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An Application Markov Chain Monte Carlo for Bayes' methods in Estimating Parameter of Exponential Distribution

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ABSTRACT

The goal of this work is to estimate the parameter from exponential distribution by maximum likelihood, Bayes, Markov Chain Monte Carlo, and applied Bayes with Markov Chain Monte Carlo methods. The performance of these method is considered by the hypothesis testing when the estimator is provided an asymptotically unbiased estimator. Data is stimulated from exponential distribution with true parameter at 0.5, 2, 5, and 10, and the sample size at 30, 50, 100, 200, and 300. The results are shown that the maximum likelihood, Markov Chain Monte Carlo, and applied Bayes with Markov Chain Monte Carlo methods are provided an asymptotically unbiased estimator when the sample sizes is large especially the large true parameter . However the Bayes method is depended on the prior distribution, so the most estimators are not asymptotically unbiased estimator.

The estimated parameter (λ) of gamma distribution is using Maximum Likelihood (ML), Bayes', Markov Chain Monte Carlo (MCMC), and applied Bayes' with Markov Chain Monte Carlo (Bayes'-MCMC) methods. The Bayes' method is depended on the prior distribution, so gamma distribution is considered the parameter for Bayes' estimator. The t-test is used for testing that the true parameter is equal the mean of estimated parameter, while it means the performant method for estimating parameter. Data is simulated from gamma distribution by setting the shape parameter (α) as 4,5,6,7, and 8 and the scale parameter or called true parameter (λ) as 2 based on the sample sizes (n) 30, 50 and 70. The results are found that ML, Bayes', MCMC is a good performance to estimate parameter in most all cases except moderate sample size and small shape parameter. The Bayes'-MCMC method outperforms all others when shape parameter and sample sizes are large.

Keywords: Bayes Method; Exponential Distribution; Gamma Distribution; Maximum Likelihood Method; Markov Chain Monte Carlo Method;

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Chapter 1

Exponential Estimation

1.1 Introduction

The important field of statistics is statistical inference which is a process to study some properties about a population or called a parameter by estimating a sample from a population or called an estimator. The common estimators are used by social and science data such as mean, proportion ,standard deviation ,and variance. In generally, statistical inference includes the parameter estimation and hypothesis testing.

Parameter estimation consists of point estimation and interval estimation. Point estimation is to estimate a parameter by computing from the sample data, and to obtain a single value. This estimation will be inaccurate, if the random sample is not a good representation of the population. The properties of a good point estimator are unbiasedness, sufficiency, completeness, and minimum variance unbiased estimator. There are several point estimation methods such as the moments method, the maximum likelihood method, the minimum chi-square method, the least squares method, and the Bayes method.

In this case, we were interested the maximum likelihood method because the most estimators are presented in a class of a minimum variance unbiased estimator[1]. The Bayes method uses both a probability distribution and prior probability distribution to approach a posterior probability distribution[2] which is obtained a Bayes estimator. However, it is fairly difficult to demonstrate a posterior distribution from a probability distribution and a prior distribution. The method of Markov Chain Monte Carlo (MCMC) method [3] was invented Monte Carlo method at Los Alamos, New Mexico, USA. A Markov chain is a process to generate a random number from a computer program that can be thought as MCMC. The posterior distribution from the Bayes method can work with MCMC by Gibbs sampling[4] algorithm that will produce a MCMC estimator.

In this paper, we interest the experimental outcomes that occurred randomly for the counts of events within intervals of time and space. The exponential distribution is a continuous distribution that observes the time between events in Poisson process. The

parameter of the exponential distribution (λ) is the mean number of events per an interval of time, often called the rate parameter. It can be reversed into the mean of time per events ($\theta = 1/\lambda$), often called the scale parameter. We set out to find exponential parameter estimators by using the Maximum Likelihood method, the Bayes method, the Markov Chain Monte Carlo (MCMC) method, and applied Bayes with Markov Chain Monte Carlo Methods (Bayes- MCMC). The data is simulated from an exponential distribution with varying true parameters and sample sizes.

1.2 Methods for Parameter Estimation

The parameter estimation of the exponential distribution consists of three methods.

2.1.1 Maximum Likelihood (ML) Method

The ML method is common method to approximate a parameter. The concept of ML method is to start from the likelihood function of random variable X.

Let $X_1,\ldots,X_n$ be independent and identically distributed (iid) random variables via exponential distribution with parameter λ. The probability function of $x_i \mid \lambda$ or $f(x_i \mid \lambda)$ is denoted following

$$f(x;\lambda) = \lambda e^{-\lambda x}, x > 0.$$

Hence, the likelihood function is used the multiplication of probability function as

$$L(\lambda) = \prod_{i=1}^{n} f(x_i \mid \lambda).$$

The ML estimator of the parameter λ is shown by

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i},$$

$$\ln L(\lambda) = n\ln(\lambda) - \lambda \sum_{i=1}^{n} x_i,$$

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0,$$

$$\lambda = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}},$$

and $\hat{\lambda} = \frac{1}{\bar{x}}$.

