HAMILTONIAN PATHS AND CIRCUITS IN SOME CAYLEY DIGRAPHS OF SEMIGROUPS

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บทคัดย่อ

สำหรับกึ่งกรุปจำกัด S และเซต A เป็นเซตย่อยของ S ที่ไม่เป็นเซตว่าง ไดกราฟเคย์เลย์ $Cay(S,A)$ คือไดกราฟที่ S เป็นเซตของจุดและ $\{(s,sa) : s \in S$ และ $a \in A\}$ เป็นเซตของเส้นเชื่อม วิทยานิพนธ์นี้เราได้เงื่อนไขที่เพียงพอและจำเป็นบางประการของ S และ A ซึ่ง $|A| \leq 2$ ที่มีวิถีแฮมมิลโทเนียนหรือวงจรแฮมมิลโทเนียนใน $Cay(S,A)$

นอกจากนี้เราได้หาเงื่อนไขบางประการที่ทำให้ไดกราฟ $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ โดยที่ $|A| \leq 2$ มีวงจรแฮมมิลโทเนียน

คำสำคัญ : กึ่งกรุป ไดกราฟเคย์เลย์ วงจรแฮมมิลโทเนียน วิถีแฮมมิลโทเนียน

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Abstract

For a finite semigroup S and a nonempty subset A of S, the Cayley digraph $Cay(S,A)$ is the digraph with vertex set S and edge set $\{(s,sa) : s \in S \text{ and } a \in A\}$. In this thesis, we obtain some necessary and sufficient conditions of S and A such that $|A| \leq 2$, there exists a Hamiltonian paths or a Hamiltonian circuits in $Cay(S,A)$.

Moreover, we find some conditions that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$, $|A| \leq 2$, there exists a Hamiltonian circuits.

Keywords : Semigroup, Cayley digraph, Hamiltonian circuit, Hamiltonian path

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Sornsawan Meechana

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Chapter 1

Introduction

Hamiltonian paths and Hamiltonian circuits are the mathematical field of graph theory which are interested. The problem of finding the Hamiltonian path and the Hamiltonian circuit are NP-complete. They can apply to many problems, such as, pizza delivery, mail delivery, traveling sales man, garbage pick up, bus service/limosine service and reading gas meters. Furthermore, they have been appied to biology[5].

In 1878, Arthur Cayley introduced definition of the Cayley digraph of group. For G is a finite group and A is a set of generators of G. The Cayley digraph of G with respect to A, denoted by $Cay(G, A)$ to be the directed graph with vertex set G and arc set $\{(g, ga) : g \in G \text{ and } a \in A\}$. Moreover, Cayley studied properties of a group such as commutativity and the multiplication table can be recovered from the Cayley digraph.

A well known conjecture of Lovasz states[9] that every Cayley digraph is a Hamiltonian path. There are many researchers that studied about this conjecture.

In 1976, Nathanson[13] said that the finite group G is generated by two element a and b, such that $a^2 = b^3 = e$. If $|G| \geqslant 9|ab^2|$, then the Cayley digraph $Cay(G, \{a, b\})$ does not have a Hamiltonian path.

In 1978, Holszyński and Strube[7] showed that every connected Cayley digraph on any abelian group has a Hamiltonian path.

In 1982, Dragon[10] proved that if either G is a finite abelian group or a semidirect product of a cyclic group of prime order by a finite abelian of odd order, then every connected Cayley digraph of G is hamiltonian.

In 1983, Marušić[10] showed that if either G is a finite abelian group or a semidirect product of a cyclic group of prime order by a finite abelian group of odd order, then every connected Cayley digraph of G is Hamiltonian circuit.

In 2009, Pak and Radoičić[14] said that every finite group G of size $|G| \geq 3$ has a generating set A of size $|A| \leq log_2|G|$, such that the corresponding Cayley digraph (G, A) contains a Hamiltonian circuit.

This conjecture have never been solved before. However, people studied some property and some condition of G and A that $Cay(G, A)$ has a Hamiltonian path, as follow

Problem :

(1) For what generating sets does the groups have a Hamiltonian path or a Hamiltonian circuit in Cayley digraph?

(2) Which group G have the property that for all generating set A for S, $Cay(G, A)$ contains a Hamiltonian path or a Hamiltonian circuit?

At the same time, Bohdan Zenlinka[19] introduced the Cayley digraph of a semigroup, for a finite semigroup S and a nonempty subset A of S, the Cayley digraph of S with respect to A, denoted by $Cay(S, A)$ to be the directed graph with vertex set S and arc set $\{(s, sa) : s \in S \text{ and } a \in A\}$.

Shoufeng[17] showed that the vertex-transitive, connected and undirected finite Cayley graph of semigroup are isomorphic to Cayley graphs of groups, and all finite vertex-transitive Cayley graphs of inverse semigroups are isomorphic to Cayley graphs of groups.

In 2015, Suksumran and Panma[15] said that S is a right simple semigroup and $A \subseteq S$, a Cayley digraph $Cay(S, A)$ is strongly connected if and only if $\langle A \rangle = S$. The digraph is strongly connected, this means the digraph may have a Hamiltonian path.

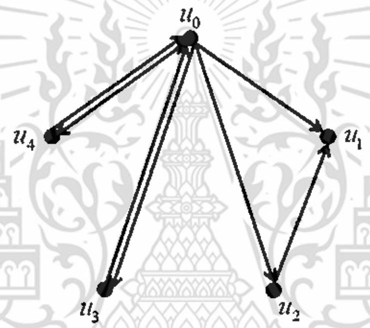

Figure 1.1: A digraph D_1

From Figure 1.1, We can see that a digraph D_1 is strongly connected but it is not contain a Hamiltonian path. Hence, we will consider some necessary and sufficient conditions of S and A that $Cay(S, A)$ has Hamiltonian paths or Hamiltonian circuits.

In particular S of $\mathbb{Z}_m \oplus \mathbb{Z}_n$, Gallian[4] has proved that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(0,1),(1,0)\})$ has a Hamiltonian circuit when n divides m but $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(0,1),(1,0)\})$ doesn't have a Hamiltonian circuit when n and m are relatively prime and greater than 1. We will show some conditions of m, n and A in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ for some $|A| \leq 2$, such that there exist a Hamiltonian circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$.

There are 5 chapters in this thesis. Chapter 1, we provide about Hamiltonian circuits and Hamiltonian paths in digraphs and Cayley digraphs. Chapter 2, we introduce basics, definitions, theorem and examples about semigroup, group and directed graph. Moreover, we also introduce Cayley digraph. Chapter 3, we present some necessary and sufficient conditions of S and A that $Cay(S, A)$ has a Hamiltonian path and a Hamiltonian circuit when $|A| \leq 2$. Chapter 4, we find some conditions of m, n and A, $|A| \leq 2$, in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ such that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ contains a Hamiltonian circuit. Chapter 5, we conclude our theorem and corollary.

Chapter 2

Preliminaries

In this chapter, we introduce basics and examples of semigroup, group and digraph. We also present a Cayley digraph.

2.1 Semigroup and group

Definition 2.1. A semigroup is a system $(S, \cdot)$ consisting of a nonempty set S together with a binary operation $\cdot$ satisfying the conditions $\forall x, y \in S, (x \cdot y) \cdot z = x \cdot (y \cdot z)$.

If there is no ambiguity we may write S and xy for $(S, \cdot)$ and $x \cdot y$, respectively. Furthermore, if there is an identity element $e \in S$ which $a \cdot e = e \cdot a = a$ for all $a \in S$ and for which $a^{-1} \cdot a = a \cdot a^{-1} = e$ then S is a group.

Definition 2.2. A semigroup S is commutative if $x \cdot y = y \cdot x$ holds for all $x, y \in S$.

