COVERING NUMBER FOR CLUSTERING IN QUASI-METRIC SPACE

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บทคัดย่อ

การจัดกลุ่ม คือการจัดให้จุดต่างๆ ที่มีคุณลักษณะคล้ายกันอยู่กลุ่มเดียวกัน เรียกกลุ่มที่จัดว่า คลัสเตอร์ การแบ่งกลุ่มจะแบ่งจาก n จุด เป็น k กลุ่มข้อมูล โดยจะหาจุดศูนย์กลาง ของแต่ละกลุ่มข้อมูล k จุด กับเซตปกคลุม ของแต่ละกลุ่ม ในงานวิจัยนี้ เราจะเสนอวิธีการหา จุดศูนย์กลางซึ่งจะใช้เป็นจุดปกคลุมของแต่ละกลุ่มข้อมูลได้อย่างไร เมื่อให้แต่ละกลุ่มมีระยะปกคลุมมีขนาด ϵ เพื่อปกคลุมเซต S เมื่อ $S \subset \mathbb{R}^2$ และใช้เมตริก L_1, L_2 และ L_∞ ในการหาระยะและแสดงสูตรเพื่อหาจำนวนจุดที่ปกคลุมของแต่ละกลุ่มข้อมูลบนปริภูมิเมตริก และนิยามปริภูมิเมตริกกึ่งเสมือนโดยใช้ปริภูมิเมตริก และหาสูตรเพื่อหาจำนวนจุดที่ปกคลุมของแต่ละกลุ่มข้อมูลบนปริภูมิเมตริกกึ่งเสมือน

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Abstract

Clustering is the task of partitioning the points into natural groups called clusters. The partitions of n points into k clusters will find k-center points of each cluster with the cover set. In this thesis, we present how to find the center points and the covering number by giving ϵ-cover for a covering space S where $S \subset \mathbb{R}^2$ with metric L_1, L_2 and L_∞. We present the formula for the covering number in metric space and define the quasi-metric space by giving metric space. We obtain the formula for the covering number in a quasi-metric space.

Keywords : covering number, metric spaces, quasi-metric space, clustering.

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Supawan Yena

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Research Motivation

A clustering algorithm is to cluster data points. Data points within the same cluster are similar to each other but data points in different clusters are dissimilar. Depending on the data and desired cluster characteristics, there are many types of clustering paradigms such as representative-based, hierarchical, density-based, graph-based, and spectral clustering. Clustering has been applied in a wide variety of fields: psychology, social, sciences, biology, statistics, pattern recognition, information retrieval, machine learning, and data mining. Distance measures are use in many clustering methods. The popular distance measure is metric space. In a typical clustering problem, we have a set of n input points and then partition the points into k clusters by the metric. We select a center point for each cluster. The distance is considered from each point to the center of belonging cluster. Then we minimize the maximum of these distances. The problem is called the k-center problem of clustering. The problem is corresponding with the ϵ-cover problem or the covering number. The covering number is the number of the point with distances of the covering number at the center point of a set equal to ϵ. Therefore, the covering number can confirm that k center point minimizes a maximum of distances which is equal to ϵ. Normally, metric space is used to measure the distance but the problem in the real world is asymmetry so quasi-metric is used instead of metric. In this thesis, we focus on the problem of the covering number and find the formula in metric space and quasi-metric space.

1.2 Objectives of the study

1) To study the step for finding the center point or the covering number of a set in metric space.

2) To show a form of the covering number in metric L_1, L_2 and L_∞.

3) To define quasi-metric space by using metric space.

4) To describe the covering number in quasi-metric space.

1.3 Scope of the study

Clustering is a common problem in the analysis of large data sets. Approximation of k-center and asymmetric k-center are NP-hard. We describe the basic idea of the covering number in metric L_1, L_2 and L_∞ and explain the covering number in quasi-metric by using some conditions to defined a quasi-metric by using a metric.

1.4 Benefits of the study

1) To provide the further mathematical method for clustering.
2) To fine form of the covering number in metric space L_1, L_2 and L_∞ with ϵ-cover.
3) To explain the covering number in quasi-metric distances.
4) To apply the covering number in Minimax Facility Location problems

1.5 Research Methodology

1) Study data mining and data analysis.
2) Study research papers and textbooks on metric space and quasi-metric space.
3) Study research papers of clustering in metric space.
4) Determine the goals and the scope of this research.
5) Study clustering in quasi-metric space.
6) Conclude the results and write the thesis.

Table 1.1: Research Methodology

Activity	Time frame																						
	2015				2016												2017						
	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12	1	2	3	4	5	6	7
step 1	■	■	■	■	■	■	■	■															
step 2				■	■	■	■	■	■	■	■	■											
step 3										■	■	■	■	■	■	■							
step 4											■	■	■	■	■	■	■						
step 5															■	■	■	■	■	■			
step 6																		■	■	■	■	■	■

Chapter 2

Preliminaies

Many mathematical issues need manipulating and dominant collections of random variables indexed by sets with an associated infinite. Any finite set can be measured in terms of its cardinality. The size of a group of infinitely several components is called covering numbers. Covering numbers are number of center points of a set which data point takes group nearest the center point. Set of each group is called the cluster. The k-center problem is a problem of clustering. The goal of the k-center problem is to choose a set of k points to serve as centers and to assign all the points to the centers. Then the maximum distance of any point in its center is small as possible. The ϵ-cover problem is to find a number of centers that make the cluster to cover all the points when the maximum distance is ϵ.

Clustering or cluster analysis is the task of grouping a set of objects (clusters). The object of a group is similar to another group and different from the objects in other groups. Formally, the clustering structure is represented by a set of subsets C_i of S where S is a set of objects and C_i is cluster such that $S = \cup_{i=1}^{k} C_i$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Consequently, any instance in S belongs to exactly one and only one subset [2].

Many clustering methods use distance measures to determine the distance between two instances x_i and x_j denote by $d(x_i, x_j)$. We have to describe the kind of space in which the data contains many distance functions in metric space.

2.1 Metric space

Definition 2.1. [3] A metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ for any $x, y, z \in X$ if it has the following properties:

- Positiveness: $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
- Symmetry: $d(x, y) = d(y, x)$.
- Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

The mapping d is called metric on X or distance function on X and (X, d) is called a **metric space**.

Example 2.1. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$. To show that d is a metric space on $\mathbb{R}$, we verify only triangle inequality, while the other properties are obviously satisfied. For any $x, y, z \in \mathbb{R}$, we obtain $d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$. Therefore, (X, d) is a metric space.

The distance function or Metric which we used in this research:

- L_1 (Taxicab): $d(x,y) = \|x-y\|_1 = \sum_{i=1}^{n} |x_i - y_i|$,
- L_2 (Euclidean): $d(x,y) = \|x-y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$,
- L_∞ (Supremum): $d(x,y) = \|x-y\|_\infty = \max |x_i - y_i|$,

where n is a dimension of space. All above distance functions are metric on $\mathbb{R}^n$ such that $(\mathbb{R}^n, L_1)$, $(\mathbb{R}^n, L_2)$ and $(\mathbb{R}^n, L_\infty)$ are metric space.

The open and closed set is to provide basic concepts about cover set.

Definition 2.2. Open and closed Set [4] Let (X,d) be a metric space, $x_0 \in X$ and $r > 0$. The **open ball** with center x_0 and radius r is the set $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$. The **closed ball** with center x_0 and radius r is the set $\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$.

Definition 2.3. [4] Let (X,d) be a metric space. The **unit ball** around some point $x \in X$ is the set of points with the distance at most 1 from x, $\{y \in X : d(x,y) \leq 1\}$.

The unit ball is a set of points in which the distance less than or equal to one with a fixed center point. The center point in space $\mathbb{R}^2$ and the unit ball with metric L_1, L_∞ and L_2 on $\mathbb{R}^2$ are shown in figure 2.1 .

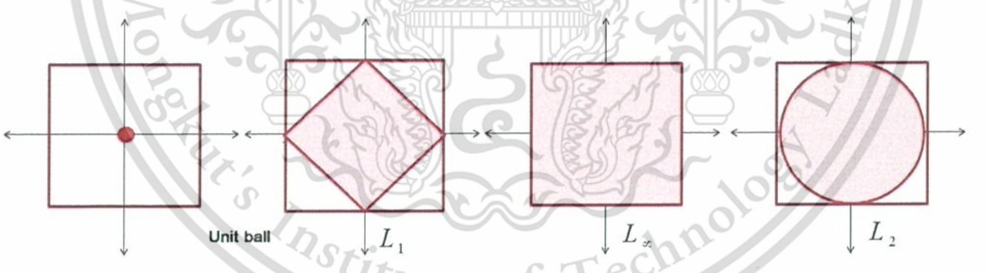

Figure 2.1: The unit balls $\overline{B}((0,0),1)$ on $\mathbb{R}^2$

Notice that the characteristic shape of the L_∞ metric is a box, while the characteristic shape of the L_1 metric is a diamond. Similarly, the characteristic shape of the L_2 metric is the sphere.

Some distance metrics does not satisfy the symmetry property which called asymmetry distances or quasi-metric.

2.2 Quasi-metric space

Definition 2.4. [3] A quasi-metric on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$ if it has the following properties:

- $d(x, y) = d(y, x) = 0$ if and only if $x = y$.
- $d(x, y) \leq d(x, z) + d(z, y)$.

The mapping d is called a quasi-metric on X. A pair (X, d) is called a **quasi-metric space**.

Example 2.2. Let $\alpha > 0$. If $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is defined by

$$d(x, y) = \{ \begin{array}{c} x - y; x \geq y \\ \alpha(y - x); y > x \end{array}$$

then d is an quasi-metric.
For $x, y, z \in \mathbb{R}$. We consider the properties:

1. It is clear that $d(x, y) \geq 0$ thus d satisfies the positiveness.
2. We will show that d satisfies the asymmetry.
   Case 1: $x \geq y$.
   We get $d(x, y) = x - y$,$d(y, x) = \alpha(x - y)$. Thus d is asymmetric.
   Case 2: $y \geq x$.
   We get $d(x, y) = \alpha(y - x)$ $d(y, x) = y - x$. Thus d is asymmetry.
3. We will show that d satisfies the triangle inequality.
   Case 1: $z \leq y \leq x$.
   We get $d(x, z) = x - z = (x - y) + (y - z) = d(x, y) + d(y, z)$.
   Case 2: $y \leq z \leq x$.
   We get $d(x, z) = x - z \leq x - y + \alpha\ (z - y) = d(x, y) + d(y, z)$.
   Case 3: $y \leq x \leq z$.
   We get $d(x, z) = \alpha(z - x) \leq (x - y) + \alpha\ (z - y) = d(x, y) + d(y, z)$.
   Case 4: $z \leq x \leq y$.
   We get $d(x, z) = x - z \leq \alpha\ (y - x) + (y - z) = d(x, y) + d(y, z)$.
   Case 5: $x \leq z \leq y$.
   We get $d(x, z) = \alpha\ (z - x) \leq \alpha\ (y - x) + (y - z) = d(x, y) + d(y, z)$.
   Case 6: $x \leq y \leq z$.
   We get $d(x, z) = \alpha\ (z - x) = \alpha(z - x) + \alpha(y - y) = \alpha(y - x) + \alpha(z - y) = d(x, y) + d(y, z)$.

Therefore, $(\mathbb{R}, d)$ is a quasi-metric space.

The symmetry property can be added in quasi-metric with the definition of symmetrization and weight function.

Definition 2.5. Symmetrization [3] Let (X, d) be a quasi-metric space. The function $\theta : X \times X \to \mathbb{R}^+$ while $\theta(x, y) = \frac{1}{2}[d(x, y) + d(y, x)]$ is called **symmetrization** of d.

Definition 2.6. Weight function [3] Let (X, d) be a quasi-metric space. The quasi-metric d is called **weightable quasi-metric** if there exists a weight function $w : M \to [0, \infty)$ that satisfies

$$d(x, y) + w(x) = d(y, x) + w(y) \text{ for all } x, y \in X.$$

We define quasi-metric by metric with this theorem.

Theorem 2.3. [3] Let (M, d) be any quasi-metric space. Then d is weightable if and only if there exists $w : M \to [0, \infty)$ such that

$$d(x, y) = \rho(x, y) + \tfrac{1}{2}[w(y) - w(x)] \text{ for all } x, y \in M,$$

where ρ is the symmetrized distance of d. Moreover, we have

$$\tfrac{1}{2}\left[w(x) - w(y)\right] \leq \rho(x, y) \text{ for all } x, y \in M.$$

Example 2.4. Let M be any set and d is a quasi-metric which

$$d(x, y) = \sum_{i=1}^{d} |(x_i - y_i)| + \frac{1}{2}\left[w(x) - w(y)\right].$$

Notice that the metric satisfies $\rho(x, d) = L_1 = \sum_{i=1}^{n} |(x_i - y_i)|$ and weighted function is $\frac{1}{2}\left[w(x) - w(y)\right]$; $w(x) > w(y)$. We will show that (M, d) is a quasi-metric space. For $x, y, z \in \mathbb{R}$, if $d(x, y)$ is a quasi-metric then it satisfies two properties.

- First property: It is clear that $d(x, y) \geq 0$ and $d(x, y) \neq d(y, x)$ since $\frac{1}{2}[w(x) - w(y)] \neq \frac{1}{2}[w(y) - w(x)]$ where $w(x) > w(y)$.
- Second property: We have to show that $d(x, y)$ is corresponding with the triangle inequality property,

$$d(x, y) = \sum_{i=1}^{d} |(x_i - y_i)| + \frac{1}{2}[w(x) - w(y)] \qquad ; w(x) > w(y),$$

$$d(y, z) = \sum_{i=1}^{d} |(y_i - z_i)| + \frac{1}{2}[w(y) - w(z)] \qquad ; w(y) > w(z),$$

$$d(x, z) = \sum_{i=1}^{d} |(x_i - z_i)| + \frac{1}{2}[w(x) - w(z)] \qquad ; w(x) > w(z),$$

and

$$
\begin{aligned}
d(x,y)+d(y,z) &= \sum_{i=1}^{d}|(x_i-y_i)|+\frac{1}{2}[w(x)-w(y)]+\sum_{i=1}^{d}|(y_i-z_i)|+\frac{1}{2}[w(y)-w(z)] \\
&\geq \sum_{i=1}^{d}|(x_i-z_i)|+\frac{1}{2}[w(x)-w(y)+w(y)-w(z)] \\
&= \sum_{i=1}^{d}|(x_i-z_i)|+\frac{1}{2}[w(x)-w(z)].
\end{aligned}
$$

Thus $d(x,z) \leq d(x,y)+d(y,z)$.

