รายงานการวิจัยฉบับสมบูรณ์

การประมาณพารามิเตอร์ของการแจกแจงปัวซงโดยวิธีความควรจะเป็นสูงสุด วิธีมาร์คอฟ เชน มอนติคาร์โล และวิธีของเบส์

The Parameter Estimation of Poisson Distribution by Using Maximum Likelihood, Markov Chain Monte Carlo, and Bayes methods

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Research Title : The Parameter Estimation of Poisson Distribution by Using Maximum Likelihood, Markov Chain Monte Carlo, and Bayes methods

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ABSTRACT

The objective of this research is to test a hypothesis that the means of Poisson parameter estimations obtained from Maximum Likelihood, Markov Chain Monte Carlo, and Bayes method were not different from the true parameters. Data was simulated from a Poisson distribution with the true λ parameter set at 0.5, 2, 5, 10, and 20 and the sample size at 5, 10, 30,50, 100, and 200. The results are as follows: the Maximum Likelihood method produced means of parameter estimations that were not perceivably different from the true parameters in all cases. On the other hand, the Markov Chain Monte Carlo and the Bayes methods produced dissimilar estimations to the true λ parameters when the sample sizes and the true λ parameters were small. Additionally, the maximum likelihood method produced minimum mean square errors when the sample sizes and the true parameters were small while the Markov Chain Monte Carlo and Bayes method did so when the sample sizes and the true parameters were large.

The interval estimation is evaluated by Maximum Likelihood (ML) method, Markov Chain Monte Carlo (MCMC) method, and Bayes method from a point estimation to estimate confidence interval. The confidence coefficients are approximated by considering the proportion when the upper and lower of confidence interval are covered the true parameters. If the Confidence Coefficients (CC) are greater than the fixed confidence interval, the Average Width (AW) of the confidence interval will focus the performance of these methods. In this case, the data is generated by Monte Carlo process depended on Poisson distribution with true parameters 0.5, 2, 5, 10, and 20, sample sizes 30, 50, 100, 300, and 500, and the 90%, 95%, and 99% confidence interval. The performance of three methods is compared by the CC and the AW values. The output is showed that ML method outperforms the other methods when true parameter is small values (0.5) for all sample sizes. For the large sample sizes (100,300,500), MCMC method is reasonably performed when true parameter is 5. Furthermore, the Bayes method is a good performance in most cases.

Keywords: Bayes Method; Maximum Likelihood Method; Markov Chain Monte Carlo Method; Poisson Distribution.

CONTENTS

Chapter 1

Point Estimation

1.1 Introduction

Today, tremendous amount of new data is generated at an increasing rate. It is usually not possible to collect and measure all units of data in a population because of restrictions in time, budget, and labor. Any characteristics of a population are parameters of population. Since it is not possible to acquire and study all units of data in a population, we only collect and study a small part, a sample, of the population instead. We use a sample to estimate the true parameter of the population. We call it an estimator. In inferential statistics, a realized value of an estimator is used to describe the population.

Point estimation is a part of inferential statistics that uses a sample to get an estimator for interpreting the population. Some properties of a good point estimator are unbiasedness, sufficiency, completeness, and minimum variance unbiased estimator. There are several point estimation methods such as the moments method, the maximum likelihood method, the minimum chi-square method, the least squares method, and the Bayes method. We were interested in the maximum likelihood method because its estimator is a class of a minimum variance unbiased estimator. We were also interested in the Bayes method because it uses both a prior probability distribution and a posterior probability distribution to find an estimator. However, it is fairly difficult to demonstrate a posterior distribution from a probability distribution and a prior distribution. Thus, we also considered the Markov Chain Monte Carlo method [1]. It can overcome this particular problem because it uses only the posterior distribution to make statements about parameter estimation.

In this case, we were interested the experimental outcomes that occur randomly for the counts of events within intervals of time and space. The Poisson distribution is a discrete distribution that observe the counts of event in a given interval of time. The parameter of the Poisson distribution is the mean number of events per an interval of time. We set out to find Poisson parameter estimators by using the Maximum Likelihood (ML) method, the Markov Chain Monte Carlo (MCMC) method, and the Bayes method.

The data. we used were simulated from a Poisson distribution with varying true parameters and sample sizes.

1.2 Methods for Parameter Estimation

The parameter estimation of the Poisson distribution consist of the following three methods.

1.2.1 Maximum Likelihood (ML) Method

The ML method is the most popular techniques for deriving an estimator because it is simple to understand and to calculate the estimators. The solution of this method is established from the maximum of the likelihood function.

Let $X_1,\ldots,X_n$ be independent and identically distributed (iid) random variables following a Poisson distribution with parameter λ, and $f(x_i\,|\,\lambda)$ denote the probability density function of $x_i\,|\,\lambda$. Then,

$$f(x_i\,|\,\lambda) \;=\; \frac{e^{-\lambda}\lambda^{x_i}}{x_i\,!} \quad , \quad x_i = 0,1,2,\ldots \quad .$$

Hence, the likelihood function is

$$L(\lambda) \;=\; \prod_{i=1}^{n} f(x_i\,|\,\lambda).$$

The ML estimator of the parameter λ is solved as follows:

$$L(\lambda) \;=\; \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i\,!},$$

$$\ln L(\lambda) \;=\; -n\lambda + \sum_{i=1}^{n} x_i \ln\lambda - \ln\left(\prod_{i=1}^{n} x_i\,!\right),$$

$$\frac{\partial \ln L(\lambda)}{\partial\lambda} \;=\; -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} \;=\; 0\,,$$

$$\lambda = \frac{\sum_{i=1}^{n} x_i}{n}, \text{ and}$$

$$\hat{\lambda} = \bar{x}.$$

The second derivative of $\ln L(\lambda)$ is

$$\frac{\partial^2 \ln L(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2} < 0.$$

Therefore, the ML estimator is $\hat{\lambda}_{MLE} = \bar{x}$.

1.2.2 Bayes Method

Let $X_1,\ldots,X_n$ be iid random variables following a Poisson distribution with parameter λ, and $f(x_i \mid \lambda)$ denote the probability density function of $x_i \mid \lambda$.

The likelihood function can be written as

$$L(\lambda) = \prod_{i=1}^{n} f(x_i \mid \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}.$$

The prior distribution of λ is a gamma distribution with parameters a,b defined as Gamma(a,b) or rewritten as

$$g(\lambda \mid a,b) = \frac{\lambda^{a-1} e^{-\frac{\lambda}{b}}}{\Gamma(a) b^a}, \quad \lambda > 0.$$

The Poisson and gamma distributions are the conjugate distribution because the Poisson distribution lists the parameter λ which is similar to the format of a gamma distribution.