Definition 2.3. A group is commutative called abelian group.

Definition 2.4. The order of a semigroup S is denoted by $|S|$.

Definition 2.5. A nonempty subset I of S is called a right[left] ideal of S if for any $s \in S$ and $a \in I$, $as \in I[sa \in I]$.

Definition 2.6. A semigroup S is said to be right[left] simple provided that it contains no proper right[left] ideals.

We can check that S is right simple if and only if $aS = S$ for all a in S if and only if the linear equation $ax = b$ in the variable x possesses a solution in S for all $a, b \in S$.

Definition 2.7. A subsemigroup of S is a nonempty subset T of S such that $\forall x, y \in T, xy \in T$.

Definition 2.8. Let A be a nonempty subset of a semigroup S. The subsemigroup of S generated by A, denoted by $\langle A \rangle$, is the set of element of S that can be expressed as finite products of elements in A.

If A is such that $\langle A \rangle = S$, then A is called a generating set of S, S is generated by A.

Definition 2.9. A semigroup S is cyclic semigroup if it is generated by a single element, denoted by $S = \langle \{a\} \rangle$ where $a \in S$ and a called a generator of S.

Definition 2.10. Assume $(S, \cdot)$ is a semigroup.

(i) $z \in S$ is a zero of S if $x \cdot z = z \cdot x = z$ holds for all $x \in S$.

(ii) An element n of S is an identity of S if $n \cdot x = x \cdot n = x$ holds for all $x \in S$.

Example 2.1. Let $S = \{0, 1, 2, 3\}$ and defined the operation $\cdot$ on S by

•	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	2
3	0	1	2	3

Let $T = \{1, 2, 3\}$. We can see that $T \subseteq S$ and $x \cdot y \in T$ for all $x, y \in T$. Then T is a subsemigroup of S. Furthermore, 0 is a zero element of S and 3 is an identity element of S.

Definition 2.11. Let G_1, G_2 be groups. The external direct product of G_1, G_2 written as $G_1 \oplus G_2$, is the set of cartesian product of G_1 and G_2 which the operation is component wise.

Example 2.2. Consider $\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$, we can see that $\langle(1,1)\rangle$ is a generator of $\mathbb{Z}_2 \oplus \mathbb{Z}_3$.

2.2 Digraph

Definition 2.12. A digraph D is a finite nonempty set objected called vertices together with a set of ordered pairs of distinct vertices of D called arcs or directed edges.

Definition 2.13. The vertex set of D is denoted by $V(D)$ and the arc set is denoted by $E(D)$.

In Figure 2.1, $V(D) = \{a, b, c\}$ and $E(D) = \{(b, a), (c, a), (c, b)\}$.

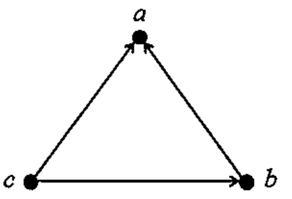

Figure 2.1: A digraph D

Definition 2.14. A spanning subgraph of H is a subgraph of D which $V(H) = V(D)$ and $E(H) \subseteq E(D)$.

Definition 2.15. A $u_1 - u_k$ directed walk is an alternating sequence $u_1, e_1, u_2, \dots, e_k, u_k$ of vertices and arcs or sequence $u_1, u_2, \dots, u_{k-1}, u_k$, beginning with u_1 and ending with u_k.

Definition 2.16. For the $u_1 - u_k$ directed walk is a $u_1 - u_k$ directed path when the vertices $u_1, u_2, \dots, u_{k-1}, u_k$ are distinct.

Definition 2.17. A directed circuit is a closed directed path.

Definition 2.18. A directed path and a directed circuit in a digraph D containing every vertex of D is called a Hamiltonian path and a Hamiltonian circuit.

Definition 2.19. A digraph D is strongly connected(or strong) if for every pair u, v of vertices D have a directed path.

Definition 2.20. For (u, v) is an arc of a digraph D, we call vertex u is adjacent to vertex v.

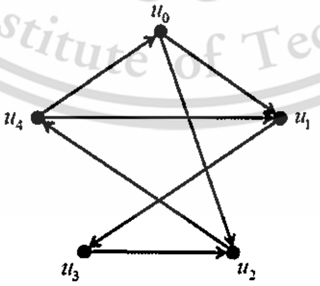

Figure 2.2: A digraph D_2

From Figure 2.2, there is the $u_1 - u_4$ directed walk, alternating sequence u_1, u_3, u_2, u_4. The sequence u_3, u_2, u_4, u_0, u_1 is the directed path. The $u_3 - u_1$ directed path containing every vertex in D_2 then D_2 has a Hamiltonian path. Moreover, D_2 has a Hamiltonian circuit.

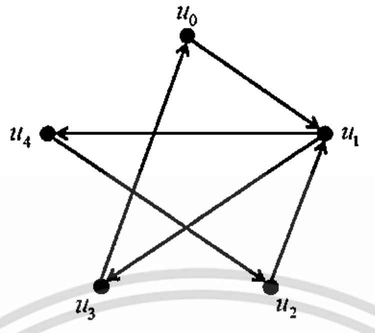

Figure 2.3: A digraph D_3

From Figure 2.3, the digraph D_3 is strongly connected. At vertex u_0 is adjacent to vertex u_1, vertex u_1 is adjacent to u_3, vertex u_2 is adjacent to vertex u_1, vertex u_3 is adjacent to vertex u_0 and vertex u_4 is adjacent to vertex u_2.

Definition 2.21. For finite semigroup S and a nonempty subset A of S, we define the Cayley digraph of S with respect to A, denoted by $Cay(S, A)$ to be the directed graph with vertex set S and arc set $\{(s, sa) : s \in S \text{ and } a \in A\}$.

In this thesis, we do not consider loop or multiple edges in $Cay(S, A)$.

Example 2.3. From Figure 2.4, we show $Cay(\mathbb{Z}_5, \{1\})$.

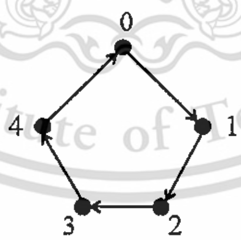

Figure 2.4: $Cay(\mathbb{Z}_5, \{1\})$

Example 2.4. Let $S = \{0, 1, 2\}$ and define the operation $\cdot$ on S by

$\bullet$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

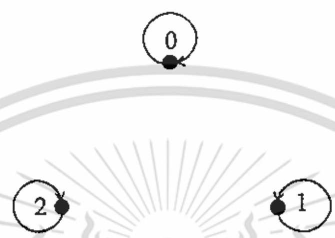

Figure 2.5: $Cay(S, \{0\})$

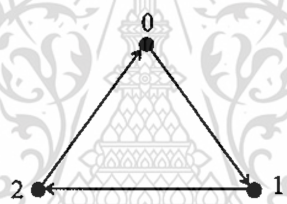

Figure 2.6: $Cay(S, \{1\})$

Figure 2.7: $Cay(S, \{2\})$

Theorem 2.5. [15]S is a right simple semigroup and $A \subseteq S$. A Cayley graph $Cay(S, A)$ is strong if and only if $\langle A \rangle = S$.

Theorem 2.6. [7]Every connected Cayley digraphs of abelian group has a Hamiltonian path.

Chapter 3

Hamiltonian Paths and Circuits in some Cayley Digraph of Semigroups

In this chapter, we find some necessary and sufficient conditions of S and A that $Cay(S,A)$ has a Hamiltonian path and a Hamiltonian circuit. First, we will show a Hamiltonian path and a Hamiltonian circuit in $Cay(S,A)$ when $|A| = 1$.

Theorem 3.1. S is a cyclic semigroup which generated by a if and only if $Cay(S,\{a\})$ has a Hamiltonian path. Furthermore, S is a right simple cyclic semigroup if and only if $Cay(S,\{a\})$ has a Hamiltonian circuit.

