TRACY-SINGH CONVOLUTION PRODUCT FOR INTEGRABLE MATRIX-VALUED FUNCTIONS AND APPLICATION TO LINEAR MATRIX CONVOLUTION EQUATIONS

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หัวข้อวิทยานิพนธ์	ผลคูณเทรซี-ซิงห์คอนโวลูชันสำหรับฟังก์ชันค่าเมทริกซ์ที่หาปริพันธ์ได้และการประยุกต์กับสมการคอนโวลูชันเมทริกซ์เชิงเส้น
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บทคัดย่อ

ในงานวิจัยนี้ เราขยายแนวคิดของผลคูณโครเนกเคอร์คอนโวลูชันไปสู่ผลคูณเทรซีซิงค์คอนโวลูชันสำหรับฟังก์ชันค่าเมทริกซ์ที่หาปริพันธ์ได้ เราพิจารณาศึกษาคุณสมบัติเชิงพีชคณิตของผลคูณดังกล่าวที่เกี่ยวข้องกับการดำเนินการเชิงพีชคณิต เรายังพิจารณาตัวทำ(ปฏิ)สลับที่คอนโวลูชันและการยกกำลังเทรซีซิงค์คอนโวลูชัน ยิ่งกว่านั้น เราประยุกต์ผลคูณเทรซีซิงค์คอนโวลูชันในการหาผลเฉลยของสมการคอนโวลูชันเมทริกซ์เชิงเส้น

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Abstract

In this research, we extend the notion of Kronecker convolution product to the Tracy-Singh convolution product of integrable matrix-valued functions. We investigate its algebraic properties involving certain algebraic operations. We also discuss convolution (anti)commutators and Tracy-Singh convolution powers. Moreover, we apply the Tracy-Singh convolution product for solving linear matrix convolution equations.

Keywords : convolution, Tracy-Singh convolution product, linear matrix convolution equation, vector operator.

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Jirapud Limthanakul

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Chapter 1

Introduction

1.1 Inception and importance

In mathematical analysis, convolution is an operation assigned to each pair of (Riemann) integrable real-valued functions f and g given by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t \geqslant 0.$$

Convolution turns out to be important in many areas of mathematics, such as, differential equations and probability. It was remarkable that convolution has applications in various scientific fields that include control and system theory, image and signal processing, mathematical physics, etc.

On the other hand in linear and multilinear algebra, there are many kinds of matrix products which are of interest in both theory and applications. Recall that the Kronecker product of two real matrices $A = [a_{ij}]$ and B is defined by

$$A \otimes B = [a_{ij}B]_{ij},$$

that is, each (i, j)-th block of $A \otimes B$ is given by $a_{ij}B$.
The Tracy-Singh product, introduced by [14], is defined for two partitioned matrices $A = [A_{ij}]_{ij}$ and $B = [B_{kl}]_{kl}$ by

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij}.$$

Note that if A has one block, their Tracy-Singh product is reduced to the so-called block Kronecker product [8]. When both factors have only one block, $A \boxtimes B$ reduces to $A \otimes B$. See more information about Tracy-Singh product and related topics in [10, 11, 13, 15].

In the last decade, Kilicman and Al Zhour developed further kinds of matrix convolution products for matrix-valued functions. These include Kronecker convolution product [5], Hadamard convolution product [6] and box convolution product [15]. In [1], systems of linear matrix convolution equations can be reduced to a simple matrix convolution equation by using Kronecker convolution product, and then such systems can be solved by using Laplace transform. Itrative procedures for linear matrix convolution equations are proposed in [15, 1].

In this research, we extend the notion of Kronecker convolution product to the Tracy-Singh convolution product of integrable matrix-valued functions. We investigate its algebraic properties involving certain algebraic operations. We also discuss convolution (anti)commutators and Tracy-Singh convolution powers. Moreover, we apply the Tracy-Singh convolution product for solving linear matrix convolution equations.

1.2 Objectives

1) To investigate algebraic properties of Tracy-Singh convolution product for matrix-valued functions.

2) To apply algebraic properties of Tracy-Singh convolution product for solving linear matrix convolution equations.

1.3 Scopes of the study

First, we define the Tracy-Singh convolution product of integrable matrix-valued functions. Then we investigate its algebraic properties involving certain algebraic operations. We also define convolution (anti)commutators and Tracy-Singh convolution powers, and deduce their algebraic properties. Finally, we apply the Tracy-Singh convolution product for solving linear matrix convolution equations.

1.4 Benefits

To obtain a tool for solving linear matrix convolution equations.

1.5 Research methodology

1) Study convolutions for real-valued functions from textbooks.

2) Study convolution product of integrable matrix-valued functions from research papers.

3) Study Tracy-Singh product of matrices from research papers.

4) Determine objectives and scope of the research.

5) Define the Tracy-Singh product and investigate algebraic properties involving certain algebraic operations.

6) Define convolution (anti)commutators and Tracy-Singh convolution powers and deduce their algebraic properties.

7) Apply the Tracy-Singh convolution product for solving linear matrix convolution equation.

8) Conclude the results, make suggestions for further works and write the thesis.

Table 1.1*:* The research schedule

Activity	Time frame							
	2015		2016				2017	
	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	↔							
Step 2		↔	↔					
Step 3			↔	↔				
Step 4					↔			
Step 5						↔		
Step 6							↔	
Step 7							↔	
Step 8								↔

Chapter 2

Preliminaries

The purpose of this chapter is to provide basic concept and tools in matrix theory and used in the research.

The first and the second sections deal with preliminaries in matrix theory. We collect fundamental properties of the convolution product of real-valued functions, Kronecker product of complex matrices and the Tracy-Singh product, as a generalized Tracy-Singh convolution product for Riemann or continuous matrix-valued functions.

2.1 Convolution of real-valued functions

In what follows, let $\Omega = [0,\infty)$ or $\Omega = [0,b]$ for some $b > 0$. Denote by $M_{m,n}(\mathbb{R})$ the set of m-by-n real matrices. When $m = n$, we abbreviate $M_{m,n}(\mathbb{R})$ to $M_n(\mathbb{R})$. Recall that a matrix-valued function $A : \Omega \to M_{m,n}(\mathbb{R}), A(t) = [a_{ij}(t)]$ is said to be integrable (continuous) if the real-valued function a_{ij} is integrable (continuous, respectively) for each $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Denote by $\mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$ and $\mathcal{C}(\Omega, M_{m,n}(\mathbb{R}))$ the set of integrable (continuous, resp.) matrix-valued functions form Ω to $M_{m,n}(\mathbb{R})$.

For any integrable functions $f,g,h : \Omega \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, the following properties are straightforward to verify:

$$f*(g+h) = f*g+f*h, \tag{2.1}$$

$$(f+g)*h = f*h+g*h, \tag{2.2}$$

$$\alpha(f*g) = (\alpha f)*g = f*(\alpha g). \tag{2.3}$$

If, in addition, f,g,h are continuous, then by changing variables we have

$$f*g = g*f, \tag{2.4}$$

$$(f*g)*h = f*(g*h). \tag{2.5}$$

Example 2.1. Consider $f(t) = t^a$ and $g(t) = t^b$, $t \geq 0$ where $a, b \in \mathbb{N}$.

$$\begin{aligned}
(f * g)(t) &= \int_0^t x^a (t-x)^b dx \\
&= \int_0^t x^a \sum_{i=0}^{b} (-1)^i \binom{b}{i} t^{b-i} x^i dx \\
&= \sum_{i=0}^{b} (-1)^i \binom{b}{i} t^{b-i} \int_0^t x^{i+a} dx \\
&= \sum_{i=0}^{b} (-1)^i \binom{b}{i} t^{b-i} \frac{t^{i+a+1}}{i+a+1} \\
&= \sum_{i=0}^{b} (-1)^i \binom{b}{i} \frac{t^{a+b+1}}{i+a+1}.
\end{aligned}$$

Example 2.2. Consider $f(t) = \cos at$ and $g(t) = \sin bt$, $t \geq 0$ where $a, b \in \mathbb{N} - \{0\}$.

case $a \neq b$

$$(f * g)(t) = \int_0^t \cos ax \sin b(t-x) dx$$

By using Trigonometric identity, we obtain

$$\begin{aligned}
\int_0^t \cos ax \sin b(t-x) dx &= \frac{1}{2} \int_{bt}^{at} \frac{\sin u}{a-b} du - \frac{1}{2} \int_{bt}^{at} \frac{\sin u}{a+b} du \\
&= -\frac{1}{2} \frac{\cos u}{a-b} \Big|_{bt}^{at} + \frac{1}{2} \frac{\cos u}{a+b} \Big|_{bt}^{at} \\
&= \frac{-1}{2(a-b)} (\cos at - \cos bt) + \frac{1}{2(a+b)} (\cos at - \cos bt) \\
&= \frac{-b}{a^2 + b^2} (\cos at - \cos bt).
\end{aligned}$$

case $a = b$

$$(f * g)(t) = \int_0^t \cos ax \sin a(t-x) dx$$

By using Trigonometric identity, we obtain

$$\begin{aligned}
\int_0^t \cos ax \sin a(t-x) dx &= \frac{1}{2} \int_0^t \sin at dx - \frac{1}{2} \int_0^t \sin a(2x - t) dx \\
&= \frac{1}{2} x \sin at \Big|_0^t + \frac{1}{4a} \cos a(2x-t) \Big|_0^t \\
&= \frac{1}{2} t \sin at + \frac{1}{4a} (\cos at - \cos at) \\
&= \frac{t \sin at}{2}.
\end{aligned}$$

Convolutions of elementary functions can be obtained via integration by parts and changes of variables. The following table illustrates the resulting convolutions.

Table 2.1: The table of convolution product

$f(t)$	$g(t)$	$(f*g)(t)$
t^a	t^b	$\sum_{i=0}^{b}(-1)^i\binom{b}{i}\frac{t^{b+a+1}}{i+a+1}$ $a,b\in\mathbb{N}$
$\sin at$	$\sin bt$	$\frac{-a\sin bt+b\sin bt}{b^2-a^2}$ $a\neq b$ $\frac{-bt\cos bt+b\sin at}{2b}$ $a=b\neq 0$
$\sin at$	$\cos bt$	$\frac{-b(\cos bt-\cos at)}{b^2-a^2}$ $a\neq b$ $\frac{t\sin bt}{2}$ $a=b\neq 0$
$\sin at$	e^{bt}	$\frac{e^{bt}b-b\cos at+a\sin at}{b^2+a^2}$ $a^2+b^2\neq 0$
$\cos at$	$\cos bt$	$\frac{b\sin bt-a\sin at}{a^2-b^2}$ $a\neq b$ $\frac{bt\cos bt+\sin bt}{2b}$ $a=b\neq 0$
$\cos at$	e^{bt}	$\frac{e^{bt}b-b\cos at+a\sin at}{a^2+b^2}$ $a^2+b^2\neq 0$
e^{at}	e^{bt}	$\frac{e^{bt}-e^{at}}{a-b}$ $a\neq b$ $e^{bt}t$ $a=b$

2.2 Convolution of matrix-valued functions

Definition 2.3. The convolution product between an integrable function $f:\Omega\to\mathbb{R}$ and an integrable matrix-valued function $A:\Omega\to M_{m,n}(\mathbb{R})$ is defined to be the function $f*A:\Omega\to M_{m,n}(\mathbb{R})$, $(f*A)(t)=[f(t)*a_{ij}(t)]$. For convenience, we may write $f(t)*A(t)$ instead of $(f*A)(t)$.

Definition 2.4. The (usual) convolution product of two integrable matrix-valued functions $A:\Omega\to M_{m,n}(\mathbb{R})$, $A(t)=[a_{ij}(t)]$ and $B:\Omega\to M_{n,p}(\mathbb{R})$, $B(t)=[b_{ij}(t)]$ is defined to be the matrix-valued function $A*B:\Omega\to M_{m,p}(\mathbb{R})$,

$$(A*B)(t) = \left[\sum_{k=1}^{n} a_{ik}(t)*b_{kj}(t)\right]\in M_{m,p}(\mathbb{R}) \quad \text{for each} \quad t\in\Omega.$$

For convenience, we may write $A(t)*B(t)$ instead of $(A*B)(t)$.

Example 2.5. Consider

$$A(t)=\begin{bmatrix}\sin t & t^2\\ -1 & t-1\end{bmatrix},\qquad B(t)=\begin{bmatrix}e^t & \sin t\\ 1 & t\end{bmatrix}$$

The convolution product of A and B is

$$\begin{aligned}(A*B)(t) &= \begin{bmatrix}\sin t*e^t+t^2*1 & \sin t*\sin t+t^2*t\\ -1*e^t+(t-1)*1 & -1*\sin t+(t-1)*t\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{2}(e^t-\cos t-\sin t+\frac{t^3}{3}) & \frac{1}{2}(-t\cos t+\sin t+\frac{t^4}{12})\\ 1-e^t-t+\frac{t^2}{2} & -1+\cos t-\frac{t^2}{2}+\frac{t^6}{6}.\end{bmatrix}\end{aligned}$$

Theorem 2.6. Let A,B and C be $n\times n$ integrable matrix-valued functions on Ω. Then for any scalar α and β

$$((\alpha A+\beta B)*C)(t)=\alpha(A*C)(t)+\beta(B*C)(t).$$

Theorem 2.7. Let A,B be $n\times n$ continuous matrix-valued functions on Ω. Then

$$\begin{aligned}((A*B)*C)(t) &= (A*(B*C))(t),\\ (A*B)^T(t) &= B^T(t)*A^T(t).\end{aligned}$$

2.3 Kronecker convolution product

Definition 2.8. Let $A=[a_{ij}]\in M_{m,n}$ and $B=[b_{ij}]\in M_{q,p}$. The Kronecker product define by

$$A\otimes B=[a_{ij}B]_{ij}\in M_{mq,np}$$

Example 2.9. Consider

$$A=\begin{bmatrix}1 & 2\\ -1 & 4\end{bmatrix},\qquad B=\begin{bmatrix}3 & 2\\ 4 & 1\end{bmatrix}$$

The Kronecker product of A and B is

$$A \otimes B = \begin{bmatrix} 1B & 2B \\ -1B & 4B \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 & 4 \\ 4 & 1 & 8 & 2 \\ -3 & -2 & 12 & 8 \\ -4 & -1 & 16 & 4 \end{bmatrix}.$$

Definition 2.10. The Kronecker convolution product of two integrable matrix-valued functions $A : \Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ and $B : \Omega \to M_{p,q}(\mathbb{R})$, is defined to be the function $A \circledast B : \Omega \to M_{mp,nq}(\mathbb{R})$,

$$(A \circledast B)(t) = [a_{ij}(t) * B(t)]_{ij} \quad \text{for each} \quad t \in \Omega$$

That is, each (i,j)-th block of $(A \circledast B)(t)$ is given by $a_{ij}(t) * B(t)$. For convenience, we may write $A(t) \circledast B(t)$ for $(A \circledast B)(t)$.