Therefore, (M,d) is a quasi-metric space.

Definition 2.7. Closure [5] Let (X,d) be a quasi-metric space and $y \in Y \subset X$. The **closure** of $\{y\}$ in X is defined by the set $Cl_X\{y\} = \{x \in X : d(x,y) = 0\}$.

Definition 2.8. The best approximation [5] Given $d(p,Y) = inf\{d(p,y) \mid y \in Y\}$. Let (X,d) be a quasi-metric space. Given $Y \subset X$ and $p \in X$. An element $y_0 \in Y$ such that $d(p,Y) = d(p,y_0)$ is said to be an **element of the best approximation** to ρ. Let Y be a (nonempty) subset of a quasi-metric space (X,d). For each $p \neq y$, we denote $P_Y(p)$ to be **the set of all the best approximation** to ρ by elements of Y.

2.3 The k-center problem

The goal of k-center problem is to choose a set of k points to serve as centers and to assign all the points to the centers so the maximum distance of any point to its center is small as possible.

The k-center problem is defined as follows: given a set S of n points in a d-dimensional metric space $(\mathbb{R}^d,\rho)$ and an integer k, we compute a k-clustering of the smallest possible size. The k-center problem can be formulated as covering S by k congruent disks (under the ρ-metric) of the smallest possible size. If centers of clusters are required to be a subset of the input points, the problem is called the discrete k-center problem.

At any metric space (X,ρ). The k-center problem: given a set S and an integer k. What is the smallest value of ϵ for which you can find an ϵ-cover of S as size k?

The k-center problem algorithms:

Input: finite set $S \subset X$ and the integer k.

Output: $S \subset X$ with $|T| = k$.

Goal: minimize cost$(T) = max_{x \in S}\rho(x,T)$.

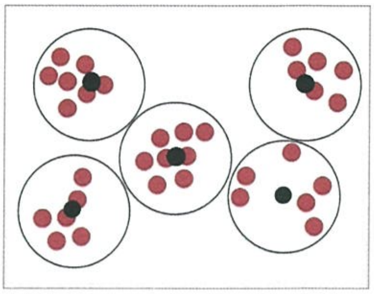

Figure 2.2: Five clusters

2.3.1 Farthest-first traversal

A basic fact about the k-center problem is NP-hard. There is no efficient algorithm that returns the correct answer. How ever, there is a good algorithm called farthest first traversal.

Farthest-first traversal algorithms:
Input: choose any $z \in S$ and give set $T = \{z\}$.
Output: while $|T| < k$:

1. $z = \arg\max_{x \in S} \rho(x, T)$,
2. $T = \cup\{z\}$.

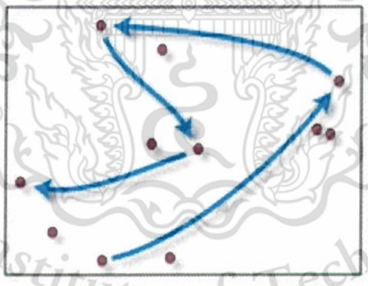

Figure 2.3: $|T| = 5$ center points

The algorithm builds a solution T at one point at a time. It starts with any point, and then iteratively adds in the point furthest from the ones chosen so far.

Farthest-first traversal takes time $o(k\,|S|)$, which is fairly efficient. The solution might not be perfect, but it is always close to the optimal solution.

If T is the solution returned by farthest-first traversal, and T^* is the optimal solution, then cost(T) ≤ 2cost(T^*).

2.4 Covering number in metric space

Definition 2.9. [1] Let be any metric space (X, ρ). For any $\epsilon > 0$, an **ϵ-cover** of a set $S \subset X$ be any set $T \subset X$ such that $\sup_{x \in S} \rho(x, T) \leq \epsilon$ and the cardinality of set T is called **the covering number**.

Here $\rho(x, T)$ is the distance from the point x to the closest point in set T, that is $\rho(x, T) = \inf_{z \in T} \rho(x, z)$.

Definition 2.10. [5] Let (X, d) be a metric space and $M \subset X$. For $x \in X$, **set of best approximation** is define by

$$P_M(x) = \{z \in M : d(x, z) = d(x, M)\}$$

where $d(x, M) = inf\{d(x, y), y \in M\}$. Any $z \in P_M(X)$ is called **the point of the best approximation** for x from M.

In other words, an ϵ-cover of S is a (typically smaller) set of points T which constitute the best approximation to S at most ϵ away from T. Otherwise, the ϵ-covering number is defined as the smallest number of balls of radius ϵ whose the union contains S.

Example 2.5. [1] In metric space, we set up $S = [-1, 1]^2$. There are the covering number at ϵ-cover when $\epsilon = 1$ by metric L_1, L_2 and L_∞.

Case1: In metric L_∞ and $\epsilon = 1$. The covering number has only one point as shown in figure 2.4.

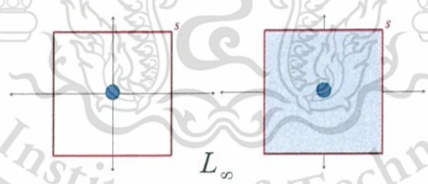

Figure 2.4: 1-cover by L_∞

Case2: In metric L_1 and $\epsilon = 1$. The covering number has four points as shown in figure 2.5.

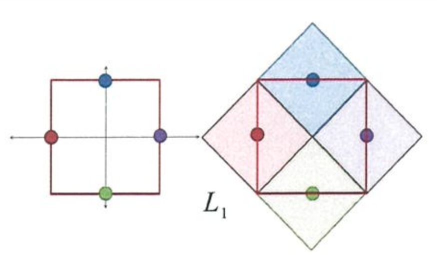

Figure 2.5: 1-cover by L_1

Case3: In metric L_2 and $\epsilon = 1$. The covering number does not exists at point but the space can be covered only one point where $\epsilon = \sqrt{2}$ as shown in figure 2.6.

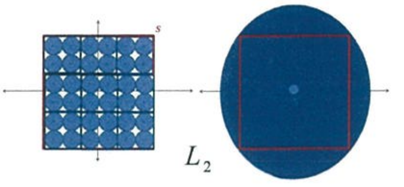

Figure 2.6: $\sqrt{2}$-cover by L_2

2.4.1 Computing covering numbers

In a metric space (X, ρ), the ϵ-covering number of a set $S \subset X$ is the size of its smallest ϵ-cover. Specifically, The covering number is define by

$$N(S, \epsilon) = min\{|T| : T \text{ is an } \epsilon\text{-cover of } S\}.$$

Farthest-first traversal is used to approximate k-center points:
Choose any $z \in S$ and give set $T = \{z\}$.
While $max_{x \in S}\rho(x, T) > \epsilon$, we have

1. $z = argmax_{x \in S}\rho(x, T)$,
2. $T = T \cup \{z\}$.

Return the value $|T|$.
The returned value $|T|$ satisfies the property

$$N(S, \varepsilon) \leq |T| \leq N(S, \tfrac{\varepsilon}{2}).$$

This is not a strong guarantee.

2.5 Voronoi regions or the cover set

The representatives T induce a Voronoi partition of $\mathbb{R}^d$: a decomposition of $\mathbb{R}^d$ into k convex cells, each corresponding to some $z \in T$ and containing the region of space whose the nearest representative is z. The partition induces an optimal clustering of the data set $S = \cup_{z \in T} C_z$, where

$$C_z = \{x \in S : \text{the closest representative } xisz\}.$$

Thus the k-means cost function can be written as

$$\text{cost}(T) = \sum_{z \in T} \sum_{x \in C_z} \|x - z\|^2.$$

In analyzing algorithms, we consider suboptimal partitions of S. To this end, cost function of set is define by

$$\text{cost}(C_1,\ldots,C_k;z_1,\ldots,z_k)=\sum_{j=1}^{k}\sum_{x\in C_j}\|x-z_j\|^2.$$

The k-means algorithm converges to a local optimum of its cost function.

Chapter 3

Covering Number in Metric spaces

3.1 A method to find the covering number in metric spaces

We determine the covering number of set $S = [-1,1]^2 \subset \mathbb{R}^2$ as the following step:

(i) To create the closed ball $B(x, \epsilon)$ of a corner point in set S. The corner point is $(1,1)$, $(-1,-1)$, $(1,-1)$ and $(-1,1)$ as shown in figure 3.1 (left).

(ii) To cover each point which the point is nearest to the other balls as shown in figure 3.1 (right).

(iii) To iterative the step 1 and 2 if some regions are uncovered.

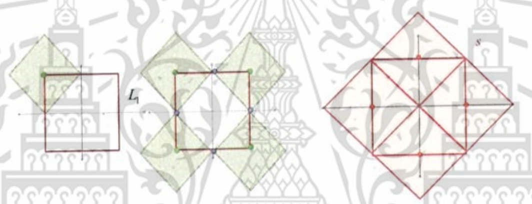

Figure 3.1: The closed ball of the corner point the covers each center point

If we make cluster C_i at the center point in set T then $\bigcup_{i=1}^{|T|} C_i \geq S$ or $\bigcup_{i=1}^{|T|} C_i \cap S = S$ and area of $\bigcup_{i=1}^{|T|} C_i$ is more than an equal area of set S.

Suppose that $P_1, P_2, \ldots,$ and P_k are center points. If the union of ϵ- ball at $P_1, P_2, \ldots,$ and P_k cover set S then the area of all clusters covered by $P_1, P_2, \ldots,$ and P_k intersect space S is greater than or equal to the space S.

3.1.1 1-cover

Remark 3.1. The space $S = [-1,1]^2$ and the covering number for 1-cover has the following:

(i) Metric L_∞: The covering number has one point which it is an origin.

(ii) Metric L_1: The covering number has four points.

(iii) Metric L_2: The covering number does not exist.

Theorem 3.2. Let $(\mathbb{R}^d, L_\infty)$ be a metric space. If $\epsilon = 1$ then the covering number has one point which is an origin.

Proof. By definition2.9, if $\epsilon = 1$ then $T = \{(0,0)\}$. Moreover, if $\rho(x,T) = \rho(x,(0,0)) \leq 1$ for all $x \in S$ then $\sup_{x \in S}\rho(x,T) \leq 1$. Therefore, the covering number is $|T| = 1$ at $T = \{(0,0)\}$.

Example 3.3. For 1-cover. Let $(\mathbb{R}^d, L_\infty)$ be metric space. We will show that the point of set in the theorem 3.2 is cover point in the metric space at $\epsilon = 1$. By method to find the covering number in metric space:
First, we will find all center points, by considering each ball at point $(-1,1), (1,1), (1,-1)$ and $(-1,-1)$.

1. At the point $x = (-1,1)$, the ball is the set
   $\overline{B}((-1,1),1) = \{y \in S \mid d((y_1,y_2),(-1,1)) = \max(|y_1-(-1)|, |y_2-1|) = 1\}$.
2. At the point $x = (1,1)$, the ball is the set
   $\overline{B}((1,1),1) = \{y \in S \mid d((y_1,y_2),(1,1)) = \max(|y_1-1|, |y_2-1|) = 1\}$.
3. At the point $x = (-1,-1)$, the ball is the set
   $\overline{B}((-1,-1),1) = \{y \in S \mid d((y_1,y_2),(-1,-1)) = \max(|y_1-(-1)|, |y_2-(-1)|) = 1\}$.
4. At the point $x = (1,-1)$, the ball is the set
   $\overline{B}((1,-1),1) = \{y \in S \mid d((y_1,y_2),(1,-1)) = \max(|y_1-1|, |y_2-(-1)|) = 1\}$.

The set of the closet point in the other balls is $T = \{(0,0)\}$.

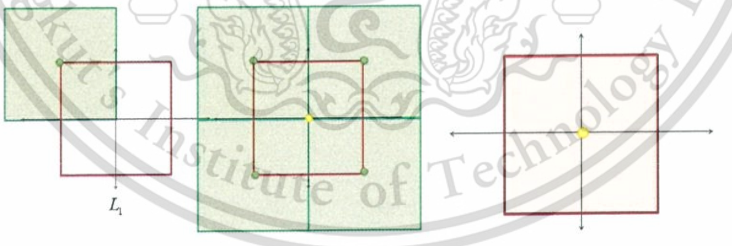

Figure 3.2: Finding the covering number of metric L_∞

If we make cluster of set S where $T = \{(0,0)\}$ with metric L_∞ at $\epsilon = 1$ then we have one cluster, C_1. We get $C_1 = S$, so $|T| = 1$ is the covering number. The area of $C_1 = 4$ equal the area of set $S = 4$.

Theorem 3.4. Let $(\mathbb{R}^d, L_1)$ be a metric space. If $\epsilon = 1$ then the covering number has four points at $T = \{(0,1),(0-1),(1,0),(-1,0)\}$.

Proof. By definition2.9, if $\epsilon = 1$ then $T = \{(0,1),(0-1),(1,0),(-1,0)\}$. Moreover, if $\rho(x,T) \leq 1$ for all $x \in S$ then $\sup_{x \in S} \rho(x,T) \leq 1$. Therefore, the covering number is $|T| = 4$ at $T = \{(0,1),(0-1),(1,0),(-1,0)\}$.

Example 3.5. For 1-cover. Let $(\mathbb{R}^d, L_1)$ be metric space. We will show that the point of set in the theorem 3.4 are cover points in the metric space at $\epsilon = 1$. By method to find the covering number in metric space:
First, we will find all center points, by considering each ball at point $(-1,1),(1,1),(1,-1)$ and $(-1,-1)$.

1. At the point $x = (-1,1)$, the ball is the set
   $\overline{B}((-1,1),1) = \{y \in S \; | d((y_1,y_2),(-1,1)) = |y_1-(-1)| + |y_2-1| = 1\}$.
2. At the point $x = (1,1)$, the ball is the set
   $\overline{B}((1,1),1) = \{y \in S \; | d((y_1,y_2),(1,1)) = |y_1-1| + |y_2-1| = 1\}$.
3. At the point $x = (-1,-1)$, the ball is the set
   $\overline{B}((-1,-1),1) = \{y \in S \; | d((y_1,y_2),(-1,-1)) = |y_1-(-1)| + |y_2-(-1)| = 1\}$.
4. At the point $x = (1,-1)$, the ball is the set
   $\overline{B}((1,-1),1) = \{y \in S \; | d((y_1,y_2),(1,-1)) = |y_1-1| + |y_2-(-1)| = 1\}$.