Proof. Let S be a cyclic semigroup of order k and generated by a. Then $S = \{a, a^2, a^3, \ldots ,a^k\}$ and $a^i \neq a^j$ for all $i,j \in \{1,2,\ldots,k\}$ and $i \neq j$. Since $a \cdot a = a^2 \in S$, vertex a is adjacent to vertex a^2. Since $a^2 \cdot a = a^3 \in S$, vertex a^2 is adjacent to vertex a^3. Continue this process, we have a dipath from a to a^k. Then $Cay(S,\{a\})$ has a Hamiltonian path. Conversely, assume that $Cay(S,\{a\})$ has a Hamiltonian path but $S \neq \langle\{a\}\rangle$. Let H be a Hamiltonian path in $Cay(S,\{a\})$. There exists $a_0 \in \langle\{a\}\rangle$ and $b \in S\setminus\{a\}$ such that vertex a_0 is adjacent to vertex b in dipath H. That is $b = a_0 \in \langle\{a\}\rangle$, contradiction. Therefore $S = \langle\{a\}\rangle$.

Furthermore, assume S is a right simple cyclic semigroup generated by a. There is a Hamiltonian path, $a, a^2, \ldots, a^k$, in $Cay(S,\{a\})$. From Theorem 2.3, we have $Cay(S,\{a\})$ is strongly connected digraph. So, vertex a^k must have a dipath to vertex a. This means, there is a Hamiltonian circuit in $Cay(S,\{a\})$. Conversely, suppose $Cay(S,\{a\})$ has a Hamiltonian circuit. Then $Cay(S,\{a\})$ has a Hamiltonian path. This implies that $S = \langle\{a\}\rangle$ and $a^{k+1} = a$. Let $b \in S$ then $b = a^m$ for some $1 \leq m \leq k$ and $b \cdot S = a^m\{a^1, a^2, a^3, \ldots, a^k\} = \{a^{m+1}, a^{m+2}, a^{m+3}, \ldots, a^{m+k}\} = S$, that is $b \cdot S = S$ for all $b \in S$. Hence S is a right simple cyclic semigroup. □

Example 3.2. Let $S = \{0,1,2,3\}$ and defined the operation $\cdot$ on S by

•	0	1	2	3
0	0	0	2	2
1	0	0	2	2
2	2	2	0	0
3	2	2	0	1

Then $(S, \cdot)$ is a semigroup. Since $3^2 = 1$, $3^3 = 2$ and $3^4 = 0$, S is a cyclic semigroup generated by 3 then $Cay(S, \{3\})$ has a Hamiltonian path as show in Figure 3.1 and 3.2.

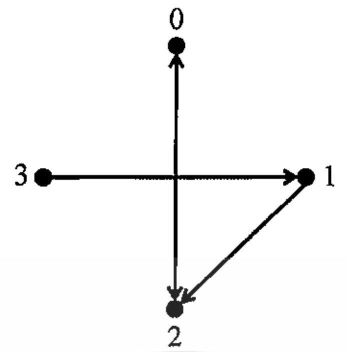

Figure 3.1: $Cay(S, \{3\})$

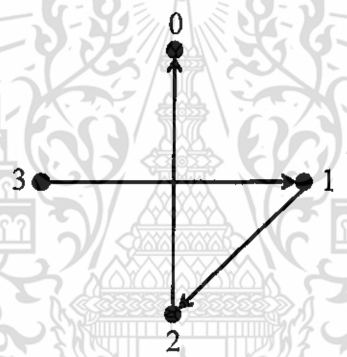

Figure 3.2: A Hamiltonian Path in $Cay(S, \{3\})$

From the prove of Theorem 3.1, we can see that the Hamiltonian path of $Cay(S, \{a\})$ is $a, a^2, a^3, \ldots, a^k$. This mean if S has no a generator, $\langle\{a\}\rangle \neq S$, then $Cay(S, \{a\})$ has no a Hamiltonian path. Thus, there is not a Hamiltonain circuit in $Cay(S, \{a\})$.

Example 3.3. Let $S = \{0,1,2,3\}$ and defined the operation $\cdot$ on S by

$\bullet$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	1	0
3	0	0	0	3

Then $(S,\cdot)$ is a semigroup. We can see that S is a semigroup which is not cyclic. Consider $Cay(S,\{a\})$ for all $a \in S$, as follow in Figure 3.3 - 3.6. Hence, $Cay(S,\{a\})$ for all $a \in S$ has no a Hamiltonian path.

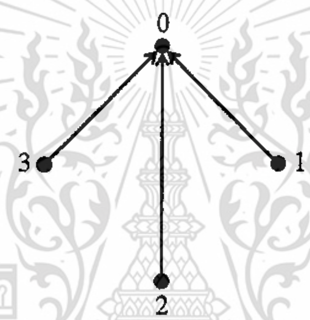

Figure 3.3: $Cay(S,\{0\})$

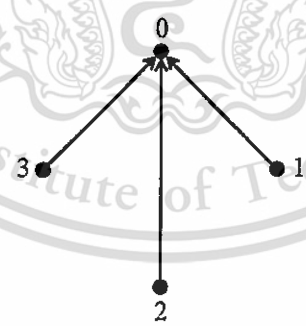

Figure 3.4: $Cay(S,\{1\})$

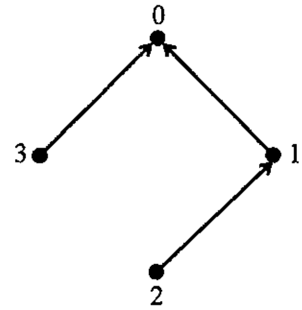

Figure 3.5: $Cay(S, \{2\})$

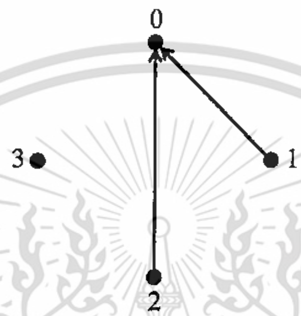

Figure 3.6: $Cay(S, \{3\})$

A cyclic semigroup S is sufficient condition of S for $Cay(S, \{a\})$ has a Hamiltonian path. Moreover, a right simple cyclic semigroup S is necessary condition of S for $Cay(S, \{a\})$ has a Hamiltonian circuit but it is not sufficient condition of S for $Cay(S, \{a\})$ has a Hamiltonian path.

Notice that for any semigroup S and $a \neq b$, $a, b \in S$. It is easy to see that $Cay(S, \{a\})$ is a spanning subgraph of $Cay(S, \{a, b\})$. The process of finding a Hamiltonian path and a Hamiltonian circuit in $Cay(S, \{a, b\})$ are more complicated than $Cay(S, \{a\})$. Next, we will show some conditions of S and A in $Cay(S, A)$, there exits a Hamiltonian path and a Hamiltonian circuit when $|A| = 2$.

Corollary 3.4. If S is a right simple cyclic semigroup generated by a then $Cay(S, \{a, b\})$ has a Hamiltonian circuit for all $b \in S$.

Proof. Let $b = a^m$. Suppose S is a right simple cyclic semigroup generated by a. From Theorem 3.1, there is a Hamiltonian circuit in $Cay(S, \{a\})$. Since $Cay(S, \{a\})$ is a spanning subgraph of $Cay(S, \{a, b\})$ then $Cay(S, \{a, b\})$ for all $b \in S$ has a Hamiltonian circuit. □

Remark For A is nonempty subset of S is containing $\{a\}$. If S is a right simple cyclic semigroup generated by a then $Cay(S, A)$ has a Hamiltonian circuit.

