)\}$$

and $\dim_{\mathbb{C}} V = 36$.

Example 2.2. Let $g = 2^2 5^5 11^2$ and $h = 2^2 5^3 11^4$. Here,

$$p_1 = 2, q_1 = 5, w_1 = 2, a_1 = 2, b_1 = 5, c_1 = 3, d_1 = 2, e_1 = 4.$$

Then $\gcd(g,h) = 2^2 5^3 11^2 = 60500$, $[g,h] = 2^2 5^5 11^4 = 183012500$,
$x = \min\left(\left\lfloor \frac{5}{5-3} \right\rfloor, \left\lfloor \frac{4}{4-2} \right\rfloor \right) = 2$. By Theorem 1.4(i), we have

$$\begin{aligned}\varphi(n) &= \sum_{k=1}^{2} c_k \ell_k(n) + \sum_{k=1}^{2} \sum_{i=1}^{\frac{\gcd(g,h)^k}{[g,h]^{k-1}}-1} N_i^k B_i^k(n) \\ &= c_1 \ell_1(n) + c_2 \ell_2(n) + \sum_{i=1}^{60499} N_i^1 B_i^1(n) + \sum_{i=1}^{19} N_i^2 B_i^2(n),\end{aligned}$$

where

$$n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1} - 1(\mathrm{mod}\ \gcd(g,h)^k),$$

$c_k, N_i^k \in \mathbb{C}$ $(k = 1, 2)$,

$$B_i^k(n) = 1 \text{ for } n \equiv i[g,h]^{k-1}, i[g,h]^{k-1}+1, \ldots, (i+1)[g,h]^{k-1} - 1\,(\mathrm{mod}\ \gcd(g,h)^k)\}$$

and $B_i^k(n) = 0$ otherwise. By Theorem 1.4(ii), a basis of V is

$$\{n, \ell_2(n), B_1^1(n), \ldots, B_{605499}^1(n), B_1^2(n), \ldots, B_{19}^2(n)\}$$

and $\dim_{\mathbb{C}} V = \sum_{k=1}^{2} \frac{(g,h)^k}{[g,h]^{k-1}} = 60500 + 20 = 60520.$

References

[1] H. Delange, *Sur les function q-additive ou q-multiplicatives*, Acta Arith. **21** (1972), 285–298.

[2] A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives domées*, Acta Arith. **13** (1968), 259–265.

[3] J.-C. Puchta and J.Spilker, *Function that are both g- and h-additive*, Arch. Math. **80** (2003), 264–270.

[4] T. Toshimitsu, *Strongly q-additive functions and algebraic independence*, Tokyo J.Math. **21** (1998), 107–117.

[5] Y. Uchida, *On p- and q-additive functions*, Tokyo J.Math. **22** (1999), 83–87.

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1. Seekam, N., Laohakosol, V. and Mavecha, S. 2018."Explicit determination of functions which are additive with respect to two parameters" Proceedings of Annual Pure and Applied Mathematics Conference ,Chulalongkorn University, Bangkok, Thailand, 30-31 May and 1 June 2018. pp.128-136.