The set of the closet point in the other balls is $T = \{(0,1),(0-1),(1,0),(-1,0)\}$.

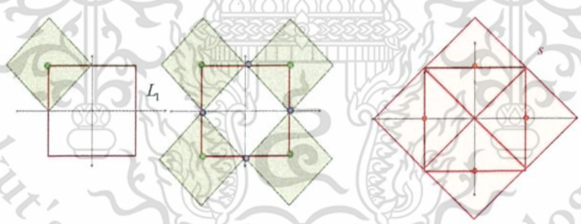

Figure 3.3: Finding the covering number of metric L_1

If we make cluster of set S where $T = \{(0,1),(0-1),(1,0),(-1,0)\}$ with metric L_1 at $\epsilon = 1$ then we have four cluster, C_1, C_2, C_3 and C_4. We get $C_1 \cup C_2 \cup C_3 \cup C_4 \geq S$, so $|T| = 4$ is the covering number. The area of $C_1 \cup C_2 \cup C_3 \cup C_4 = 8$ more than the area of set $S = 4$.

Example 3.6. Let $(\mathbb{R}^d, L_2)$ be a metric space . If $\epsilon = 1$ then the covering number does not exist because we can not make cluster.
For 1-cover with metric L_2. We will find all center points, by considering each ball at point $(-1,1),(1,1),(1,-1)$ and $(-1,-1)$.

1. At the point $x = (-1,1)$, the ball is the set
   $\overline{B}((-1,1),1) = \{y \in S \; | d((y_1,y_2),(-1,1)) = \sqrt{|y_1-(-1)| + |y_2-1|} = 1\}$.

2. At the point $x = (1,1)$, the ball is the set
$\overline{B}((1,1),1) = \{y \in S \mid d((y_1,y_2),(1,1)) = \sqrt{|y_1-1|+|y_2-1|} = 1\}$.

3. At the point $x = (-1,-1)$, the ball is the set
$\overline{B}((-1,-1),1) = \{y \in S \mid d((y_1,y_2),(-1,-1)) = \sqrt{|y_1-(-1)|+|y_2-(-1)|} = 1\}$.

4. At the point $x = (1,-1)$, the ball is the set
$\overline{B}((1,-1),1) = \{y \in S \mid d((y_1,y_2),(1,-1)) = \sqrt{|y_1-1|+|y_2-(-1)|} = 1\}$.

The set of the closet point in the other balls is $T = \{(0,1),(0-1),(1,0),(-1,0)\}$. If we make cluster of set S where $T = \{(0,1),(0-1),(1,0),(-1,0)\}$ with metric L_2 at $\epsilon = 1$ then we have four cluster C_1, C_2, C_3, C_4 and we get $C_1 \cup C_2 \cup C_3 \cup C_4 \geq S$ but $C_1 \cap C_2 \cap C_3 \cap C_4 \neq \emptyset$ as shown in figure 3.4. The covering number does not exist.

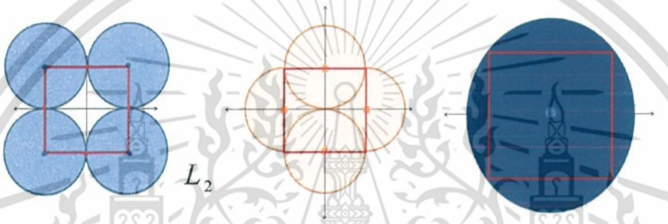

Figure 3.4: 1-cover does not exist but set S can be covered by $\epsilon = \sqrt{2}$

3.1.2 $\frac{1}{n}$-cover

Remark 3.7. The space $S = [-1,1]^2$ and the covering number for $\frac{1}{n}$-cover for all $n \in \mathbb{N}$ and $n > 1$:

(i) Metric L_∞: The covering number is equal to n^2 or $|T| = n^2$.

(ii) Metric L_1: The covering number is equal to $\sum_{k=1}^{n} 4k$ or $|T| = \sum_{k=1}^{n} 4k = 4\sum_{k=1}^{n} k = \frac{4(n)(n+1)}{2} = 2n^2 + 2n = 2n(2n+1)$.

(iii) Metric L_2: The covering number does not exist.

Theorem 3.8. Let $(\mathbb{R}^d, L_\infty)$ be a metric space. If $\epsilon = \frac{1}{2}$ then the covering number are four points at $T = \left\{(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right\}$.

Proof. By definition2.9, if $\epsilon = \frac{1}{2}$ Then $T = \left\{(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right\}$. If $\rho(x,T) \leq \frac{1}{2}$ for all $x \in S$. Then $\sup_{x\in S}\rho(x,T) \leq \frac{1}{2}$. Therefore, the covering number is $|T| = 4 = 2^2$ at $T = \left\{(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right\}$.

Example 3.9. For $\frac{1}{2}$-cover. Let $(\mathbb{R}^d, L_\infty)$ be a metric space. We will show that the point of set in the theorem 3.8 are cover points in the metric space at $\epsilon = \frac{1}{2}$. By method to find the covering number in metric space:
First, we will find all center points by considering each ball at point $(-1,1), (1,1), (1,-1)$ and $(-1,-1)$.

1. At the point $x = (-1,1)$, the ball is the set
$\overline{B}((-1,1), \frac{1}{2}) = \{y \in S \mid d((y_1,y_2),(-1,1)) = \max(|y_1 - (-1)|, |y_2 - 1|) = \frac{1}{2}\}$.

2. At the point $x = (1,1)$, the ball is the set
$\overline{B}((1,1), \frac{1}{2}) = \{y \in S \mid d((y_1,y_2),(1,1)) = \max(|y_1 - 1|, |y_2 - 1|) = \frac{1}{2}\}$.

3. At the point $x = (-1,-1)$, the ball is the set
$\overline{B}((-1,-1), \frac{1}{2}) = \{y \in S \mid d((y_1,y_2),(-1,-1)) = \max(|y_1 - (-1)|, |y_2 - (-1)|) = \frac{1}{2}\}$.

4. At the point $x = (1,-1)$, the ball is set
$\overline{B}((1,-1), \frac{1}{2}) = \{y \in S \mid d((y_1,y_2),(1,-1)) = \max(|y_1 - 1|, |y_2 - (-1)|) = \frac{1}{2}\}$.

The set of the closet point in the other ball is $T = \left\{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\right\}$, $|T| = 4 = 2^2$.

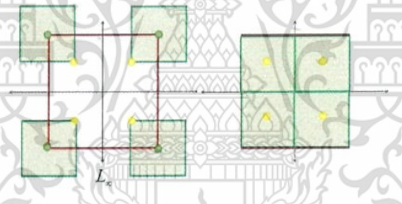

Figure 3.5: The covering number of metric L_∞ ($\frac{1}{2}$-cover)

If we make cluster of set S where $T = \left\{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\right\}$ with metric L_∞ at $\epsilon = \frac{1}{2}$ then we have four cluster C_1, C_2, C_3 and C_4. We get $C_1 \cup C_2 \cup C_3 \cup C_4 \geq S$, so $|T| = 4$ is the covering number. The area of $C_1 \cup C_2 \cup C_3 \cup C_4 = 4$ is equal to the area of set $S = 4$.

Theorem 3.10. Let $(\mathbb{R}^d, L_\infty)$ be a metric space . If $\epsilon = \frac{1}{3}$ then the covering number are nine points at $T = \left\{(-\frac{2}{3}, \frac{2}{3}), (-\frac{2}{3}, -\frac{2}{3}), (\frac{2}{3}, \frac{2}{3}), (\frac{2}{3}, -\frac{2}{3}), (0, \frac{2}{3}), (0,0), (-\frac{2}{3}, 0), (0, -\frac{2}{3}), (\frac{2}{3}, 0)\right\}$
.

Proof. By definition2.9, if $\epsilon = \frac{1}{3}$ Then $T = \left\{(-\frac{2}{3}, \pm\frac{2}{3}), (\frac{2}{3}, \pm\frac{2}{3}), (0, \pm\frac{2}{3}), (\pm\frac{2}{3}, 0), (0,0)\right\}$.
If $\rho(x,T) \leq \frac{1}{3}$ for all $x \in S$. then $\sup_{x \in S} \rho(x,T) \leq \frac{1}{3}$. Therefore, the covering number is $|T| = 9 = 3^2$ at $T = \left\{(-\frac{2}{3}, \pm\frac{2}{3}), (\frac{2}{3}, \pm\frac{2}{3}), (0, \pm\frac{2}{3}), (\pm\frac{2}{3}, 0), (0,0)\right\}$.

Example 3.11. For $\frac{1}{3}$-cover. Let $(\mathbb{R}^d, L_\infty)$ be a metric space.We will show that the point of set in the theorem 3.10 are cover points in the metric space at $\epsilon = \frac{1}{3}$ by

method to find the covering number in metric space:
First, we will find all center points by considering each ball at point $(-1,1),(1,1),(1,-1)$ and $(-1,-1)$.

1. At the point $x=(-1,1)$, the ball is the set
   $\overline{B}((-1,1),\frac{1}{3}) = \{y \in S \mid d((y_1,y_2),(-1,1)) = \max(|y_1-(-1)|,|y_2-1|) = \frac{1}{3}\}$.

2. At the point $x=(1,1)$, the ball is the set
   $\overline{B}((1,1),\frac{1}{3}) = \{y \in S \mid d((y_1,y_2),(1,1)) = \max(|y_1-1|,|y_2-1|) = \frac{1}{3}\}$.

3. At the point $x=(-1,-1)$, the ball is the set
   $\overline{B}((-1,-1),\frac{1}{3}) = \{y \in S \mid d((y_1,y_2),(-1,-1)) = \max(|y_1-(-1)|,|y_2-(-1)|) = \frac{1}{3}\}$.

4. At the point $x=(1,-1)$, the ball is the set
   $\overline{B}((1,-1),\frac{1}{3}) = \{y \in S \mid d((y_1,y_2),(1,-1)) = \max(|y_1-1|,|y_2-(-1)|) = \frac{1}{3}\}$.

The set of the closet point in the other ball is $T_1 = \left\{(-\frac{2}{3},\frac{2}{3}),(-\frac{2}{3},-\frac{2}{3}),(\frac{2}{3},\frac{2}{3}),(\frac{2}{3},-\frac{2}{3})\right\}$.

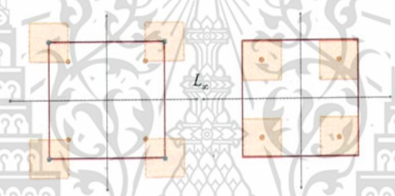

Figure 3.6: First round for finding the covering number of metric L_∞ ($\frac{1}{3}$- cover)

T_1 is not the covering set S. We consider another cover point from the remaining region of set S. Now new corner points are in set A such that
$A = \{(-\frac{1}{3},1),(\frac{1}{3},\frac{1}{3}),(-1,\frac{1}{3}),(-1,-\frac{1}{3}),(-\frac{1}{3},-\frac{1}{3}),(-\frac{1}{3},-1),(\frac{1}{3},-1),(\frac{1}{3},-\frac{1}{3}),(1,-\frac{1}{3}),(1,\frac{1}{3}),$ $(\frac{1}{3},\frac{1}{3}),(\frac{1}{3},1)\}$, and $\overline{B}(x,\frac{1}{3}) = \{y \in S, x \in A \mid d((y_1,y_2),x) = \frac{1}{3}\}$.
The set of the closet point in other balls is $T_2 = \left\{(0,\frac{2}{3}),(0,0),(-\frac{2}{3},0),(0,-\frac{2}{3}),(\frac{2}{3},0)\right\}$
Therefore, the covering number is $|T_1 \cup T_2| = |3^2| = 9 = 4+5$.

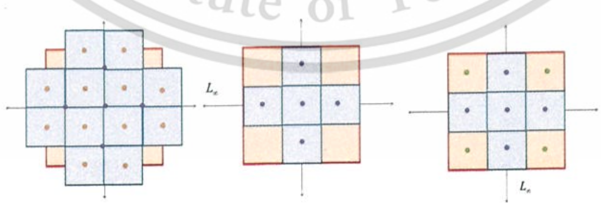

Figure 3.7: Second round for finding the covering number of metric L_∞ ($\frac{1}{3}$-cover)

If we make cluster of set S where $T = \left\{(-\frac{2}{3},\pm\frac{2}{3}),(\frac{2}{3},\pm\frac{2}{3}),(0,\pm\frac{2}{3}),(\pm\frac{2}{3},0),(0,0)\right\}$ with metric L_∞ at $\epsilon = \frac{1}{3}$ then we have nine cluster of $C_1, C_2, \ldots$ and C_9. We get $C_1 \cup C_2 \cup \ldots \cup C_9 \geq S$, so $|T| = 9$ is the covering number. The area of $C_1 \cup C_2 \cup \ldots \cup C_9 = 4$ is equal to the area of set $S = 4$.

Theorem 3.12. Let $(\mathbb{R}^d, L_1)$ be a metric space . If $\epsilon = \frac{1}{2}$ then the covering number are 12 points at $T = \Big\{(-\frac{1}{2}, 1), (-1, \frac{1}{2}), (-1, -\frac{1}{2}), (-\frac{1}{2}, -1), (\frac{1}{2}, 1), (1, \frac{1}{2}), (1, -\frac{1}{2}), (\frac{1}{2}, -1), (0, \frac{1}{2}), (-\frac{1}{2}, 0), (0, -\frac{1}{2})(\frac{1}{2}, 0)\Big\}$.

Proof. By definition2.9, if $\epsilon = \frac{1}{2}$ Then $T = \Big\{(-\frac{1}{2}, \pm 1), (-1, \pm\frac{1}{2}), (\frac{1}{2}, \pm 1), (1, \pm\frac{1}{2}), (0, \pm\frac{1}{2}), (\pm\frac{1}{2}, 0)\Big\}$. If $\rho(x, T) \leq \frac{1}{2}$ for all $x \in S$. Then $\sup_{x \in S} \rho(x, T) \leq \frac{1}{2}$. Therefore, the covering number is $|T| = 12 = 4(1) + 4(2)$ at $T = \Big\{(-\frac{1}{2}, \pm 1), (-1, \pm\frac{1}{2}), (\frac{1}{2}, \pm 1), (1, \pm\frac{1}{2}), (0, \pm\frac{1}{2}), (\pm\frac{1}{2}, 0)\Big\}$.