Example 2.11. Consider

$$A(t) = \begin{bmatrix} \sin t & t^2 \\ -1 & t-1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} e^t & \sin t \\ 1 & t \end{bmatrix}$$

The Kronecker convolution product of A and B is

$$(A \circledast B)(t) = \begin{bmatrix} \sin t * \begin{bmatrix} e^t & \sin t \\ 1 & t \end{bmatrix} & t^2 * \begin{bmatrix} e^t & \sin t \\ 1 & t \end{bmatrix} \\ -1 * \begin{bmatrix} e^t & \sin t \\ 1 & t \end{bmatrix} & (t-1) * \begin{bmatrix} e^t & \sin t \\ 1 & t \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \sin t * e^t & \sin t * \sin t & t^2 * e^t & t^2 * \sin t \\ \sin t * 1 & \sin t * 1 & t^2 * 1 & t^2 * t \\ -1 * e^t & -1 * \sin t & (t-1) * e^t & (t-1) * \sin t \\ -1 * 1 & -1 * t & (t-1) * 1 & (t-1) * t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^t - \cos t - \sin t) & \frac{1}{2}(-t\cos t + \sin t) & -2 + 2e^t - 2t - t^2 & -1 + t^2 + 2\cos t \\ 1 - \cos t & t - \sin t & \frac{t^3}{3} & \frac{t^4}{12} \\ 1 - e^t & -1 + \cos t & -t & -1 + t + \cos t - \sin t \\ -t & \frac{-t^2}{2} & -t + \frac{t^2}{2} & \frac{-t^2}{2} + \frac{t^3}{6} \end{bmatrix}.$$

Lemma 2.12 ([5]). Let A, B and C be compatibly integrable matrix-valued functions on Ω. Then for any scalars α, β we have

(i) $A \circledast (B + C) = (A \circledast B) + (A \circledast C)$,

(ii) $(A + B) \circledast C = (A \circledast C) + (B \circledast C)$,

(iii) $(\alpha A) \circledast B = \alpha(A \circledast B) = A \circledast (\alpha B)$.

Lemma 2.13 ([5]). Let A, B and C be compatibly continuous matrix-valued functions on Ω and for any integrable function $\phi : \Omega \to \mathbb{R}$, we have

$$(A \circledast B) \circledast C = A \circledast (B \circledast C), \tag{2.6}$$

$$(\phi * A) \circledast B = \phi * (A \circledast B) = A \circledast (\phi * B), \tag{2.7}$$

$$(A \circledast B)^T = A^T \circledast B^T. \tag{2.8}$$

Theorem 2.14 ([5]). Let A, B and C be $n \times n$ continuous integrable matrices on Ω. Then

$$(A(t) \circledast B(t)) * (C(t) \circledast D(t)) = (A(t) * C(t)) \circledast (B(t) * D(t)).$$

2.4 Tracy-Singh product of complex matrices

Definition 2.15 ([11]). Consider matrices A and B of order $m \times n$ and $p \times q$, respectively. Let $A = [A_{ij}]$ be partitioned with A_{ij} of order $m_i \times n_j$ as the (i, j)th submatrix and let $B = [B_{kl}]$ be partitioned with B_{kl} of order $p_k \times q_l$ as the (k, l)th submatrix where

$$\sum_{i=1}^{r} m_i = m, \quad \sum_{j=1}^{s} n_j = n, \quad \sum_{k=1}^{t} p_k = p, \quad \sum_{l=1}^{u} q_l = q.$$

The **Tracy-Singh product** of A and B, denoted as $A \boxtimes B$, is defined by

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij}, \tag{2.9}$$

where $A \boxtimes B$ is of order $mp \times nq$.

Example 2.16. Consider

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 7 & 8 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 9 \end{bmatrix},$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = \begin{bmatrix} 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 4 & 7 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 5 & 8 \\ 6 & 9 \end{bmatrix}.$$

The Tracy-Singh product of A and B is

$$A \boxtimes B = \begin{bmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} & \begin{bmatrix} 4 & 7 & 8 & 14 \\ 16 & 28 & 20 & 35 \end{bmatrix} & \begin{bmatrix} 3 \\ 6 \end{bmatrix} & \begin{bmatrix} 12 & 21 \\ 24 & 42 \end{bmatrix} \\ \begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} & \begin{bmatrix} 5 & 8 & 10 & 16 \\ 6 & 9 & 12 & 18 \\ 20 & 32 & 25 & 40 \\ 24 & 36 & 30 & 45 \end{bmatrix} & \begin{bmatrix} 6 \\ 9 \\ 12 \\ 18 \end{bmatrix} & \begin{bmatrix} 15 & 24 \\ 18 & 27 \\ 30 & 48 \\ 36 & 54 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 \end{bmatrix} & \begin{bmatrix} 28 & 49 & 32 & 56 \end{bmatrix} & \begin{bmatrix} 9 \end{bmatrix} & \begin{bmatrix} 36 & 63 \end{bmatrix} \\ \begin{bmatrix} 14 & 16 \\ 21 & 24 \end{bmatrix} & \begin{bmatrix} 35 & 56 & 40 & 64 \\ 42 & 63 & 48 & 72 \end{bmatrix} & \begin{bmatrix} 18 \\ 27 \end{bmatrix} & \begin{bmatrix} 45 & 72 \\ 54 & 81 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 & 7 & 8 & 14 & 3 & 12 & 21 \\ 4 & 5 & 16 & 28 & 20 & 25 & 6 & 24 & 42 \\ 2 & 4 & 5 & 8 & 10 & 16 & 6 & 15 & 24 \\ 3 & 6 & 6 & 9 & 12 & 18 & 9 & 18 & 27 \\ 8 & 10 & 21 & 32 & 25 & 40 & 12 & 30 & 48 \\ 12 & 15 & 24 & 36 & 30 & 45 & 18 & 36 & 54 \\ 7 & 8 & 28 & 49 & 32 & 56 & 9 & 36 & 63 \\ 14 & 16 & 35 & 56 & 40 & 64 & 18 & 45 & 72 \\ 21 & 24 & 42 & 63 & 48 & 72 & 27 & 54 & 81 \end{bmatrix}.$$

Example 2.17. Consider

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & 3 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix},$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 3 & 6 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 9 \end{bmatrix}.$$

The Tracy-Singh product of A and B is

$$A \boxtimes B = \begin{bmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} & \begin{bmatrix} 7 \\ 8 \end{bmatrix} & \begin{bmatrix} 2 & 8 & 3 & 12 \\ 4 & 20 & 6 & 15 \end{bmatrix} & \begin{bmatrix} 14 & 21 \\ 16 & 24 \end{bmatrix} \\ \begin{bmatrix} 3 & 6 \end{bmatrix} & \begin{bmatrix} 9 \end{bmatrix} & \begin{bmatrix} 6 & 12 & 9 & 18 \end{bmatrix} & \begin{bmatrix} 18 & 27 \end{bmatrix} \\ \begin{bmatrix} 4 & 16 \\ 8 & 20 \\ 7 & 28 \\ 14 & 35 \\ 12 & 24 \\ 21 & 42 \end{bmatrix} & \begin{bmatrix} 28 \\ 32 \\ 49 \\ 56 \\ 36 \\ 63 \end{bmatrix} & \begin{bmatrix} 5 & 20 & 6 & 24 \\ 10 & 25 & 12 & 30 \\ 8 & 32 & 9 & 36 \\ 16 & 40 & 12 & 45 \\ 15 & 30 & 18 & 36 \\ 24 & 48 & 27 & 54 \end{bmatrix} & \begin{bmatrix} 35 & 42 \\ 40 & 48 \\ 56 & 63 \\ 64 & 72 \\ 45 & 54 \\ 72 & 81 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 7 & 2 & 8 & 3 & 12 & 14 & 21 \\ 2 & 5 & 8 & 4 & 20 & 6 & 15 & 16 & 24 \\ 3 & 6 & 9 & 6 & 12 & 9 & 18 & 18 & 27 \\ 4 & 16 & 28 & 5 & 20 & 6 & 24 & 35 & 42 \\ 8 & 20 & 32 & 10 & 25 & 12 & 30 & 40 & 48 \\ 7 & 28 & 49 & 8 & 32 & 9 & 36 & 56 & 63 \\ 14 & 35 & 56 & 16 & 40 & 12 & 45 & 64 & 72 \\ 12 & 24 & 36 & 15 & 30 & 18 & 36 & 45 & 54 \\ 21 & 42 & 63 & 24 & 48 & 27 & 54 & 72 & 81 \end{bmatrix}.$$

Proposition 2.18. Let A, B, C and D be compatibly partitioned matrices and $\alpha \in \mathbb{C}$. Then

(i) $(\alpha A) \boxtimes B = A \boxtimes (\alpha B) = \alpha(A \boxtimes B)$,

(ii) $(A + C) \boxtimes B = A \boxtimes B + C \boxtimes B$,

(iii) $A \boxtimes (B + D) = A \boxtimes B + A \boxtimes D$,

(iv) $(A \boxtimes B)^* = A^* \boxtimes B^*$.

2.5 Vector operator

Definition 2.19. For each matrix $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ we associate the vector vec$(A) \in \mathbb{C}^{mn}$ and rvec$(A) \in \mathbb{C}^{mn}$ respectively defined by

$$\text{vec}(A) = [a_{11}, ..., a_{m1}, a_{12}, ..., a_{m2}, ..., a_{1n}, ..., a_{mn}]^T$$

and

$$\text{rvec}(A) = [a_{11}, ..., a_{1m}, a_{21}, ..., a_{2m}, ..., a_{m1}, ..., a_{mn}]^T.$$

Example 2.20. Consider

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Then

$$\text{vec}(A) = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \\ 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \text{rvec}(A) = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 2 \\ 4 \\ 6 \end{bmatrix}.$$

Theorem 2.21 ([5]). Let A, X, and B be $n \times n$ integrable matrix-valued functions on Ω Then

(i) $Vec[A(t) * X(t) * B(t)] = [B^T(t) \circledast A(t)] * VecX(t)$,

(ii) $Vec[A(t)X(t) * B(t)] = [B^T(t) \otimes A(t)] * VecX(t)$,

Chapter 3

Tracy-Singh Convolution Products

In this chapter, we introduce the notion of Tracy-Singh convolution product for matrix-valued functions and investigate its algebraic properties. This kind of convolution product is associative and compatible with certain algebraic operations, namely, the addition, scalar multiplication, scalar convolution product, usual convolution product and transposition.

3.1 Tracy-Singh convolution product of matrix-valued functions

Definition 3.1. Let $A \in \mathcal{C}(\Omega, M_{m,n}(\mathbb{R}))$ and $B \in \mathcal{C}(\Omega, M_{p,q}(\mathbb{R}))$. We partition $A(t) = [A_{ij}(t)]_{ij}$ and $B(t) = [B_{kl}(t)]_{kl}$, for all $t \in \Omega$. Here each $A_{ij}(t)$ is of order $m_i \times n_j$ and $B_{kl}(t)$ is of order $p_k \times q_l$, where

$$\sum m_i = m, \quad \sum n_j = n, \quad \sum p_k = p, \quad \sum q_l = q$$

We define the **Tracy-Singh convolution product** $A \boxast B : \Omega \to M_{mp,nq}(\mathbb{R})$ by

$$(A \boxast B)(t) = [[A_{ij}(t) \circledast B_{kl}(t)]_{kl}]_{ij} \quad \text{for each} \quad t \in \Omega$$

For convenience, we write $A(t) \boxast B(t)$ for $(A \boxast B)(t)$.

Example 3.2. Consider

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}$$

where

$$A_{11}(t) = \begin{bmatrix} \sin t & t^2 \\ -1 & t \end{bmatrix}, \quad A_{12}(t) = \begin{bmatrix} -t \\ 1 \end{bmatrix}, \quad A_{21}(t) = \begin{bmatrix} \cos t & t \end{bmatrix}, \quad A_{22}(t) = \begin{bmatrix} 1 \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}$$

where

$$B_{11}(t) = \begin{bmatrix} -1 \end{bmatrix}, \quad B_{12}(t) = \begin{bmatrix} t & -\sin t \end{bmatrix}, \quad B_{21}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}, \quad B_{22}(t) = \begin{bmatrix} \sin t & t \\ \cos t & 1 \end{bmatrix}.$$

The Tracy-Singh convolution product of $A(t)$ and $B(t)$ is

$$A(t) \boxast B(t)$$

$$= \begin{bmatrix} A_{11}(t) \circledast B_{11}(t) & A_{11}(t) \circledast B_{12}(t) & A_{12}(t) \circledast B_{11}(t) & A_{12}(t) \circledast B_{12}(t) \\ A_{11}(t) \circledast B_{21}(t) & A_{11}(t) \circledast B_{22}(t) & A_{12}(t) \circledast B_{21}(t) & A_{12}(t) \circledast B_{22}(t) \\ A_{21}(t) \circledast B_{11}(t) & A_{21}(t) \circledast B_{12}(t) & A_{22}(t) \circledast B_{11}(t) & A_{22}(t) \circledast B_{12}(t) \\ A_{21}(t) \circledast B_{21}(t) & A_{21}(t) \circledast B_{22}(t) & A_{22}(t) \circledast B_{21}(t) & A_{22}(t) \circledast B_{22}(t) \end{bmatrix}$$

$$= \begin{bmatrix} -1+cost & \frac{-t^3}{3} & t-sint & \frac{1}{2}(tcost-sint) & \frac{t^4}{12} & 2-t^2-2cost & \frac{t^2}{2} & \frac{-t^3}{6} & t-sint \\ t & \frac{-t^2}{2} & \frac{-t^2}{2} & 1-cost & \frac{t^3}{6} & t-sint & -t & \frac{t^2}{2} & -1+cost \\ t-sint & \frac{t^4}{12} & \frac{1}{2}(-tcost+sint) & t-sint & -2+t^2+2cost & \frac{t^4}{12} & \frac{-t^3}{6} & -t+sint & \frac{-t^3}{6} \\ 0 & 0 & \frac{1}{2}tsint & 1-cost & 2(t-sint) & \frac{t^3}{3} & 0 & -1+cost & \frac{-t^2}{2} \\ \frac{-t^2}{2} & \frac{t^3}{6} & -1+cost & \frac{-t^2}{2} & t-sint & \frac{t^3}{6} & \frac{t^2}{2} & 1-cost & \frac{t^2}{2} \\ 0 & 0 & -sint & -t & 1-cost & \frac{t^2}{2} & 0 & sint & t \\ -sint & \frac{-t^2}{2} & 1-cost & -\frac{1}{2}tsint & \frac{t^3}{6} & -t+sint & -t & \frac{t^2}{2} & -1+cost \\ 1-cost & \frac{t^3}{6} & \frac{1}{2}tsint & 1-cost & t-sint & \frac{t^3}{6} & \frac{t^2}{2} & 1-cost & \frac{t^2}{2} \\ 0 & 0 & \frac{1}{2}(tcost+sint) & sint & 1-cost & \frac{t^2}{2} & 0 & sint & t \end{bmatrix}.$$

Theorem 3.3. Let A, B, C and D be compatibly partitioned integrable matrix-valued functions on Ω. Then

(i) $A \boxast (B+C) = (A \boxast B) + (A \boxast C)$,

(ii) $(A+B) \boxast C = (A \boxast C) + (A \boxast C)$,

(iii) $(\alpha A) \boxast B = A \boxast (\alpha B) = \alpha(A \boxast B)$ for any $\alpha \in \mathbb{R}$.