Theorem 3.5. Let S be a semigroup with zero 0. If $S = \langle\{0, a\}\rangle$ and $S \setminus \{0\}$ is a subsemigroup of S then $Cay(S, \{0, a\})$ has a Hamiltonian path.

Proof. Let S be a semigroup have zero element and $S = \langle\{0, a\}\rangle$ which $S \setminus \{0\}$ is a subsemigroup of S. Since $S = \langle\{0, a\}\rangle$ and $x \cdot 0 = 0$ for all $x \in S$, so $S \setminus \{0\} = \langle\{a\}\rangle$. From Theorem 3.1, there is a Hamiltonian path in $Cay(S \setminus \{0\}, \{a\})$. Since $y \cdot 0 = 0$ for all $y \in S$ then vertex y is adjacent to vertex 0 for all $y \in S$. Hence, $Cay(S, \{0, a\})$ has a Hamiltonian path. $\square$

The process of finding a Hamiltonian path in $Cay(S, \{0, a\})$ start at vertex a and then we generate by a. Repeating at $n - 2$ times, so we have a Hamiltonian path in $Cay(S \setminus \{0\}, \{a\})$. Finally, we generate by 0. Thus there are hamiltonian path in $Cay(S, \{0, a\}$.

Example 3.6. Let $S = \{0, 1, 2, 3\}$ and defined the operation $\cdot$ on S by

•	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	3	3
2	0	1	1	3	3
3	0	3	3	1	1
4	0	3	3	1	2

Then $(S, \cdot)$ is a semigroup with 0. Consider the $Cay(S, \{0, 4\})$, we have a Hamiltonian path in $Cay(S, \{0, 4\})$.

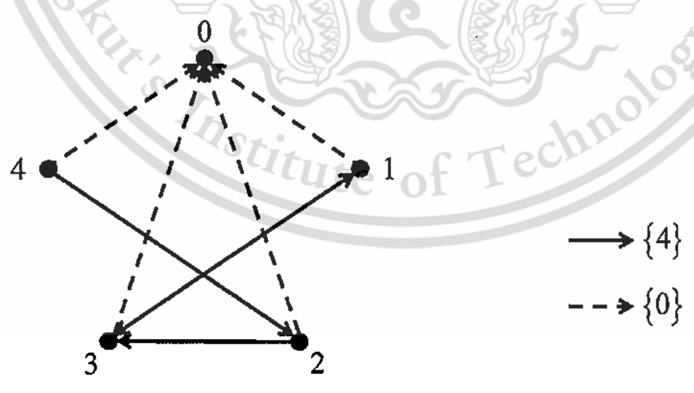

Figure 3.7: $Cay(S, \{0, 4\})$

A semigroup S with 0, $A \subseteq S$ and $0 \in A$. We have that $x \cdot 0 = 0$ for all $x \in S$, vertex x is adjacent to vertex 0. Hence there is not a Hamiltonian circuit in $Cay(S, A)$.

Corollary 3.7. Let S be a semigroup with zero 0 and $A \subseteq S$. If $0 \in A$, then $Cay(S, A)$ has no a Hamiltonian circuit.

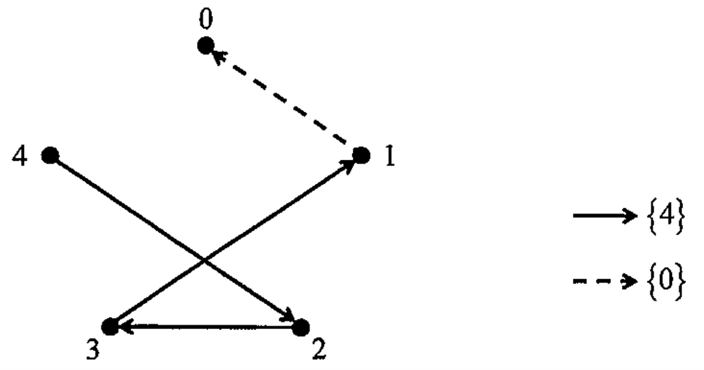

Figure 3.8: A Hamiltonian Path in $Cay(S, \{0, 4\})$

Theorem 3.8. Let S be a semigroup with identity e. If $S = \langle\{e, a\}\rangle$ for some $a \in S$ then $Cay(S, \{e, a\})$ has a Hamiltonian path.

Proof. Let S be a semigroup have identity element, $|S| = k$ and $S = \langle\{e, a\}\rangle$ for some $a \in S$. Since $S = \langle\{e, a\}\rangle$ for some $a \in S$ and $e \cdot x = x$ for all $x \in S$. Thus $S \setminus \{e\} = \langle\{a\}\rangle$, there is $e, a, a^2, \ldots, a^k$. Therefore $Cay(S, \{e, a\})$ has a Hamiltonian path. □

Processing to find a Hamiltonian path in $Cay(S, \{e, a\})$, starting at vertex e and then we generated by a. Repeating at $n - 1$ times, then there is a Hamiltonian path in $Cay(S, \{e, a\})$.

Example 3.9. Let $S = \{0, 1, 2, 3\}$ and defined the operation $\cdot$ on S by

•	0	1	2	3	4
0	0	0	0	3	3
1	0	0	1	3	3
2	0	1	2	3	4
3	3	3	3	0	0
4	3	3	4	0	1

Then $(S, \cdot)$ is a semigroup. Since $S = \langle\{2, 4\}\rangle$, we obtain that $Cay(S, \{2, 4\})$ has a Hamiltonian path such that 2 is an identity element of S.

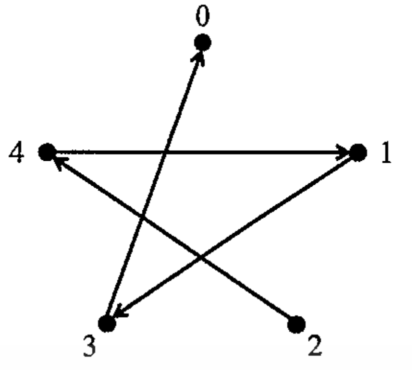

Figure 3.9: A Hamiltonian Path of $Cay(S, \{2,4\})$

A semigroup S with e, $A \subseteq S$, $a \cdot b \neq e$ for all $a, b \in S \setminus \{e\}$ and $e \in A$. Since $e \cdot x = x$ for all $x \in S$, vertex e is adjacent to vertex x. Thus, there is not a Hamiltonian circuit in $Cay(S, A)$.

Corollary 3.10. Let S be a semigroup with identity e and $A \subseteq S$. If $e \in A$ and $a \cdot b \neq e$ for all $a, b \in S \setminus \{e\}$, then $Cay(S, A)$ has no a Hamiltonian circuit.

Chapter 4

Hamiltonian Circuits in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ for some $|A| \leq 2$

Every connected Cayley digraphs of abelian group has a Hamiltonian path but it is not guarantee that there exists a Hamiltonian circuit. In this chapter, we obtain some conditions of m, n and A, for some $|A| \leq 2$, in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ that contains a Hamiltonian circuit.

Theorem 4.1. For $\mathbb{Z}_m \oplus \mathbb{Z}_n$, $gcd(m,n) = 1$, $gcd(a,m) = 1$ and $gcd(b,n) = 1$ if and only if $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(a,b)\})$ has a Hamiltonian circuit.