The posterior distribution of λ given x_i is

$$h(\lambda \mid x_i) = \frac{f(x_i \mid \lambda) g(\lambda \mid a,b)}{\int f(x_i \mid \lambda) g(\lambda \mid a,b)\, d\lambda}$$

$$= \frac{\dfrac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\dfrac{\lambda^{a-1}e^{-\frac{\lambda}{b}}}{\Gamma(a)b^{a}}}{\int_0^\infty \dfrac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\dfrac{\lambda^{a-1}e^{-\frac{\lambda}{b}}}{\Gamma(a)b^{a}}d\lambda}$$

$$= \frac{e^{-(n+\frac{1}{b})\lambda}\lambda^{n\bar{x}+a-1}}{\int_0^\infty e^{-(n+\frac{1}{b})\lambda}\lambda^{n\bar{x}+a-1}d\lambda}.$$

The term $\int_0^\infty e^{-(n+\frac{1}{b})\lambda}\lambda^{n\bar{x}+a-1}d\lambda$ is in the form of a gamma function and can be rewritten as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$
$$= (\alpha-1)\Gamma(\alpha-1)$$

since

$$\int_0^\infty e^{-(n+\frac{1}{b})\lambda}\lambda^{n\bar{x}+a-1}d\lambda = \frac{1}{(n+\frac{1}{b})^{n\bar{x}+a-2}}\int_0^\infty e^{-(n+\frac{1}{b})\lambda}[(n+\frac{1}{b})\lambda]^{(n\bar{x}+a-1)}d(n+\frac{1}{b})\lambda$$

$$= \frac{\Gamma(n\bar{x}+a)}{(n+\frac{1}{b})^{n\bar{x}+a-2}}.$$

Hence, the posterior distribution is written as

$$h(\lambda|x_i) = \frac{e^{-(n+\frac{1}{b})\lambda}\lambda^{n\bar{x}+a-1}}{\dfrac{\Gamma(n\bar{x}+a)}{(n+\frac{1}{b})^{n\bar{x}+a-2}}}.$$

The posterior mean of λ is obtained through the following sequence of derivation:

$$E(\lambda \mid x_i) = \frac{\int_0^\infty \lambda e^{-(n+\frac{1}{b})\lambda} \lambda^{n\bar{x}+a-1} d\lambda}{\frac{\Gamma(n\bar{x}+a)}{(n+\frac{1}{b})^{n\bar{x}+a-2}}} = \frac{\int_0^\infty e^{-(n+\frac{1}{b})\lambda} \lambda^{n\bar{x}+a} d\lambda}{\frac{\Gamma(n\bar{x}+a)}{(n+\frac{1}{b})^{n\bar{x}+a-2}}}$$

$$= \frac{\frac{\Gamma(n\bar{x}+a+1)}{(n+1/b)^{n\bar{x}+a-1}}}{\frac{\Gamma(n\bar{x}+a)}{(n+1/b)^{n\bar{x}+a-2}}} = \frac{(n\bar{x}+a)\Gamma(n\bar{x}+a)\,(n+1/b)^{n\bar{x}+a-2}}{\Gamma(n\bar{x}+a)(n+1/b)^{n\bar{x}+a-1}}$$

$$= \frac{(n\bar{x}+a)b}{nb+1}.$$

The Bayes estimator is

$$\hat{\lambda}_{Bayes} = \frac{(n\bar{x}+a)b}{nb+1}.$$

The quality of the estimated λ is measured through a loss function. For example, the squared error loss function is

$$L(\lambda, \hat{\lambda}(x)) = \left[\lambda - \hat{\lambda}(x)\right]^T \left[\lambda - \hat{\lambda}(x)\right].$$

The posterior Bayes risk is

$$\int L(\lambda, \hat{\lambda}(x))\, f(\lambda \mid x)\, d\lambda = \int \left[\lambda - \hat{\lambda}(x)\right]^T \left[\lambda - \hat{\lambda}(x)\right] f(\lambda \mid x) d\lambda.$$

The Posterior Bayes risk estimator is writte

$$\frac{\partial}{\partial \hat{\lambda}} \int L(\lambda, \hat{\lambda}(x)) f(\lambda \mid x) d\lambda = \frac{\partial}{\partial \hat{\lambda}} \int \left[\lambda - \hat{\lambda}(x)\right]^T \left[\lambda - \hat{\lambda}(x)\right] f(\lambda \mid x) d\lambda,$$

$$= -2\int \left[\lambda - \hat{\lambda}(x)\right] f(\lambda \mid x) d\lambda = 0,$$

$$\Rightarrow \int \lambda f(\lambda \mid x) d\lambda = \int \hat{\lambda}(x) f(\lambda \mid x) d\lambda,$$

$$\Rightarrow \hat{\lambda}_{bayes} = \int \lambda(x) f(\lambda \mid x) d\lambda,$$

$$\Rightarrow \hat{\lambda}_{bayes} = E(\lambda \mid x).$$

The posterior Bayes risk estimator becomes the Bayes estimator.

1.2.3 Markov Chain Monte Carlo Method

Bayesian analysis treats all parameters as random, assigns prior distributions to characterize knowledge about parameter values, and uses the posterior distribution given the observed data as the basis of inference. Often the posterior distribution is a complicated models with many parameters, so statisticians have developed simulated methods to generate samples from the posterior distribution, namely the Markov Chain Monte Carlo (MCMC) method. Gibbs Sampling ([2]-[4]) is a popular MCMC method that generates values which are always moving to new values, and most importantly, does not require a specification of proposed distributions.

We carried out the Gibbs Sampling by means of a software package known as WinBUGS (Bayesian Inference Using Gibbs Sampling) introduced by Spiegelhalter *et al.*,[5]. We use the MCMC samples of the parameter obtained via WinBUGS to compute approximate posterior summaries as the posterior distribution.

Let $X_1,...,X_n$ be iid random variables following a Poisson distribution with parameter λ, and let λ be a random variable of gamma distribution with parameters a and b. The estimated parameters are λ, a, and b.

The algorithm of Gibbs sampling from Markov Chain Monte Carlo [7] proceeds as follows.

1. Set initial values $a^{(t)}$ from an exponential distribution with parameters 1 and $b^{(t)}$ from the gamma distribution with parameter $(0.1,1)$.
   Notice that a,b are the parameters of the gamma distribution and that the values of a,b are greater than zero, which is supported the exponential and the gamma distribution.

2. For $t=1,2,...,T$ update $a^{(t)}$ and $b^{(t)}$.
3. Generate $\lambda^{(t)}$ from the posterior distribution function based on the gamma distribution with the parameters $a^{(t)}$ and $b^{(t)}$ following 1.
4. Plot the density of the posterior distribution function.
5. Calculate the mean, the median, and the standard deviation from the posterior distribution function.

For each chain, the first 2000 iterations were discarded and the last 5000 iterations were used to obtain the posterior distribution of the parameters.

Thus, the MCMC estimator is

$$\hat{\lambda}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}\lambda^{(t)}.$$

Moreover, the MCMC method is obtaining a and b and approximating as follows:
$\hat{a}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}a^{(t)}$, and $\hat{b}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}b^{(t)}$.

The Bayes estimator is used $\hat{a}_{MCMC}$ and $\hat{b}_{MCMC}$ to compute the parameters λ as follows:

$$\hat{\lambda}_{Bayes} = \frac{(n\bar{x} + \hat{a}_{MCMC})\hat{b}_{MCMC}}{n\,\hat{b}_{MCMC} + 1}.$$

1.3 Research Scope

This section discusses a simulation study for investigating the performance of the ML method, the Markov Chain Monte Carlo method, and the Bayes method. In this case, the estimators are

$$\hat{\lambda}_{MLE} = \bar{x},$$

$$\hat{\lambda}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}\lambda^{(t)},$$

$$\hat{\lambda}_{Bayes} = \frac{(n\bar{x} + \hat{a}_{MCMC})\hat{b}_{MCMC}}{n\,\hat{b}_{MCMC} + 1}.$$

1.3.1 The procedure for the simulation study

In order to simulate the random variable X_i's that follow Poisson distributions with the true parameter λ of 0.5, 2, 5, 10, and 20, we proceed as follows.

1.3.1.1 Prior distribution is defined as the gamma distribution with parameters a and b.

1.3.1.2 The sample sizes are considered at n = 5, 10, 30, 50, 100, and 200.

1.3.1.3 The data is generated 500 times in each case with the R program [7].

1.3.2 A Test Statistic

A statistic is used to whether the means of a parameters is different from the true values of the parameter. In this case, the hypotheses are

$H_0 : \mu_{\hat{\lambda}} = \lambda$ and $H_1 : \mu_{\hat{\lambda}} \neq \lambda$.