The second derivative of $\ln L(\lambda)$ is

$$\frac{\partial^2 \ln L(\lambda)}{\partial \lambda^2} = -\frac{n}{\lambda^2} < 0.$$

The second derivative is a negative, then the estimator means the maximum value.

Therefore, the ML estimator is $\hat{\lambda}_{ML} = \frac{1}{\bar{x}}$.

1.2.2 Bayes Method

From the ML method, we know that the likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} f(x_i \mid \lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}.$$

The prior distribution of λ is a gamma distribution with parameter a,b denoted by Gamma (a,b) or rewritten as

$$g(\lambda; a,b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \qquad , \lambda > 0.$$

The resulting of posterior distribution is a gamma distribution that implied by the conjugate distribution from exponential distribution and gamma distribution as

$$h(\lambda \mid x_i) \propto L(\lambda) g(a,b \mid \lambda),$$

$$h(\lambda \mid x_i) \propto \left(\lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \right) \left(\frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \right),$$

$$\propto \frac{b^a}{\Gamma(a)} \lambda^{n+a-1} e^{-(n\bar{x}+b)\lambda}.$$

The posterior distribution can be concluded that

$$\lambda \mid x_i \sim gamma(n+a, n\bar{x}+b).$$

Therefore, the Bayes estimator is

$$\hat{\lambda}_{BAYES} = E(\lambda \mid x_i) = \frac{n+a}{n\bar{x}+b}.$$

1.2.3 Markov Chain Monte Carlo Method

The Markov Chain Monte Carlo (MCMC)[5] method is operated by sequentially sampling parameter values from a Markov Chain. The Gibbs sampling[4] is an algorithm of MCMC that can be generated random values from the posterior distribution. We carry out the WinBUGS [6] program to obtain the estimator from posterior distribution function. The MCMC estimators can be computed by

1. Generating a sample $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(T)}$ from a posterior distribution $h(\lambda \mid x_i)$.
2. Calculating the sample mean of $\lambda^{(t)}$ by simply calculating the quantity by

$$\hat{\lambda}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} \lambda^{(t)},$$

where $\lambda_i^{(t)}$ is generated from posterior distribution based on the gamma distribution at parameter $a^{(t)}$ and $b^{(t)}$, and T is an iteration of posterior distribution function.

From the Bayes method, the Bayes estimator is

$$\hat{\lambda}_{Bayes} = \frac{n+a}{n\bar{x}+b}.$$

It can be seen that the Bayes estimator depends on the parameter a and b, so the MCMC method can be approximated by the Gibbs sampling algorithm following

$$\hat{a}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} a^{(t)} \quad , \quad \hat{b}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} b^{(t)},$$

where $a^{(t)}$ is generated from exponential distribution and $b^{(t)}$ is generated from gamma distribution. Notice that a, b are the parameters of the exponential and gamma distribution

because the values of a,b are greater than zero, which is supported the exponential and the gamma distribution.

Then we use this benefit to apply Bayes with MCMC method or called Bayes-MCMC estimator. Furthermore, we get the Bayes-MCMC estimator as

$$\hat{\lambda}_{Bayes-MCMC} = \frac{n+\hat{a}_{MCMC}}{n\bar{x}+\hat{b}_{MCMC}}.$$

1.3. Research Scope

1.3.1 The random variables X is simulated from the exponential distribution with the true parameter λ of 0.5, 2, 5, and 10 shown in Figure1.

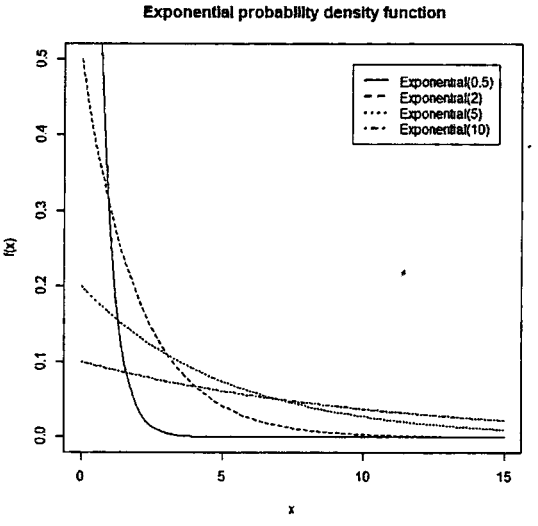

Figure1: The curve of exponential distribution with the true parameter λ of 0.5, 2, 5, and 10.