Proof. Let $\mathbb{Z}_m \oplus \mathbb{Z}_n = \langle (a,b) \rangle$. In $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(a,b)\})$, there exist alternating sequence of directed walk which $(0,0), (a,b), \ldots, ((mn-1)a, (mn-1)b), ((mn)a, (mn)b)$ and $(0,0), (a,b), \ldots, ((mn-1)a \ mod \ m, (mn-1)b \ mod \ n), ((mn)a \ mod \ m, (mn)b \ mod \ n)$ are equivalence. Moreover, $(\mathbb{Z}_m \oplus \mathbb{Z}_n, \cdot)$ is cyclic group, so this directed walk is closed directed path. The proof is complete. Conversely, assume $gcd(m,n) \neq 1$, $gcd(a,m) \neq 1$ and $gcd(b,n) \neq 1$. We can see that $\langle (a,b) \rangle$ is not a generator of $\mathbb{Z}_m \oplus \mathbb{Z}_n$. Then $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(a,b)\})$ does not contain a Hamiltonian circuit. □

Next, we will show the condition of m and n when $\langle (a,b) \rangle = \langle (1,1) \rangle$ and $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$ has a Hamiltonian circuit.

Corollary 4.2. There exists a Hamiltonian circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$, if $gcd(m,n) = 1$.

Proof. Let (i,j) be any vertex in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$. Since $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$ is cyclic group then $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$ contain a path of length $mn-1$. Furthermore, it contains a Hamiltonian circuit, alternating sequence $(i,j), (i+1, j+1), \ldots, (i+mn, j+mn) = (i,j)$.

□

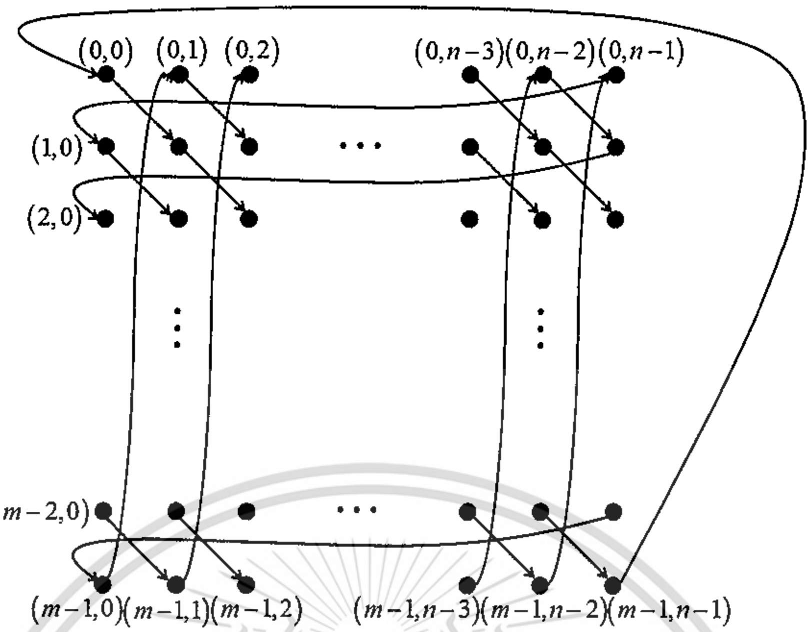

Figure 4.1: A Hamiltonian Circuit of $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$

In [4], we consider the Hamiltonian circuit of $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ such that $\langle A \rangle = \mathbb{Z}_m \oplus \mathbb{Z}_n$, $|A| = 2$. $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(0,1),(1,0)\})$ has a Hamiltonian circuit when n divides m but $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(0,1),(1,0)\})$ does not have a Hamiltonian circuit when n and m are relatively prime and greater than 1.

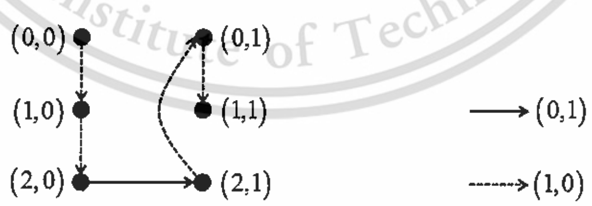

Figure 4.2: A Hamiltonian Path in $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}_2, \{(0,1),(1,0)\})$

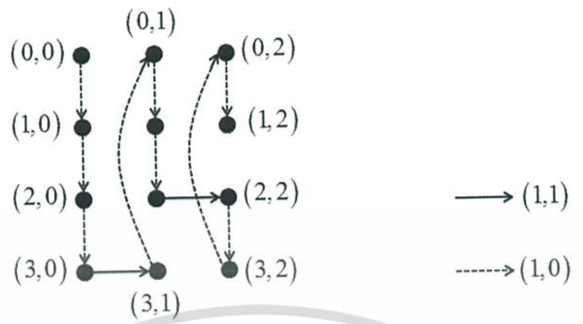

Figure 4.3: A Hamiltonian Path in $Cay(\mathbb{Z}_4 \oplus \mathbb{Z}_3, \{(0,1),(1,0)\})$

From Figure 4.2 and 4.3, $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}_2, \{(0,1),(1,0)\})$ and $Cay(\mathbb{Z}_4 \oplus \mathbb{Z}_3, \{(0,1),(1,0)\})$ have a Hamiltonian path but have no a Hamiltonian circuit. We will find the generating set of $\mathbb{Z}_m \oplus \mathbb{Z}_n$ for $m, n \geq 2$ that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(a,b),(c,d)\})$ has Hamiltonian circuit.

Theorem 4.3. For $m, n \geq 2$, there exist a Hamiltonian circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(1,0)\})$.

Proof. Let $\{(1,1),(1,0)\}$ be generating set of $\mathbb{Z}_m \oplus \mathbb{Z}_n$, dark arc be generator by $(1,1)$ and dash arc be generator by $(1,0)$. We will show that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(1,0)\})$ has a Hamiltonian circuit, the following cases:
Case 1 $m = n$, starting at $(0,0)$, use the generator $(1,1)$ to move crosswise down to $(1,1)$ until to $(n-1, n-1)$. Next, use the generator $(1,0)$ to move vertically up to $(0, n-1)$ and use the generator $(1,1)$ to move crosswise down to $(1,0)$. Then use the generator $(1,1)$ to move crosswise down to $(2,1)$ until to $(n-1, n-2)$ and use the generator $(1,0)$ to move vertically up to $(0, n-2)$. Next, use the generator $(1,1)$ to move crosswise down to $(1, n-1)$ until to $(2,0)$. Keep this process up until $(n-1, 0)$. Finally, complete the circuit by use the generator $(1,0)$ moving vertically up to $(0,0)$.

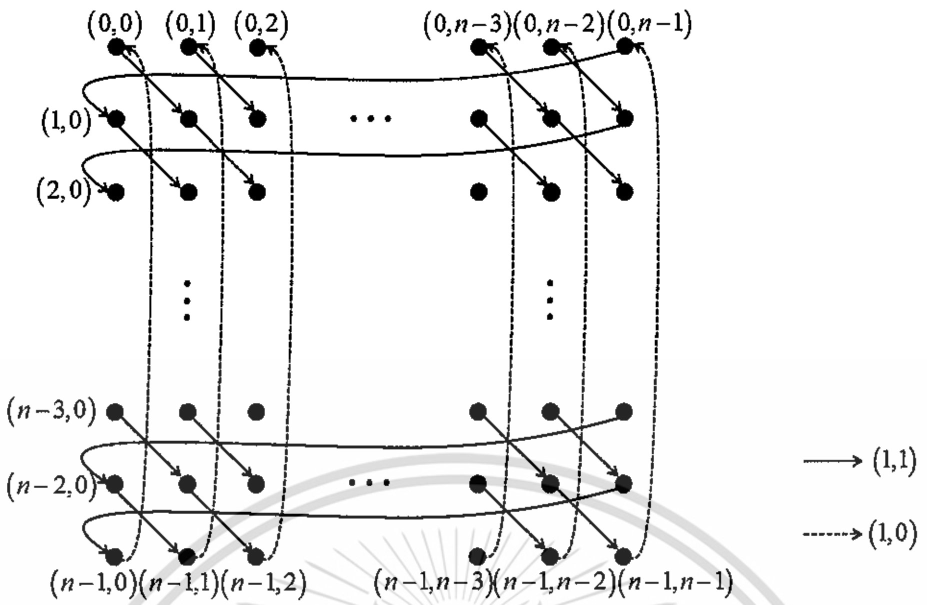

Figure 4.4: A Hamiltonian Circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(1,0)\})$

Case 2 $m \neq n$, starting at $(0,0)$, use the generator $(1,0)$ to move vertically down to $(1,0)$ until to $(m-2,0)$. Next, use the generator $(1,1)$ to move crosswise down to $(n-1,1)$. Then, use the generator $(1,0)$ to move vertically up to $(0,1)$. Keep this process up until $(m-1,0)$. Finally, complete the circuit by use the generator $(1,0)$ moving vertically up to $(0,0)$.