The t statistic is computed as follows:

$$t = \frac{\bar{\hat{\lambda}} - \lambda}{s_{\hat{\lambda}} / \sqrt{n}},$$

where $s_{\hat{\lambda}} = \sqrt{\frac{\sum_{j=1}^{500} (\hat{\lambda}_j - \bar{\hat{\lambda}})^2}{n-1}}$, and df = n-1.

For the level of significance at α, we will reject H_0 if $|t| > t_{\alpha/2,n-1}$.

Hence, a lower and upper bounds of (1-α)100% confidence interval are computed by

$$\lambda = \bar{\hat{\lambda}} \pm t_{\alpha/2,n-1} \frac{s_{\hat{\lambda}}}{\sqrt{n}}.$$

1.3.3 The Criterion for data analysis

The Mean Square Error (MSE) is the criterion to indicate the performance of parameter estimation from these methods and is computed by

$$MSE = \frac{\sum_{j=1}^{500} (\lambda - \hat{\lambda}_j)^2}{500}.$$

1.4 The Results

In this section, the parameter estimation with Poisson distribution is appeared from generated data with the true parameters and the sample sizes in previous section. Tables 1-3 showed the results in the form of tables and histograms. The first and the second columns of these tables represented the sample sizes and the true parameters from simulated data. A mean, a standard deviation, a lower and upper bounds of 95% confidence interval were given in the next four columns. The last two columns of these tables listed the t statistics and p-values for hypothesis testing.

By observing the p-values, the results appear as follows:

1. ML method

From Table 1, the ML method indicated that the means of the estimated parameters were not different from the true parameters in all cases. The histograms of the estimated parameters of all true parameters were presented in Figures 1-5. The histograms follow a normal distribution for large sample sizes.

2. MCMC Method

The p-values of the MCMC estimators from Table 2 indicated that the means of the estimated parameters were not different from the true parameters when $\lambda = 0.5$ at n = 5, 10, 30 ,and 50 and when $\lambda = 2$ at n = 10. The histograms of the estimated parameters of all true parameters were presented in Figures 6-10. Similar to the results from the ML method, the histograms follow a normal distribution.

3. Bayes Method

The p-values of the Bayes estimators from Table 3 show that the means of the estimated parameters were not different from the true parameters when $\lambda = 0.5$ at n = 5, 10 , and 30 and when $\lambda = 2$ at n = 5 and 10. The histograms of the estimated parameters of all of the true parameters were presented in Figures 11-15. Similar to the results from the previous two methods, the histograms follow a normal distribution.

Table 1. The mean, standard deviation (S.D.), lower confidence interval (LCI), upper confidence interval (UCI), t statistics (t), and p-values obtained via the ML method.

Sample sizes	λ	Mean	S.D.	LCI	UCI	t	p-values
n = 5	0.5	0.4924	0.3073	0.4653	0.5194	-0.5530	0.5805
	2	2.0228	0.6809	1.9629	2.0826	0.7487	0.4544
	5	5.0344	1.0808	4.9394	5.1293	0.7117	0.4770
	10	9.9056	1.4126	9.7814	10.0297	-1.4943	0.1357
	20	20.0956	1.8971	19.9289	20.2623	1.1268	0.2604
n= 10	0.5	0.5204	0.2337	0.4998	0.5409	1.9512	0.0515
	2	2.0186	0.4478	1.9792	2.0579	0.9287	0.3535
	5	5.0408	0.6998	4.9793	5.1022	1.3036	0.1930
	10	9.9790	1.0070	9.8905	10.0674	-0.4663	0.6412
	20	20.0206	1.4705	19.8913	20.1498	0.3132	0.7542
n= 30	0.5	0.5086	0.1355	0.4966	0.5205	1.4185	0.1567
	2	2.0006	0.2671	1.9771	2.0241	0.0558	0.9555
	5	4.9948	0.4209	4.9578	5.0318	-0.2727	0.7852
	10	10.0226	0.5786	9.9717	10.0734	0.8733	0.3829
	20	20.0502	0.7891	19.9809	20.1196	1.4243	0.1550
n= 50	0.5	0.5011	0.0984	0.4925	0.5098	0.2634	0.7924
	2	1.9960	0.2098	1.9776	2.0145	-0.4176	0.6764
	5	4.9969	0.3270	4.9682	5.0256	-0.2079	0.8354
	10	10.0187	0.4348	9.9802	10.0572	0.9508	0.3391
	20	20.0075	0.5807	19.9564	20.0585	0.2895	0.7723

n= 100	0.5	0.5016	0.0733	0.4952	0.5081	0.5063	0.6129
	2	2.0085	0.1371	1.9965	2.0206	1.3986	0.1626
	5	5.0163	0.2139	4.9975	5.0351	1.7090	0.0879
	10	10.0102	0.3063	9.9832	10.0371	0.7444	0.4571
	20	20.0006	0.4478	19.9612	20.0399	0.0300	0.9761
n = 200	0.5	0.5006	0.0497	0.4962	0.5050	0.2919	0.7705
	2	2.0042	0.0981	1.9956	2.0128	0.9709	0.3321
	5	5.0057	0.1569	4.9919	5.0195	0.8176	0.4140
	10	10.0140	0.2270	9.9904	10.0303	1.0270	0.3049
	20	19.9739	0.3169	19.9460	20.0017	-1.8404	0.0663

* indicates significance level at 5%

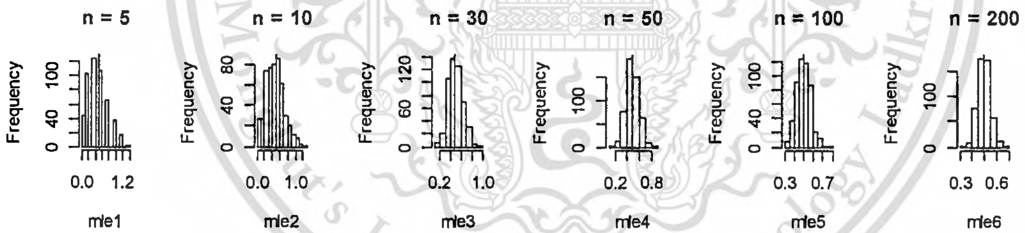

Fig.1. Histograms of estimated parameters λ with ML method when $\lambda = 0.5$.

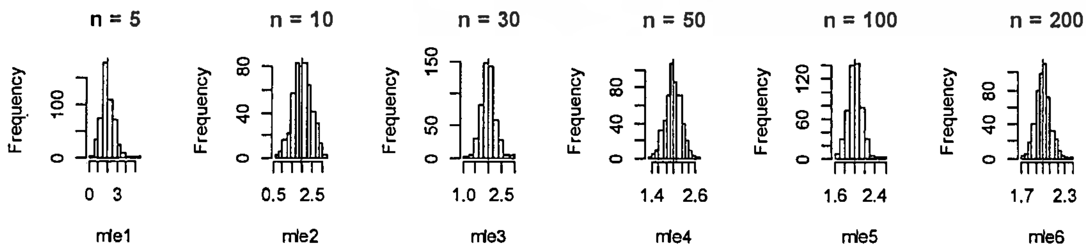

Fig.2. Histograms of estimated parameters λ with ML method when $\lambda = 2$.

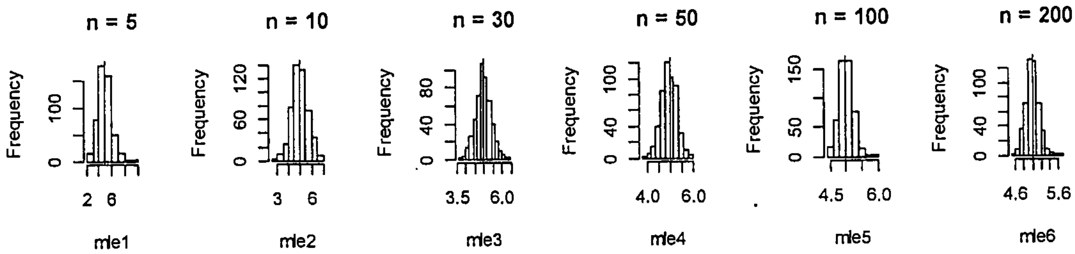

Fig.3. Histograms of estimated parameters λ with ML method when $\lambda = 5$.