Example 3.13. For $\frac{1}{2}$-cover. Let $(\mathbb{R}^d, L_1)$ be a metric space. We will show that the point of set in the theorem 3.12 are cover points in the metric space at $\epsilon = \frac{1}{2}$ by method to find the covering number in metric space:
First, we will find all center points by considering each ball at point $(-1, 1), (1, 1), (1, -1)$ and $(-1, -1)$.

1. At the point $x = (-1, 1)$, the ball is the set
   $\overline{B}((-1, 1), \frac{1}{2}) = \{y \in S | d((y_1, y_2), (-1, 1)) = (|y_1 - (-1)| + |y_2 - 1|) = \frac{1}{2}\}$.
2. At the point $x = (1, 1)$, the ball is the set
   $\overline{B}((1, 1), \frac{1}{2}) = \{y \in S | d((y_1, y_2), (1, 1)) = |y_1 - 1| + |y_2 - 1|) = \frac{1}{2}\}$.
3. At the point $x = (-1, -1)$, the ball is the set
   $\overline{B}((-1, -1), \frac{1}{2}) = \{y \in S | d((y_1, y_2), (-1, -1)) = (|y_1 - (-1)| + |y_2 - (-1)|) = \frac{1}{2}\}$.
4. At the point $x = (1, -1)$, the ball is the set
   $\overline{B}((1, -1), \frac{1}{2}) = \{y \in S | d((y_1, y_2), (1, -1)) = (|y_1 - 1| + |y_2 - (-1)|) = \frac{1}{2}\}$.

The set of the closet point in the other ball is
$T_1 = \Big\{(-\frac{1}{2}, 1), (-1, \frac{1}{2}), (-1, -\frac{1}{2}), (-\frac{1}{2}, -1), (\frac{1}{2}, 1), (1, \frac{1}{2}), (1, -\frac{1}{2}), (\frac{1}{2}, -1)\Big\}$.

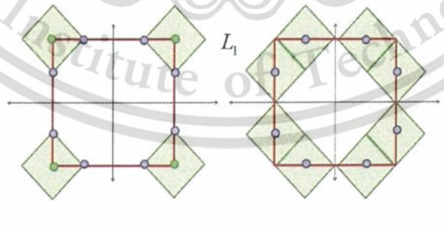

Figure 3.8: First round for finding the covering number of metric L_1 ($\frac{1}{2}$-cover)

T_1 is not the covering set S. We consider another cover point from the remaining region of set S. Now new corner points are in set A such that $A = \{(0, 1), (1, 0), (-1, 0), (0, -1)\}$, and $\overline{B}(x, \frac{1}{2}) = \{y \in S, x \in A | d((y_1, y_2), x) = \frac{1}{2}\}$.
The set of the closet point in other balls is $T_2 = \Big\{(0, \frac{1}{2}), (-\frac{1}{2}, 0), (0, -\frac{1}{2})(\frac{1}{2}, 0)\Big\}$.
Therefore, the covering number is $|T_1 \cup T_2| = \sum_{k=1}^{2} 4k = 4 + 8 = 12$.

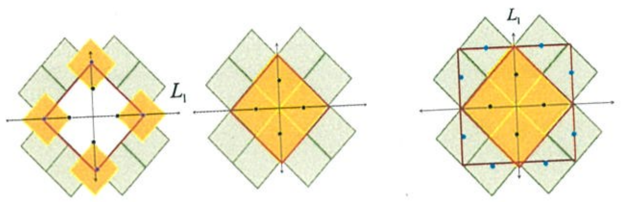

Figure 3.9: Second round for finding the covering number of metric L_1 ($\frac{1}{2}$-cover)

If we make cluster of set S where $T = \left\{(-\frac{1}{2}, \pm 1), (-1, \pm\frac{1}{2}), (\frac{1}{2}, \pm 1), (1, \pm\frac{1}{2}), (0, \pm\frac{1}{2}), (\pm\frac{1}{2}, 0)\right\}$ with metric L_1 at $\epsilon = \frac{1}{2}$ then we have twelve cluster $C_1, C_2, \ldots$ and C_{12}. We get $C_1 \cup C_2 \cup \ldots \cup C_{12} \geq S$, so $|T| = 9$ is covering number. The area of $C_1 \cup C_2 \cup \ldots \cup C_{12} = 6$ is more than to the area of set $S = 4$.

Theorem 3.14. Let be $(\mathbb{R}^d, L_1)$ be a metric space. If $\epsilon = \frac{1}{3}$ then the covering number are 24 points at $T = \left\{(-\frac{2}{3}, 1), (-1, \frac{2}{3}), (-1, -\frac{2}{3}), (-\frac{2}{3}, -1), (\frac{2}{3}, 1), (1, \frac{2}{3}), (1, -\frac{2}{3}), (\frac{2}{3}, -1), (0, 1), (-\frac{1}{3}, \frac{2}{3}), (-\frac{2}{3}, \frac{1}{3}), (-1, 0), (-\frac{1}{3}, -\frac{2}{3}), (-\frac{2}{3}, -\frac{1}{3}), (0, -1), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}), (1, 0), (\frac{1}{3}, -\frac{2}{3}), (\frac{2}{3}, -\frac{1}{3}), (-\frac{1}{3}, 0), (0, -\frac{1}{3}), (\frac{1}{3}, 0), (0, \frac{1}{3})\right\}$.

Proof. By definition2.9, if $\epsilon = \frac{1}{3}$ Then $T = \left\{(-\frac{2}{3}, \pm 1), (-1, \pm\frac{2}{3}), (\frac{2}{3}, \pm 1), (1, \pm\frac{2}{3}), (0, \pm 1), (-\frac{1}{3}, \pm\frac{2}{3}), (-\frac{2}{3}, \pm\frac{1}{3}), (\pm 1, 0), (\frac{2}{3}, \pm\frac{1}{3}), (\pm\frac{1}{3}, 0), (0, \pm\frac{1}{3}), (\frac{1}{3}, \pm\frac{2}{3})\right\}$. If $\rho(x, T) \leq \frac{1}{3}$ for all $x \in S$. Then $\sup_{x \in S} \rho(x, T) \leq \frac{1}{3}$. Therefore, the covering number is $|T| = 24 = 4(1)+4(2)+4(3)$ at $T = \left\{(-\frac{2}{3}, \pm 1), (-1, \pm\frac{2}{3}), (\frac{2}{3}, \pm 1), (1, \pm\frac{2}{3}), (0, \pm 1), (-\frac{1}{3}, \pm\frac{2}{3}), (-\frac{2}{3}, \pm\frac{1}{3}), (\pm 1, 0), (\frac{2}{3}, \pm\frac{1}{3}), (\pm\frac{1}{3}, 0), (0, \pm\frac{1}{3}), (\frac{1}{3}, \pm\frac{2}{3})\right\}$.

Example 3.15. For $\frac{1}{3}$-cover. Let $(\mathbb{R}^d, L_1)$ be a metric space. We will show that the point of set in the theorem 3.14 are cover points in the metric space at $\epsilon = 1$. By method to find the covering number in metric space:
First, we will find all center points by consider each ball at point $(-1, 1), (1, 1), (1, -1)$ and $(-1, -1)$.

1. At the point $x = (-1, 1)$, the ball is the set
   $\overline{B}((-1, 1), \frac{1}{3}) = \{y \in S | d((y_1, y_2), (1, 1)) = |y_1 - (-1)| + |y_2 - 1|) = \frac{1}{3}\}$.

2. At the point $x = (1, 1)$, the ball is the set
   $\overline{B}((1, 1), \frac{1}{3}) = \{y \in S | d((y_1, y_2), (1, 1)) = |y_1 - 1| + |y_2 - 1|) = \frac{1}{3}\}$.

3. At the point $x = (-1, -1)$, the ball is the set
   $\overline{B}((-1, -1), \frac{1}{3}) = \{y \in S | d((y_1, y_2), (-1, -1)) = (|y_1 - (-1)| + |y_2 - (-1)|) = \frac{1}{3}\}$.

4. At the point $x = (1, -1)$, the ball is the set
   $\overline{B}((1, -1), \frac{1}{3}) = \{y \in S | d((y_1, y_2), (1, -1)) = (|y_1 - 1| + |y_2 - (-1)|) = \frac{1}{3}\}$.

The set of the closet point in the other ball is
$T_1 = \left\{(-\frac{2}{3},1),(-1,\frac{2}{3}),(-1,-\frac{2}{3}),(-\frac{2}{3},-1),(\frac{2}{3},1),(1,\frac{2}{3}),(1,-\frac{2}{3}),(\frac{2}{3},-1)\right\}$.

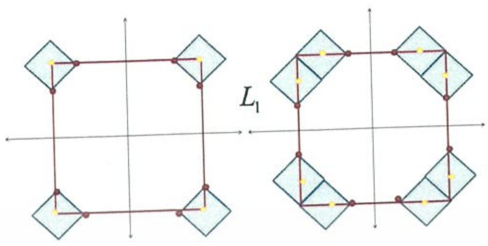

Figure 3.10: First round for finding the covering number of metric L_1 ($\frac{1}{3}$-cover)

T_1 is not the covering set S. We consider another cover point from the remaining region of set S. Now new corner points are in set A such that
$A = \{(-\frac{1}{3},1),(-1,\frac{1}{3}),(-1,-\frac{1}{3}),(-\frac{1}{3},-1),(\frac{1}{3},1),(1,\frac{1}{3}),(1,-\frac{1}{3}),(\frac{1}{3},-1)\}$, and
$\overline{B}(x,\frac{1}{3}) = \{y \in S, x \in A | d((y_1,y_2),x) = \frac{1}{3}\}$.
The set of the the closet point in other balls is
$T_2 = \left\{(0,1),(-\frac{1}{3},\frac{2}{3}),(-\frac{2}{3},\frac{1}{3}),(-1,0),(-\frac{1}{3},-\frac{2}{3}),(-\frac{2}{3},-\frac{1}{3}),(0,-1),(\frac{1}{3},\frac{2}{3}),(\frac{2}{3},\frac{1}{3}),(1,0),(\frac{1}{3},-\frac{2}{3}),(\frac{2}{3},-\frac{1}{3})\right\}$.

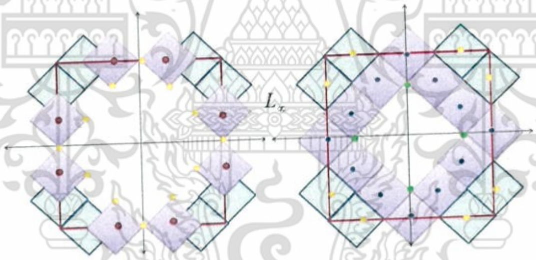

Figure 3.11: Second round for finding the covering number of metric L_1 ($\frac{1}{3}$-cover)

T_2 is not the covering set S. We consider another cover point from the remaining region of set S.
Now new corner points are in set A' such that $A' = \{(-\frac{2}{3},0),(0,-\frac{2}{3}),(\frac{2}{3},0),(0,\frac{2}{3})\}$, and
$\overline{B}(x,\frac{1}{3}) = \{y \in S, x \in A' | d((y_1,y_2),x) = \frac{1}{3}\}$.
The set of the closet point in other balls is $T_3 = \{(-\frac{1}{3},0),(0,-\frac{1}{3}),(\frac{1}{3},0),(0,\frac{1}{3})\}$.

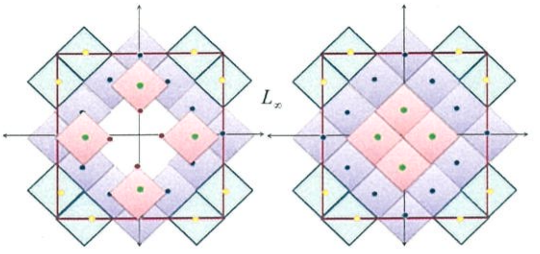

Figure 3.12: Third round for finding the covering number of metric L_1 ($\frac{1}{3}$-cover)

Therefore, the covering number is $|T_1 \cup T_2 \cup T_3| = \sum_{k=1}^{3} 4k = 4 + 8 + 12 = 24$.
If we make cluster of set S where $T = \{(-\frac{2}{3}, \pm 1), (-1, \pm\frac{2}{3}), (\frac{2}{3}, \pm 1), (1, \pm\frac{2}{3}), (0, \pm 1), (-\frac{1}{3}, \pm\frac{2}{3}), (-\frac{2}{3}, \pm\frac{1}{3}), (\pm 1, 0), (\frac{2}{3}, \pm\frac{1}{3}), (\pm\frac{1}{3}, 0), (0, \pm\frac{1}{3}), (\frac{1}{3}, \pm\frac{2}{3})\}$. with metric L_1 at $\epsilon = \frac{1}{3}$ then we have twenty-four cluster $C_1, C_2, \ldots$ and C_{24}. We get $C_1 \cup C_2 \cup \ldots \cup C_{24} \geq S$, so $|T| = 9$ is covering number. The area of $C_1 \cup C_2 \cup \ldots \cup C_{24} = \frac{32}{3}$ is more than to the area of set $S = 4$.

One center point in space $S = [-1, 1]^2$

On the other hand, we can change the ϵ-cover problem to the k-center problem when choosing k points for cover set S and find the smallest ϵ to cover set S.

Remark 3.16. Let space $S = [-1, 1]^2$. The covering number is equal to 1 at ϵ-cover.

(i) Metric L_∞ and $k = 1$: the smallest ϵ to cover set S is 1.

(ii) Metric L_1 and $k = 1$: the smallest ϵ to cover set S is 2.

(iii) Metric L_2 and $k = 1$: the smallest ϵ to cover set S is $\sqrt{2}$.