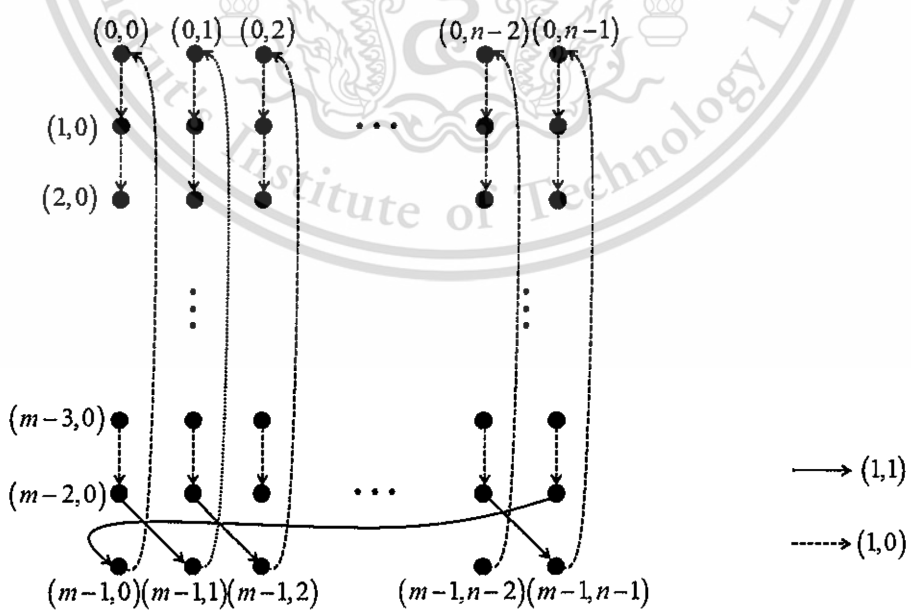

Figure 4.5: A Hamiltonian Circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(1,0)\})$

□

Theorem 4.4. For $m, n \geq 2$, there exist a Hamiltonian circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(0,1)\})$.

Proof. Let $\{(1,1),(0,1)\}$ be generating set of $\mathbb{Z}_m \oplus \mathbb{Z}_n$, dark arc be generator by $(1,1)$ and dash arc be generator by $(0,1)$. We will show that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(0,1)\})$ has a Hamiltonian circuit, the following Figure 4.6.

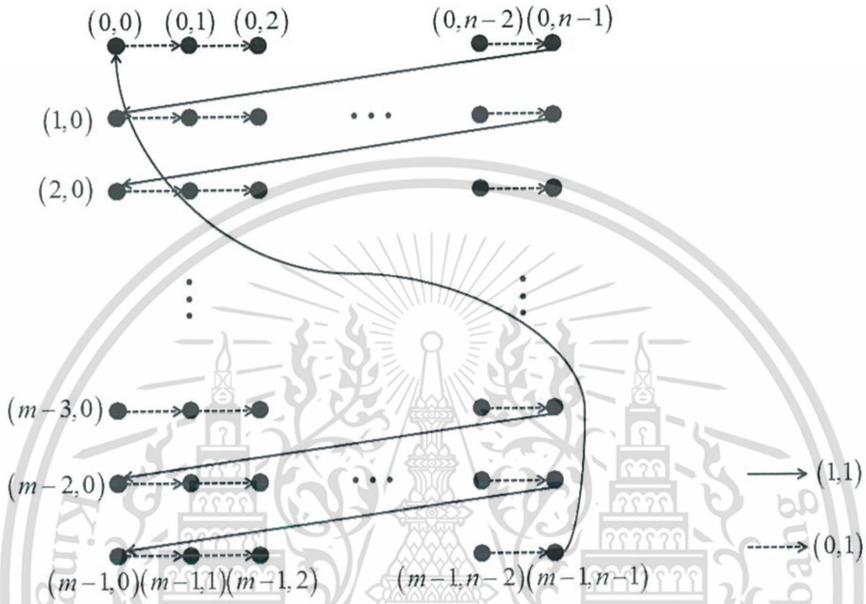

Figure 4.6: A Hamiltonian Circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(0,1)\})$

□

Chapter 5

Conclusion

For $A = \{a\}$, $Cay(S, \{a\})$ has a Hamiltonian path if and only if S is a cyclic semigroup generated by a. Furthermore, $Cay(S, \{a\})$ has a Hamiltonian circuit if and only if S is a right simple cyclic semigroup.

For $A = \{a, b\}$ and $a \neq b$, we have

1. If S is a right simple cyclic semigroup generated by a then $Cay(S, \{a, b\})$ for all $b \in S$ has a Hamiltonian circuit.
2. If b is a zero element of S, $S = \langle\{0, a\}\rangle$ and $S \setminus \{0\}$ is a subsemigroup of S then $Cay(S, \{0, a\})$ has a Hamiltonian path.
3. If b is a identity element of S and $S = \langle\{e, a\}\rangle$ for some $a \in S$ then $Cay(S, \{e, a\})$ has a Hamiltonian path.
4. If A contains zero or identity element of S then $Cay(S, A)$ has no a Hamiltonian circuit.

Moreover, in this case $S = \mathbb{Z}_m \oplus \mathbb{Z}_n$, we have

1. $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(a, b)\})$ has a Hamiltonian circuit if and only if $gcd(m, n) = 1$, $gcd(a, m) = 1$ and $gcd(b, n) = 1$.
2. $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1, 1), (0, 1)\})$ and $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1, 1), (1, 0)\})$ has a Hamiltonian circuit for all $m, n \geq 2$.

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Appendix

Appendix A

Proceedings The 8th Sciece Research Conference. 30-31 May 2016. University Of Phayao.

วงจรฮามิลโทเนียนในบางเคย์เลย์ไดกราฟ $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$

The Hamiltonian Circuit in Some Cayley Digraphs $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$

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บทคัดย่อ

สำหรับกรุปจำกัด G และเซต A เป็นสับเซตที่ไม่เซตว่างของ G เคย์เลย์ไดกราฟ $Cay(G,A)$ คือไดกราฟ โดยที่ G เป็นเซตของจุดและ $\{(g,ga): g \in G$ และ $a \in A\}$ เป็นเซตของเส้นเชื่อม ในงานวิจัยนี้เราได้หาวงจรฮามิลโทเนียนในบางเคย์เลย์ไดกราฟ $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ สำหรับบาง $|A| \le 2$

คำสำคัญ: กรุป, เคย์เลย์ไดกราฟ, วงจรฮามิลโทเนียน

Abstract

For a finite G and a nonempty subset A of G, the Cayley digraph Cay(G,A) is the digraph with vertex set G and edge set $\{(g,ga): g \in G \text{ and } a \in A\}$. In this paper, we find the Hamiltonian circuit in some Cayley digraph Cay$(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ for some $|A| \le 2$.