Moreover, if A and B are continuous, then for any continuous function $\phi : \Omega \to \mathbb{R}$, we have

$$\phi * (A \boxast B) = (\phi * A) \boxast B = A \boxast (\phi * B), \tag{3.1}$$

$$(A \boxast B)^T = A^T \boxast B^T. \tag{3.2}$$

Proof. Write $A(t) = [A(t)_{ij}], B(t) = [B(t)_{kl}]$ and $C(t) = [C(t)_{kl}]$, for each $t \in \Omega$. By using Lemma 2.12, we have

$$\begin{aligned} (A \boxast (B+C))(t) &= [[A_{ij}(t) \circledast (B_{kl}(t) + C_{kl}(t))]_{kl}]_{ij} \\ &= [[A_{ij}(t) \circledast B_{kl}(t) + A_{ij}(t) \circledast C_{kl}(t)]_{kl}]_{ij} \\ &= [[A_{ij}(t) \circledast B_{kl}(t)]_{kl} + [A_{ij}(t) \circledast C_{kl}(t)]_{kl}]_{ij} \\ &= [[A_{ij}(t) \circledast B_{kl}(t)]_{kl}]_{ij} + [[A_{ij}(t) \circledast C_{kl}(t)]_{kl}]_{ij} \\ &= (A \boxast B)(t) + (A \boxast C)(t), \end{aligned}$$

$$
\begin{aligned}
((A+B)\boxast C)(t) &= [[(A_{ij}(t)+B_{ij}(t))\circledast C_{kl}(t)]_{kl}]_{ij}\\
&= [[(A_{ij}(t)\circledast C_{kl}(t)+B_{ij}(t)\circledast C_{kl}(t)]_{kl}]_{ij}\\
&= [[(A_{ij}(t)\circledast C_{kl}(t)]_{kl}+[B_{ij}(t)\circledast C_{kl}(t)]_{kl}]_{ij}\\
&= [[(A_{ij}(t)\circledast C_{kl}(t)]_{kl}]_{ij}+[[B_{ij}(t)\circledast C_{kl}(t)]_{kl}]_{ij}\\
&= (A\boxast C)(t)+(B\boxast C)(t),\\
((\alpha A)\boxast B)(t) &= [[\alpha A_{ij}(t)\circledast B_{kl}(t)]_{kl}]_{ij}\\
&= [[A_{ij}(t)\circledast \alpha B_{kl}(t)]_{kl}]_{ij} = (A\boxast(\alpha B))(t)\\
&= [[A_{ij}(t)\circledast \alpha B_{kl}(t)]_{kl}]_{ij}\\
&= [[\alpha(A_{ij}(t)\circledast B_{kl}(t))]_{kl}]_{ij}\\
&= [\alpha[A_{ij}(t)\circledast B_{kl}(t)]_{kl}]_{ij}\\
&= \alpha[[A_{ij}(t)\circledast B_{kl}(t)]_{kl}]_{ij} = \alpha(A\boxast B)(t).
\end{aligned}
$$

Hence, we obtain the properties (i-iii).

Property (3.1) and (3.2) can be proved by using (2.7) and (2.8) as follows:

$$
\begin{aligned}
(\phi*(A\boxast B))(t) &= [[\phi*(A_{ij}(t)\circledast B_{kl}(t))]_{kl}]_{ij}\\
&= [[(\phi*A_{ij}(t))\circledast B_{kl}(t)]_{kl}]_{ij} \;=\; ((\phi*A)\boxast B)(t)\\
&= [[A_{ij}(t)\circledast(\phi*B_{kl}(t))]_{kl}]_{ij} \;=\; (A\boxast(\phi*B))(t),\\
(A(t)\boxast B(t))^T &= [[A_{ij}(t)\circledast B_{kl}(t)]_{kl}]_{ij}^T\\
&= [[A_{ji}(t)\circledast B_{kl}(t)]_{kl}^T]_{ij}\\
&= [[A_{ji}^T(t)\circledast B_{lk}^T(t)]_{kl}]_{ij}\\
&= [[[A_{ij}(t)]^T\circledast[B_{kl}(t)]^T]_{kl}]_{ij}\\
&= A^T(t)\boxast B^T(t).
\end{aligned}
$$

Hence, we obtain the properties (3.1) and (3.2). □

Lemma 3.4. Let $A\in\mathcal{I}(\Omega,M_{m,n}(\mathbb{R}))$ and $B\in\mathcal{I}(\Omega,M_{p,q}(\mathbb{R}))$. Then

$$
A(t)\boxast B(t) \;=\; [A_{ij}(t)\boxast B(t)]_{ij} \;=\; \begin{bmatrix} A_{11}(t)\boxast B(t) & \cdots & A_{1n}(t)\boxast B(t)\\ \vdots & \ddots & \vdots\\ A_{m1}(t)\boxast B(t) & \cdots & A_{mn}(t)\boxast B(t)\end{bmatrix}.
$$

That is, each (i,j)-th block of $A(t)\boxast B(t)$ is given by $A_{ij}(t)\boxast B(t)$, regardless of how to partition $A(t)$.

Proof. By Definition of the Tracy-Singh convolution product, we have

$A(t) \boxast B(t)$

$$= \begin{bmatrix} \begin{bmatrix} A_{11}(t) \circledast B_{11}(t) & \cdots & A_{11}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{11}(t) \circledast B_{p1}(t) & \cdots & A_{11}(t) \circledast B_{pq}(t) \end{bmatrix} & \cdots & \begin{bmatrix} A_{1n}(t) \circledast B_{11}(t) & \cdots & A_{1n}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{1n}(t) \circledast B_{p1}(t) & \cdots & A_{1n}(t) \circledast B_{pq}(t) \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} A_{m1}(t) \circledast B_{11}(t) & \cdots & A_{m1}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{m1}(t) \circledast B_{p1}(t) & \cdots & A_{m1}(t) \circledast B_{pq}(t) \end{bmatrix} & \cdots & \begin{bmatrix} A_{mn}(t) \circledast B_{11}(t) & \cdots & A_{mn}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{mn}(t) \circledast B_{p1}(t) & \cdots & A_{mn}(t) \circledast B_{pq}(t) \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}(t) \boxast \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} & \cdots & A_{1n}(t) \boxast \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} \\ \vdots & \ddots & \vdots \\ A_{m1}(t) \boxast \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} & \cdots & A_{mn}(t) \boxast \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}(t) \boxast B(t) & \cdots & A_{1n}(t) \boxast B(t) \\ \vdots & \ddots & \vdots \\ A_{m1}(t) \boxast B(t) & \cdots & A_{mn}(t) \boxast B(t) \end{bmatrix}.$$

□

Example 3.5. Consider

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}$$

where

$$A_{11}(t) = \begin{bmatrix} \sin t & t^2 \\ -1 & t \end{bmatrix}, \quad A_{12}(t) = \begin{bmatrix} -t \\ 1 \end{bmatrix}, \quad A_{21}(t) = \begin{bmatrix} \cos t & t \end{bmatrix}, \quad A_{22}(t) = \begin{bmatrix} 1 \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}$$

where

$$B_{11}(t) = \begin{bmatrix} -1 \end{bmatrix}, \quad B_{12}(t) = \begin{bmatrix} t & -\sin t \end{bmatrix}, \quad B_{21}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}, \quad B_{22}(t) = \begin{bmatrix} \sin t & t \\ \cos t & 1 \end{bmatrix}.$$

By Example 3.2 and Lemma 3.4, we have

$$A(t) \boxast B(t) = \begin{bmatrix} A_{11}(t) \boxast B(t) & A_{12}(t) \boxast B(t) \\ A_{21}(t) \boxast B(t) & A_{22}(t) \boxast B(t) \end{bmatrix}$$

$$= \begin{bmatrix}
-1+cost & \frac{-t^3}{3} & t-sint & \frac{1}{2}(tcost-sint) & \frac{t^4}{12} & 2-t^2-2cost & \frac{t^2}{2} & \frac{-t^3}{6} & t-sint \\
t & \frac{-t^2}{2} & \frac{-t^2}{2} & 1-cost & \frac{t^3}{6} & t-sint & -t & \frac{t^2}{2} & -1+cost \\
t-sint & \frac{t^4}{12} & \frac{1}{2}(-tcost+sint) & t-sint & -2+t^2+2cost & \frac{t^4}{12} & \frac{-t^3}{6} & -t+sint & \frac{-t^3}{6} \\
0 & 0 & \frac{1}{2}tsint & 1-cost & 2(t-sint) & \frac{t^3}{3} & 0 & -1+cost & \frac{-t^2}{2} \\
\frac{-t^2}{2} & \frac{t^3}{6} & -1+cost & \frac{-t^2}{2} & t-sint & \frac{t^3}{6} & \frac{t^2}{2} & 1-cost & \frac{t^2}{2} \\
0 & 0 & -sint & -t & 1-cost & \frac{t^2}{2} & 0 & sint & t \\
-sint & \frac{-t^2}{2} & 1-cost & -\frac{1}{2}tsint & \frac{t^3}{6} & -t+sint & -t & \frac{t^2}{2} & -1+cost \\
1-cost & \frac{t^3}{6} & \frac{1}{2}tsint & 1-cost & t-sint & \frac{t^3}{6} & \frac{t^2}{2} & 1-cost & \frac{t^2}{2} \\
0 & 0 & \frac{1}{2}(tcost+sint) & sint & 1-cost & \frac{t^2}{2} & 0 & sint & t
\end{bmatrix}.$$

Theorem 3.6. Let $A \in \mathcal{C}(\Omega, M_{m,n}(\mathbb{R}))$, $B \in \mathcal{C}(\Omega, M_{p,q}(\mathbb{R}))$ and $C \in \mathcal{C}(\Omega, M_{v,w}(\mathbb{R}))$. Then

$$(A \boxast B) \boxast C = A \boxast (B \boxast C). \tag{3.3}$$

Proof. We have by using Lemma 2.13 that for each $t \in \Omega$,

$$\begin{aligned}
(A(t) \boxast B(t)) \boxast C(t) &= [[A_{ij}(t) \circledast B_{kl}(t)]_{kl}]_{ij} \boxast C(t) \\
&= [[[(A_{ij}(t) \circledast B_{kl}(t)) \circledast C_{tu}(t)]_{tu}]_{kl}]_{ij} \\
&= [[[A_{ij}(t) \circledast (B_{kl}(t) \circledast C_{tu}(t))]_{tu}]_{kl}]_{ij} \\
&= A(t) \boxast [[B_{kl}(t) \circledast C_{tu}(t)]_{tu}]_{kl} \\
&= A(t) \boxast (B(t) \boxast C(t)).
\end{aligned}$$

Hence, we obtain the associativity property (3.3). □

Recall that the direct sum of matrix-valued functions $A_i : \Omega \to M_{m_i,n_i}(\mathbb{R})$, $i = 1, \ldots, k$, is defined by

$$\bigoplus_{i=1}^{k} A_i : \Omega \to M_{m,n}(\mathbb{R}), \qquad \bigoplus_{i=1}^{k} A_i(t) = \begin{bmatrix} A_1(t) & 0 & \ldots & 0 \\ 0 & A_2(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_k(t) \end{bmatrix}.$$

Here, $m = \sum m_i$ and $n = \sum n_i$.

Example 3.7. Consider

$$A_1(t) = \begin{bmatrix} \sin t & \cos t \\ 1 & e^t \\ t & -1 \end{bmatrix}, \quad A_2(t) = \begin{bmatrix} e^t & t & t^2 \\ \cos t & 1 & -1 \end{bmatrix}.$$

Then

$$\bigoplus_{i=1}^{2} A_i = \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} = \begin{bmatrix} \sin t & \cos t & 0 & 0 & 0 \\ 1 & e^t & 0 & 0 & 0 \\ t & -1 & 0 & 0 & 0 \\ 0 & 0 & e^t & t & t^2 \\ 0 & 0 & \cos t & 1 & -1 \end{bmatrix} \in M_{5,5}.$$

Proposition 3.8. For each $i = 1, \ldots, k$, let A_i and B be integrable matrix-valued functions on Ω. Then

$$\left(\bigoplus_{i=1}^{k} A_i \right) \boxtimes B = \bigoplus_{i=1}^{k} (A_i \boxtimes B).$$

Proof. By induction, we may assume that $k = 2$. By Lemma 3.4, we have that for each $t \in \Omega$,

$$\begin{aligned} (A_1(t) \oplus A_2(t)) \boxtimes B(t) &= \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} \boxtimes B(t) \\ &= \begin{bmatrix} A_1(t) \boxtimes B(t) & 0 \\ 0 & A_2(t) \boxtimes B(t) \end{bmatrix} \\ &= (A_1(t) \boxtimes B(t)) \oplus (A_2(t) \boxtimes B(t)). \end{aligned}$$

□

The next property, called the mixed product property, shows that the Tracy-Singh convolution product is compatible with the usual convolution product.