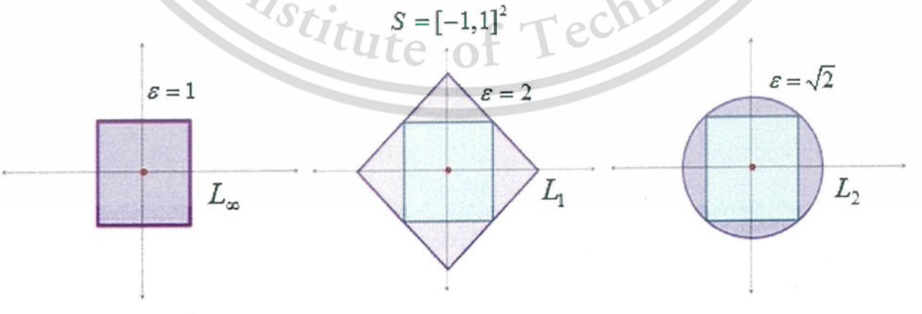

Figure 3.13: The covering number is equal to 1

3.1.3 The covering number in space $S = [-a, a]^2$

Remark 3.17. The covering number of set $S = [-1, 1]$. The maximum distance is equal to 1. The set can be covered where $\epsilon = 1$ or $\epsilon = \frac{1}{n}$. Moreover, For space $S = [-a, a]^2$. The maximum distance is equal to a. The set can be covered where $\epsilon = a$ or $\epsilon = \frac{a}{k}$, where $1 < k < a$; $k \in \mathbb{N}$. Notice the distance between any center points are less than or equal to 2ϵ.

The set $S = [-a, a]^2$ and a-cover. There are the covering number as the follows:

- Metric L_∞: The covering number is only one point at origin.
- Metric L_1: The covering number has four points.
- Metric L_2: The covering number does not exist.

The set $S = [-a, a]^2$ and $\frac{a}{k}$-cover. There are the covering number as the follows:

- Metric L_∞: The covering number is $(ak)^2$ or $|T| = (ak)^2$.
- Metric L_1: The covering number is $\sum_{j=1}^{ak} 4j$ or $|T| = \sum_{j=1}^{ak} 4j = 4\sum_{j=1}^{ak} j = 4\left(\frac{ak(ak+1)}{2}\right) = 2(ak)^2 + 2(ak) = 2ak(ak+1)$.
- Metric L_2: The covering number does not exist.

The space $S = [-a, a]^2$, the covering number is equal to 1 at ϵ-cover:

- Metric L_∞ and $k = 1$: the smallest ϵ to cover set S is a.
- Metric L_1 and $k = 1$: the smallest ϵ to cover set S is $2a$.
- Metric L_2 and $k = 1$: the smallest ϵ to cover set S is $a\sqrt{2}$.

Chapter 4

Covering Number in Quasi-metric space

Theorem 4.1. [3] Let (M, d) be any quasi-metric space. Then d is weightable if and only if there exists w: $M \to [0, \infty)$ such that

$$d(x, y) = \rho(x, y) + \tfrac{1}{2}[w(y) - w(x)] \text{ for all } x, y \in M,$$

where ρ is the symmetrized distance of d. Moreover, we have

$$\tfrac{1}{2}[w(x) - w(y)] \leq \rho(x, y) \text{for all } x, y \in M.$$

More generally, a metric space (X, d) does not satisfies the symmetry condition called a quasi-metric space. We consider the covering number in a quasi-metric space. Let (M, d) be quasi-metric space defined by metric space as follows

$$d(x, y) = \rho(x, y) + \tfrac{1}{2}[w(y) - w(x)].$$

Example 4.2. Let (M, d) be a quasi-metric space. The function given by

$$d(y, x) = \max|x_i - y_i| + \tfrac{1}{2}[w(x) - w(y)] \; ; w(x) > w(y).$$

where $\rho = L_\infty = \max|x_i - y_i|$ and $\max|x_i - y_i| \geq \frac{1}{2}[w(x) - w(y)]$.

For $x, y, z \in \mathbb{R}$. We show that d is an quasi-metric space on $\mathbb{R}$. We need to verify the asymmetry and triangle inequality, while the other properties are obviously satisfied.

- We will show that d is asymmetry. We have $d(x, y) \neq d(y, x)$ since $\frac{1}{2}[w(y) - w(x)] \neq \frac{1}{2}[w(x) - w(y)]$; $w(x) > w(y)$.
- We will show that d satisfies the triangle inequality. For all $w(x) > w(y) > w(z)$. We have

$$d(x, y) = \max|x_i - y_i| + \frac{1}{2}[w(y) - w(x)], \tag{4.1}$$

$$d(y, z) = \max|y_i - z_i| + \frac{1}{2}[w(z) - w(y)], \tag{4.2}$$

$$d(x, z) = \max|x_i - z_i| + \frac{1}{2}[w(z) - w(x)]. \tag{4.3}$$

Next, We consider $d(x, y) + d(y, z)$, defined by

$$\begin{aligned} &d(x, y) + d(y, z) \\ &= \max|x_i - y_i| + \tfrac{1}{2}[w(y) - w(x)] + \max|y_i - z_i| + \tfrac{1}{2}[w(z) - w(y)] \\ &\geq \max|x_i - z_i| + \tfrac{1}{2}[w(y) - w(x) + w(z) - w(y)] \\ &= \max|x_i - z_i| + \tfrac{1}{2}[w(z) - w(x)]. \end{aligned}$$

Thus $d(x,z) \leq d(x,y) + d(y,z)$.

Therefore, (M,d) is a quasi-metric space.

Example 4.3. Let (M,d) be a quasi-metric space. The function given by

$$d(y,x) = \sum_{i=1}^{d} |(x_i - y_i)| + \tfrac{1}{2}\left[w(x) - w(y)\right] \; ; \; w(x) > w(y).$$

where $\rho = L_1 = {}_{i=1}^{d} \sum |(x_i - y_i)|$ and $\sum_{i=1}^{d} |(x_i - y_i)| \geq \frac{1}{2}[w(x) - w(y)]$.

For $x, y, z \in \mathbb{R}$. We show that d is an quasi-metric space on $\mathbb{R}$. We need to verify the asymmetry and triangle inequality, while the other properties are obviously satisfied.

- We will show that d is asymmetry. We have $d(x,y) \neq d(y,x)$ since $\frac{1}{2}[w(y) - w(x)] \neq \frac{1}{2}[w(x) - w(y)]; w(x) > w(y)$.
- We will show that d satisfies the triangle inequality. For all $w(x) > w(y) > w(z)$. We have

$$d(x,y) = \sum_{i=1}^{d} |(x_i - y_i)| + \frac{1}{2}[w(y) - w(x)], \tag{4.4}$$

$$d(y,z) = \sum_{i=1}^{d} |(y_i - z_i)| + \frac{1}{2}[w(z) - w(y)], \tag{4.5}$$

$$d(x,z) = \sum_{i=1}^{d} |(x_i - z_i)| + \frac{1}{2}[w(z) - w(x)]. \tag{4.6}$$

Next, We consider $d(x,y) + d(y,z)$, defined by

$$\begin{aligned} & d(x,y) + d(y,z) \\ &= \sum_{i=1}^{d} |(x_i - y_i)| + \tfrac{1}{2}[w(y) - w(x)] + \sum_{i=1}^{d} |(y_i - z_i)| + \tfrac{1}{2}[w(z) - w(y)] \\ &\geq \sum_{i=1}^{d} |(x_i - z_i)| + \tfrac{1}{2}[w(y) - w(x) + w(z) - w(y)] \\ &= \sum_{i=1}^{d} |(x_i - z_i)| + \tfrac{1}{2}[w(z) - w(x)]. \end{aligned}$$

Thus $d(x,z) \leq d(x,y) + d(y,z)$.

Therefore, (M,d) is a quasi-metric space.

4.1 1-cover in quasi-metric space.

From a quasi-metric space (M,d) and $M = S = [-1,1]^2 \subset \mathbb{R}^2$ where $w(x) > w(y)$, the weight function is less than or equal to the symmetrized distance function p and the maximum distance function which is equal to one.

Remark 4.4.

- Case1: If the weight function $\frac{1}{2}[w(x)-w(y)]=0$, then $w(x)=w(y)$.
  We have the covering number of 1-cover in a quasi-metric space which is equal to 1-cover in a metric space.
- Case2: If the weight function $\frac{1}{2}[w(x)-w(y)]\neq 0$ and $\frac{1}{2}[w(x)-w(y)]=\frac{1}{n}$, then $w(x)=w(y)+\frac{2}{n}$ and $\rho(x,y)=1-\frac{1}{n}$.
  We have the covering number of 1-cover in a quasi-metric space which is equal to $\left(1-\frac{1}{n}\right)$-cover in a metric space.

Example 4.5. Let (M,d) be a quasi-metric space. The function d given by

$$d(y,x)=\max|x_i-y_i|+\frac{1}{2}[w(x)-w(y)]\ ;\ w(x)>w(y).$$

We consider 1-cover where $M=\mathbb{R}^2$, $S=[-1,1]^2$ and determining $\frac{1}{2}[w(x)-w(y)]<1$.
Case1: If thee weight function $\frac{1}{2}[w(x)-w(y)]=\frac{1}{2}$ then $w(x)=w(y)+1$.
We have

$$\frac{1}{2}[w(y)-w(x)]=\frac{1}{2}[w(y)-(w(y)+1)]=\frac{1}{2}(-1)=\frac{-1}{2}.$$

Thus $\max|x_i-y_i|=\frac{1}{2}$. We find all center points by considering each ball at radius equal to $\frac{1}{2}$ with the points $(-1,1),(1,1),(1,-1)$ and $(-1,-1)$.

1. At the point $x=(-1,1)$, the ball is the set
   $\overline{B}((-1,1),\frac{1}{2})=\{y\in S|d((y_1,y_2),(-1,1))=\max(|y_1-(-1)|,|y_2-1|)=\frac{1}{2}\}$.
2. At the point $x=(1,1)$, the ball is the set
   $\overline{B}((1,1),\frac{1}{2})=\{y\in S|d((y_1,y_2),(1,1))=\max(|y_1-1|,|y_2-1|)=\frac{1}{2}\}$.
3. At the point $x=(-1,-1)$, the ball is the set
   $\overline{B}((-1,-1),\frac{1}{2})=\{y\in S|d((y_1,y_2),(-1,-1))=\max(|y_1-(-1)|,|y_2-(-1)|)=\frac{1}{2}\}$.
4. At the point $x=(1,-1)$, the ball is set
   $\overline{B}((1,-1),\frac{1}{2})=\{y\in S|d((y_1,y_2),(1,-1))=\max(|y_1-1|,|y_2-(-1)|)=\frac{1}{2}\}$.

The set of the closet point in other balls is $T=\left\{(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right\}$ and $|T|=4=2^2$.

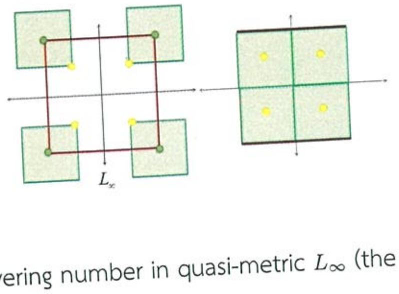

Figure 4.1: The covering number in quasi-metric L_∞ (the weight function $=\frac{1}{2}$)

Therefore, the covering number in a quasi-metric space is $|T| = 4 = 2^2$.

Case2: If weight function $\frac{1}{2}[w(x) - w(y)] = \frac{1}{3}$ then $w(x) = w(y) + \frac{2}{3}$.
We have

$$\tfrac{1}{2}[w(y) - w(x)] = \tfrac{1}{2}[w(y) - (w(y) + \tfrac{2}{3})] = \tfrac{1}{2}(-\tfrac{2}{3}) = \tfrac{-1}{3}.$$

Thus $\max|x_i - y_i| = \frac{2}{3}$. We find all center points by considering each ball at radius equal to $\frac{2}{3}$ with points $(-1,1), (1,1), (1,-1)$ and $(-1,-1)$.

1. At the point $x = (-1,1)$, the ball is the set
$\overline{B}((-1,1), \frac{2}{3}) = \{y \in S | d((y_1,y_2),(-1,1)) = \max(|y_1 - (-1)|, |y_2 - 1|) = \frac{2}{3}\}$.
2. At the point $x = (1,1)$, the ball is the set
$\overline{B}((1,1), \frac{2}{3}) = \{y \in S | d((y_1,y_2),(1,1)) = \max(|y_1 - 1|, |y_2 - 1|) = \frac{2}{3}\}$.
3. At the point $x = (-1,-1)$, the ball is the set
$\overline{B}((-1,-1), \frac{2}{3}) = \{y \in S | d((y_1,y_2),(-1,-1)) = \max(|y_1 - (-1)|, |y_2 - (-1)|) = \frac{2}{3}\}$.
4. At the point $x = (1,-1)$, the ball is set
$\overline{B}((1,-1), \frac{2}{3}) = \{y \in S | d((y_1,y_2),(1,-1)) = \max(|y_1 - 1|, |y_2 - (-1)|) = \frac{2}{3}\}$.

The set of the closet point in other balls is $T = \left\{(\frac{-1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}), (\frac{1}{3}, -\frac{1}{3})\right\}$.

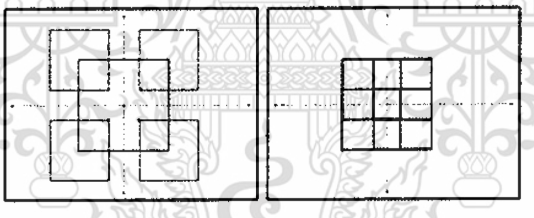

Figure 4.2: 1-cover in quasi does not exist the covering number (the weight function $=\frac{1}{3}$).

Example 4.6. Let (M,d) be a quasi-metric space. The function d is defined by

$$d(x,y) = \sum_{i=1}^{d} |(x_i - y_i)| + \tfrac{1}{2}[w(x) - w(y)]; w(x) > w(y).$$

We consider 1-cover where $M = \mathbb{R}^2$, $S = [-1,1]^2$ and determining $\frac{1}{2}[w(x) - w(y)] < 1$.
Case1: If weight function $\frac{1}{2}[w(x) - w(y)] = \frac{1}{2}$ then $w(x) = w(y) + 1$.
We have

$$\tfrac{1}{2}[w(y) - w(x)] = \tfrac{1}{2}[w(y) - (w(y) + 1)] = \tfrac{1}{2}(-1) = \tfrac{-1}{2}.$$

Thus $\sum_{i=1}^{d} |(x_i - y_i)| = \frac{1}{2}$. We find all center points by considering each ball at points $(-1,1), (1,1), (1,-1)$ and $(-1,-1)$.

1. At the point $x = (-1,1)$, the ball is the set
$\overline{B}((-1,1), \frac{1}{2}) = \{y \in S \ | d((y_1,y_2),(-1,1)) = (|y_1 - (-1)| + |y_2 - 1|) = \frac{1}{2}\}$.