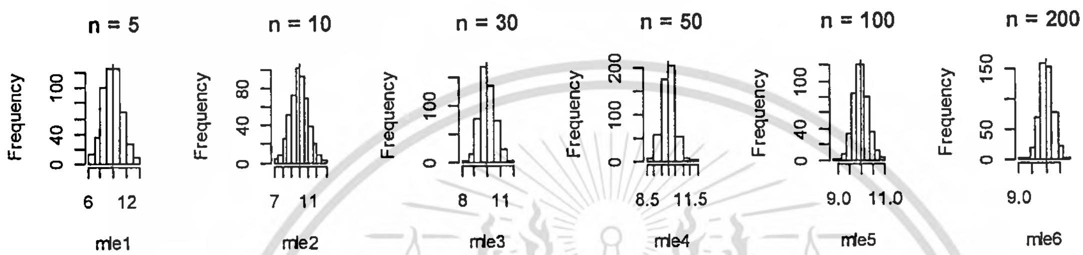

Fig.4. Histograms of estimated parameters λ with ML method when $\lambda = 10$.

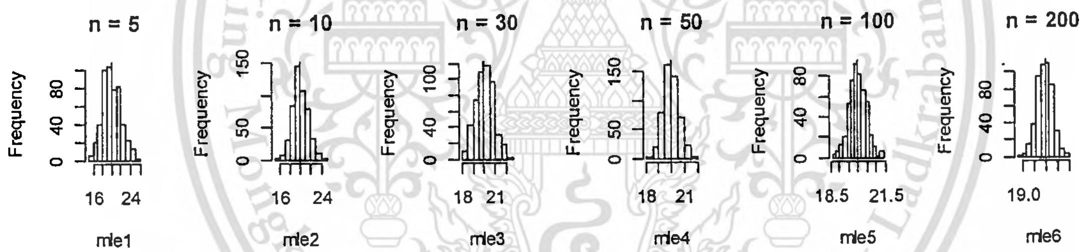

Fig.5. Histograms of estimated parameters λ with ML method when $\lambda = 20$.

Table 2. The mean, standard deviation (S.D.), lower confidence interval (LCI), upper confidence interval (UCI), t statistic (t), and p-values obtained via the MCMC method.

Sample sizes	λ	Mean	S.D.	LCI	UCI	t	p-values
n = 5	0.5	0.5567	0.3084	0.5295	0.5838	4.1100	0.0000*
	2	2.0567	0.6690	1.9980	2.1155	1.8981	0.0582
	5	5.0431	1.0791	4.9483	5.1379	0.8942	0.3717
	10	9.9058	1.4062	9.7822	10.0293	-1.4978	0.1348
	20	20.0675	1.9119	19.8995	20.2355	0.7896	0.4301
n= 10	0.5	0.5573	0.2327	0.5368	0.5778	5.5088	0.0000*
	2	2.0399	0.4470	2.0006	2.0792	1.9984	0.0462*
	5	5.0444	0.6975	4.9831	5.1057	1.4246	0.1549
	10	9.9756	1.0131	9.8866	10.0646	-0.5372	0.5924
	20	20.0145	1.4717	19.8851	20.1438	0.2202	0.8258
n= 30	0.5	0.5218	0.1361	0.5099	0.5338	3.5921	0.0003*
	2	2.0048	0.2671	1.9813	2.0283	0.4071	0.6841
	5	4.9987	0.4208	4.9618	5.0357	-0.0640	0.9490
	10	10.0242	0.5785	9.9733	10.0750	0.9356	0.3500
	20	20.0590	0.7912	19.9895	20.1285	1.6686	0.0950
n= 50	0.5	0.5094	0.0984	0.5007	0.5180	2.1387	0.0329*
	2	1.9956	0.2095	1.9772	2.0140	-0.4629	0.6436
	5	4.9997	0.3268	4.9710	5.0284	-0.0193	0.9846
	10	10.0177	0.4390	9.9791	10.0562	0.9018	0.3674
	20	20.0145	0.5827	19.9633	20.0657	0.5584	0.5768

n= 100	0.5	0.5053	0.0736	0.4988	0.5118	1.6620	0.1054
	2	2.0082	0.1367	1.9962	2.0202	1.3507	0.1774
	5	5.0178	0.2130	4.9990	5.0365	1.8702	0.0620
	10	10.0083	0.3062	9.9813	10.0352	0.6066	0.5444
	20	19.9993	0.4491	19.9598	20.0388	-0.0323	0.9742
n = 200	0.5	0.5019	0.0493	0.4976	0.5062	0.8785	0.3801
	2	2.0052	0.0987	1.9966	2.0139	1.1959	0.2323
	5	5.0042	0.1564	4.9905	5.0180	0.6096	0.5424
	10	10.0089	0.2266	9.9889	10.0288	0.8776	0.3806
	20	19.9729	0.3168	19.9450	20.0007	-1.9103	0.0560

* indicates significance level at 5%

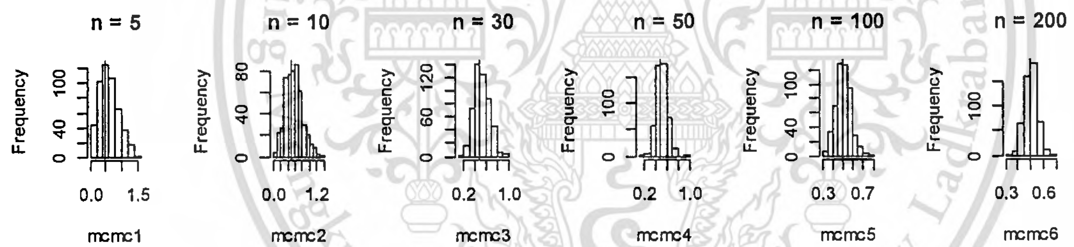

Fig.6. Histograms of estimated parameters λ with MCMC method when $\lambda = 0.5$.

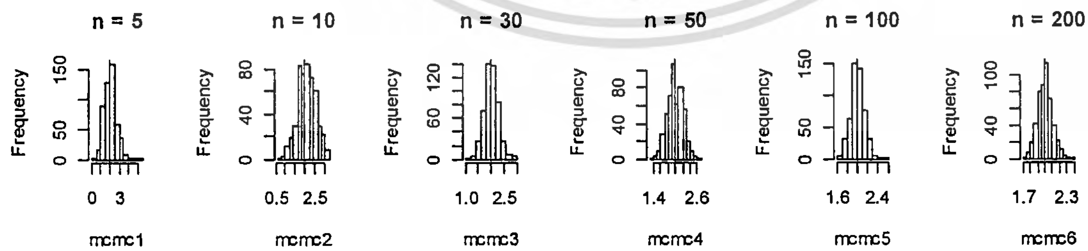

Fig .7. Histograms of estimated parameters λ with MCMC method when $\lambda = 2$.

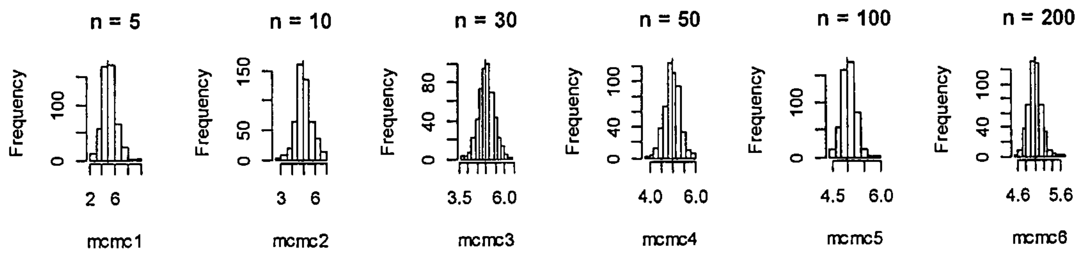

Fig.8. Histograms of estimated parameters λ with MCMC method when $\lambda = 5$.