1.3.2 The prior distribution is defined as the gamma distribution with parameter $a=2$ and $b=1$ shown in Figure 2.

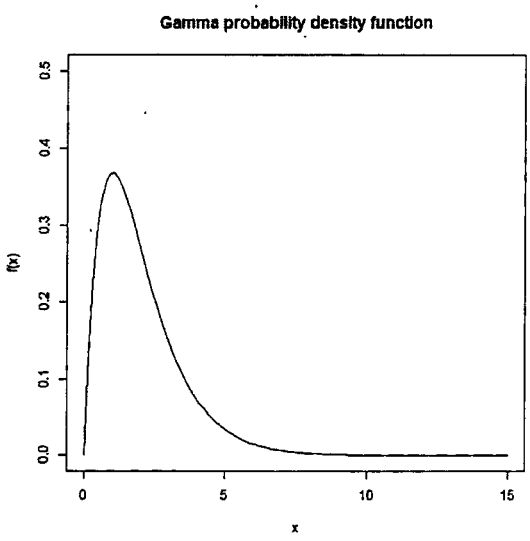

Figure2: The curve of gamma distribution with parameter $a = 2$ and $b = 1$.

1.3.3 The sample sizes are considered at n = 30, 50, 100, 200, and 300.

1.3.4 The data is generated 500 times in each case with the R program [7].

1.4 Statistical Hypothesis Testing

A t-statistic is adopted to test the mean of a parameter is different from the true parameter (λ_0). The hypothesizes are defined by

$$H_0 : \mu_{\hat{\lambda}} = \lambda_0 \text{ ,and } H_1 : \mu_{\hat{\lambda}} \neq \lambda_0.$$

The t statistic is computed as follows:

$$t = \frac{\bar{\hat{\lambda}} - \lambda_0}{S_{\hat{\lambda}} / \sqrt{m}},$$

where $S_{\hat{\lambda}} = \sqrt{\frac{\sum_{j=1}^{m}\left(\hat{\lambda}_j - \bar{\hat{\lambda}}\right)^2}{m-1}}$, $\bar{\hat{\lambda}}$ is a sample mean of independent observation, and m is a generating times (500). We will reject H_0, if $|t| > t_{\alpha/2,m-1}\frac{S_{\hat{\lambda}}}{\sqrt{m}}$ when the level of significance is set at α.

1.5 The Results

The parameter estimation with exponential distribution is approximated from simulated data with the true parameters and the sample sizes. For Tables 1-4, the first and the second columns of these tables presented the true parameters from simulated data and the sample sizes. The next four columns are shown in estimating parameter by a mean, a standard deviation, a lower and upper bounds of 95% confidence interval. The t statistic and p-value for hypothesis testing are listed in the last two columns. By observing the p-values, the results appear as follows:

1. For Table 1, ML method is appeared that the p-value is less than the level of significance for all λ at $n=30, \lambda=0.5$ at $n=100$, $\lambda=2$ at $n=300$, and $\lambda=10$ at $n=50,300$. It's indicated that the remainder of the estimated parameter are asymptotically unbiased estimates for 12 from 20 situations.
2. For Table 2, the p-value of Bayes provides asymptotically unbiased estimates of estimator for $\lambda=2$ at $n=50,100,200$, and $\lambda=5$ at $n=300$. The Bayes method indicated that most remainder of the estimated parameters were not different from the true parameters for 4 from 20 situations.
3. For Table 3, MCMC method appeared that the p-value is less than the level of significance for all λ at $n=30, \lambda=0.5$ at $n=100$, $\lambda=5$ at $n=200$, and $\lambda=10$ at $n=50,300$. It's indicated that the remainder of the estimated parameter are asymptotically unbiased estimates for 12 from 20 situations.
4. For Table 4, the p-values of Bayes-MCMC estimator is less than the level of significance for all λ at $n=30, \lambda=0.5$ at $n=100$, $\lambda=2$ at $n=300$, and $\lambda=10$ at $n=50,300$. It's indicated that the remainder of the estimated parameter are asymptotically unbiased estimates for 12 from 20 situations.

Table 1 : The true parameter (λ), sample sizes, mean, standard deviation (SD.), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value obtained via the ML method.