2. At the point $x = (1, 1)$, the ball is the set
$\overline{B}((1,1), \frac{1}{2}) = \{y \in S \; |d((y_1, y_2), (1,1)) = |y_1 - 1| + |y_2 - 1|) = \frac{1}{2}\}$.

3. At the point $x = (-1, -1)$, the ball is the set
$\overline{B}((-1,-1), \frac{1}{2}) = \{y \in S \; |d((y_1, y_2), (-1,-1)) = (|y_1 - (-1)| + |y_2 - (-1)|) = \frac{1}{2}\}$.

4. At the point $x = (1, -1)$, the ball is the set
$\overline{B}((1,-1), \frac{1}{2}) = \{y \in S \; |d((y_1, y_2), (1,-1)) = (|y_1 - 1| + |y_2 - (-1)|) = \frac{1}{2}\}$.

The set of the closet point in the other ball is
$T_1 = \left\{(-\frac{1}{2}, 1), (-1, \frac{1}{2}), (-1, -\frac{1}{2}), (-\frac{1}{2}, -1), (\frac{1}{2}, 1), (1, \frac{1}{2}), (1, -\frac{1}{2}), (\frac{1}{2}, -1)\right\}$.

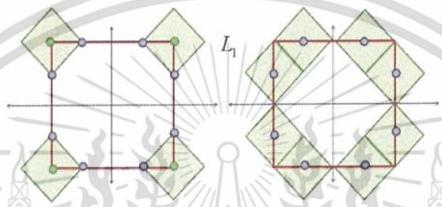

Figure 4.3: First round for finding the covering number of quasi-metric L_1 (1-cover)

T_1 is not the covering set S. We consider another cover point from the remaining region of set S. Now new corner points are in set A such that
$A = \{(0,1), (1,0), (-1,0), (0,-1)\}$, and $B(x, \frac{1}{2}) = \{y \in S, x \in A | d((y_1, y_2), x) = \frac{1}{2}\}$.
The set of the closet point in other balls is $T_2 = \left\{(0, \frac{1}{2}), (-\frac{1}{2}, 0), (0, -\frac{1}{2})(\frac{1}{2}, 0)\right\}$.
Therefore, the covering number in a quasi-metric space is $|T_1 \cup T_2| = \sum_{k=1}^{2} 4k = 4 + 8 = 12$.

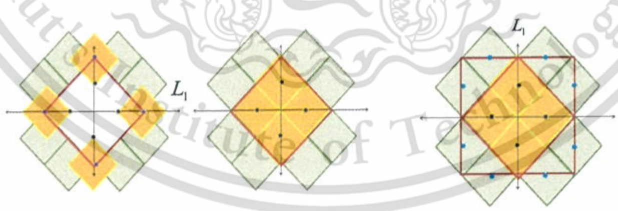

Figure 4.4: Second round for finding the covering number of quasi-metric at 1-coverL_1 (1-cover)

Case2: If weight function $\frac{1}{2}[w(x) - w(y)] = \frac{1}{3}$ then $w(x) = w(y) + \frac{2}{3}$.
We have

$$\frac{1}{2}[w(y) - w(x)] = \frac{1}{2}\left[w(y) - (w(y) + \frac{2}{3})\right] = \frac{1}{2}(-\frac{2}{3}) = \frac{-1}{3}.$$

Thus $\max|x_i - y_i| = \frac{2}{3}$. We find all center points by considering each ball at radius equal to $\frac{2}{3}$ with point $(-1, 1), (1, 1), (1, -1)$ and $(-1, -1)$.

1. At the point $x = (-1, 1)$, the ball is the set
$\overline{B}((-1,1), \frac{2}{3}) = \{y \in S | d((y_1, y_2), (-1, 1)) = \frac{2}{3}\}$.

2. At the point $x = (1, 1)$, the ball is the set
$\overline{B}((1,1), \frac{2}{3}) = \{y \in S | d((y_1, y_2), (1, 1)) = \frac{2}{3}\}$.

3. At the point $x = (-1, -1)$, the ball is the set
$\overline{B}((-1,-1), \frac{2}{3}) = \{y \in S | d((y_1, y_2), (-1, -1)) = \frac{2}{3}\}$.

4. At the point $x = (1, -1)$, the ball is the set
$\overline{B}((1,-1), \frac{2}{3}) = \{y \in S | d((y_1, y_2), (1, -1)) = \frac{2}{3}\}$.

The set of the closet point in other balls is $T = \Big\{(\frac{-1}{3}, 1), (-1, \frac{1}{3}), (\frac{1}{3}, 1), (1, \frac{1}{3}), (-\frac{1}{3}, -1), (-1, -\frac{1}{3}), (\frac{1}{3}, -1), (1, -\frac{1}{3})\Big\}$.

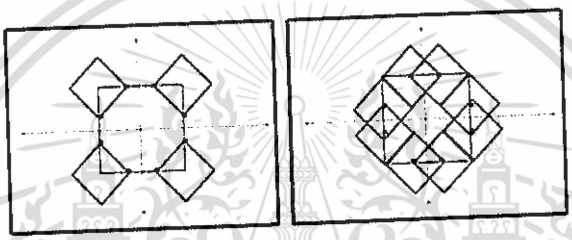

Figure 4.5: 1-cover in quasi does not exist covering number (the weight function $=\frac{1}{3}$)

4.2 ϵ-cover in quasi-metric space.

From a quasi-metric space (M, d), we consider ϵ-cover in a quasi-metric space on $M = S = [-a, a]^2 \subset \mathbb{R}^2$ where $w(x) \geqslant w(y)$. The maximum distance function is equal to ϵ.

Remark 4.7.
Case1: If the weight function $\frac{1}{2}[w(x) - w(y)] = 0$, then $w(x) = w(y)$.
We have the covering number of ϵ-cover in a quasi-metric space which is equal to ϵ-cover in a metric space.
Proof. Let (M, d) be any quasi-metric space which is defined by metric space. If we have the covering number then $\frac{1}{2}[w(x) - w(y)] = 0$ then $\rho(x, y) = a$. By remark, We have the ϵ cover is equal to a.

Case2: If the weight function $\frac{1}{2}[w(x) - w(y)] \neq 0$ and $\frac{1}{2}[w(x) - w(y)] = \frac{\epsilon}{k}$; $k > 1$, then $w(x) = w(y) + \frac{2\epsilon}{k}$ and $\rho(x, y) = \epsilon - \frac{\epsilon}{k}$.
We have the covering number of ϵ-cover in a quasi-metric space which is equal to $\left(\epsilon - \frac{\epsilon}{k}\right)$-cover in a metric space.

Proof. Let (M,d) be any quasi-metric space which is defined by metric space. If we have the covering number then $\rho(x,y) \geq \frac{1}{2}\left[w(x)-w(y)\right]$.

- If $\rho(x,y) = \frac{1}{2}\left[w(x)-w(y)\right]$ then we get the maximum of the weight function $\frac{1}{2}\left[w(x)-w(y)\right]$ to cover. Since, the set $S=[-a,a]^2$ have the maximum ϵ cover is equal to a. We get $a+a>a$ it is impossible, so the maximum the weight function $\frac{1}{2}\left[w(x)-w(y)\right]$ to cover is equal to $\frac{a}{2}$. Therefore, $\rho(x,y)=\frac{a}{2}$.
- If $\rho(x,y) > \frac{a}{2}$ then $2\rho < a$. By step to cover, we have the cover point. If $B(center\ point, \rho(x,y)) \cap B(center\ point, \rho(x,y)) = \emptyset$ then the diameter of ball is equal to $2\rho(x,y)$. Thus $2\rho(x,y) > \epsilon = a$ but $2\rho(x,y) < a$. Since we have the cover point where $\rho < \frac{a}{2}$ it is contradiction, so the covering number does not exist at the weight function less than $\frac{a}{2}$.

ε-cover in a metric space θ on S when $\sup_{x\in S}\theta(x,y)$ is	ε-cover in a quasi-metric space d on S when		
	$\sup_{x\in S} d(y,x) =$	$\rho(x,y)+\frac{1}{2}[w(x)-w(y)]$	
		$\rho(x,y)$	$\frac{1}{2}[w(x)-w(y)]$
1	1	1	0
		$\frac{1}{2}$	$\frac{1}{2}$
2	2	2	0
		1	1
⋮	⋮	⋮	⋮
ε	ε	ε	0
		$\frac{\varepsilon}{2}$	$\frac{\varepsilon}{2}$

Figure 4.6: The covering number in metric and quasi-metric space

Chapter 5

Conclusions and Suggestions

5.1 Conclusions

In a metric space, we can find the covering number by considering the ball at ϵ-cover. Furthermore, a quasi-metric space cannot cover. However, we can find the covering number on a quasi-metric space by considering the condition for determining the distance only one direction. For example, $d(x,y)$ always more than $d(y,x)$. We define a quasi-metric space by using metric space.

The covering number on a quasi-metric space is related to a metric space. We have the covering number of ϵ-cover in a quasi-metric space which is equal to $\left(\epsilon - \frac{\epsilon}{k}\right)$-cover in a metric space.

5.2 Suggestions

The covering number can be applied to the location problem such as signal, broadcast, satellite, WIFI, etc. The current step for computing the covering number cannot applied for solving in another metric space due to a limitation in the geometry of a metric. Hence, some metric such as L_3, L_4, etc. Some metric might be used by another step. Future study will focus on this issue.

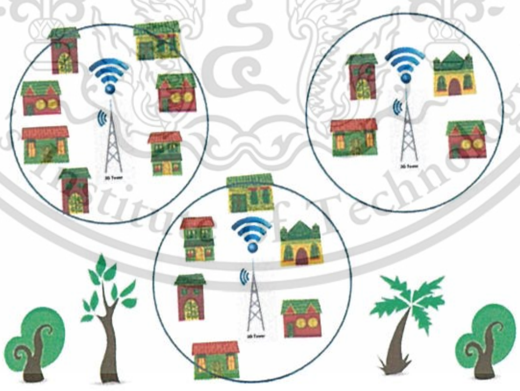

Figure 5.1: Application of the covering number

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Appendix

Appendix A

The 22nd Annual Meeting in Mathematics (AMM 2017)
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The covering number for metric and quasi-metric space

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Abstract

In this paper, we present how to find the center points and the covering number by given ϵ-cover for covering space S, $S \subset \mathbb{R}^2$ with metric L_1, L_2 and L_∞. We found the formula for the covering number in metric space. The paper also defines the quasi-metric space by given metric space. We have got the formula for the covering number in quasi-metric space.

Keywords: covering number, ϵ-cover, metric spaces, quasi-metric spaces.

2010 MSC: Primary 57M10; Secondary 54E35.

1 Introduction

The many applied mathematics issues need manipulating and dominant collections of random variables indexed by sets with an associated infinite a variety of parts. Whereas any finite set will be measured in terms of it cardinality, measure the size of a group of infinitely several components, which is called covering numbers. Covering numbers is a number of center points of a set which data point takes group nearest center point and set of each group is called cluster. The k-center problem is a problem of clustering, the goal of k-center problem is choosing a set of k points to serve as centers and to assign all the points to the centers, so that the maximum distance of any point in its center are as small as possible. The ϵ-cover problem is to find the number of centers that make cluster to cover all the points when the maximum distance are ϵ.

Clustering or cluster analysis is grouping a set of objects (clusters). The object of a group is similar to another and different from the objects in other groups. Formally, the clustering structure is represented as a set of subsets C_i of S, S is set of objects and C_i is cluster such that $S = \cup_{i=1}^k C_i$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Consequently, any instance in S belongs to exactly one and only one subset [2].

Many clustering methods use distance measures to determine the distance between two instances x_i, x_j denote as $d(x_i, x_j)$. We need to describe the kind of space in which the data are contained which many distance functions can be defined in metric space.

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2 Preliminaries

2.1 Metric space

Definition 2.1. [3] A metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ for any $x, y, z \in X$ if it has the properties:

- Positive definiteness: $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$.
- Symmetry: $d(x,y) = d(y,x)$.
- Triangle inequality: $d(x,y) \leq d(x,z) + d(z,y)$.

The mapping d is called metric on X or distance function on X and (X,d) is called **metric space.**

Example 2.2. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by $d(x,y) = |x-y|$ is a metric on $\mathbb{R}$

To show that d is a metric space on $\mathbb{R}$ we need verify only triangle inequality, while the other properties are obviously satisfied. We have $d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z)$ for any $x, y, z \in \mathbb{R}$. Therefore, (X,d) is a metric space.

Three metrics will be used in this research such as

- L_1 (Taxicab): $d(x,y) = \|x-y\|_1 = \sum_{i=1}^{n} |x_i - y_i|$,
- L_2 (Euclidean) : $d(x,y) = \|x-y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$,
- L_∞ (Supremum): $d(x,y) = \|x-y\|_\infty = \max |x_i - y_i|$,

where n is dimension of space and all above distance functions is metric on $\mathbb{R}^n$ that is $(\mathbb{R}^n, L_1)$, $(\mathbb{R}^n, L_2)$, $(\mathbb{R}^n, L_\infty)$ are metric space.

2.2 Open and Closed Set

Definition 2.3. [4] Let (X,d) be a metric space, $x_0 \in X$ and $r > 0$. The Open ball with center x_0 and radius r is the set $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$. The closed ball with center x_0 and radius r is the set $\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$.

Definition 2.4. [4] Let (X,d), the unit ball around some point $x \in X$ is the set of point of distance at most 1 from x, $\{y \in X : d(x,y) \leq 1\}$.

Unit ball is a set of points is the distance less than or equal to one with fixed center point. The figure 1 shown center point in space $\mathbb{R}^2$ and unit ball with metric L_1, L_∞, L_2 on $\mathbb{R}^2$.

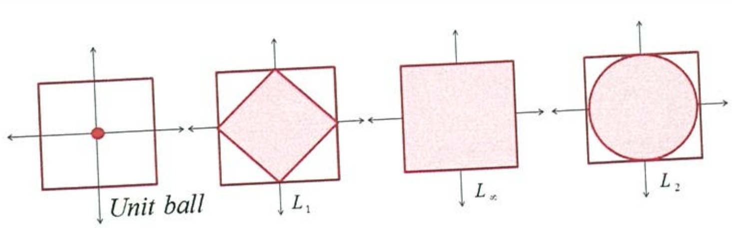

Figure 1: The unit balls $\overline{B}((0,0),1)$ on $\mathbb{R}^2$

Notice that the characteristic shape of the L_∞ metric is a box, while that of the L_1 metric is a diamond. Similarly, the characteristic shape of the L_2 metric is the sphere.