Theorem 3.9. Let $A \in C(\Omega, M_{m,n}(\mathbb{R}))$, $B \in C(\Omega, M_{n,s}(\mathbb{R})), C \in C(\Omega, M_{p,q}(\mathbb{R}))$ and $D \in C(\Omega, M_{q,r}(\mathbb{R}))$. Then

$$(A * B) \boxtimes (C * D) = (A \boxtimes C) * (B \boxtimes D). \tag{3.4}$$

Proof. Write $A(t) = [A_{ij}(t)]_{i,j=1}^{m,n}$, $B(t) = [B_{ij}(t)]_{i,j=1}^{n,s}$, $C(t) = [C_{kl}(t)]_{k,l=1}^{p,q}$ and $D(t) = [D_{kl}(t)]_{k,l=1}^{q,r}$. By using Theorem 2.14 and block-matrix algebra, we have that for each

$t \in \Omega$,

$$
\begin{aligned}
&(A(t) * B(t)) \circledast (C(t) * D(t)) \\
&= \left[\sum_{u=1}^{n} A_{iu}(t) * B_{uj}(t)\right]_{ij} \boxast \left[\sum_{t=1}^{q} C_{kt}(t) * D_{tl}(t)\right]_{kl} \\
&= \left[\left[\left(\sum_{u=1}^{n} A_{iu}(t) * B_{uj}(t)\right) \circledast \left(\sum_{t=1}^{q} C_{kt}(t) * D_{t}(t)\right)\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{u=1}^{n}\left(\sum_{t=1}^{q} (A_{iu}(t) * B_{uj}(t)) \circledast (C_{kt}(t) * D_{tl}(t))\right)\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{u=1}^{n}\sum_{t=1}^{q} (A_{iu}(t) * B_{uj}(t)) \circledast (C_{kt}(t) * D_{tl}(t))\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{u=1}^{n}\sum_{t=1}^{q} (A_{iu}(t) \circledast C_{kt}(t)) * (B_{uj}(t) \circledast D_{tl}(t))\right]_{kl}\right]_{ij} \\
&= [[A_{ij}(t) \circledast C_{kl}(t)]_{kl}]_{ij} * [[B_{ij}(t) \circledast D_{kl}(t)]_{kl}]_{ij} \\
&= (A(t) \boxast C(t)) * (B(t) \boxast D(t)).
\end{aligned}
$$

Hence, we arrive at the property (3.4). □

3.2 Commutator of convolution product

Definition 3.10. For $A, B \in \mathcal{I}(\Omega, M_n(\mathbb{R}))$, the **convolution commutator** and the **convolution anticommutator** of A and B are respectively defined as follows:

$$
\begin{aligned}
[A, B]_{*} &= A * B - B * A, \\
[A, B]_{*,+} &= A * B + B * A.
\end{aligned}
$$

Example 3.11. Consider

$$
A(t) = \begin{bmatrix} \sin t & t^2 \\ -1 & t \end{bmatrix}, \quad B(t) = \begin{bmatrix} \sin t & t \\ \cos t & 1 \end{bmatrix}.
$$

Then

$$
[A, B]_{*} = \begin{bmatrix} \sin t & t^2 \\ -1 & t \end{bmatrix} * \begin{bmatrix} \sin t & t \\ \cos t & 1 \end{bmatrix} - \begin{bmatrix} \sin t & t \\ \cos t & 1 \end{bmatrix} * \begin{bmatrix} \sin t & t^2 \\ -1 & t \end{bmatrix}
$$

$$
= \begin{bmatrix} \sin t * \sin t + t^2 * \cos t & \sin t * t + t^2 * 1 \\ -1 * \sin t + t * \cos t & -1 * t + t * 1 \end{bmatrix} - \begin{bmatrix} \sin t * \sin t + t * (-1) & \sin t * t^2 + t * t \\ \cos t * \sin t + 1 * (-1) & \cos t * t^2 + 1 * t \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{(-t\cos t + \sin t)}{2} + 2(t - \sin t) & t - \sin t + \frac{t^3}{3} \\ -1 + \cos t + 1 - \cos t & \frac{-t^2}{2} + \frac{t^2}{2} \end{bmatrix} - \begin{bmatrix} \frac{(-t\cos t + \sin t)}{2} + \frac{-t^2}{2} & -2 + t^2 + 2\cos t + \frac{t^3}{6} \\ \frac{t \sin t}{2} + (-t) & 2(t - \sin t) + \frac{t^2}{2} \end{bmatrix}
$$

$$
= \begin{bmatrix} 2(t - \sin t) + \frac{t^2}{2} & t - \sin t + \frac{t^3}{6} + 2 - t^2 - 2\cos t \\ -\frac{t \sin t}{2} + t & -2(t - \sin t) - \frac{t^2}{2} \end{bmatrix}.
$$

and

$$[A,B]_{*,+} = \begin{bmatrix} \sin t & t^2 \\ -1 & t \end{bmatrix} * \begin{bmatrix} \sin t & t \\ \cos t & 1 \end{bmatrix} + \begin{bmatrix} \sin t & t \\ \cos t & 1 \end{bmatrix} * \begin{bmatrix} \sin t & t^2 \\ -1 & t \end{bmatrix}$$

$$= \begin{bmatrix} \sin t * \sin t + t^2 * \cos t & \sin t * t + t^2 * 1 \\ -1 * \sin t + t * \cos t & -1 * t + t * 1 \end{bmatrix} + \begin{bmatrix} \sin t * \sin t + t * (-1) & \sin t * t^2 + t * t \\ \cos t * \sin t + 1 * (-1) & \cos t * t^2 + 1 * t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(-t\cos t + \sin t)}{2} + 2(t - \sin t) & t - \sin t + \frac{t^3}{3} \\ -1 + \cos t + 1 - \cos t & \frac{-t^2}{2} + \frac{t^2}{2} \end{bmatrix} + \begin{bmatrix} \frac{(-t\cos t + \sin t)}{2} + \frac{-t^2}{2} & -2 + t^2 + 2\cos t + \frac{t^3}{6} \\ \frac{t\sin t}{2} + (-t) & 2(t - \sin t) + \frac{t^2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2(t - \sin t) - \frac{t^2}{2} & t - \sin t + \frac{t^3}{2} - 2 + t^2 + 2\cos t \\ \frac{t\sin t}{2} - t & 2(t - \sin t) + \frac{t^2}{2} \end{bmatrix}.$$

Corollary 3.12. Let $A, B, C, D \in \mathcal{I}(\Omega, M_n(\mathbb{R}))$. Then

$$[A \circledast B, C \circledast D]_* = \frac{1}{2}([A,C]_* \circledast [B,D]_{*,+} + [A,C]_{*,+} \circledast [B,D]_*), \tag{3.5}$$

$$[A \circledast B, C \circledast D]_{*,+} = \frac{1}{2}([A,C]_* \circledast [B,D]_* + [A,C]_{*,+} \circledast [B,D]_{*,+}). \tag{3.6}$$

Proof. By using Theorem 3.9, we obtain

$$\begin{aligned} [A,C]_* \circledast [B,D]_{*,+} &= (A*C - C*A) \circledast (B*D + D*B) \\ &= (A*C) \circledast (B*D) + (A*C) \circledast (D*B) \\ &\quad - (C*A) \circledast (B*D) - (C*A) \circledast (D*B), \\ [A,C]_{*,+} \circledast [B,D]_* &= (A*C + C*A) \circledast (B*D - D*B) \\ &= (A*C) \circledast (B*D) - (A*C) \circledast (D*B) \\ &\quad + (C*A) \circledast (B*D) - (C*A) \circledast (D*B). \end{aligned}$$

Putting them together yields

$$\begin{aligned} &[A,C]_* \circledast [B,D]_{*,+} + [A,C]_{*,+} \circledast [B,D]_* \\ &\quad = 2((A*C) \circledast (B*D) - (C*A) \circledast (D*B)) \\ &\quad = 2((A \circledast B) * (C \circledast D) - (C \circledast D) * (A \circledast B)) \\ &\quad = 2[A \circledast B, C \circledast D]_*. \end{aligned}$$

Similarly, we arrive at the property (3.6). ☐

Corollary 3.13. Let $A, B, C, D \in \mathcal{I}(\Omega, M_n(\mathbb{R}))$. If $[A,C]_* = 0$ and $[B,D]_* = 0$, then

$$[A \circledast B, C \circledast D]_* = 0. \tag{3.7}$$

Proof. Suppose $A(t) * C(t) = C(t) * A(t)$ and $B(t) * D(t) = D(t) * B(t)$, we have

$$\begin{aligned}
[A(t) \boxast B(t), C(t) \boxast D(t)] &= (C(t) \boxast D(t)) * (B(t) \boxast C(t)) - (A(t) \boxast B(t)) * (A(t) \boxast B(t)) \\
&= (A(t) * C(t)) \boxast (B(t) * D(t)) - (C(t) * A(t)) \boxast (D(t) * B(t)) \\
&= (C(t) * A(t)) \boxast (D(t) * B(t)) - (C(t) * A(t)) \boxast (D(t) * B(t)) \\
&= 0.
\end{aligned}$$

Hence, we obtain the property (3.7). □

3.3 Tracy-Singh convolution power

Definition 3.14. The k-th Tracy-Singh convolution power $A^{\boxast k}$ is defined for all positive integers k by

$$A^{\boxast 1} = A \quad \text{and} \quad A^{\boxast (k+1)} = A^{\boxast k} \boxast A.$$

Example 3.15. Consider

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \end{bmatrix}$$

where

$$A_{11}(t) = \begin{bmatrix} \sin t \\ 1 \end{bmatrix}, \quad A_{12}(t) = \begin{bmatrix} e^t \\ \cos t \end{bmatrix}.$$

By using Definition 3.14 and Lemma 3.4, we obtain

$$\begin{aligned}
A^{\boxast 2}(t) &= A^{\boxast 1}(t) \boxast A(t = A(t) \boxast A(t) \\
&= \begin{bmatrix} A(t) \boxast A_{11}(t) & A(t) \boxast A_{12}(t) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A_{11}(t) & A_{12}(t) \end{bmatrix} \boxast A_{11}(t) & \begin{bmatrix} A_{11}(t) & A_{12}(t) \end{bmatrix} \boxast A_{12}(t) \end{bmatrix} \\
&= \begin{bmatrix} A_{11}(t) \circledast A_{11}(t) & A_{12}(t) \circledast A_{11}(t) & A_{11}(t) \circledast A_{12}(t) & A_{12}(t) \circledast A_{12}(t) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \sin t \\ 1 \end{bmatrix} \circledast \begin{bmatrix} \sin t \\ 1 \end{bmatrix} & \begin{bmatrix} e^t \\ \cos t \end{bmatrix} \circledast \begin{bmatrix} \sin t \\ 1 \end{bmatrix} & \begin{bmatrix} \sin t \\ 1 \end{bmatrix} \circledast \begin{bmatrix} e^t \\ \cos t \end{bmatrix} & \begin{bmatrix} e^t \\ \cos t \end{bmatrix} \circledast \begin{bmatrix} e^t \\ \cos t \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-t\sin t + \sin t}{2} & \frac{e^t - \cos t - \sin t}{2} & \frac{e^t - \cos t - \sin t}{2} & e^t t \\ 1 - \cos t & -1 + e^t & \frac{t \sin t}{2} & \frac{e^t - \cos t + \sin t}{2} \\ 1 - \cos t & \frac{t \sin t}{2} & -1 + e^t & \frac{e^t - \cos t + \sin t}{2} \\ t & \sin t & \sin t & \frac{t\cos t + \sin t}{2} \end{bmatrix}.
\end{aligned}$$

Corollary 3.16. For any $A \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$ and $B \in \mathcal{I}(\Omega, M_{n,q}(\mathbb{R}))$, we have

$$A^{\boxast r} * B^{\boxast r} = (A * B)^{\boxast r} \tag{3.8}$$

for any $r \in \mathbb{N}$.

Proof. The proof is by induction on r, we have known that from Theorem 3.8 that

$$\begin{aligned}(A(t))^{\circledast 2} * (B(t))^{\circledast 2} &= (A(t) \circledast A(t)) * (B(t) \circledast B(t)) = (A(t) * B(t)) \circledast (A(t) * B(t)) \\ &= (A(t) * B(t))^{\circledast 2}.\end{aligned}$$

i.e., the claim (3.8) is true for $r = 2$. Now, assume that the property 3.8 hold for a positive integer r. By applying the mixed product property (Theorem 3.9), we have

$$\begin{aligned}(A^{\circledast(r+1)}(t)) * (B^{\circledast(+1)}(t)) &= (A^{\circledast r}(t) \circledast A(t)) * (B^{\circledast r}(t) \circledast B(t)) \\ &= (A^{\circledast r}(t) * B^{\circledast r}(t)) \circledast (A(t) * B(t)) \\ &= (A(t) * B(t))^{\circledast r} \circledast (A(t) * B(t)) \\ &= (A(t) * B(t))^{\circledast(r+1)}.\end{aligned}$$

This implies that (3.8) is true for $r + 1$. □

Corollary 3.17. For any $A \in \mathcal{I}(\Omega, M_n(\mathbb{R}))$ and $B \in \mathcal{I}(\Omega, M_p(\mathbb{R}))$, we have

$$(A \circledast B)^{*r} = A^{*r} \circledast B^{*r}. \tag{3.9}$$

for any $r \in \mathbb{N}$.

Proof. The proof is by induction on r, we have known that from Theorem 3.8 that

$$(A(t) \circledast B(t))^{*2} = (A(t) \circledast B(t)) * (A(t) \circledast B(t)) = (A(t) * A(t)) \circledast (B(t) * B(t)) = (A^{*2}(t)) \circledast (B^{*2}(t))$$

i.e., the claim (3.9) is true for $r = 2$. Now, assume that the property 3.9 hold for a positive integer r. By applying the mixed product property (Theorem 3.9), we have

$$\begin{aligned}(A(t) \circledast B(t))^{r+1} &= (A(t) \circledast B(t))^r * (A(t) \circledast B(t)) \\ &= (A^r(t) \circledast B^r(t)) * (A(t) \circledast B(t)) \\ &= (A^r(t) * A(t)) \circledast (B^r(t) \circledast B(t)) \\ &= (A^{r+1}(t)) \circledast (B^{r+1}(t)).\end{aligned}$$

This implies that (3.9) is true for $r + 1$. □

Chapter 4

Vector-block Operator with Tracy-Singh Convolution Products

In this chapter, we discuss how to reduce linear matrix convolutions to simple ones by vectorization.