λ	n	Mean	SD	LCI	UCI	t-test	p-value
0.5	30	0.52327	0.10342	0.51419	0.53236	5.03230	0.00000*
	50	0.50504	0.07218	0.49870	0.51138	1.56200	0.11890
	100	0.50947	0.05432	0.50470	0.51424	3.89810	0.00011*
	200	0.50196	0.03486	0.49890	0.50503	1.25980	0.20830
	300	0.50078	0.02826	0.49830	0.50327	0.62035	0.53530
λ	n	Mean	SD	LCI	UCI	t-test	p-value
2	30	2.06140	0.41826	2.02465	2.09815	3.28240	0.00110*
	50	2.02242	0.28734	1.99717	2.04767	1.74460	0.08167
	100	2.01530	0.20215	1.99754	2.03306	1.69260	0.09116
	200	2.00388	0.14062	1.99152	2.01623	0.61627	0.53800
	300	2.01067	0.11764	2.00033	2.02101	2.02800	0.04309*
5	30	5.22359	0.98596	5.13696	5.31022	5.07080	0.00000*
	50	5.05802	0.72299	4.99449	5.12155	1.79440	0.07335
	100	5.03556	0.49942	4.99168	5.07944	1.59200	0.11200
	200	5.02852	0.34130	4.99853	5.05851	1.86850	0.06228
	300	5.02602	0.30585	4.99915	5.05290	1.90260	0.05767
10	30	10.51217	1.97508	10.33863	10.68571	5.79850	0.00000*
	50	10.27710	1.52058	10.14349	10.41070	4.07480	0.00005*
	100	10.05077	1.00491	9.96247	10.13906	1.12960	0.25920
	200	10.04100	0.68526	9.98079	10.10121	1.33780	0.18160
	300	10.08159	0.62561	10.02662	10.13656	2.91630	0.00370*

* indicates significance level at 5 %

Table 2: The true parameter (λ), sample sizes, mean, standard deviation (SD.), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value obtained via the Bayes method.

λ	n	Mean	SD	LCI	UCI	t-test	p-value
0.5	30	0.54823	0.10625	0.53890	0.55757	10.15000	0.00000*
	50	0.51989	0.07351	0.51343	0.52635	6.04920	0.00000*
	100	0.51700	0.05484	0.51218	0.52181	6.93000	0.00000*
	200	0.50571	0.03503	0.50263	0.50879	3.64360	0.00030*
	300	0.50328	0.02835	0.50079	0.50577	2.58650	0.00998*
2	30	2.05242	0.38659	2.01845	2.08639	3.03190	0.00256*
	50	2.02003	0.27516	1.99585	2.04421	1.62770	0.10420
	100	2.01461	0.19800	1.99721	2.03201	1.64970	0.09964
	200	2.00374	0.13921	1.99151	2.01597	0.60082	0.54820
	300	2.01055	0.11684	2.00029	2.02082	2.01970	0.04395*
5	30	4.72462	0.74941	4.65877	4.79047	-8.21660	0.00000*
	50	4.76904	0.61451	4.71504	4.82303	-8.40430	0.00000*
	100	4.88784	0.46107	4.84733	4.92835	-5.43940	0.00000*
	200	4.95370	0.32789	4.92489	4.98251	-3.15770	0.00169*
	300	4.97586	0.29774	4.94970	5.00203	-1.81260	0.07049
10	30	8.24898	1.12150	8.15044	8.34752	-34.91200	0.00000*
	50	8.83881	1.07575	8.74429	8.93333	-24.13700	0.00000*
	100	9.30783	0.84219	9.23383	9.38183	-18.37800	0.00000*
	200	9.65456	0.62750	9.59942	9.70969	-12.31000	0.00000*
	300	9.81765	0.58929	9.76587	9.86943	-6.91930	0.00000*

* indicates significance level at 5 %

Table 3: The true parameter (λ), sample sizes, mean, standard deviation (SD.), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value obtained via the MCMC method.

λ	n	Mean	SD	LCI	UCI	t-test	p-value
0.5	30	0.52336	0.10304	0.51431	0.53241	5.06890	0.00000*
	50	0.50500	0.07206	0.49866	0.51133	1.55030	0.12170
	100	0.50877	0.05409	0.50402	0.51352	3.62510	0.00032*
	200	0.50142	0.03484	0.49836	0.50448	0.91313	0.36160
	300	0.50074	0.02832	0.49825	0.50323	0.58606	0.55810
2	30	2.05737	0.41773	2.02066	2.09407	3.07070	0.00225*
	50	2.02102	0.28787	1.99573	2.04632	1.63290	0.10310
	100	2.01042	0.20178	1.99269	2.02815	1.15510	0.24860
	200	2.00452	0.14084	1.99214	2.01689	0.71715	0.47360
	300	2.00884	0.11742	1.99853	2.01916	1.68390	0.09282
5	30	5.21944	0.98479	5.13291	5.30597	4.98270	0.00000*
	50	5.05068	0.72128	4.98731	5.11406	1.57120	0.11680
	100	5.01978	0.49786	4.97603	5.06352	0.88833	0.37480
	200	5.03161	0.34171	5.00159	5.06164	2.06870	0.03909*
	300	5.01991	0.30546	4.99308	5.04675	1.45780	0.14550
10	30	10.50400	1.97062	10.33085	10.67715	5.71890	0.00000*
	50	10.26062	1.51944	10.12711	10.39413	3.83540	0.00014*
	100	10.02524	1.00350	9.93707	10.11341	0.56245	0.57410
	200	10.04483	0.68586	9.98456	10.10509	1.46140	0.14450
	300	10.06836	0.62440	10.01349	10.12322	2.44790	0.01471*

* indicates significance level at 5 %

Table 4: The true parameter (λ), sample sizes, mean, standard deviation (SD.), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value obtained via the applied Bayes with MCMC method.