2.3 Quasi-metric space

Definition 2.5. [3] A quasi-metric on a set X will be a function $d : X \times X \to \mathbb{R}^+$ for all $x, y, z \in X$ if it has the properties:

- $d(x,y) = d(y,x) = 0$ if and only if $x = y$.
- $d(x,y) \leq d(x,z) + d(z,y)$.

The mapping d is called quasi-metric on X and a pair (X,d) is called **quasi-metric space.**

Symmetrization
[3] Let (X,d) be a quasi-metric space. The function $\theta : X \times X \to \mathbb{R}^+$ while $\theta(x,y) = \frac{1}{2}[d(x,y) + d(y,x)]$ is called symmetrization of d.

Weight function
[3] Let (X,d) be a quasi-metric space. The quasi-metric d is called weightable quasi-metric if there exists a weight function $w : M \to [0,\infty)$, satisfies

$$d(x,y) + w(x) = d(y,x) + w(y) \text{ for all } x,y \in X.$$

Theorem 2.6. [3] *Let (M,d) be any quasi-metric space. Then d is weightable if and only if there exists $w : M \to [0,\infty)$ such that*

$$d(x,y) = \rho(x,y) + \tfrac{1}{2}[w(y) - w(x)] \text{ for all } x,y \in M,$$

where ρ is the symmetrized distance of d. Moreover, we have

$$\tfrac{1}{2}\left[w(x) - w(y)\right] \leq \rho(x,y) \text{for all } x,y \in M.$$

From theorem2.5 we define quasi-metric by metric as the following example.

Example 2.7. Let M be any set and d is a quasi-metric which

$$d(x,y) = \sum_{i=1}^{d} |(x_i - y_i)| + \frac{1}{2}\left[w(x) - w(y)\right]; w(x) > w(y).$$

By theorem2.5: metric is $\rho(x,d) = L_1 = \sum_{i=1}^{n} |(x_i - y_i)|$ and weighted function is $\frac{1}{2}\left[w(x) - w(y)\right]$.
We will show that (M,d) is a quasi-metric space.

Let be $x, y, z \in \mathbb{R}$. If $d(x,y)$ is quasi-metric then it is satisfy two properties.

- First property: It clearly $d(x,y) \geq 0$ and $d(x,y) \neq d(y,x)$ since $\frac{1}{2}[w(x) - w(y)] \neq \frac{1}{2}[w(y) - w(x)]$ where $w(x) > w(y)$.
- Second property: We have to show the triangle inequality property,

$$d(x,y) = \sum_{i=1}^{d} |(x_i - y_i)| + \frac{1}{2}[w(x) - w(y)]; w(x) > w(y)$$
$$d(y,z) = \sum_{i=1}^{d} |(y_i - z_i)| + \frac{1}{2}[w(y) - w(z)]; w(y) > w(z)$$
$$d(x,z) = \sum_{i=1}^{d} |(x_i - z_i)| + \frac{1}{2}[w(x) - w(z)]; w(x) > w(z)$$

and

$$d(x,y)+d(y,z) = \sum_{i=1}^{d}|(x_i-y_i)| + \frac{1}{2}[w(x)-w(y)] + \sum_{i=1}^{d}|(y_i-z_i)| + \frac{1}{2}[w(y)-w(z)]$$
$$\geq \sum_{i=1}^{d}|(x_i-z_i)| + \frac{1}{2}[w(x)-w(y)+w(y)-w(z)]$$
$$= \sum_{i=1}^{d}|(x_i-z_i)| + \frac{1}{2}[w(x)-w(z)]$$

so $d(x,z) \leq d(x,y)+d(y,z)$.
Since d satisfies two properties.
Therefore, (M,d) is quasi-metric space.

2.4 Covering number in metric space

Definition 2.8. [1] Let any metric space (X,ρ). For any $\epsilon > 0$, an **ϵ-cover** of a set $S \subset X$ is defined to be any set $T \subset X$ such that $\sup_{x\in S}\rho(x,T) \leq \epsilon$ and cardinality of set T is called **the covering number**.

Here $\rho(x,T)$ is the distance from point x to the closest point in set T, that is $\rho(x,T) = \inf_{z\in T}\rho(x,z)$.

Example 2.9. In metric space and set $S = [-1,1]^2$. There are the covering number at ϵ-cover when $\epsilon = 1$ by L_1, L_2, L_∞ metric.
Case1[1] In metric L_∞ and $\epsilon = 1$. The covering number is only just one point as the figure 2.

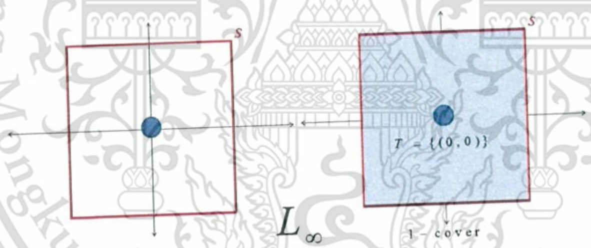

Figure 2: 1-cover by L_∞

Case2[1] In metric L_1 and $\epsilon = 1$. The covering number has four points as the figure 3.

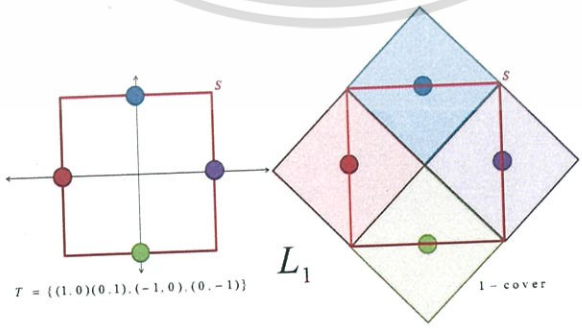

Figure 3: 1-cover by L_1

Case3 In metric L_2 and $\epsilon = 1$. The covering number dose not exists any point but the space can be covering only one point where $\epsilon = \sqrt{2}$ as the figure 4.

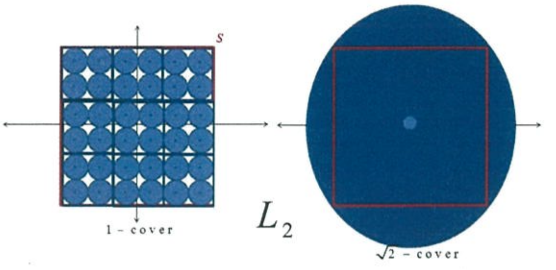

Figure 4: $\sqrt{2}$-cover by L_2

3 Main Results

3.1 Find the covering number in metric space

We will find the covering number of set $S = [-1, 1]^2$ subset of $\mathbb{R}^2$ as following the step

(i) Create the closed ball $B(x, \epsilon)$ of corner point of set S that is point $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$ as figure 5(left).

(ii) Cover each point which the point is the nearest the other balls as figure 5(right).

(iii) Iterative the step 1 and 2 if there are some regions uncover.

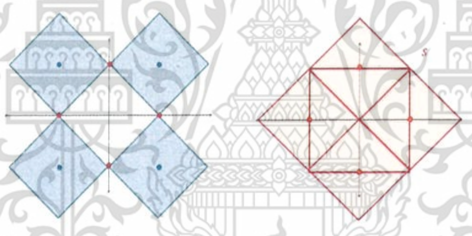

Figure 5: Show closed ball of corner point and cover in each center points

Remark 3.1. Space $S = [-1, 1]^2$, the covering number for 1-cover with metric L_∞ has only one point which it is an origin.

Remark 3.2. Space $S = [-1, 1]^2$, the covering number for 1-cover with metric L_1 there are four points (can not cover by one point).

Remark 3.3. Space $S = [-1, 1]^2$, the covering number for 1-cover with metric L_2 doses not to exist to on any point because one point must contain in only one cover but space can cover by one point where $\epsilon = \sqrt{2}$.

Example 3.4. Give the space $S = [-1, 1]^2$, we consider **1-cover** in metric space.

1. For 1-cover with metric L_∞.
   We find all center points, by consider each ball at point $(-1, 1), (1, 1), (1, -1), (-1, -1)$.

   (a) At point $x = (-1, 1)$, ball is set $B((-1, 1), 1)$
   $= \{y \in S | d((y_1, y_2), (-1, 1)) = \max(|y_1 - (-1)|, |y_2 - 1|) = 1\}$.

   (b) At point $x = (1, 1)$, ball is set $B((1, 1), 1)$
   $= \{y \in S | d((y_1, y_2), (1, 1)) = \max(|y_1 - 1|, |y_2 - 1|) = 1\}$.

   (c) At point $x = (-1, -1)$, ball is set $B((-1, -1), 1)$
   $= \{y \in S | d((y_1, y_2), (-1, -1)) = \max(|y_1 - (-1)|, |y_2 - (-1)|) = 1\}$.

(d) At point $x = (1, -1)$, ball is set $B((1, -1), 1)$
$= \{y \in S | d((y_1, y_2), (1, -1)) = \max(|y_1 - 1|, |y_2 - (-1)|) = 1\}$.

The set of closet point in the other ball is $T = \{(0, 0)\}$.

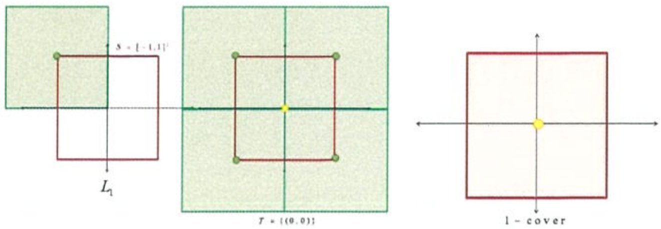

Figure 6: covering number of metric L_∞

2. For 1-cover with metric L_1
We find all center points, by consider each ball at point $(-1, 1), (1, 1), (1, -1), (-1, -1)$.

(a) At point $x = (-1, 1)$, ball is set $B((-1, 1), 1)$
$= \{y \in S | d((y_1, y_2), (-1, 1)) = |y_1 - (-1)| + |y_2 - 1| = 1\}$.

(b) At point $x = (1, 1)$, ball is set $B((1, 1), 1)$
$= \{y \in S | d((y_1, y_2), (1, 1)) = |y_1 - 1| + |y_2 - 1| = 1\}$.

(c) At point $x = (-1, -1)$, ball is set $B((-1, -1), 1)$
$= \{y \in S | d((y_1, y_2), (-1, -1)) = |y_1 - (-1)| + |y_2 - (-1)| = 1\}$.

(d) At point $x = (1, -1)$, ball is set $B((1, -1), 1)$
$= \{y \in S | d((y_1, y_2), (1, -1)) = |y_1 - 1| + |y_2 - (-1)| = 1\}$.

The set of closet point in the other ball is $T = \{(0, 1), (0 - 1), (1, 0), (-1, 0)\}$.

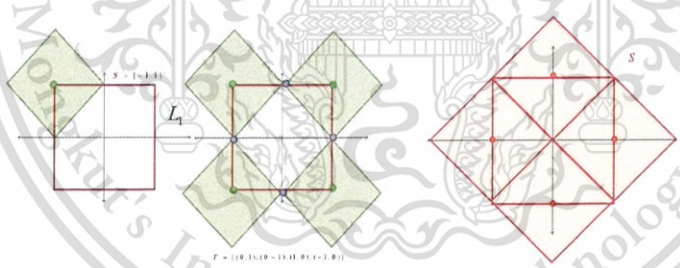

Figure 7: covering number of metric L_1

3. For 1-cover with metric L_2
We find all center points, by consider each ball at point $(-1, 1), (1, 1), (1, -1), (-1, -1)$.

(a) At point $x = (-1, 1)$, ball is set $B((-1, 1), 1)$
$= \{y \in S | d((y_1, y_2), (-1, 1)) = \sqrt{|y_1 - (-1)| + |y_2 - 1|} = 1\}$.

(b) At point $x = (1, 1)$, ball is set $B((1, 1), 1)$
$= \{y \in S | d((y_1, y_2), (1, 1)) = \sqrt{|y_1 - 1| + |y_2 - 1|} = 1\}$.

(c) At point $x = (-1, -1)$, ball is set $B((-1, -1), 1)$
$= \{y \in S | d((y_1, y_2), (-1, -1)) = \sqrt{|y_1 - (-1)| + |y_2 - (-1)|} = 1\}$.

(d) At point $x = (1, -1)$, ball is set $B((1, -1), 1)$
$= \{y \in S | d((y_1, y_2), (1, -1)) = \sqrt{|y_1 - 1| + |y_2 - (-1)|} = 1\}$.

The set of closet point in the other ball is $T = \{(0, 1), (0 - 1), (1, 0), (-1, 0)\}$.

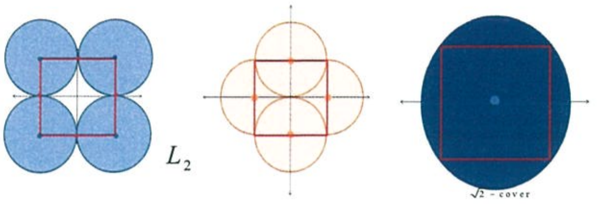

Figure 8: 1-cover dose not exist but set S can be cover by $\epsilon = \sqrt{2}$

Remark 3.5. The space $S = [-1,1]^2$ with metric L_∞ and $\epsilon = \dfrac{1}{n}$, the covering number is equal to n^2 or $|T| = n^2$.

Remark 3.6. The space $S = [-1,1]^2$ with metric L_1 and $\epsilon = \dfrac{1}{n}$, the covering number is equal to $\sum_{k=1}^{n} 4k$ or $|T| = \sum_{k=1}^{n} 4k$.

Remark 3.7. The space $S = [-1,1]^2$ with metric L_2 and $\epsilon = \dfrac{1}{n}$, the covering number is equal to $|T| = |S|$.

Example 3.8. Given the space $S = [-1,1]^2$, we consider **ϵ-cover** in metric space where $\epsilon < 1$ and distance between any center point are 2ϵ, so maximum of ϵ is $\frac{1}{2}$. It means $\epsilon = \dfrac{1}{n}$ for all $n > 1$.