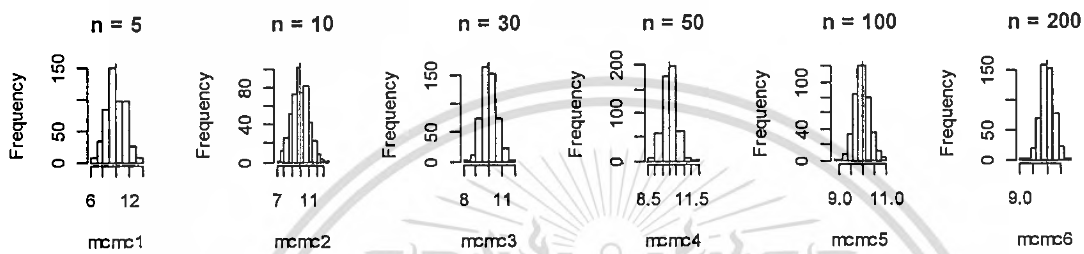

Fig.9. Histograms of estimated parameters λ with MCMC method when $\lambda = 10$.

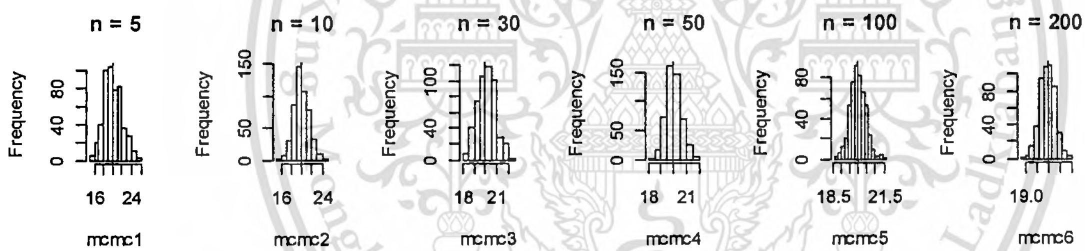

Fig.10. Histograms of estimated parameters λ with MCMC method when $\lambda = 20$.

Table 3. The mean, standard deviation (S.D.), lower confidence interval (LCI), upper confidence interval (UCI), t statistic (t), and p-values obtained via the Bayes method.

Sample sizes	λ	Mean	S.D.	LCI	UCI	t	p-values
n = 5	0.5	0.5566	0.3069	0.5297	0.5836	4.1290	0.0000*
	2	2.0625	0.6718	2.0035	2.1216	2.0825	0.0378*
	5	5.0482	1.0754	4.9537	5.1427	1.0023	0.3167
	10	9.9043	1.4101	9.7804	10.0282	-1.5268	0.1300
	20	20.0830	1.8977	19.9162	20.2497	0.9781	0.3285
n= 10	0.5	0.5554	0.2323	0.5349	0.5758	5.3332	0.0000*
	2	2.0408	0.4449	2.0017	2.0799	2.0535	0.0405*
	5	5.0495	0.6976	4.9882	5.1108	1.5866	0.1132
	10	9.9796	1.0064	9.8912	10.0680	-0.4523	0.6513
	20	20.0169	1.4697	19.8878	20.1461	0.2582	0.7964
n= 30	0.5	0.5207	0.1351	0.5088	0.5326	3.4304	0.0006*
	2	2.0080	0.2665	1.9846	2.0315	0.6787	0.4977
	5	4.9977	0.4204	4.9607	5.0346	-0.1205	0.9041
	10	10.0229	0.5785	9.9720	10.0737	0.8853	0.3764
	20	20.0484	0.7890	19.9791	20.1177	1.3734	0.1702
n= 50	0.5	0.5084	0.0983	0.4997	0.5170	1.9126	0.0563
	2	2.0004	0.2095	1.9820	2.0188	0.0442	0.9648
	5	4.9988	0.3268	4.9701	5.0275	-0.0810	0.9355
	10	10.0188	0.4382	9.9803	10.0573	0.9662	0.3364
	20	20.0067	0.5808	19.9557	20.0577	0.2600	0.7950

n= 100	0.5	0.5053	0.0732	0.4989	0.5117	1.6316	0.1034
	2	2.0107	0.1370	1.9987	2.0228	1.7596	0.0790
	5	5.0172	0.2139	4.9984	5.0360	1.8000	0.0724
	10	10.0101	0.3063	9.9832	10.0371	0.7436	0.4575
	20	20.0002	0.4478	19.9608	20.0395	0.0110	0.9913
n = 200	0.5	0.5024	0.0497	0.4980	0.5068	1.0995	0.2721
	2	2.0053	0.0980	1.9967	2.0139	1.2239	0.2216
	5	5.0061	0.1569	4.9923	5.0199	0.8755	0.3817
	10	10.0104	0.2270	9.9905	10.0304	1.0318	0.3027
	20	19.9737	0.3169	19.9459	20.0016	-1.8505	0.0648

* indicates significance level at 5%

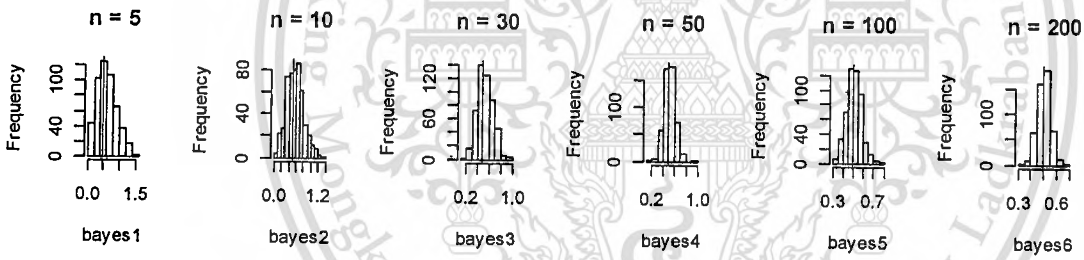

Fig.11. Histograms of estimated parameters λ with Bayes method when $\lambda = 0.5$.

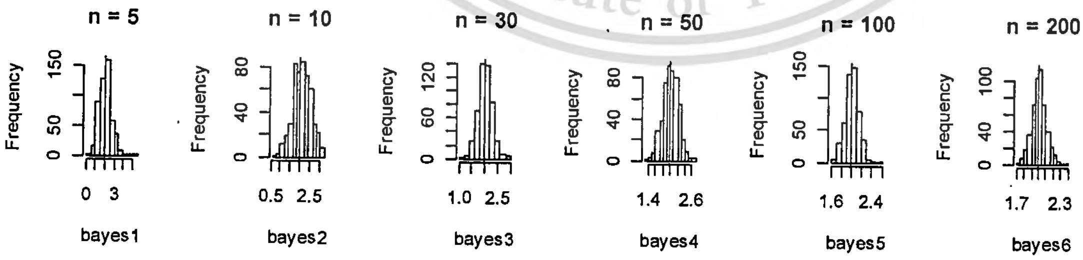

Fig.12. Histograms of estimated parameters λ with Bayes method when $\lambda = 2$.