4.1 Vector-block operator

Definition 4.1. Let $A(t) = [A_{ij}(t)]$ where A_{ij} is of order $m_i \times n_j$ $(i = 1, .., p, j = 1, .., q)$ So $A(t)$ is of order $mp \times nq$, we defined vector-block column operator and vector-block row operator by

$$\text{Vecb}_c\, A(t) = [\text{vec}\, A_{11}(t), .., \text{vec}\, A_{m1}(t), \text{vec}\, A_{12}(t), .., \text{vec}\, A_{m2}(t), .., \text{vec}\, A_{1n}(t), .., \text{vec}\, A_{mn}(t)]^T,$$

and

$$\text{Vecb}_r\, A(t) = [\text{rvec}\, A_{11}(t), .., \text{rvec}\, A_{1n}(t), \text{rvec}\, A_{21}(t), .., \text{rvec}\, A_{2n}(t), .., \text{rvec}\, A_{m1}(t), .., \text{rvec}\, A_{mn}(t)]^T.$$

respectively.

Remark 4.2. The vector-block operator is a bijection.

Example 4.3. Consider

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 7 & 8 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 9 \end{bmatrix}.$$

Then, vector-block column operator and vector-block row operator of A is

$$\text{Vecb}_c\, A(t) = \begin{bmatrix} \text{vec}\, A_{11}(t) \\ \text{vec}\, A_{21}(t) \\ \text{vec}\, A_{12}(t) \\ \text{vec}\, A_{22}(t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 7 \\ 8 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 9 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

and

$$\text{Vecb}_r\, A(t) = \begin{bmatrix} \text{rvec}\, A_{11}(t) \\ \text{rvec}\, A_{12}(t) \\ \text{rvec}\, A_{21}(t) \\ \text{rvec}\, A_{22}(t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 7 \\ 8 \end{bmatrix} \\ \begin{bmatrix} 9 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 3 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}.$$

4.2 Tracy-Singh convolution product and vector-block operator

Theorem 4.4. Let $A(t) = [A(t)]_{i,j}^{m;n}, B(t) = [B(t)]_{k,l}^{p;q}$ and $C(t) = [C(t)]_{u,v}^{m;q}$ continuous matrix-valued functions and $X(t) \in M_{n,p}$. Then

$$\text{Vecb}_r(A(t) * X(t) * B(t)) = (B^T(t) \boxast A(t)) * \text{Vecb}_r\, X(t), \qquad (4.1)$$

$$\text{Vecb}_c(A(t) * X(t) * B(t)) = (A(t) \boxast B^T(t)) * \text{Vecb}_c\, X(t). \qquad (4.2)$$

Proof. To prove (4.1), consider

$$A(t)*(X(t)*B(t)) = \begin{bmatrix} row_1(A(t)) \\ row_2(A(t)) \\ \vdots \\ row_m(A(t)) \end{bmatrix} * X(t)*B(t)$$

$$= \begin{bmatrix} A_1(t) \\ A_2(t) \\ \vdots \\ A_m(t) \end{bmatrix} * X(t)*B(t)$$

$$= \begin{bmatrix} A_1(t)*X(t)*B(t) \\ A_2(t)*X(t)*B(t) \\ \vdots \\ A_m(t)*X(t)*B(t) \end{bmatrix}.$$

The row k-th of $A(t)$ is given by

$$\begin{bmatrix} A_{k1}(t) & A_{k2}(t) & \cdots & A_{kj} \end{bmatrix}.$$

We have

$$\begin{aligned}
(A(t)*X(t)*B(t))_k &= A_k(t)*X(t)*B(t) \\
&= \left(\begin{bmatrix} A_{k1}(t) & A_{k2}(t) & \cdots & A_{kj}(t) \end{bmatrix} * \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_j(t) \end{bmatrix} \right) * B(t) \\
&= (A_{k1}(t)*X_1(t) + A_{k2}(t)*X_2(t) + \cdot + A_{kj}(t)*X_j(t))*B(t) \\
&= \left(\sum_{j=1}^{n} A_{kj}(t)*X_j(t) \right) * B(t) \\
&= \sum_{j=1}^{n} A_{kj}(t)*X_j(t)*B(t).
\end{aligned}$$

Hence, $(A(t)*X(t)*B(t))_k = \sum_{j=1}^{n} A_{kj}(t)*X_j(t)*B(t)$.

Consider

$$\begin{aligned}
\text{vec}\,(A(t)*X(t)*B(t))_k &= \text{vec}\left(\sum_{j=1}^{n} A_{kj}(t)*X_j(t)*B(t) \right) \\
&= \sum_{j=1}^{n} \text{vec}\,(A_{kj}(t)*X_j(t)*B(t)) \\
&= \sum_{j=1}^{n} \left(B^T(t) \circledast A_{kj}(t)\right) * \text{vec}\, X_j(t)
\end{aligned}$$

$$= \left(B^T(t) \circledast A_{k1}(t)\right) * \operatorname{vec} X_1(t) + \left(B^T(t) \circledast A_{k2}(t)\right) * \operatorname{vec} X_2(t) + \cdots + \left(B^T(t) \circledast A_{kj}(t)\right) * \operatorname{vec} X_j(t)$$
$$= \left[B^T(t) \circledast A_{k1}(t) \quad B^T(t) \circledast A_{k2}(t) \quad \cdots \quad B^T(t) \circledast A_{kj}(t)\right] * \operatorname{Vecb}_r X(t)$$

we obtain

$$\operatorname{vec}(A(t) * X(t) * B(t))_k = \left[B^T(t) \circledast A_{k1}(t) \quad B^T(t) \circledast A_{k2}(t) \quad \cdots \quad B^T(t) \circledast A_{kj}(t)\right] * \operatorname{Vecb}_r X(t)$$

That is

$$\begin{aligned}
\operatorname{Vecb}_r(A(t) * X(t) * B(t)) &= \begin{bmatrix} \operatorname{vec}(A(t) * X(t) * B(t))_1 \\ \operatorname{vec}(A(t) * X(t) * B(t))_2 \\ \vdots \\ \operatorname{vec}(A(t) * X(t) * B(t))_k \end{bmatrix} \\
&= \begin{bmatrix} \left[B^T(t) \circledast A_{11}(t) \quad B^T(t) \circledast A_{12}(t) \quad \cdots \quad B^T(t) \circledast A_{1j}(t)\right] * \operatorname{Vecb}_r X(t) \\ \left[B^T(t) \circledast A_{21}(t) \quad B^T(t) \circledast A_{22}(t) \quad \cdots \quad B^T(t) \circledast A_{2j}(t)\right] * \operatorname{Vecb}_r X(t) \\ \vdots \\ \left[B^T(t) \circledast A_{k1}(t) \quad B^T(t) \circledast A_{k2}(t) \quad \cdots \quad B^T(t) \circledast A_{kj}(t)\right] * \operatorname{Vecb}_r X(t) \end{bmatrix} \\
&= \begin{bmatrix} B^T(t) \circledast A_{11}(t) & B^T(t) \circledast A_{12}(t) & \cdots & B^T(t) \circledast A_{1j}(t) \\ B^T(t) \circledast A_{21}(t) & B^T(t) \circledast A_{22}(t) & \cdots & B^T(t) \circledast A_{2j}(t) \\ \vdots & & & \\ B^T(t) \circledast A_{k1}(t) & B^T(t) \circledast A_{k2}(t) & \cdots & B^T(t) \circledast A_{kj}(t) \end{bmatrix} * \operatorname{Vecb}_r X(t) \\
&= (B^T(t) \boxast A(t)) \operatorname{Vecb}_r X(t).
\end{aligned}$$

Hence, $\operatorname{Vecb}_r(A(t) * X(t) * B(t)) = (B^T(t) \boxast A(t)) \operatorname{Vecb}_r X(t)$. □

Proof. Prove of (4.2), we have by using the fact that $\operatorname{Vecb}_c A(t) = \operatorname{Vecb}_r A^T(t)$ and (4.1)

$$\begin{aligned}
\operatorname{Vecb}_c(A(t) * X(t) * B(t)) &= \operatorname{Vecb}_r(A(t) * X(t) * B(t))^T \\
&= \operatorname{Vecb}_r(B^T(t) * X^T(t) * A^T(t)) \\
&= (A(t) \boxast B^T(t)) * \operatorname{Vecb}_r X^T(t) \\
&= (A(t) \boxast B^T(t)) * \operatorname{Vecb}_c X(t).
\end{aligned}$$

□

Theorem 4.5. Let A, B, C and X be compatibly partitioned continuous matrix-valued functions on Ω. Then The following statements are equivalent:

(i) $A(t) * X(t) * B(t) = C(t)$,

(ii) $(B^T(t) \boxast A(t)) * \operatorname{Vecb}_r X(t) = \operatorname{Vecb}_r C(t)$,

(iii) $(A(t) \boxast B^T(t)) * \operatorname{Vecb}_c X(t) = \operatorname{Vecb}_c C(t)$.

More generally, the following statements are equivalent:

(iv) $\sum_{i=1}^{n}(A_i(t) * X(t) * B_i(t)) = C(t)$,

(v) $(\sum_{i=1}^{n}(B_i^T(t) \boxtimes A_i(t)) * \text{Vecb}_r X(t) = \text{Vecb}_r C(t)$,

(vi) $(\sum_{i=1}^{n}(A_i(t) \boxtimes B_i^T(t)) * \text{Vecb}_c X(t) = \text{Vecb}_c C(t)$.

Proof. (iv) ⇔ (v). The proof is by induction on n. We have known that from Theorem 4.4 that

$$\begin{aligned}
\sum_{i=1}^{2} A_i(t) * X(t) * B_i(t) &= C(t)\\
\text{Vecb}_r\left(\sum_{i=1}^{2} A_i(t) * X(t) * B_i(t)\right) &= \text{Vecb}_r C(t)\\
\sum_{i=1}^{2} \text{Vecb}_r(A_i(t) * X(t) * B_i(t)) &= \text{Vecb}_r C(t)\\
\sum_{i=1}^{2}(B_i^T(t) \boxtimes A_i(t)) * \text{Vecb}_r X(t) &= \text{Vecb}_r C(t)\\
\left(\sum_{i=1}^{2} B_i^T(t) \boxtimes A_i(t)\right) * \text{Vecb}_r X(t) &= \text{Vecb}_r C(t)
\end{aligned}$$

i.e., the claim (v) is true for $n = 2$. Now, assume that (v) hold for a positive integer n. By applying Theorem 4.4, we have

$$\begin{aligned}
\sum_{i=1}^{n+1} A_i(t) * X(t) * B_i(t) &= C(t)\\
\sum_{i=1}^{n} A_i(t) * X(t) * B(t) + A_{n+1}(t) * X(t) * B_{n+1}(t) &= C(t)\\
\text{Vecb}_r\left(\sum_{i=1}^{n} A_i(t) * X(t) * B(t)\right) + \text{Vecb}_r(A_{n+1}(t) * X(t) * B_{n+1}(t)) &= \text{Vecb}_r C(t)\\
\left(\sum_{i=1}^{n} B_i^T(t) \boxtimes A_i(t)\right) * \text{Vecb}_r X(t) + \left(B_{n+1}^T(t) \boxtimes A_{n+1}(t)\right) * \text{Vecb}_r X(t) &= \text{Vecb}_r C(t)\\
\left(\sum_{i=1}^{n+1} B_i^T(t) \boxtimes A_i(t)\right) * \text{Vecb}_r X(t) &= \text{Vecb}_r C(t)
\end{aligned}$$

This implies that (v) is true for $n + 1$. □

Proof. (iv) ⇔ (vi). The proof is by induction on n. We have known that from Theorem 4.4 that

$$\begin{aligned}
\sum_{i=1}^{2} A_i(t) * X(t) * B_i(t) &= C(t) \\
\text{Vecb}_c\left(\sum_{i=1}^{2} A_i(t) * X(t) * B_i(t)\right) &= \text{Vecb}_c\, C(t) \\
\sum_{i=1}^{2} \text{Vecb}_c(A_i(t) * X(t) * B_i(t)) &= \text{Vecb}_c\, C(t) \\
\sum_{i=1}^{2} (A_i(t) \circledast B_i^T(t)) * \text{Vecb}_c\, X(t) &= \text{Vecb}_c\, C(t) \\
\left(\sum_{i=1}^{2} A(t) \circledast B_i^T(t)\right) * \text{Vecb}_c\, X(t) &= \text{Vecb}_c\, C(t)
\end{aligned}$$

i.e., the claim (vi) is true for $n = 2$. Now, assume that (vi) hold for a positive integer n. By applying Theorem 4.4, we have

$$\begin{aligned}
\sum_{i=1}^{n+1} A_i(t) * X(t) * B_i(t) &= C(t) \\
\sum_{i=1}^{n} A_i(t) * X(t) * B(t) + A_{n+1}(t) * X(t) * B_{n+1}(t) &= C(t) \\
\text{Vecb}_c\left(\sum_{i=1}^{n} A_i(t) * X(t) * B(t)\right) + \text{Vecb}_c(A_{n+1}(t) * X(t) * B_{n+1}(t)) &= \text{Vecb}_c\, C(t) \\
\left(\sum_{i=1}^{n} A_i(t) \circledast B_i^T(t)\right) * \text{Vecb}_c\, X(t) + (A_{n+1}(t) \circledast B_{n+1}^T(t)) * \text{Vecb}_c\, X(t) &= \text{Vecb}_c\, C(t) \\
\left(\sum_{i=1}^{n+1} A_i(t) \circledast B_i^T(t)\right) * \text{Vecb}_c\, X(t) &= \text{Vecb}_c\, C(t)
\end{aligned}$$

This implies that (vi) is true for $n + 1$. □

Theorem 4.6. Let $A(t) = [A(t)]_{i,j}^{m,n}$ and $B(t) = [B(t)]_{r,s}^{n,q}$ be given. Then

$$\text{Vecb}_c(A(t) * B(t)) = \left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c\, B(t), \tag{4.3}$$