- For $\dfrac{1}{n}$-cover with metric L_∞
  1. $\frac{1}{2}-$cover
     We find all center points by consider each ball at point $(-1,1),(1,1),(1,-1),(-1,-1)$.
     (a) At point $x = (-1,1)$, ball is set $B((-1,1),\frac{1}{2})$
     $= \{y \in S | d((y_1,y_2),(-1,1)) = \max(|y_1-(-1)|,|y_2-1|) = \frac{1}{2}\}$.
     (b) At point $x = (1,1)$, ball is set $B((1,1),\frac{1}{2})$
     $= \{y \in S | d((y_1,y_2),(1,1)) = \max(|y_1-1|,|y_2-1|) = \frac{1}{2}\}$.
     (c) At point $x = (-1,-1)$, ball is set $B((-1,-1),\frac{1}{2})$
     $= \{y \in S | d((y_1,y_2),(-1,-1)) = \max(|y_1-(-1)|,|y_2-(-1)|) = \frac{1}{2}\}$.
     (d) At point $x = (1,-1)$, ball is set $B((1,-1),\frac{1}{2})$
     $= \{y \in S | d((y_1,y_2),(1,-1)) = \max(|y_1-1|,|y_2-(-1)|) = \frac{1}{2}\}$.

     The set of closet point in the other ball is $T = \left\{(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right\}$, $|T| = 4 = 2^2$

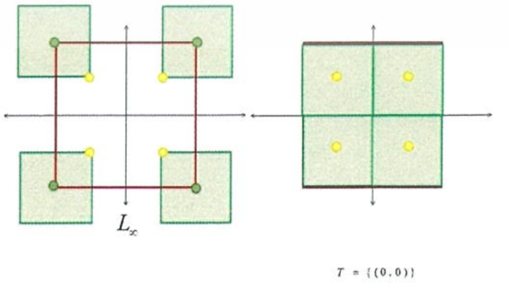

Figure 9: covering number of metric L_∞ ($\frac{1}{2}$ − cover)

2. $\frac{1}{3}-$cover
   We find all center points by consider each ball at point $(-1,1),(1,1),(1,-1),(-1,-1)$.
   (a) At point $x=(-1,1)$, ball is set $B((-1,1),\frac{1}{3})$
   (b) At point $x=(1,1)$, ball is set $B((1,1),\frac{1}{3})$
   (c) At point $x=(-1,-1)$, ball is set $B((-1,-1),\frac{1}{3})$
   (d) At point $x=(1,-1)$, ball is set $B((1,-1),\frac{1}{3})$
   The set of closet point in the other ball is $T1=\left\{(-\frac{2}{3},\frac{2}{3}),(-\frac{2}{3},-\frac{2}{3}),(\frac{2}{3},\frac{2}{3}),(\frac{2}{3},-\frac{2}{3})\right\}$

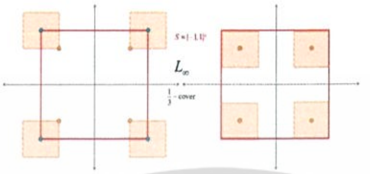

Figure 10: covering number of metric L_∞ ($\frac{1}{3}$ – cover)

   $T1$ is not covering set S. We have to consider another cover point from remain region of set S. Now new corner points are in set A such that
   $A=\{(-\frac{1}{3},1),(\frac{1}{3},\frac{1}{3}),(-1,\frac{1}{3}),(-1,-\frac{1}{3}),(-\frac{1}{3},-\frac{1}{3}),(-\frac{1}{3},-1),(\frac{1}{3},-1),(\frac{1}{3},-\frac{1}{3}),(1,-\frac{1}{3}),(1,\frac{1}{3}),(\frac{1}{3},\frac{1}{3}),(\frac{1}{3},1)\}$
   $B(x,\frac{1}{3})=\{y\in S,x\in A|d((y_1,y_2),x)=\frac{1}{3}\}$
   The set of closet point in the other ball is $T2=\left\{(0,\frac{2}{3}),(0,0),(-\frac{2}{3},0),(0,-\frac{2}{3}),(\frac{2}{3},0)\right\}$
   Therefore, the covering number is $|T1\cup T2|=|3^2|=9=4+5$

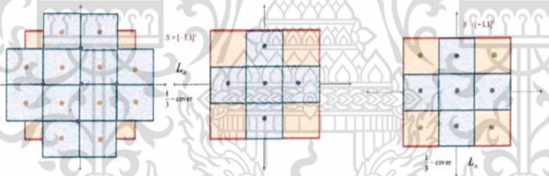

Figure 11: covering number of metric L_∞ ($\frac{1}{3}$ – cover)

- . For $\dfrac{1}{n}$-cover with metric L_1

1. $\frac{1}{2}-$cover
   We find all center points by consider each ball at point $(-1,1),(1,1),(1,-1),(-1,-1)$.
   (a) At point $x=(-1,1)$, ball is set $B((-1,1),\frac{1}{2})$
   $=\{y\in S|d((y_1,y_2),(-1,1))=(|y_1-(-1)|+|y_2-1|)=\frac{1}{2}\}$.
   (b) At point $x=(1,1)$, ball is set $B((1,1),\frac{1}{2})$
   $=\{y\in S|d((y_1,y_2),(1,1))=|y_1-1|+|y_2-1|)=\frac{1}{2}\}$.
   (c) At point $x=(-1,-1)$, ball is set $B((-1,-1),\frac{1}{2})$
   $=\{y\in S|d((y_1,y_2),(-1,-1))=(|y_1-(-1)|+|y_2-(-1)|)=\frac{1}{2}\}$.
   (d) At point $x=(1,-1)$, ball is set $B((1,-1),\frac{1}{2})$
   $=\{y\in S|d((y_1,y_2),(1,-1))=(|y_1-1|+|y_2-(-1)|)=\frac{1}{2}\}$.
   The set of closet point in the other ball is
   $T1=\left\{(-\frac{1}{2},1),(-1,\frac{1}{2}),(-1,-\frac{1}{2}),(-\frac{1}{2},-1),(\frac{1}{2},1),(1,\frac{1}{2}),(1,-\frac{1}{2}),(\frac{1}{2},-1)\right\}$
   $T1$ is not covering set S. We have to consider another cover point from remain region of set S. Now new corner points are in set A such that $A=\{(0,1),(1,0),(-1,0),(0,-1)\}$

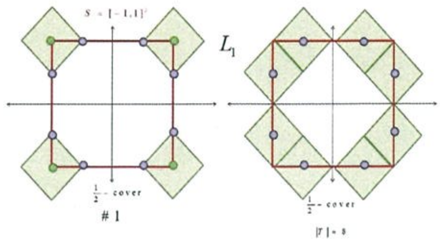

Figure 12: covering number of metric L_1 ($\frac{1}{2}$ − cover)

$B(x, \frac{1}{2}) = \{y \in S, x \in A | d((y_1, y_2), x) = \frac{1}{2}\}$
The set of closet point in the other ball is $T2 = \Big\{(0, \frac{1}{2}), (-\frac{1}{2}, 0), (0, -\frac{1}{2})(\frac{1}{2}, 0)\Big\}$
Therefore, the covering number is $|T1 \cup T2| = \sum_{k=1}^{2} 4k = 4 + 8 = 12$

Figure 13: covering number of metric L_1 ($\frac{1}{2}$ − cover)

2. . $\frac{1}{3}$−cover
   We find all center points by consider each ball at point $(-1, 1), (1, 1), (1, -1), (-1, -1)$.
   (a) At point $x = (-1, 1)$, ball is set $B((-1, 1), \frac{1}{3})$
   (b) At point $x = (1, 1)$, ball is set $B((1, 1), \frac{1}{3})$
   (c) At point $x = (-1, -1)$, ball is set $B((-1, -1), \frac{1}{3})$
   (d) At point $x = (1, -1)$, ball is set $B((1, -1), \frac{1}{3})$
   The set of closet point in the other ball is
   $T1 = \Big\{(-\frac{2}{3}, 1), (-1, \frac{2}{3}), (-1, -\frac{2}{3}), (-\frac{2}{3}, -1), (\frac{2}{3}, 1), (1, \frac{2}{3}), (1, -\frac{2}{3}), (\frac{2}{3}, -1)\Big\}$

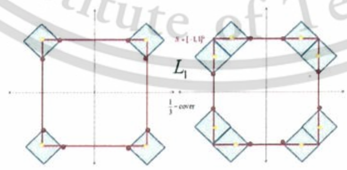

Figure 14: covering number of metric L_1 ($\frac{1}{3}$ − cover)

$T1$ is not covering set S. We have to consider another cover point from remain region of set S. Now new corner points are in set A such that
$A = \{(-\frac{1}{3}, 1), (-1, \frac{1}{3}), (-1, -\frac{1}{3}), (-\frac{1}{3}, -1), (\frac{1}{3}, 1), (1, \frac{1}{3}), (1, -\frac{1}{3}), (\frac{1}{3}, -1)\}$
$B(x, \frac{1}{3}) = \{y \in S, x \in A | d((y_1, y_2), x) = \frac{1}{3}\}$
The set of closet point in the other ball is
$T2 = \Big\{\{(0, 1), (-\frac{1}{3}, \frac{2}{3}), (-\frac{2}{3}, \frac{1}{3}), (-1, 0), (-\frac{1}{3}, -\frac{2}{3}), (-\frac{2}{3}, -\frac{1}{3}), (0, -1), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}), (1, 0), (\frac{1}{3}, -\frac{2}{3}), (\frac{2}{3}, -\frac{1}{3})\Big\}$
$T2$ is not covering set S. We have to consider another cover point from remain region of set S.

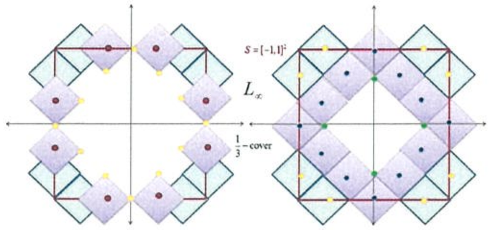

Figure 15: covering number of metric L_1 ($\frac{1}{3}$ − cover)

Now new corner points are in set A' such that $A' = \{(-\frac{2}{3}, 0), (0, -\frac{2}{3}), (\frac{2}{3}, 0), (0, \frac{2}{3})\}$
$B(x, \frac{1}{3}) = \{y \in S, x \in A' | d((y_1, y_2), x) = \frac{1}{3}\}$
The set of closet point in the other ball is $T3 = \{(-\frac{1}{3}, 0), (0, -\frac{1}{3}), (\frac{1}{3}, 0), (0, \frac{1}{3})\}$

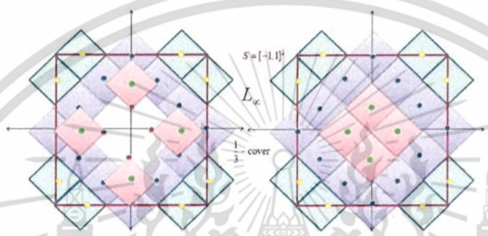

Figure 16: covering number of metric L_1 ($\frac{1}{3}$ − cover)

Therefore, the covering number is $|T1 \cup T2 \cup T3| = \sum_{k=1}^{3} 4k = 4 + 8 + 12 = 24$

Remark 3.9. **Covering number in space $S = [-a, a]^2$**
The covering number of set $S = [-1, 1]$ and $\epsilon \leq 1$, the $\epsilon = 1$ given the maximum distance and for $S = [-a, a]^2$, the $\epsilon = a$ given the maximum distance. The set can be cover where $\epsilon = a$ or $\epsilon = \frac{a}{k}$, where $1 < k < a$ and distance between any center point is less than or equal to 2ϵ.
The set $S = [-a, a]^2$ and a-cover. The covering number is the following

- Metric L_∞: The covering number is only one point at origin,
- Metric L_1: The covering number are four points,
- Metric L_2: It is not cover by $\epsilon = a$. This space will cover by $\epsilon = a\sqrt{2}$ and origin is only center point.

The set $S = [-a, a]^2$ and $\frac{a}{k}$-cover. The covering number is following.

- Metric L_∞: The covering number are $(ak)^2$ or $|T| = (ak)^2$.
- Metric L_1: The covering number are $\sum_{j=1}^{ak} 4j$ or $|T| = \sum_{j=1}^{ak} 4j$.
- Metric L_2: The covering number are $|T| = |S|$

3.2 Covering Number in quasi metric space

Let (M, d) be quasi-metric space defined by metric space,

$$d(x, y) = \rho(x, y) + \tfrac{1}{2}[w(y) - w(x)] \text{ for all } x, y \in M,$$

1-cover in quasi-metric space

From(M, d), we consider 1-cover in quasi-metric space on $M = S = [-1, 1]^2 \subset \mathbb{R}^2$ where $w(x) > w(y)$, weighted function is less than or equal to symmetrized distance function and the maximum distance function is equal to one.

- **Case1** Let weighted function $\frac{1}{2}[w(x) - w(y)] = 0$, so $w(x) = w(y)$.
  We have got the covering number of 1-cover in quasi-metric space and it is equal to 1-cover in metric space.

- **Case2** Let weighted function $\frac{1}{2}[w(x) - w(y)] = \frac{1}{n}$, so $w(x) = w(y) + \frac{2}{n}$ and symmetrized distance function is $d(x, y) = 1 - \frac{1}{n}$.
  We have got the covering number of 1-cover in quasi-metric space and it is equal to $\left(1 - \frac{1}{n}\right)$-cover in metric space.

ϵ-cover in quasi-metric space

From (M, d) we consider ϵ-cover in quasi-metric space on $M = S = [-a, a]^2 \subset \mathbb{R}^2$ where $w(x) > w(y)$, weighted function is less than or equal to symmetrized distance and the maximum distance function is equal to ϵ

- **Case1** Let weighted function $\frac{1}{2}[w(x) - w(y)] = 0$, so $w(x) = w(y)$
  We have got the covering number of ϵ-cover in quasi-metric space and it is equal toϵ-cover in metric space.

- **Case2** Let weighted function $\frac{1}{2}[w(x) - w(y)] = \frac{\epsilon}{k}$; $k > 1$, so $w(x) = w(y) + \frac{2\epsilon}{k}$ and symmetrized distance function is $d(x, y) = \epsilon - \frac{\epsilon}{k}$.
  We have got the covering number of ϵ-cover in quasi-metric space and it is equal to $\left(\epsilon - \frac{\epsilon}{k}\right)$-cover in metric space.

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