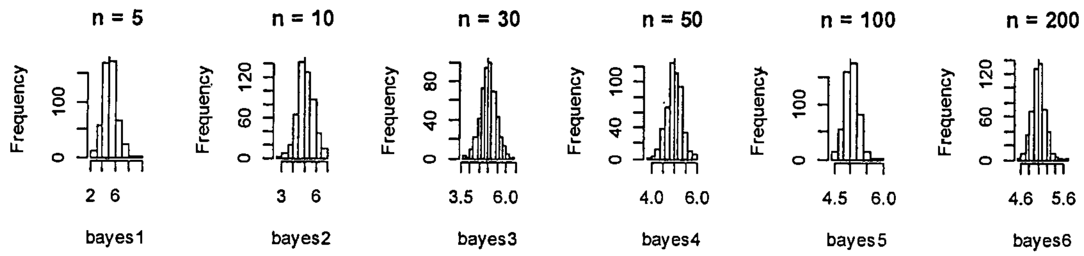

Fig.13. Histograms of estimated parameters λ with Bayes method when λ = 5.

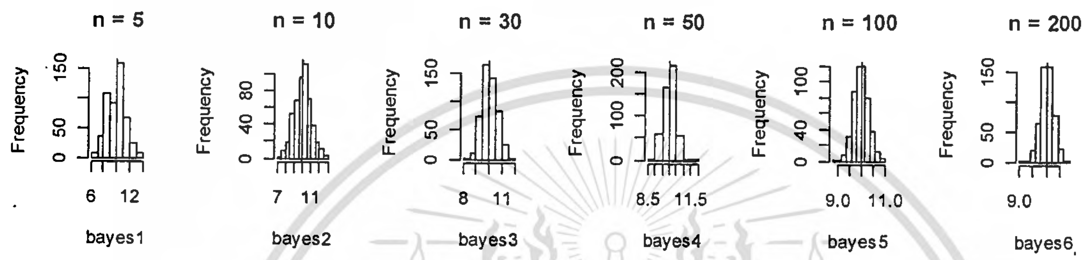

Fig.14. Histograms of estimated parameters λ with Bayes method when λ = 10.

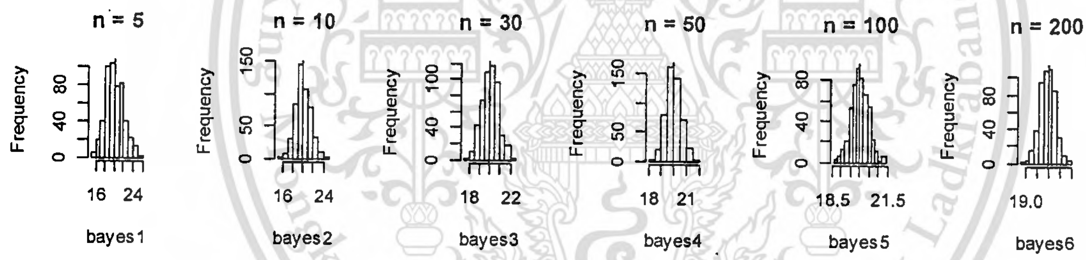

Fig.15. Histograms of estimated parameters λ with Bayes method when λ = 20.

Table 5 listed the method of parameter estimation when the MSEs is minimum

m depended on the sample sizes and the true parameters λ. As shown in Table 5, it appears that the following holds.

- For $\lambda = 0.5$, the ML method obtained a minimum MSE for all sample sizes.
- For $\lambda = 2$, the ML method yielded the lowest MSE when n = 5, 10, 30, and 200 whereas the MCMC method yielded the lowest MSE when n = 50 and 100 .
- For $\lambda = 5$, the ML method yielded the lowest MSE when n = 5, 10, 30, and 100 whereas the MCMC method yielded the lowest MSE when n = 200.
- For $\lambda = 10$, the ML method obtained a minimum MSE when n = 30. However, even for n = 10, 50, 100, and 200, the MCMC method produced minimum

values of the MSEs. For the smallest sample sizes (n=5), the Bayes method yielded reasonably good estimates in terms of minimizing the MSE.

- For $\lambda = 20$, the ML method yielded the lowest MSE when n = 5, 10, 100, and 200 whereas the Bayes method yielded the lowest MSE when n = 30 and 50 .

Table 5. The minimum values for parameter estimation.

n	λ				
	0.5	2	5	10	20
n = 5	ML	ML	ML	Bayes	MCMC
n = 10	ML	ML	ML	MCMC	MCMC
n = 30	ML	ML	ML	ML	Bayes
n = 50	ML	MCMC	ML	MCMC	Bayes
n = 100	ML	MCMC	ML	MCMC	Bayes
n = 200	ML	ML	MCMC	MCMC	Bayes

1.5 Conclusions

In this paper, we analyzed the Poisson parameter estimation by testing hypothesis and computing a MSE from the ML, the MCMC, and the Bayes method. Through a simulation study, the means of estimated parameters from the ML method were not different from the true parameters in all cases. However, in some cases, the MSEs obtained via the ML method were not minimum compared to the other two mwthods, but it worked well for small sample sizes. For the MCMC and the Bayes methods, the hypothesis tests showed that the means of estimator were different from the true parameters when the sample sizes and the true parameters show large values that the results of the MSE are similar to the other two methods. We would recommend users to use the MCMC and the Bayes method where the sample sizes and parameters were large.

For parameter estimation with Poisson distribution, if we did not dentify the prior distribution, the ML method work reasonably well. On the other hand, when the prior distribution is identified, the MCMC and the Bayes method performed better.

Chapter 2

Interval Estimation

2.1 Introduction

The estimation is a part of statistical inference for studying data from sample which is described the characteristic population. The estimation contains point estimation and interval estimation. The point estimation is to estimate an estimator from sample that referred to a population parameter. The estimator presents a single values or point, but the interval estimation presents a range which is constructed along with point estimation to show the reliability of the estimation.

Confidence interval is a type of interval estimation to approximate the range of a population parameter. Interval estimation consists of lower confidence interval and upper confidence interval which depended on the level of confidence and standard deviation. The level of confidence interval is a range of probability that captured this population parameter. When the population parameter hold on the narrow confidence intervals, it can be concluded that the estimator is a high accuracy.

In this case, we interested the discrete random variable in form of Poisson distribution. The population parameter defined an average number of successes that occurred in a specified region of time and space. A Poisson random variable is a number of successes in a length, an area, a period of time such as the number of accident occurring at the express way per day.

The interval estimation depends on the point estimation which is referred to the population parameter by the estimator. The Maximum Likelihood (ML) method is widely used a popular method to estimate the population parameter because the estimator is to be a class of uniformly minimum variance unbiased estimator [8]. Moreover, the Markov Chain Monte Carlo (MCMC) [6] method can produce the estimator of prior distribution and posterior distribution by generating sample from these functions. But the posterior distribution is provided by Bayesian model, the prior distribution can be used to evaluate this estimator. The Bayes method is also to interest since the distribution of the population parameter or called prior distribution and posterior distribution are used to approximate estimator[9]. The Bayes method is so complicate for computing estimator, then the MCMC method is used to help for estimating the population parameter.

For this reason, we propose the MLE, MCMC, and Bayes methods to estimate confidence interval with Poisson distribution using simulation studies.

2.2 Research Scope

2.2.1 Let random variable X be independent and identically distributed (iid) random variables following a Poisson distribution with parameter λ, and the probability density function is

$$f(x_i \mid \lambda) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \quad , \quad x_i = 0,1,2,\ldots \quad .$$

2.2.2 Let the prior distribution of λ be a gamma distribution with parameters a,b, and the probability density function is

$$g(\lambda \mid a,b) = \frac{\lambda^{a-1}e^{-\frac{\lambda}{b}}}{\Gamma(a)b^{a}} \quad , \quad \lambda > 0.$$

The gamma distribution shows the parameter format as the Poisson distribution or called conjugate distribution.