Keywords : Cayley digraph, Group, Hamiltonian circuit

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1. Introduction

Finding Hamiltonian circuit is interesting and very difficult which can apply to many problems , such as, pizza delivery, mail delivery, traveling salesman, garbage pickup, bus service/limousine service and reading gas meters. Moreover it can apply to biology(see[1],[2]).

A *group* is a system $(G,\cdot)$ consisting of a nonempty set G together with a binary operation $\cdot$ satisfying the condition $\forall x,y\in G\,(x\cdot y)\cdot z=x\cdot(y\cdot z)$. If there is no ambiguity we may write G and xy for $(G,\cdot)$ and $x\cdot y$, respectively. There is an identity element $e\in G$ which $a\cdot e=e\cdot a=a$ for all $a\in G$ and for each $a\in G$, there is an inverse element $a^{-1}\in G$ which $a^{-1}\cdot a=a\cdot a^{-1}=e$. The subgroup of G generated by A is denoted by $\langle A\rangle$ and consist of the element of G that can be expressed as finite products of elements in A. If A is such that $\langle A\rangle=G$, then A is called a generating set of G , G is generated by A. A group G is *cyclic* if it is generated by a single element, which we denote by $G=\langle a\rangle$. Let G_1,G_2 be a finite collection of groups. The external direct product of G_1,G_2 written as $G_1\oplus G_2$, is the set of Cartesian product of G_1 and G_2 which the operation is component wise.

A digraph D is a finite nonempty set of objected called vertices together with a set of ordered pairs of distinct vertices of D. Any such pair (u,v) is called arc and directed edge and will usually be denoted uv. For vertices u and v in a digraph D, a $u-v$ *directed walk* in D is a finite, alternating sequence $u=u_0,e_1,u_1,\ldots,u_{k-1},e_k,u_k=v$ of vertices and arcs or sequence $u=u_0,u_1,\ldots,u_{k-1},u_k=v$, beginning with u and ending with v, such that either $e_i=(u_{i-1},u_i)$ or $e_i=(u_i,u_{i-1})$ for $i=1,2,\ldots,k$. If the vertices $u_0,u_1,\ldots,u_k$ are distinct, then the $u-v$ directed walk is a $u-v$ *directed path*. If a directed path is closed then it is a *directed circuit*. A digraph D is *strongly connected (or strong)* if for every pair u and v of D have directed path. A path and circuit in a graph D containing every vertex of D is called a *Hamiltonian path* and *Hamiltonian circuit*. Graph in this paper have neither loops nor multiple edges and are finite.

For finite group G and a nonempty subset A of G, we define the *Cayley digraph* of G with respect to A, denoted by $Cay(G,A)$, to be the directed graph with vertex set G and edge set $\{(g,ga):g\in G \text{ and } a\in A\}$.

In this paper, there exists a Hamiltonian circuit in cyclic group of $Cay(\mathbb{Z}_m\oplus\mathbb{Z}_n,\{(a,b)\})$ and $Cay(\mathbb{Z}_m\oplus\mathbb{Z}_n,A)$ for some set A where $|A|=2$.

2. Main Results

Every connected Cayley digraphs of abelian group has a Hamiltonian path but it is not guarantee that there exists a Hamiltonian circuit. We will show that $Cay(\mathbb{Z}_m\oplus\mathbb{Z}_n,\{(a,b)\})$ there exists a Hamiltonian circuit.

Theorem 2.1 If $\gcd(m,n)=1$ then $(\mathbb{Z}_m\oplus\mathbb{Z}_n,\cdot)$ is cyclic group which generating set is $\{(a,b)|\gcd(a,n)=1 \text{ and } \gcd(b,n)=1\}$ that there exists a Hamiltonian circuit.

Proof In $Cay(\mathbb{Z}_m\oplus\mathbb{Z}_n,\{(a,b)\})$, there exist alternating sequence of directed walk which

$$(0,0),(a,b),\ldots,((mn-1)a,(mn-1)b),((mn)a,(mn)b)$$

and $(0,0),(a,b),\ldots,((mn-1)a \bmod m,(mn-1)b \bmod n),((mn)a \bmod m,(mn)b \bmod n)=(0,0)$ are equivalence. Moreover, $(\mathbb{Z}_m\oplus\mathbb{Z}_n,\cdot)$ is cyclic group, so this directed walk is closed directed path. The proof is complete.

□

Next, when $\langle(a,b)\rangle=\langle(1,1)\rangle$, we have $\langle(1,1)\rangle=\mathbb{Z}_m\oplus\mathbb{Z}_n$ and there exists a Hamiltonian circuit in $Cay(\mathbb{Z}_m\oplus\mathbb{Z}_n,\{(1,1)\})$.

Corollary 2.2 If $\gcd(m,n)=1$, then there exists a Hamiltonian circuit in $Cay(\mathbb{Z}_m\oplus\mathbb{Z}_n,\{(1,1)\})$.

Proof Let (i,j) be any vertex in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$. Since $(\mathbb{Z}_m \oplus \mathbb{Z}_n, \cdot)$ is cyclic group then $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$ contain a path of length $mn-1$. Furthermore, it contains a Hamiltonian circuit, alternating sequence

$$(i,j),(i+1,j+1),\ldots,(i+mn,i+mn)=(i,j).$$

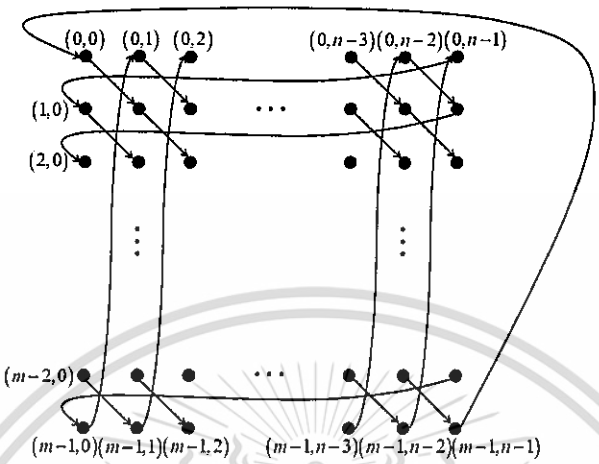

Figure 1 $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1)\})$

□

We consider the Hamiltonian circuit of $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, A)$ such that $\langle A \rangle = \mathbb{Z}_m \oplus \mathbb{Z}_n$, $|A| = 2$. $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(0,1),(1,0)\})$ has a Hamiltonian circuit when n divides m but $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(0,1),(1,0)\})$ does not have a Hamiltonian circuit when n and m are relatively prime and greater than 1[3], shown in figure 2.

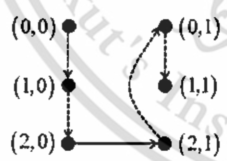

Figure 2 $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}_2, \{(0,1),(1,0)\})$

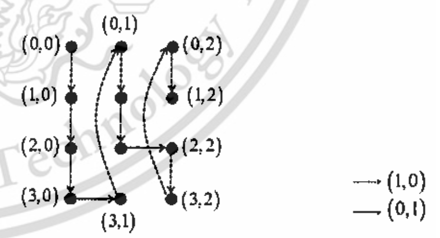

Figure 3 $Cay(\mathbb{Z}_4 \oplus \mathbb{Z}_3, \{(0,1),(1,0)\})$

From figure 2,3 $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}_2, \{(0,1),(1,0)\})$ and $Cay(\mathbb{Z}_4 \oplus \mathbb{Z}_3, \{(0,1),(1,0)\})$ has a Hamiltonian path but has no a Hamiltonian circuit. We will find the generating set of $\mathbb{Z}_m \oplus \mathbb{Z}_n$ for all m,n that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(a,b),(c,d)\})$ has Hamiltonian circuit.

Theorem 2.3 For $m,n \geq 2$, there exist a Hamiltonian circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(1,0)\})$.