Proof. (4.3) Let $A(t) = [A(t)]_{i,j}^{m,n}$ and $B(t) = [B(t)]_{r,s}^{n,q}$. By Definition 4.1, we have

$$\begin{aligned}
\text{Vecb}_c(A(t) * B(t)) &= \begin{bmatrix} (A(t) * B(t))_1 \\ \vdots \\ (A(t) * B(t))_q \end{bmatrix} \\
&= \begin{bmatrix} A(t) & 0 & \cdots & 0 \\ 0 & A(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(t) \end{bmatrix} * \begin{bmatrix} (B(t))_1 \\ \vdots \\ (B(t))_q \end{bmatrix} \\
&= \left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c\, B(t).
\end{aligned}$$

□

Theorem 4.7. Let $A(t) = [A(t)]_{i,j}^{m;n}$ and $X(t) = [X(t)]_{r,s}^{n;q}$. Then The following statements are equivalent:

$$A(t) * X(t) = C(t), \tag{4.4}$$

$$\left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c X(t) = \text{Vecb}_c C(t). \tag{4.5}$$

Proof. (4.4) ⇔ (4.5). Let Q_k denote the k-th block column of matrix Q. we have

$$\begin{aligned}
\text{Vecb}_c(A(t) * X(t)) &= \begin{bmatrix} (A(t) * X(t))_1 \\ \vdots \\ (A(t) * X(t))_q \end{bmatrix} = \begin{bmatrix} A(t) * X_1(t) \\ \vdots \\ A(t) * X_q(t) \end{bmatrix} \\
&= \begin{bmatrix} A(t) & 0 & \cdots & 0 \\ 0 & A(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A(t) \end{bmatrix} * \begin{bmatrix} X_1(t) \\ \vdots \\ X_q(t) \end{bmatrix} \\
&= (A(t) \oplus A(t) \oplus \cdots \oplus A(t)) * \text{Vecb}_c X(t) \\
&= \left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c X(t).
\end{aligned}$$

□

Theorem 4.8. Let $A(t) = [A(t)]_{i,j}^{n;q}$ and $X(t) = [X(t)]_{r,s}^{m;n}$. Then the following statements are equivalent:

$$X(t) * A(t) = C(t), \tag{4.6}$$

$$\left(\bigoplus_{i=1}^{q} A^T(t)\right) * \text{Vecb}_r X(t) = \text{Vecb}_r C(t). \tag{4.7}$$

Proof. (4.6) ⇔ (4.7) Let Q_k denote the k-th block column of matrix Q, and we know that $\text{Vecb}_r\, A(t) = \text{Vecb}_c\, A(t)^T$. Then

$$\begin{aligned}
\text{Vecb}_r(X(t) * A(t)) &= \text{Vecb}_c(X(t) * A(t))^T \\
&= \text{Vecb}_c\, A^T(t) * X^T(t) \\
&= \begin{bmatrix} (A^T(t)^* X^T(t))_1 \\ \vdots \\ (A^T(t) * X^T(t))_q \end{bmatrix} = \begin{bmatrix} A^T(t) * X_1^T(t) \\ \vdots \\ A^T(t) * X_q^T(t) \end{bmatrix} \\
&= \begin{bmatrix} A^T(t) & 0 & \cdots & 0 \\ 0 & A^T(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A^T(t) \end{bmatrix} * \begin{bmatrix} X_1^T(t) \\ \vdots \\ X_q^T(t) \end{bmatrix} \\
&= (A^T(t) \oplus A^T(t) \oplus \cdots \oplus A^T(t)) * \text{Vecb}_c\, X^T(t) \\
&= \left(\bigoplus_{i=1}^{q} A^T(t) \right) * \text{Vecb}_c\, X^T(t) \\
&= \left(\bigoplus_{i=1}^{q} A^T(t) \right) * \text{Vecb}_r\, X(t).
\end{aligned}$$

□

Theorem 4.9. Let $A(t) = [A(t)]_{i,j}^{m,n}$ and $X(t) = [X(t)]_{r,s}^{n,q}$. Then The following statements are equivalent:

$$A(t) * X(t) = C(t), \tag{4.8}$$

$$\left(\bigoplus_{i=1}^{q} A(t) \right) * \text{Vecb}_r\, X^T(t) = \text{Vecb}_r\, C^T(t). \tag{4.9}$$

Proof. (4.8) ⇔ (4.9). Taking the transpose of both side of (4.8), we obtain

$$\begin{aligned}
(A(t) * X(t))^T &= C^T(t) \\
X^T(t) * A^T(t) &= C^T(t).
\end{aligned}$$

By Theorem 4.8, we have

$$\left(\bigoplus_{i=1}^{q} A(t) \right) * \text{Vecb}_r\, X^T(t) = \text{Vecb}_r\, C^T(t).$$

Hence, the equations (4.8) and (4.9) are equivalent. □

Theorem 4.10. Let $A(t) = [A(t)]_{i,j}^{n,q}$ and $X(t) = [X(t)]_{r,s}^{m,n}$. Then The following statements are equivalent:

$$X(t) * A(t) = C(t), \tag{4.10}$$

$$\left(\bigoplus_{i=1}^{q} A^T(t) \right) * \text{Vecb}_c\, X^T(t) = \text{Vecb}_c\, C^T(t). \tag{4.11}$$

Proof. (4.10) ⇔ (4.11). Taking the transpose of both side of (4.10), we obtain

$$\begin{aligned}
(X(t) * A(t))^T &= C^T(t) \\
A^T(t) * X^T(t) &= C^T(t).
\end{aligned}$$

By Theorem 4.7. we have

$$\left(\bigoplus_{i=1}^{q} A^T(t)\right) * \text{Vecb}_c X^T(t) = \text{Vecb}_c C^T(t).$$

Hence, the equations (4.10) and (4.11) are equivalent. □

Corollary 4.11. Let $A(t) = [A(t)]_{i,j}^{m,n}$ and $X(t) = [X(t)]_{r,s}^{n,q}$. Then The following statements are equivalent:

$$A(t) * X(t) + X^T(t) * A^T(t) = C(t), \tag{4.12}$$

$$\left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c X(t) = \frac{1}{2}\text{Vecb}_c C(t), \tag{4.13}$$

$$\left(\bigoplus_{i=1}^{q} A^T(t)\right) * \text{Vecb}_r X^T(t) = \frac{1}{2}\text{Vecb}_r C(t). \tag{4.14}$$

Proof. (4.12) ⇔ (4.13). We have by using Theorem 4.7 that

$$\begin{aligned}
\text{Vecb}_c(A(t) * X(t) + X^T(t) * A^T(t)) &= \text{Vecb}_c C(t)\\
\text{Vecb}_c(A(t) * X(t) + \text{Vecb}_c(X^T(t) * A^T(t)) &= \text{Vecb}_c C(t)\\
\left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c X(t) + \left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c X(t) &= \text{Vecb}_c C(t)\\
2\left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_c X(t) &= \text{Vecb}_c C(t).
\end{aligned}$$

(4.12) ⇔ (4.14). We have by using Theorem 4.8 that

$$\begin{aligned}
\text{Vecb}_r(A(t) * X(t) + X^T(t) * A^T(t)) &= \text{Vecb}_r C(t)\\
\text{Vecb}_r(A(t) * X(t) + \text{Vecb}_r(X^T(t) * A^T(t)) &= \text{Vecb}_r C(t)\\
\left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_r X^T(t) + \left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_r X^T(t) &= \text{Vecb}_r C(t)\\
2\left(\bigoplus_{i=1}^{q} A(t)\right) * \text{Vecb}_r X^T(t) &= \text{Vecb}_r C(t).
\end{aligned}$$

□

Proposition 4.12. Let $A(t) \in M_{m,p}, B(t) \in M_{n,q}, X(t) \in \mathbb{R}^{pq}$, and $v(t) \in \mathbb{R}^{mn}$.

$$(B(t) \boxtimes A(t)) * x(t) = v(t) \quad \text{if and only if} \quad (A(t) \boxtimes B(t)) * x_T(t) = v_T(t).$$

where

$$x(t) = \text{Vecb}_c X(t), v(t) = \text{Vecb}_c V(t), x_T(t) = \text{Vecb}_c X^T(t) \quad \text{and} \quad v_T(t) = \text{Vecb}_c V^T(t).$$

Proof. We have by using Theorem 4.5,

$$(B(t) \boxtimes A(t)) * \text{Vecb}_c X(t) = \text{Vecb}_c V(t) \quad \text{if and only if} \quad B(t) * X(t) * A^T(t) = V(t).$$

Taking the transpose of both sides, we have

$$A(t) * X^T(t) * B^T(t) = V^T(t).$$

The proof is completed by applying Theorem 4.5. Hence

$$(A(t) \boxtimes B^{(}t)) * x_T(t) \;=\; v_T(t) \quad \text{if and only if} \quad (A(t) \boxtimes B(t)) * x(t) \;=\; v(t).$$

□

Chapter 5

Conclusions and Suggestions

5.1 Conclusions

The Tracy-Singh convolution products of integrable matrix-valued functions have the following properties:

- Right distribution over addition,
- Left distribution over addition,
- Associativity,
- Compatibility with scalar multiplication,
- Compatibility with convolution of real-valued functions,
- Compatibility with transposition,
- Compatibility with direct sum.

Tracy-Singh convolution products of integrable matrix-valued functions have relations with (anti)commutator convolutions as follows:

- $[A \boxast B, C \boxast D]_* = 0,$
- $[A \boxast B, C \boxast D]_* = \frac{1}{2}([A, C]_* \boxast [B, D]_{*,+} + [A, C]_{*,+} \boxast [B, D]_*).$

Tracy-Singh convolution powers have the following properties:

- $A^{\boxast r} * B^{\boxast r} = (A * B)^{\boxast r},$
- $(A \boxast B)^{*r} = A^{*r} \boxast B^{*r}.$

Tracy-Singh convolution products and vector-block operators have the following properties:

- $\mathrm{Vecb}_r(A(t) * X(t) * B(t)) = (B^T(t) \boxast A(t)) * \mathrm{Vecb}_r\, X(t),$
- $\mathrm{Vecb}_c(A(t) * X(t) * B(t)) = (A(t) \boxast B^T(t)) * \mathrm{Vecb}_c\, X(t).$

The vector-block operators can be apply to reduce certain linear matrix convolution equations to simple vector-matrix convolution equations.

5.2 Suggestions

Instead of convolutions of Riemann integrable functions, we may consider convolutions of another integrable functions, such as Lebesgue or Henstock integrable functions. We may investigate the Tracy-Singh convolution product for Lebesgue/ Henstock integrable matrix-valued functions.

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Appendix

Appendix A

The research paper

Tracy-Singh Convolution Product and Linear Matrix Convolution Equations

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Abstract

In this paper, we generalize the notion of Kronecker convolution product to the Tracy-Singh convolution product for integrable matrix-valued functions. We investigate its algebraic properties involving certain algebraic operations. Moreover, we apply the Tracy-Singh convolution product for solving linear matrix convolution equations.

Mathematics Subject Classification: 15A24, 15A69, 44A35
Keywords: convolution, Tracy-Singh convolution product, linear matrix convolution equation, vector operator

1 Introduction

Recall that the convolution of two integrable real-valued functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t \geqslant 0.$$

Convolution shows up in many areas of mathematics, such as, differential equations, probability and Markov process. It was remarkable that convolution has applications in various scientific fields that include control and system theory, image and signal processing, mathematical physics, etc.

On the other hand in matrix theory, there are many kinds of matrix products which are of interest in both theory and applications. Recall that the Kronecker product of two real matrices $A = [a_{ij}]$ and B is defined by

$$A \otimes B = [a_{ij}B]_{ij},$$

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that is, each (i, j)-th block of $A \otimes B$ is given by $a_{ij}B$. The Tracy-Singh product, introduced by [11], is defined for two partitioned matrices $A = [A_{ij}]_{ij}$ and $B = [B_{kl}]_{kl}$ by

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij}.$$

Note that if A has one block, their Tracy-Singh convolution product is reduced to the so-called block Kronecker product [5]. When both factors have only one block, $A \boxtimes B$ reduces to $A \otimes B$. See more information about Tracy-Singh product and related topics in [7, 8, 10, 12].

The notion of (usual) convolution product arises naturally in Markov processes. Sumita [9] applied the matrix Laguerre transform to calculate convolution products and derived a matrix renewal function. To make a dependability analysis for semi-Markov systems with finite state space, Limnios [6] established certain properties of convolution products for semi-Markov kernel matrices. The convolution product serves as a tool for analyzing semi-Markov chains, e.g., [2]. In the last decade, Kilicman and Al Zhour developed further kinds of matrix convolution products for matrix-valued functions. These include Kronecker convolution product [3], Hadamard convolution product [4] and box convolution product [12]. In [1], systems of linear matrix convolution equations can be reduced to a simple matrix convolution equation by using Kronecker convolution product and then such systems can be solved by using Laplace transform. Itrative procedures for linear matrix convolution equations are proposed in [1, 12].

In this paper, we extend the notion of Kronecker convolution product to the Tracy-Singh convolution product of integrable matrix-valued functions. This kind of convolution product is associative and compatible with certain algebraic operations, namely, the addition, scalar multiplication, scalar convolution product, usual convolution product, transposition and direct sum. Moreover, we apply the Tracy-Singh convolution product for solving linear matrix convolution equations.

The paper is organized as follows. In the next section, we setup notations and provide fundamental results about convolutions and matrix convolution products. In section 3, we introduce the Tracy-Singh convolution product for matrix-valued functions and investigate its fundamental properties involving algebraic operations. Section 4, we investigate relationship between certain kinds of vectorization, Tracy-Singh and usual convolution product. In the final Section, we discusses how to reduce linear matrix convolutions to simple ones by using vectorization.

2 Preliminaries on Convolutions and Matrix Convolution Products

In what follows, let $\Omega = [0, \infty)$ or $\Omega = [0, b]$ for some $b > 0$. Denote by $M_{m,n}(\mathbb{R})$ the set of m-by-n real matrices. When $m = n$, we abbreviate $M_{m,n}(\mathbb{R})$ to $M_n(\mathbb{R})$. Recall that a matrix-valued function $A : \Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ is said to be integrable (continuous) if the real-valued function a_{ij} is integrable (continuous, respectively) for each $i = 1, \dots, m$ and $j = 1, \dots, n$. For any set E, denote $\mathcal{I}(\Omega, E)$ and $\mathcal{C}(\Omega, E)$ the set of integrable and continuous matrix-valued functions from Ω to E, respectively.