λ	n	Mean	SD	LCI	UCI	t-test	p-value
0.5	30	0.52299	0.10326	0.51391	0.53206	4.97760	0.00000*
	50	0.50495	0.07210	0.49861	0.51128	1.53370	0.12570
	100	0.50942	0.05430	0.50465	0.51420	3.88100	0.00012*
	200	0.50200	0.03487	0.49894	0.50507	1.28560	0.19920
	300	0.50083	0.02827	0.49835	0.50332	0.65904	0.51020
2	30	2.05906	0.41756	2.02237	2.09575	3.16270	0.00166*
	50	2.02128	0.28731	1.99603	2.04652	1.65580	0.09839
	100	2.01446	0.20211	1.99670	2.03222	1.60010	0.11020
	200	2.00373	0.14060	1.99138	2.01609	0.59326	0.55330
	300	2.01078	0.11764	2.00044	2.02112	2.04880	0.04100*
5	30	5.21632	0.98453	5.12981	5.30282	4.91300	0.00000*
	50	5.05529	0.72268	4.99179	5.11879	1.71070	0.08777
	100	5.03334	0.49920	4.98948	5.07720	1.49340	0.13600
	200	5.02811	0.34124	4.99813	5.05809	1.84190	0.06608
	300	5.02632	0.30586	4.99945	5.05320	1.92450	0.05486
10	30	10.49778	1.97246	10.32447	10.67109	5.64300	0.00000*
	50	10.27088	1.51942	10.13738	10.40438	3.98640	0.00008*
	100	10.04697	1.00456	9.95870	10.13523	1.04540	0.29630
	200	10.03977	0.68519	9.97957	10.09998	1.29790	0.19490
	300	10.08190	0.62564	10.02693	10.13688	2.92730	0.00358*

* indicates significance level at 5 %

1.6 Conclusion

In this paper, we have investigated the estimated parameter of exponential distribution by hypothesis testing from ML, Bayes, MCMC, and applied Bayes with MCMC methods. Through a simulation study, ML, MCMC, and applied Bayes with MCMC methods perform a good performance than the Bayes method especially the large sample sizes. On the other hand, the Bayes method is not a good estimator because the parameter of prior distribution depends on the Bayes estimator, so the result may be changed by the parameter of prior distribution. For this problem, we would recommend users will apply Bayes with MCMC methods, if the prior distribution is not identified parameter value.

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Chapter 2

Gamma Estimation

2.1 Introduction

The estimation is a field of statistical inference which is related to the process by which one makes inferences about a population, based on information obtained from a sample. Point estimation is a part of estimation which is contained a single value and estimated the population parameter or called estimator. The properties of good estimator are unbiasedness, sufficiency, consistency, efficiency and minimum variance [1]. There are several methods to approximate estimator such as moments method, maximum likelihood method, least squares method, and Bayes' method.

The maximum likelihood method is a well known method for estimating parameter of statistical model given observation. The estimator is estimated by maximize the likelihood function given the parameter. The properties of good estimator can be found in this method. The Bayes' method mentions the probability distribution given the parameter that related with prior distribution. The relation among probability and prior distributions imply to find the posterior distribution. So the Bayes' estimator is computed from mean of posterior distribution which is depended on the parameter of prior distribution. The Markov Chain Monte Carlo method (MCMC) [2] method is solved the problem of estimating parameter on prior distribution. The Gibbs sampling is a process of MCMC to generate the value from the posterior distribution that obtained MCMC estimator. From MCMC method, we obtain the estimated parameter of prior distribution which can be applied by Bayes and MCMC to find the estimator.

From the related research, Araveeporn A [3] studied the point parameter estimation of Poisson distribution by using Maximum Likelihood, Markov Chain Monte Carlo, and Bayes methods by estimated parameter of the prior distribution from MCMC method. Thetkham A and Araveeporn A [4] studied a comparison of parameter estimation of exponential distribution using Maximum Likelihood, Bayes, Markov Chain Monte Carlo and applied Bayes with Markov Chain Monte Carlo methods, and the prior distribution of Bayes' method depended on gamma distribution.

In this research, we interested the data in term of a gamma distribution which is related the parameter of Poisson distribution. The number of successes that occurs in fixed interval of time and space is a random variable of Poisson distribution depended on average number of successes in fixed interval of time or called the parameter of Poisson distribution (λ). For

gamma distribution, the random variable is a waiting times until α th success occurs, given that the average number of success per unit of time is λ as the meaning of parameter on Poisson distribution. In this respect, the gamma distribution is related to the exponential distribution in the average waiting time until successful for the first time.

The gamma distribution is often used to model waiting times, particularly in the case of lifespan testing in which the "waiting time" until death is modeled by a gamma distribution. It is also commonly used in applied fields such as finance, civil engineering, climatology (e.g. in estimating rainfall), and econometrics.