Proof Let $\{(1,1),(1,0)\}$ be generating set of $(\mathbb{Z}_m \oplus \mathbb{Z}_n, \cdot)$, dark arc be generator by $(1,1)$ and dash arc be generator by $(1,0)$. We will show that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(1,0)\})$ has a Hamiltonian circuit, the following cases:

Case 1 $m=n$

Starting at $(0,0)$, use the generator $(1,1)$ to move crosswise down to $(1,1)$ until to $(n-1,n-1)$. Next, use the generator $(1,0)$ to move vertically up to $(0,n-1)$ and use the generator $(1,1)$ to move crosswise down to $(1,0)$. Then use the generator $(1,1)$ to move crosswise down to $(2,1)$ until to $(n-1,n-2)$ and use the generator $(1,0)$ to move vertically up to $(0,n-2)$. Next, use the generator $(1,1)$ to move crosswise down to $(1,n-1)$ until to $(2,0)$. Keep this process up until $(n-1,0)$. Finally, complete the circuit by use the generator $(1,0)$ moving vertically up to $(0,0)$.

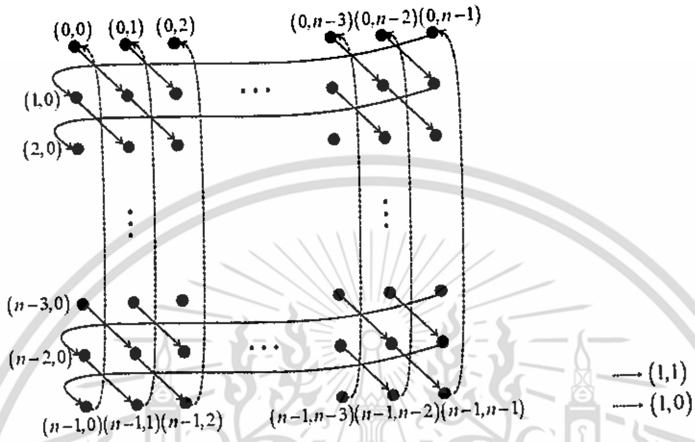

Figure 4 $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_n,\{(1,1),(1,0)\})$

Case 2 $m\neq n$

Starting at $(0,0)$, use the generator $(1,0)$ to move vertically down to $(1,0)$ until to $(m-2,0)$. Next, use the generator $(1,1)$ to move crosswise down to $(m-1,1)$. Then, use the generator $(1,0)$ to move vertically up to $(0,1)$. Keep this process up until $(m-1,0)$. Finally, complete the circuit by use the generator $(1,0)$ moving vertically up to $(0,0)$.

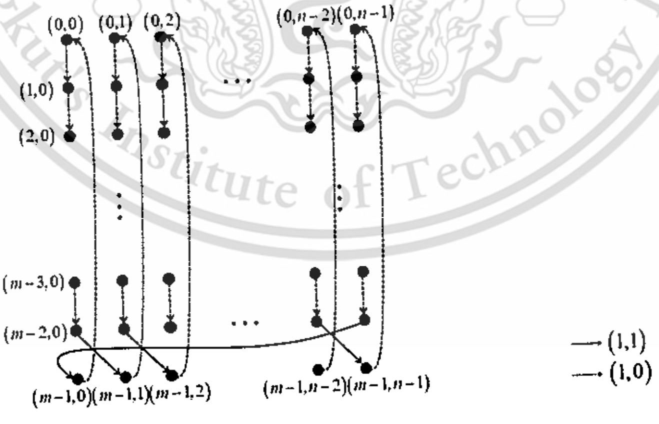

Figure 5 $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n,\{(1,1),(1,0)\})$

□

Theorem 2.4 For $m,n\geq 2$, there exist a Hamiltonian circuit in $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n,\{(1,1),(0,1)\})$

Proof Let $\{(1,1),(0,1)\}$ be generating set of $(\mathbb{Z}_m \oplus \mathbb{Z}_n,\cdot)$, dark arc be generator by $(1,1)$ and dash arc be generator by $(0,1)$. We will show that $Cay(\mathbb{Z}_m \oplus \mathbb{Z}_n,\{(1,1),(0,1)\})$ has a Hamiltonian circuit, the following figure 6.

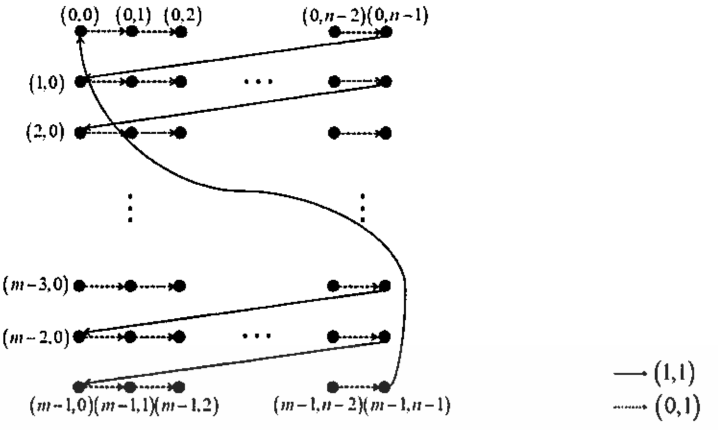

Figure 6 $Cay\left(\mathbb{Z}_m \oplus \mathbb{Z}_n, \{(1,1),(0,1)\}\right)$

□

3. Reference

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Appendix B

Table of some semigroup whose of Cayley digraph has either a Hamiltonian circuit or a Hamiltonian path

•	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

•	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

•	0	1	2
0	0	0	0
1	0	1	1
2	0	1	1

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

•	0	1	2
0	0	0	2
1	0	1	2
2	2	2	0

•	0	1	2
0	0	1	1
1	1	0	0
2	1	0	0

•	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

•	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	1	0
3	0	0	0	1

•	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	1	0
3	0	0	0	3

•	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	1	1
3	0	0	1	1

•	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

•	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	3
3	0	0	3	2

•	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	0	1
3	0	1	1	3

•	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

•	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	0	1
3	0	1	1	3

•	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

•	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	2
3	0	0	2	2

•	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	3
3	0	0	3	2

•	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	2

•	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	2
3	0	1	2	3

•	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

•	0	1	2	3
0	0	0	0	3
1	0	0	0	3
2	0	0	2	3
3	3	3	3	0

•	0	1	2	3
0	0	0	0	3
1	0	1	1	3
2	0	1	2	3
3	3	3	3	0

•	0	1	2	3
0	0	0	2	2
1	0	0	2	2
2	2	2	0	0
3	2	2	0	1

•	0	1	2	3
0	0	0	2	2
1	0	1	2	3
2	2	2	0	0
3	2	3	0	0

•	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	3	0
3	3	3	0	2

•	0	1	2	3
0	0	1	1	1
1	1	0	0	0
2	1	0	0	0
3	1	0	0	0

•	0	1	2	3
0	0	1	2	3
1	1	3	3	0
2	1	3	3	0
3	3	0	0	1

•	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

•	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	3	3
2	0	1	1	3	3
3	0	3	3	1	1
4	0	3	3	1	2

•	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	1	4	3
3	0	3	4	2	1
4	0	4	3	1	2

•	0	1	2	3	4
0	0	0	0	3	3
1	0	0	1	3	3
2	0	1	2	3	4
3	3	3	3	0	0
4	3	3	4	0	1

•	0	1	2	3	4
0	0	0	2	2	4
1	0	1	2	3	4
2	2	2	4	4	0
3	2	3	4	4	0
4	4	4	0	0	2

•	0	1	2	3	4
0	0	0	2	3	4
1	0	1	2	3	4
2	2	2	0	4	3
3	3	3	4	2	0
4	4	4	3	0	2

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