For any integrable functions $f, g, h : \Omega \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, the following properties are straightforward to verify:

$$f * (g + h) = f * g + f * h, \tag{2.1}$$

$$(f + g) * h = f * h + g * h, \tag{2.2}$$

$$\alpha(f * g) = (\alpha f) * g = f * (\alpha g). \tag{2.3}$$

If, in addition, f, g, h are continuous, then by changing variables we have

$$f * g = g * f, \tag{2.4}$$

$$(f * g) * h = f * (g * h). \tag{2.5}$$

Definition 1. The convolution product between an integrable function $f : \Omega \to \mathbb{R}$ and an integrable matrix-valued function $A : \Omega \to M_{m,n}(\mathbb{R})$ is defined to be the function

$$f * A : \Omega \to M_{m,n}(\mathbb{R}), (f * A)(t) = [f(t) * a_{ij}(t)].$$

For convenience, we may write $f(t) * A(t)$ instead of $(f * A)(t)$.

Definition 2. The (usual) convolution product of two integrable matrix-valued functions $A : \Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ and $B : \Omega \to M_{n,p}(\mathbb{R})$, $B(t) = [b_{ij}(t)]$ is defined to be the matrix-valued function $A * B : \Omega \to M_{m,p}(\mathbb{R})$ such that

$$(A * B)(t) = \left[\sum_{k=1}^{n} a_{ik}(t) * b_{kj}(t)\right] \in M_{m,p}(\mathbb{R}) \quad \text{for each} \quad t \in \Omega.$$

For convenience, we may write $A(t) * B(t)$ instead of $(A * B)(t)$.

Definition 3. The Kronecker convolution product of two integrable matrix-valued functions $A : \Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ and $B : \Omega \to M_{p,q}(\mathbb{R})$ is defined to be the function

$A \circledast B : \Omega \to M_{mp,nq}(\mathbb{R})$ such that

$$(A \circledast B)(t) = [a_{ij}(t) * B(t)]_{ij} \quad \text{for each} \quad t \in \Omega.$$

That is, each (i,j)-th block of $(A \circledast B)(t)$ is given by $a_{ij}(t) * B(t)$. For convenience, we may write $A(t) \circledast B(t)$ for $(A \circledast B)(t)$.

Lemma 1. [3] *Let A, B and C be compatibly integrable matrix-valued functions on Ω. Then for any scalars α, β we have*

(i) $A \circledast (B + C) = (A \circledast B) + (A \circledast C)$,

(ii) $(A + B) \circledast C = (A \circledast C) + (B \circledast C)$,

(iii) $(\alpha A) \circledast B = \alpha(A \circledast B) = A \circledast (\alpha B)$,

(iv) $(A \circledast B) * (C \circledast D) = (A * C) \circledast (B * D)$.

Proof. Apply properties (2.1), (2.2) and (2.3) together with matrix algebra. □

Lemma 2. *For any compatibly continuous matrix-valued functions A, B, C on Ω and for any integrable function $\phi : \Omega \to \mathbb{R}$, we have*

$$(A \circledast B) \circledast C = A \circledast (B \circledast C), \tag{2.6}$$

$$(\phi * A) \circledast B = \phi * (A \circledast B) = A \circledast (\phi * B), \tag{2.7}$$

$$(A \circledast B)^T = A^T \circledast B^T. \tag{2.8}$$

Proof. Apply properties (2.3), (2.4) and (2.5) together with matrix algebra. □

3 Tracy-Singh Convolution Product and Algebraic Operations

In this section, we introduce the notion of Tracy-Singh convolution product for matrix-valued functions and investigate its algebraic properties.

Definition 4. Let $A \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$ and $B \in \mathcal{I}(\Omega, M_{p,q}(\mathbb{R}))$. We partition $A(t) = [A_{ij}(t)]_{ij}$ and $B(t) = [B_{kl}(t)]_{kl}$ for all $t \in \Omega$. Here each $A_{ij}(t)$ is of order $m_i \times n_j$ and $B_{kl}(t)$ is of order $p_k \times q_l$, where $\sum m_i = m$, $\sum n_j = n$, $\sum p_k = p$ and $\sum q_l = q$. We define the Tracy-Singh convolution product $A \boxast B : \Omega \to M_{mp,nq}(\mathbb{R})$ by

$$(A \boxast B)(t) = [[A_{ij}(t) \circledast B_{kl}(t)]_{kl}]_{ij} \quad \text{for each} \quad t \in \Omega. \tag{3.1}$$

For convenience, we write $A(t) \boxtimes B(t)$ for $(A \boxtimes B)(t)$.

If both A and B have only one block, then $A \boxtimes B = A \circledast B$. When the first factor consists of one block, their Tracy-Singh convolution product is reduced to the block Kronecker convolution product.

Theorem 1. *Let A, B, C and D be compatibly partitioned integrable matrix-valued functions on Ω. Then,*

(i) $A \boxtimes (B + C) = (A \boxtimes B) + (A \boxtimes C)$,

(ii) $(A + B) \boxtimes C = (A \boxtimes C) + (A \boxtimes C)$,

(iii) $(\alpha A) \boxtimes B = A \boxtimes (\alpha B) = \alpha(A \boxtimes B)$ *for any* $\alpha \in \mathbb{R}$.

Moreover, if A and B are continuous, then

(iv) $(A \boxtimes B)^T = A^T \boxtimes B^T$,

(v) $\phi * (A \boxtimes B) = (\phi * A) \boxtimes B = A \boxtimes (\phi * B)$ *for any integrable function* $\phi : \Omega \to \mathbb{R}$.

Proof. For each $t \in \Omega$, we have by Lemma 1 that

$$\begin{aligned}
(A(t) \boxtimes B(t))^T &= [[A_{ij}(t) \circledast B_{kl}(t)]_{kl}]^T_{ij} \\
&= [[A_{ji}(t) \circledast B_{kl}(t)]^T_{kl}]_{ij} \\
&= [[A^T_{ji}(t) \circledast B^T_{lk}(t)]_{kl}]_{ij} \\
&= [[[A_{ij}(t)]^T \circledast [B_{kl}(t)]^T]_{kl}]_{ij} \\
&= A^T(t) \boxtimes B^T(t).
\end{aligned}$$

Properties (i)-(iii) can be also proved by using Lemma 1. Property (v) can be proved by using (2.8) as follows:

$$\begin{aligned}
(\phi * (A \boxtimes B))(t) &= [[\phi * (A_{ij}(t) \circledast B_{kl}(t))]_{kl}]_{ij} \\
&= [[(\phi * A_{ij}(t)) \circledast B_{kl}(t)]_{kl}]_{ij} = ((\phi * A) \boxtimes B)(t) \\
&= [[A_{ij}(t) \circledast (\phi * B_{kl}(t))]_{kl}]_{ij} = (A \boxtimes (\phi * B))(t).
\end{aligned}$$

□

Lemma 3. *Let $A \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$ and $B \in \mathcal{I}(\Omega, M_{p,q}(\mathbb{R}))$. Then*

$$A(t) \boxtimes B(t) \;=\; [A_{ij}(t) \boxtimes B(t)]_{ij} \;=\; \begin{bmatrix} A_{11}(t) \boxtimes B(t) & \cdots & A_{1n}(t) \boxtimes B(t) \\ \vdots & \ddots & \vdots \\ A_{m1}(t) \boxtimes B(t) & \cdots & A_{mn}(t) \boxtimes B(t) \end{bmatrix}.$$

That is, each (i,j)-th block of $A(t) \boxtimes B(t)$ is given by $A_{ij}(t) \boxtimes B(t)$, regardless of how to partition $A(t)$.

Proof.

$$A(t) \boxtimes B(t)$$

$$= \begin{bmatrix} \begin{bmatrix} A_{11}(t) \circledast B_{11}(t) & \cdots & A_{11}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{11}(t) \circledast B_{p1}(t) & \cdots & A_{11}(t) \circledast B_{pq}(t) \end{bmatrix} & \cdots & \begin{bmatrix} A_{1n}(t) \circledast B_{11}(t) & \cdots & A_{1n}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{1n}(t) \circledast B_{p1}(t) & \cdots & A_{1n}(t) \circledast B_{pq}(t) \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} A_{m1}(t) \circledast B_{11}(t) & \cdots & A_{m1}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{m1}(t) \circledast B_{p1}(t) & \cdots & A_{m1}(t) \circledast B_{pq}(t) \end{bmatrix} & \cdots & \begin{bmatrix} A_{mn}(t) \circledast B_{11}(t) & \cdots & A_{mn}(t) \circledast B_{1q}(t) \\ \vdots & \ddots & \vdots \\ A_{mn}(t) \circledast B_{p1}(t) & \cdots & A_{mn}(t) \circledast B_{pq}(t) \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}(t) \boxtimes \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} & \cdots & A_{1n}(t) \boxtimes \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} \\ \vdots & \ddots & \vdots \\ A_{m1}(t) \boxtimes \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} & \cdots & A_{mn}(t) \boxtimes \begin{bmatrix} B_{11}(t) & \cdots & B_{1q}(t) \\ \vdots & \ddots & \vdots \\ B_{p1}(t) & \cdots & B_{pq}(t) \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}(t) \boxtimes B(t) & \cdots & A_{1n}(t) \boxtimes B(t) \\ \vdots & \ddots & \vdots \\ A_{m1}(t) \boxtimes B(t) & \cdots & A_{mn}(t) \boxtimes B(t) \end{bmatrix}. \qquad \square$$

Theorem 2. *Let $A \in \mathcal{C}(\Omega, M_{m,n}(\mathbb{R}))$, $B \in \mathcal{C}(\Omega, M_{p,q}(\mathbb{R}))$ and $C \in \mathcal{C}(\Omega, M_{v,w}(\mathbb{R}))$. Then*

$$(A \boxtimes B) \boxtimes C \;=\; A \boxtimes (B \boxtimes C). \tag{3.2}$$

Proof. We have by using Lemmas 2 and 3 that for each $t \in \Omega$,

$$\begin{aligned}(A(t) \boxast B(t)) \boxast C(t) &= [[A_{ij}(t) \circledast B_{kl}(t)]_{kl}]_{ij} \boxast C(t) \\ &= [[[(A_{ij}(t) \circledast B_{kl}(t)) \circledast C_{tu}(t)]_{tu}]_{kl}]_{ij} \\ &= [[[A_{ij}(t) \circledast (B_{kl}(t) \circledast C_{tu}(t))]_{tu}]_{kl}]_{ij} \\ &= A(t) \boxast [[B_{kl}(t) \circledast C_{tu}(t)]_{tu}]_{kl} \\ &= A(t) \boxast (B(t) \boxast C(t)).\end{aligned}$$

Hence, we obtain the associativity property (3.2). □

Recall that the direct sum of matrix-valued functions $A_i : \Omega \to M_{m_i,n_i}(\mathbb{R})$, $i = 1, \ldots, k$, is defined by

$$\bigoplus_{i=1}^{k} A_i \,:\, \Omega \to M_{m,n}(\mathbb{R}), \quad \bigoplus_{i=1}^{k} A_i(t) = \begin{bmatrix} A_1(t) & 0 & \cdots & 0 \\ 0 & A_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k(t) \end{bmatrix}.$$

Here, $m = \sum m_i$ and $n = \sum n_i$.

Proposition 1. *For each $i = 1, \ldots, k$, let A_i and B be integrable matrix-valued functions on Ω. Then*

$$\left(\bigoplus_{i=1}^{k} A_i\right) \boxast B = \bigoplus_{i=1}^{k} (A_i \boxast B).$$

Proof. By induction, we may assume that $k = 2$. By Lemma 3, we have that for each $t \in \Omega$,

$$\begin{aligned}(A_1(t) \oplus A_2(t)) \boxast B(t) &= \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} \boxast B(t) \\ &= \begin{bmatrix} A_1(t) \boxast B(t) & 0 \\ 0 & A_2(t) \boxast B(t) \end{bmatrix} \\ &= (A_1(t) \boxast B(t)) \oplus (A_2(t) \boxast B(t)).\end{aligned}$$

□

The next property, called the mixed product property, shows that the Tracy-Singh convolution product is compatible with the usual convolution product.

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Theorem 3. *Let* $A \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$, $B \in \mathcal{I}(\Omega, M_{n,s}(\mathbb{R}))$, $C \in \mathcal{I}(\Omega, M_{p,q}(\mathbb{R}))$ *and* $D \in \mathcal{I}(\Omega, M_{q,r}(\mathbb{R}))$. *Then*

$$(A * B) \circledast (C * D) = (A \circledast C) * (B \circledast D). \quad (3.3)$$

Proof. By using Lemma 1 and block-matrix algebra, we have that for each $t \in \Omega$,

$$\begin{aligned}
&(A(t) * B(t)) \circledast (C(t) * D(t)) \\
&= \left[\sum_{u=1}^{n} A_{iu}(t) * B_{uj}(t)\right]_{ij} \circledast \left[\sum_{t=1}^{q} C_{kt}(t) * D_{tl}(t)\right]_{kl} \\
&= \left[\left[\left(\sum_{u=1}^{n} A_{iu}(t) * B_{uj}(t)\right) \circledast \left(\sum_{t=1}^{q} C_{kt}(t) * D_{t}(t)\right)\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{u=1}^{n}\left(\sum_{t=1}^{q}(A_{iu}(t) * B_{uj}(t)) \circledast (C_{kt}(t) * D_{tl}(t))\right)\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{u=1}^{n}\sum_{t=1}^{q}(A_{iu}(t) * B_{uj}(t)) \circledast (C_{kt}(t) * D_{tl}(t))\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{u=1}^{n}\sum_{t=1}^{q}(A_{iu}(t) \circledast C_{kt}(t)) * (B_{uj}(t) \circledast D_{tl}(t))\right]_{kl}\right]_{ij} \\
&= [[A_{ij}(t) \circledast C_{kl}(t)]_{kl}]_{ij} * [[B_{ij}(t) \circledast D_{kl}(t)]_{kl}]_{ij} \\
&= (A(t) \circledast C(t)) * (B(t) \circledast D(t)).
\end{aligned}$$

Hence, we arrive at (3.3). □

4 Tracy-Singh Convolution Product, Usual Convolution Products, and Vectorization

In this section, we discuss relationship between certain kinds of vectorization, Tracy-Singh convolution products, and usual convolution product.