The research about the gamma distribution is studied by several literatures following : Pradhan B and Kundu D [5] developed Bayes estimation and prediction of two-parameter gamma distribution by assumed that the scale parameter has a gamma distribution prior and the shape parameter has any log-concave distribution prior. Kirimi E, Ouko A, and Kipkoech CW [6] studied Modified Moment Estimation (MME) for a two parameter gamma distribution which estimated the scale parameter using the MME and compared the performance with the Maximum Likelihood Estimates (MLE). Son YS and Oh M [7] focused Bayesian Estimation of the Two-Parameter Gamma Distribution was considered under the non informative prior.

For this reason, we interest in three methods with four estimating parameter such as maximum likelihood (ML), Bayes', Markov Chain Monte Carlo (MCMC), and applied Bayes with Markov Chain Monte Carlo (Bayes'-MCMC). We estimate the parameter of gamma distribution (λ) by using these methods that mentioned before.

2.2 Methods of Parameter Estimation

In this research, we have studied parameter estimation (λ) of gamma distribution with three methods.

2.2.1 Maximum Likelihood (ML) Method

Let $X_1,...,X_n$ are random samples via gamma distribution with parameter α and λ or called $Gamma(\alpha,\lambda)$. The shape parameter is α, and the scale parameter is λ.

The probability density function is written by

$$f(x;\alpha,\lambda)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x} & ,x>0\\ 0 & ,\text{otherwise}\end{cases},$$

where mean is $E(X)=\dfrac{\alpha}{\lambda}$, and variance is $Var(X)=\dfrac{\alpha}{\lambda^2}$.

The definition of likelihood function of gamma distribution $L(\alpha,\lambda)$ is

$$\begin{aligned}L(\alpha,\lambda)&=\prod_{i=1}^{n}f(x_i;\alpha,\lambda)\\&=\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{n}\prod_{i=1}^{n}x_i^{\alpha-1}e^{-\lambda\sum_{i=1}^{n}x_i}.\end{aligned}$$

The first step of maximum likelihood method is taken by natural logarithm (ln) on a likelihood function following

$$\begin{aligned}\ln L(\alpha,\lambda) &= \ln\left(\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{n}\prod_{i=1}^{n}x_i^{\alpha-1}e^{-\lambda\sum_{i=1}^{n}x_i}\right)\\&=\alpha\, n\ln(\lambda)-n\ln(\Gamma(\alpha))+(\alpha-1)\sum_{i=1}^{n}\ln x_i-\lambda\sum_{i=1}^{n}x_i.\end{aligned}$$

Next, the partial differential of λ is considered by

$$\begin{aligned}\frac{\partial\ln L(\alpha,\lambda)}{\partial\lambda} &= \frac{\partial}{\partial\lambda}\left(\alpha n\ln(\lambda)-n\ln(\Gamma(\alpha))+(\alpha-1)\sum_{i=1}^{n}\ln x_i-\lambda\sum_{i=1}^{n}x_i\right),\\&=\frac{n\alpha}{\lambda}-\sum_{i=1}^{n}x_i=0,\end{aligned}$$

$$\begin{aligned}\frac{n\alpha}{\lambda}-\sum_{i=1}^{n}x_i &= 0,\\ \frac{n\alpha}{\lambda} &= \sum_{i=1}^{n}x_i,\\ \alpha &= \left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\lambda,\end{aligned}$$

$$\alpha = \bar{x}\lambda,$$

$$\lambda = \frac{\alpha}{\bar{x}}.$$

The second derivative of $\ln L(\alpha,\lambda)$ is $\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial \lambda^2} = -\frac{n\alpha}{\lambda^2} < 0$, so the estimators have the maximum value. Therefore, the ML estimator of λ is $\hat{\lambda}_{ML} = \frac{\alpha}{\bar{x}}$.

2.2.2 Bayes' Method

Let the prior distribution (λ) is a gamma distribution with parameter a and b which is the conjugate distribution of gamma distribution. The prior distribution is shown that the random variable is λ in the form of gamma distribution as the following

$$f(\lambda;a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda} & , \lambda > 0 \\ 0 & , \text{otherwise} \end{cases}.$$