2.2.3 The sample sizes are considered at n = 30, 50, 100, 300, and 500.

2.2.4 The true parameter of Poisson distribution is defined as 0.5, 2, 5, 10, and 20.

2.2.5 Let the significance confidence level (α) be 3 levels at 0.1, 0.05, and 0.01 following the confidence interval (1-α)100% is to be 90%, 95%, and 99%.

2.2.6 The R program [7] is used to generate data at 500 replicates for each cases.

2.3 Methodology

2.3.1 The random variable X is generated in a class of Poisson distribution following the sample sizes, the true parameter, and the significance confidence level.

2.3.2 The methods for computing the confidence interval consist of following 3 methods:

- *Maximum Likelihood (ML) Method*

The $\hat{\lambda}_{ML}$ is ML estimator of $\bar{X}$ [9] or written as $\hat{\lambda}_{ML} = \bar{X}$, and the variance of ML estimator is given by

$$Var(\hat{\lambda}_{ML}) = Var(\bar{X}) = \frac{1}{n}Var(X) \because Var(X) = \lambda$$
$$= \frac{\lambda}{n}.$$

The population parameter (λ) is estimated by ML then

$$\lambda \Rightarrow \hat{\lambda}_{ML} = \bar{X}.$$

The confidence interval (1-α)100% of λ is approximated by

$$\lambda = \hat{\lambda}_{ML} \pm z_{\alpha/2}\sqrt{Var(\hat{\lambda}_{ML})}$$
$$= \bar{X} \pm z_{\alpha/2}\sqrt{\frac{\bar{X}}{n}}.$$

- *Markov Chain Monte Carlo (MCMC) Method*

The Markov Chain Monte Carlo (MCMC) method is operated by sequentially sampling parameter values from a Markov Chain whose stationary distribution which is desired from posterior distribution. The Gibbs sampling ([3]) is a algorithm for MCMC computing. The algorithm of Gibbs sampling can be seen at [2]. We carry out the WinBUGS Program [9] to obtain the estimating estimator from the posterior distribution function. The MCMC estimators can be computed by

$$\hat{\lambda}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}\lambda^{(t)},$$
$$Var\left(\hat{\lambda}_{MCMC}\right) = \frac{1}{T-1}\sum_{t=1}^{T}\left(\lambda^{(t)} - \bar{\hat{\lambda}}\right)^2,$$

where $\lambda^{(t)}$ is generated from the posterior distribution based on the gamma distribution at parameter $a^{(t)}$ and $b^{(t)}$, $\bar{\hat{\lambda}}$ is a sample mean of independent observation, and T is a iteration of posterior distribution function.

The confidence interval (1-α)100% of λ is written by

$$\lambda = \hat{\lambda}_{MCMC} \pm z_{\alpha/2}\sqrt{Var\left(\hat{\lambda}_{MCMC}\right)}.$$

For the MCMC method, the parameter a, and b are approximated by

$$\hat{a}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}a^{(t)} \quad ,$$
$$\hat{b}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}b^{(t)} \quad ,$$

where $a^{(t)}$ is generated from exponential distribution , and $b^{(t)}$ is generated from gamma distribution.

- *Bayes Method*

Let $X_1,\ldots,X_n$ is the random variable of Poisson distribution with parameter λ while exponential and gamma distributions are considered the prior distribution with parameter a, and b. The resulting posterior distribution is equal

$$
\begin{aligned}
h(\lambda|x_i) &\propto f(x_i|\lambda)g(\lambda|a,b) \\
&\propto \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}X_i!}\times\frac{\lambda^{a-1}e^{-\frac{\lambda}{b}}}{\Gamma(a)b^a} \\
&\propto e^{-(n+\frac{1}{b})\lambda}\lambda^{n\overline{X}+a-1}.
\end{aligned}
$$

We reach to conclusion that

$$\lambda|x_i \sim \quad gamma\left(n\overline{X}+a, n+\frac{1}{b}\right) \text{ [10].}$$

Therefore the gamma distribution is conjugate to the Poisson distribution. The posterior mean or called Bayes estimator is given by

$$E(\lambda|x_i)=\frac{n\overline{X}+a}{n+\frac{1}{b}},$$

while the posterior variance is given by

$$Var(\lambda|x_i)=\frac{n\overline{X}+a}{\left(n+\frac{1}{b}\right)^2}.$$

The confidence interval (1-α)100% of λ is written by

$$\lambda=\frac{n\overline{X}+a}{n+\frac{1}{b}}\pm z_{\alpha/2}\sqrt{\frac{n\overline{X}+a}{\left(n+\frac{1}{b}\right)^2}}.$$

Hence, the parameter a, and b are obtained from MCMC method. Recall that

$$\lambda=\frac{n\overline{X}+\hat{a}_{MCMC}}{n+\frac{1}{\hat{b}_{MCMC}}}\pm z_{\alpha/2}\sqrt{\frac{n\overline{X}+\hat{a}_{MCMC}}{\left(n+\frac{1}{\hat{b}_{MCMC}}\right)^2}}.$$

2.3.3 The estimating confidence coefficient

The confidence interval is approximated by ML, Bayes, MCMC methods at significance level 0.1, 0.05, and 0.01. If the confidence intervals cover the true parameters, we will count the number and compute the proportion denoted the confidence coefficient $(1-\hat{\alpha})$.

2.3.4 The comparison of the confidence coefficient and the fixed confidence interval

The confidence coefficient $(1-\hat{\alpha})$ is to compare with the fixed confidence interval $(1-\alpha_0)$ that we define the significance level at 0.05. If the confidence coefficient is more than the fixed confidence interval , we will perform these methods. The comparison is given by

$$1-\hat{\alpha} \geq 1-\alpha_0 = P_0 - z_{\alpha/2}\sqrt{\frac{P_0(1-P_0)}{M}},$$

where P_0 is the fixed probability given by 0.9, 0.95, and 0.99, M is the number of replications.

The fixed confidence intervals are computed by:

$$P_0 = 0.9,\ 1-\alpha_0 = 0.9-1.96\sqrt{\frac{0.9(1-0.9)}{500}} = 0.8737,$$

$$P_0 = 0.95,\ 1-\alpha_0 = 0.95-1.96\sqrt{\frac{0.95(1-0.95)}{500}} = 0.9308,$$

$$P_0 = 0.99,\ 1-\alpha_0 = 0.99-1.96\sqrt{\frac{0.99(1-0.99)}{500}} = 0.9812.$$

2.3.5 The average width of confidence interval

There are several the confidence coefficients are more than the fixed confidence interval, then the average width of confidence interval will be considered instead. The average width of confidence interval is evaluated by computing the average of difference values between upper limit and lower limit or written as $\sum_{j=1}^{500}\frac{(U_j-L_j)}{500}$, where U_j is the upper confidence interval, and L_j is the lower confidence interval.

2.4 The Results

The estimating confidence interval of population parameter with Poisson distribution is presented by the Confidence Coefficient (CC) and the Average Width (AW) at Table 1-3. The first column and the second columns of these tables are shown the sample sizes and the true parameters. The Confidence Coefficient (CC) and the Average Width (AW) are presented in the next six columns for three methods. The minimizing AW values are illustrated the performance of these methods ,but some AW values are in the blank because the confidence coefficient is less than the fixed confidence interval. By observing the CC and AW, the results appear as follow:

2.4.1 A 90% confidence interval

From Table 6, the AW of ML method is a minimum values when $\lambda = 0.5$ for all sample sizes. For MCMC method, the AW outperforms at $\lambda = 5$ with n=100, 300, and 500 , and λ= 10 ,20 with n= 500. The AW of Bayes method appears the minimum values at most cases especially when n= 30 and 50.

2.4.2 A 95% confidence interval

From Table 7, the CC and AW of ML method are a minimum values when $\lambda = 0.5$ for all sample sizes, when $\lambda = 2$ for n=100 and 500. The AW of the MCMC method shows the minimum values of the same 90% confidence interval when $\lambda = 5$ with n= 100, 300, and 500 , and λ= 10 ,20 with n= 500. The AW of Bayes method appears the minimum values at most cases especially when n=30 and 50. For $\lambda = 0.5$ and $n = 50, 100$, the CC are the same values for these methods.