Definition 5. Let $A = [a_{ij}] \in M_{mn}(\mathbb{R})$. Then, the vector-column operator and the vector-row operator of A are respectively defined by

$$\operatorname{vec} A = \begin{bmatrix} a_{11} & \cdots & a_{m1}, & \cdots, & a_{1n} & \cdots & a_{mn} \end{bmatrix}^T,$$
$$\operatorname{rvec} A = \begin{bmatrix} a_{11} & \cdots & a_{1n}, & \cdots, & a_{m1} & \cdots & a_{mn} \end{bmatrix}^T.$$

Definition 6. Let $A(t) = [A_{ij}(t)]$ where A_{ij} is of order $m_i \times n_j$ $(i = 1, .., p, j = 1, .., q)$ so that $A(t)$ is of order $mp \times nq$. We define the vector block-column operator and the vector block-row operator of A respectively by

$$\text{Vecb}_c\, A(t) = [\text{vec}\, A_{11}(t), ..., \text{vec}\, A_{m1}(t), \text{vec}\, A_{12}(t), ..., \text{vec}\, A_{m2}(t), ..., \text{vec}\, A_{1n}(t), ..., \text{vec}\, A_{mn}(t)]^T,$$
$$\text{Vecb}_r\, A(t) = [\text{rvec}\, A_{11}(t), ..., \text{rvec}\, A_{1n}(t), \text{rvec}\, A_{21}(t), ..., \text{rvec}\, A_{2n}(t), ..., \text{rvec}\, A_{m1}(t), ..., \text{rvec}\, A_{mn}(t)]^T.$$

Theorem 4. *Let $A \in \mathcal{C}(\Omega, M_{m,n}(\mathbb{R}))$, $B \in \mathcal{C}(\Omega, M_{p,q}(\mathbb{R}))$, $C \in \mathcal{C}(\Omega, M_{m,q}(\mathbb{R}))$ and $X \in \mathcal{C}(\Omega, M_{m,n}(\mathbb{R}))$ where $B(t) = [B_{kl}(t)]_{k,l=1}^{p,q}$ for every $t \in \Omega$. Then*

$$\text{Vecb}_c(A(t) * X(t) * B(t)) = (B^T(t) \boxplus A(t)) * \text{Vecb}_c\, X(t), \tag{4.1}$$
$$\text{Vecb}_r(A(t) * X(t) * B(t)) = (B(t) \boxplus A^T(t)) * \text{Vecb}_r\, X(t). \tag{4.2}$$

Proof. For each $i = 1, ..., p$, denote by $X_i(t)$, the i-th row of X. A direct calculation reveals that

$$\begin{aligned}
\text{Vecb}_c(A(t) * X(t) * B(t)) &= \begin{bmatrix} \text{vec}(A(t) * X_1(t) * B_{1k}(t)) \\ \vdots \\ \text{vec}(A(t) * X_p(t) * B_{pk}(t)) \end{bmatrix} \\
&= \begin{bmatrix} (B_{1k}^T(t)) * \text{vec}\, X_1(t) \\ \vdots \\ (B_{pk}^T(t)) * \text{vec}\, X_p(t) \end{bmatrix} \\
&= \begin{bmatrix} B_{1k}^T(t) \circledast A(t) \\ \vdots \\ B_{pk}^T(t) \circledast A(t) \end{bmatrix} * \text{Vecb}_c\, X(t) \\
&= (B^T \boxplus A(t)) * \text{Vecb}_c\, X(t).
\end{aligned}$$

The identity (4.2) follows from (4.1) and the fact that $\text{Vecb}_r\, A(t) = \text{Vecb}_c\, A^T(t)$. □

Theorem 5. *Let $A \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$ and $B \in \mathcal{I}(\Omega, M_{n,q}(\mathbb{R}))$ where $B(t) = [B_{kl}(t)]_{k,l=1}^{n,q}$ for every $t \in \Omega$. Then*

$$\text{Vecb}_c(A(t) * B(t)) = \left(\bigoplus_{i=1}^{nq} A(t)\right) * \text{Vecb}_c\, B(t), \tag{4.3}$$
$$\text{Vecb}_r(A(t) * B(t)) = \left(\bigoplus_{i=1}^{nq} B(t)\right) * \text{Vecb}_r\, A(t). \tag{4.4}$$

Proof. By direct computation, we have

$$\begin{aligned}\mathrm{Vecb}_c(A(t) * B(t)) &= \begin{bmatrix}(A(t) * B(t))_{1q} \\ \vdots \\ (A(t) * B(t))_{nq}\end{bmatrix} \\ &= \begin{bmatrix}A(t) & 0 & \cdots & 0 \\ 0 & A(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(t)\end{bmatrix} * \begin{bmatrix}B_{1q}(t) \\ \vdots \\ B_{nq}(t)\end{bmatrix} \\ &= \left(\bigoplus_{i=1}^{nq} A(t)\right) * \mathrm{Vecb}_c\, B(t).\end{aligned}$$

The identity (4.4) can be proved by using (4.3) and the fact that $\mathrm{Vecb}_r\, A(t) = \mathrm{Vecb}_c\, A^T(t)$ as follows:

$$\begin{aligned}\mathrm{Vecb}_r(A(t) * B(t)) &= \mathrm{Vecb}_c(A(t) * B(t))^T \\ &= \mathrm{Vecb}_c(B^T(t) * A^T(t)) \\ &= \left(\bigoplus_{i=1}^{nq} B^T(t)\right) * \mathrm{Vecb}_c\, A^T(t) \\ &= \left(\bigoplus_{i=1}^{nq} B^T(t)\right) * \mathrm{Vecb}_r\, A(t).\end{aligned}$$

□

5 Linear Matrix Convolution Equations

In this section, we discuss how to reduce linear matrix convolutions to simple ones by vectorization.

Theorem 6. *Let $A \in \mathcal{C}(\Omega, M_{m,n}(\mathbb{R}))$, $B \in \mathcal{C}(\Omega, M_{p,q}(\mathbb{R}))$, $C \in \mathcal{C}(\Omega, M_{m,q}(\mathbb{R}))$ and $X \in \mathcal{C}(\Omega, M_{n,p}(\mathbb{R}))$ where $B(t) = [B_{kl}(t)]_{k,l=1}^{p,q}$ for every $t \in \Omega$. Then the following statements are equivalent:*

(i) $A(t) * X(t) * B(t) = C(t)$.

(ii) $(B^T(t) \boxplus A(t)) * \mathrm{Vecb}_c\, X(t) = \mathrm{Vecb}_c\, C(t)$.

(iii) $(A(t) \boxplus B^T(t)) * \mathrm{Vecb}_r\, X(t) = \mathrm{Vecb}_r\, C(t)$.

More generally, the following equations are equivalent for any compatibly continuous matrix-valued functions A_i, B_i, C, X, $(i = 1, \ldots, n)$:

(a) $\sum_{i=1}^{n}(A_i(t) * X(t) * B_i(t)) = C(t)$.

(b) $\sum_{i=1}^{n}(B_i^T(t) \boxplus A(t)) * \text{Vecb}_c X(t) = \text{Vecb}_c C(t)$.

(c) $\sum_{i=1}^{n}(A_i(t) \boxplus B^T(t)) * \text{Vecb}_r X(t) = \text{Vecb}_r C(t)$.

Proof. It follows directly from Theorem 4 and the injectively of Vecb_r and Vecb_c. □

Theorem 7. *Let $A \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R})), C \in \mathcal{I}(\Omega, M_{m,q}(\mathbb{R}))$ and $X \in \mathcal{I}(\Omega, M_{n,q}(\mathbb{R}))$ where $X(t) = [X_{rs}(t)]_{rs=1}^{n,q}$ for every $t \in \Omega$. Then the following statements are equivalent:*

$$A(t) * X(t) = C(t), \tag{5.1}$$

$$\left(\bigoplus_{i=1}^{qm} A(t)\right) * \text{Vecb}_c X(t) = \text{Vecb}_c C(t), \tag{5.2}$$

$$\left(\bigoplus_{i=1}^{qm} A(t)\right) * \text{Vecb}_r X^T(t) = \text{Vecb}_r C^T(t). \tag{5.3}$$

Proof. Let Q_k denote the k-th block column of any matrix Q. We obtain

$$\begin{aligned}
\text{Vecb}_c(A(t) * X(t)) &= \begin{bmatrix} (A(t) * X(t))_{1k} \\ \vdots \\ (A(t) * X(t))_{mk} \end{bmatrix} = \begin{bmatrix} A(t) * X_{1k}(t) \\ \vdots \\ A(t) * X_{mk}(t) \end{bmatrix} \\
&= \begin{bmatrix} A(t) & 0 & \cdots & 0 \\ 0 & A(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A(t) \end{bmatrix} * \begin{bmatrix} X_{1k}(t) \\ \vdots \\ X_{mk}(t) \end{bmatrix} \\
&= (A(t) \oplus A(t) \oplus \cdots \oplus A(t)) * \text{Vecb}_c X(t) \\
&= \left(\bigoplus_{i=1}^{mq} A(t)\right) * \text{Vecb}_c X(t).
\end{aligned}$$

Since the vector block-column operator is injective, (5.1) and (5.2) are equivalent. (5.1) and (5.3) can be shown to be equivalent by direct computation. □

Theorem 8. *Let $A \in \mathcal{I}(\Omega, M_{n,q}(\mathbb{R})), C \in \mathcal{I}(\Omega, M_{m,q}(\mathbb{R}))$ and $X \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$ where*

$X(t) = [X_{rs}(t)]_{rs=1}^{m,n}$ *for every* $t \in \Omega$. *Then the following statements are equivalent:*

$$X(t) * A(t) = C(t), \tag{5.4}$$

$$\left(\bigoplus_{i=1}^{qm} A^T(t)\right) * \mathrm{Vecb}_r\, X(t) = \mathrm{Vecb}_r\, C(t), \tag{5.5}$$

$$\left(\bigoplus_{i=1}^{qm} A^T(t)\right) * \mathrm{Vecb}_c\, X^T(t) = \mathrm{Vecb}_c\, C^T(t). \tag{5.6}$$

Proof. Let Q_k denote the k-th block column any matrix Q. Since $\mathrm{Vecb}_r\, A(t) = \mathrm{Vecb}_c\, A(t)^T$, it follows that

$$\begin{aligned}
\mathrm{Vecb}_r(X(t) * A(t)) &= \mathrm{Vecb}_c(X(t) * A(t))^T = \mathrm{Vecb}_c\, A^T(t) * X^T(t) \\
&= \begin{bmatrix} (A^T(t) * X^T(t))_{1k} \\ \vdots \\ (A^T(t) * X^T(t))_{qk} \end{bmatrix} = \begin{bmatrix} A^T(t) * X^T_{1k}(t) \\ \vdots \\ A^T(t) * X^T_{mk}(t) \end{bmatrix} \\
&= \begin{bmatrix} A^T(t) & 0 & \cdots & 0 \\ 0 & A^T(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A^T(t) \end{bmatrix} * \begin{bmatrix} X^T_{1k}(t) \\ \vdots \\ X^T_{qk}(t) \end{bmatrix} \\
&= (A^T(t) \oplus A^T(t) \oplus \cdots \oplus A^T(t)) * \mathrm{Vecb}_c\, X^T(t) \\
&= \left(\bigoplus_{i=1}^{qm} A^T(t)\right) * \mathrm{Vecb}_c\, X^T(t) \\
&= \left(\bigoplus_{i=1}^{qm} A^T(t)\right) * \mathrm{Vecb}_r\, X(t).
\end{aligned}$$

□

Since vector block-column is injective, (5.4) and (5.5) are equivalent. (5.4) and (5.6) can be shown to be equivalent by direct commutation.

As a consequences of Theorems 7 and 8, we obtain the next corollary. It asserts that a linear matrix convolution equation of Lyapunov type can be reduced to a simple vector-matrix convolution equation.

Corollary 1. *Let* $A \in \mathcal{I}(\Omega, M_{m,n}(\mathbb{R}))$, $C \in \mathcal{I}(\Omega, M_{m,q}(\mathbb{R}))$ *and* $X \in \mathcal{I}(\Omega, M_{n,q}(\mathbb{R}))$ *where*

$X = [X_{rs}(t)]_{r,s=1}^{n,q}$. *Then the following statements are equivalent:*

$$A(t) * X(t) + X^T(t) * A^T(t) = C(t), \tag{5.7}$$

$$\left(\bigoplus_{i=1}^{mq} A(t)\right) * \operatorname{Vecb}_c X(t) = \frac{1}{2} \operatorname{Vecb}_c C(t), \tag{5.8}$$

$$\left(\bigoplus_{i=1}^{mq} A^T(t)\right) * \operatorname{Vecb}_r X^T(t) = \frac{1}{2} \operatorname{Vecb}_r C(t). \tag{5.9}$$

The final result provides an alternative way to solve the equation $(A(t) \circledast B(t)) * x(t) = v(t)$.

Proposition 2. *Let* $A \in \mathcal{I}(\Omega, M_{m,p}(\mathbb{R}))$, $B \in \mathcal{I}(\Omega, M_{n,q}(\mathbb{R}))$, $x \in \mathcal{I}(\Omega, \mathbb{R}^{pq})$, *and* $v \in \mathcal{I}(\Omega, \mathbb{R}^{mn})$. *Then the following statements are equivalent:*

(i) $(A(t) \boxtimes B(t)) * x(t) = v(t)$,

(ii) $(B(t) \boxtimes A(t)) * \hat{x}(t) = \hat{v}(t)$,

where $x(t) = \operatorname{Vecb}_c X(t)$, $v(t) = \operatorname{Vecb}_c V(t)$, $\hat{x}(t) = \operatorname{Vecb}_c X^T(t)$ *and* $\hat{v}(t) = \operatorname{Vecb}_c V^T(t)$.

Proof. We have by using Theorem 6 that the equation

$$(A(t) \boxtimes B(t)) * \operatorname{Vecb}_c X(t) = \operatorname{Vecb}_c V(t)$$

is equivalent to

$$B(t) * X(t) * A^T(t) = V(t). \tag{5.10}$$

(5.10) is also equivalent to the following:

$$A(t) * X^T(t) * B^T(t) = V^T(t).$$

Now the proof is done by applying Theorem 6 again. □

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