From the Bayes' theorem [8], the posterior distribution function is approximated by

$$h(\lambda|x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_n|\lambda)f(\lambda;a,b)}{\int_0^\infty f(x_1,\ldots,x_n|\lambda)f(\lambda;a,b)d\lambda},$$

or it can be rewritten in term of conjugate distribution as

$$h(\lambda|x_1,\ldots,x_n) \propto L(\alpha,\lambda)f(a,b|\lambda),$$

then

$$h(\lambda|x_1,\ldots,x_n) \propto \left(\frac{\lambda^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1} e^{-\lambda\sum_{i=1}^n x_i} \left(\frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}\right),$$

$$\propto \frac{b^a}{(\Gamma(\alpha))^n(\Gamma(a))}\prod_{i=1}^n x_i^{\alpha-1}\lambda^{n\alpha+a-1}e^{-\lambda\sum_{i=1}^n x_i - b\lambda},$$

$$\propto \frac{b^a}{(\Gamma(\alpha))^n(\Gamma(a))}\prod_{i=1}^n x_i^{\alpha-1}\lambda^{n\alpha+a-1}e^{-\lambda\left(\sum_{i=1}^n x_i + b\right)},$$

$$\propto \frac{b^a}{\left(\Gamma(\alpha)\right)^n \left(\Gamma(a)\right)} \prod_{i=1}^{n} x_i^{\alpha-1} \lambda^{n\alpha+a-1} e^{-\lambda(n\bar{x}+b)}.$$

The posterior distribution is presented in a class of gamma distribution as

$$h\left(\lambda \middle| x_1,\ldots,x_n\right) \sim Gamma\left(n\alpha + a, n\bar{x} + b\right).$$

Therefore, the Bayes' estimator of λ is $\hat{\lambda}_{Bayes'} = E\left(\lambda \middle| x_1,\ldots,x_n\right) = \frac{n\alpha + a}{n\bar{x} + b}$.

2.2.3 Markov Chain Monte Carlo (MCMC) Method

The MCMC method is a popular method for estimating parameter when the random variable can not define the prior distribution. This method consists of sampling random variable from Markov Chain method for estimating parameter on prior distribution, sampling random variable by Gibbs sampling algorithm [9]. The posterior distribution is used a Markov Chain and Gibbs sampling to approximate estimator from the MCMC method. We carry out the RJAGS package which provides an interface from R to the JAGS library for Bayesian data analysis [10]. JAGS uses Markov Chain Monte Carlo (MCMC) to generate a sequence of dependent samples from the posterior distribution of the parameters.

Sampling process from the MCMC method as follows :

1. Set initial value $a^{(t)}$ and $b^{(t)}$ from gamma distribution.

2. Generating $a^{(t)}$ and $b^{(t)}$ from 1., $t = 1,2,\ldots,T$, where T is a number of Gibbs sampling algorithm. The number of T is approached to population parameter at large number replications. In this case, we use 5000 replications.

3. Generating $\lambda^{(t)}$ from posterior distribution of gamma distribution with parameter $a^{(t)}$ and $b^{(t)}$ from 2.

Therefore, the Markov Chain Monte Carlo estimator of λ is $\hat{\lambda}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} \lambda^{(t)}$

Furthermore, we use this benefit of MCMC algorithm to approximate parameter (a and b) on prior distribution. From Bayes' estimator, we can apply this idea to estimate Bayes'- MCMC estimator which is written as

$$\hat{\lambda}_{Bayes'-MCMC} = \frac{n\alpha + \hat{a}_{MCMC}}{n\bar{x} + \hat{b}_{MCMC}}, \text{ where } \hat{a}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} a^{(t)}, \text{ and } \hat{b}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} b^{(t)}.$$

2.3 Research Scope

2.3.1 The random variable X is simulated data from gamma distribution with R program by setting the scale parameter or called true parameter (λ) at 2 and shape parameter (α) at 4, 5, 6, 7, and 8 shown in Figure 1.

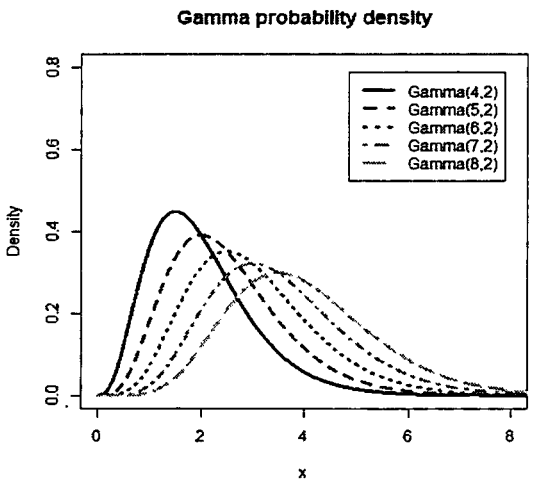

Figure 1: Gamma distribution with parameters (α, λ) as $(4,2),(5,2),(6,2),(7,2)$, and $(8,2)$.

2.3.2 The prior distribution of λ is defined as the gamma distribution with $\alpha = 2$ and $\lambda = 1$ shown in Figure 2.