2.4.3 A 99% confidence interval

From Table 8, the CC and AW of ML method are a minimum values when $\lambda = 0.5$ for all sample sizes, when $\lambda = 2$ for $n = 100$. The AW of the MCMC method shows the minimum values of the same 90% and 95% confidence interval when $\lambda = 5$ with n=100, 300, and 500 , and λ= 10 ,20 with n= 500. The AW of Bayes method appears the minimum values at most cases especially when n=30 and 50. For $\lambda = 0.5$ and $n = 30, 50$, the CC values are equal on three methods.

Table 6 : The Confidence Coefficient (CC) and Average Width (AW) obtained via 90% confidence interval.

n	λ	Methods					
		ML		MCMC		Bayes	
		CC	AW	CC	AW	CC	AW
30	0.5	0.974	0.4201	0.874	0.4282	0.874	0.4221
	2	0.912	0.8454	0.912	0.8577	0.906	0.8426
	5	0.900	1.3415	0.900	1.3383	0.900	1.3375
	10	0.890	1.8983	0.894	1.9115	0.890	1.8949
	20	0.904	2.6842	0.910	2.6840	0.904	2.6814
50	0.5	0.874	0.3265	0.906	0.3341	0.906	0.3273
	2	0.912	0.6568	0.916	0.6634	0.912	0.6554
	5	0.894	1.0412	0.894	1.0418	0.894	1.0394
	10	0.916	1.4722	0.916	1.4888	0.916	1.4707
	20	0.898	2.0793	0.898	2.0845	0.898	2.0780
100	0.5	0.894	0.2317	0.894	0.2362	0.890	0.2320
	2	0.888	0.4661	0.896	0.4670	0.896	0.4657
	5	0.896	0.7353	0.900	0.7320	0.896	0.7346
	10	0.872	-	0.880	1.0601	0.872	-
	20	0.896	1.4706	0.900	1.4884	0.896	1.4701
300	0.5	0.898	0.1346	0.908	0.1371	0.908	0.1347
	2	0.914	0.2684	0.920	0.2684	0.918	0.2683
	5	0.912	0.4250	0.904	0.4280	0.908	0.4249
	10	0.898	0.6004	0.902	0.6058	0.898	0.6003
	20	0.932	0.8494	0.932	0.8496	0.932	0.8493
500	0.5	0.886	0.1039	0.890	0.1055	0.890	0.1039
	2	0.886	0.2080	0.892	0.2103	0.886	0.2079
	5	0.902	0.3289	0.902	0.3264	0.902	0.3288
	10	0.894	0.4653	0.898	0.4642	0.894	0.4652
	20	0.884	0.6578	0.884	0.6572	0.884	0.6578

Table 7: The Confidence Coefficient (CC) and Average Width (AW) obtained via 95% confidence interval.

n	λ	Methods					
		ML		MCMC		Bayes	
		CC	AW	CC	AW	CC	AW
30	0.5	0.910	-	0.952	0.5103	0.952	0.5037
	2	0.954	1.0087	0.954	1.0227	0.950	1.0053
	5	0.956	1.6026	0.954	1.5982	0.954	1.5978
	10	0.946	2.2584	0.946	2.2745	0.946	2.2543
	20	0.958	3.1982	0.958	3.1981	0.958	3.1949
50	0.5	0.916	-	0.916	-	0.916	-
	2	0.954	0.7843	0.956	0.7922	0.954	0.7826
	5	0.934	1.2387	0.936	1.2395	0.934	1.2366
	10	0.954	1.7512	0.958	1.7711	0.954	1.7493
	20	0.960	2.4780	0.958	2.4842	0.960	2.4765
100	0.5	0.942	0.2752	0.942	0.2804	0.942	0.2756
	2	0.926	-	0.934	0.5526	0.934	0.5505
	5	0.954	0.8756	0.954	0.8414	0.960	0.8749
	10	0.946	1.2400	0.950	1.2656	0.946	1.2393
	20	0.964	1.7535	0.966	1.7751	0.964	1.7530
300	0.5	0.962	0.1600	0.970	0.1630	0.962	0.1600
	2	0.970	0.3201	0.970	0.3200	0.970	0.3199
	5	0.940	0.5059	0.938	0.5008	0.940	0.5057
	10	0.942	0.7153	0.944	0.7218	0.942	0.7152
	20	0.942	1.0120	0.942	1.0120	0.942	1.0124
500	0.5	0.942	0.1239	0.942	0.1240	0.942	0.1258
	2	0.942	0.2480	0.942	0.2508	0.942	0.2480
	5	0.952	0.3914	0.950	0.3885	0.952	0.3913
	10	0.958	0.5546	0.960	0.5533	0.958	0.5545
	20	0.943	0.7840	0.946	0.7833	0.946	0.7840

Table 8: The Confidence Coefficient (CC) and Average Width (AW) obtained via 99% confidence interval.

n	λ	Methods					
		ML		MCMC		Bayes	
		CC	AW	CC	AW	CC	AW
30	0.5	0.978	-	0.978	-	0.978	-
	2	0.984	1.3292	0.988	1.3489	0.988	1.3248
	5	0.99	2.1069	0.99	2.1020	0.99	2.1006
	10	0.986	2.9736	0.986	2.9940	0.986	2.9681
	20	0.982	4.2022	0.984	4.2021	0.982	4.1977
50	0.5	0.968	-	0.968	-	0.968	-
	2	0.998	1.0258	0.998	1.0358	0.998	1.0236
	5	0.984	1.6278	0.984	1.6284	0.984	1.6250
	10	0.992	2.2982	0.992	2.3238	0.992	2.2958
	20	0.992	3.2558	0.994	3.2642	0.992	3.2538
100	0.5	0.99	0.3652	0.994	0.3720	0.99	0.3656
	2	0.992	0.7270	0.992	0.7289	0.992	0.7271
	5	0.992	1.1505	0.992	1.1451	0.992	1.1495
	10	0.996	1.6286	0.996	0.6621	0.996	1.6277
	20	0.988	2.3037	0.988	2.3321	0.988	2.3320
300	0.5	0.994	0.2100	0.996	0.2140	0.992	0.2101
	2	0.990	0.4209	0.996	0.4208	0.994	0.4208
	5	0.994	0.6652	0.994	0.6586	0.994	0.6650
	10	0.992	0.9398	0.992	0.9484	0.992	0.9397
	20	0.988	1.3296	0.988	1.3299	0.988	1.3294
500	0.5	0.990	0.1631	0.990	0.1655	0.988	0.1631
	2	0.984	0.3263	0.986	0.3300	0.984	0.3262
	5	0.996	0.5150	0.996	0.5111	0.996	0.5150
	10	0.996	0.7281	0.996	0.7260	0.996	0.7280
	20	0.988	1.0307	0.988	1.0298	0.988	1.0307

2.5 Conclusions

In this research, we generated data from a Poisson distribution and estimated the confidence interval which is obtained confidence coefficient and average width to perform ML, MCMC, and Bayes Methods. The MCMC method is proposed to estimate population parameter of prior distribution function that can be used for estimating Bayes estimator. The posterior distribution function is related with the MCMC method to evaluate MCMC estimator. Through a simulation study, the ML method is a good performance when true parameter is small values ($\lambda = 0.5$) for all sample sizes. For the large sample sizes (n=100,300,500), MCMC method outperforms ML and Bayes when true parameter is 5. Moreover, the Bayes method is a good performance in most cases. However, the confidence coefficient and average width are equal in some case, so we would recommend user to use MCMC method because the Bayes method depends on the MCMC method.

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