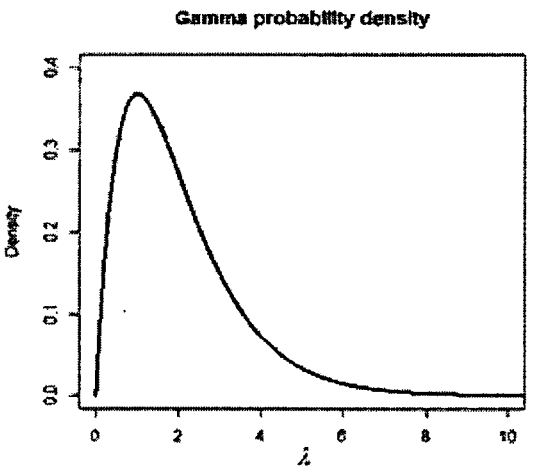

Figure 2: Gamma distribution with parameter $\alpha = 2$ and $\lambda = 1$.

2.3.3 The sample sizes (n) are studied at $30, 50$, and 70.

2.3.4 The significant confidence level (α) is defined at 2 levels as 0.01 and 0.05.

2.3.5 In this research, we use R program versions 3.3.1 that generated 1,000 replications in each situation.

2.4 Statistical Hypothesis Testing

A t statistic is applied to test the different between the mean of sample with the mean of true parameter. The hypothesis are

$$H_0 : \mu_{\hat{\lambda}} = \lambda \text{ and } H_1 : \mu_{\hat{\lambda}} \neq \lambda$$

The t statistic was computed as follows:

$$t = \frac{\bar{\hat{\lambda}} - \lambda}{S_{\hat{\lambda}} / \sqrt{m}}$$

where $\bar{\hat{\lambda}}$ is the mean of estimated parameter.

$S_{\hat{\lambda}} = \sqrt{\frac{\sum_{j=1}^{m}\left(\hat{\lambda}_j - \bar{\hat{\lambda}}_j\right)^2}{m-1}}$ is a standard deviation (SD) of sample .

$\hat{\lambda}_j$ is the estimated parameter in repeat j times.

$\bar{\hat{\lambda}}_j$ is the mean of estimated parameter in repeat j times.

m is the number of replications, here given $m = 1,000$ and $df = m - 1$.

In this case, we will reject H_0, when $|t| > t_{\alpha/2, m-1}$ and set the level of significance at α = 0.01 and 0.05. The lower confident interval (LCI), upper confident interval (UCI) of λ on confident interval $(1-\alpha)100\%$ can compute from $\lambda = \bar{\hat{\lambda}} \pm t_{\alpha/2, m-1} \frac{S_{\hat{\lambda}}}{\sqrt{m}}$.

2.5 The Results

In this section, the parameter estimation of gamma distribution by Maximum Likelihood (ML), Bayes', Markov Chain Monte Carlo (MCMC), and applied Bayes' with MCMC (Bayes'- MCMC) methods are presented in Tables 1-4.

From Table 1-3, by observing the p-values, the ML, Bayes', and MCMC estimators shown that the means of the estimated parameters do not different from the true parameters ($\lambda = 2$) as follows: $\alpha = 4$ when $n = 50$, $\alpha = 5,8$ when $n = 50,70$, $\alpha = 6,7$ in all sample sizes. The estimated parameter is a good performance in 11 situations from 15 situations.

From Table 4, the p-value of the Bayes'- MCMC estimators is larger than the significance level (0.01 and 0.05) when the mean of the estimated parameter do not different from the true parameter ($\lambda = 2$) at $\alpha = 4$ when $n = 70$, $\alpha = 5$ when $n = 50,70$ and $\alpha = 6,7,8$ in all sample sizes. The estimated parameter is a good performance in 12 situations from 15 situations.

In the part of the histograms, the estimated parameters of all the true parameters are shown in Figure 3-22. The histograms are found that the estimated value tends to the normal distribution on true parameter ($\lambda = 2$) when the sample sizes increased.

Table 1: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α) , mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by the ML method.

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01534	0.18505	2.00386	2.02682	2.62140	0.00889**
	50	2.00872	0.14230	1.99989	2.01756	1.93879	0.05281
	70	2.00825	0.11849	2.00090	2.01560	2.20196	0.02790*
5	30	2.01236	0.16623	2.00205	2.02268	2.35139	0.01890*
	50	2.00628	0.12549	1.99849	2.01406	1.58188	0.11399
	70	2.00552	0.10697	1.99888	2.01216	1.63185	0.10303
6	30	2.00858	0.15144	1.99918	2.01797	1.79090	0.07361
	50	2.00298	0.11359	1.99593	2.01003	0.82965	0.40694
	70	2.00244	0.09837	1.99633	2.00854	0.78295	0.43384
7	30	2.00815	0.13688	1.99966	2.01665	1.88332	0.05995
	50	2.00301	0.10922	0.49906	0.50245	0.87191	0.38347
	70	2.00218	0.09113	1.99653	2.00784	0.75790	0.44869
8	30	2.00832	0.13132	2.00017	2.01647	2.00392	0.04535*
	50	2.00363	0.10399	1.99717	2.01008	1.10306	0.27027
	70	2.00261	0.08427	1.99738	2.00784	0.97808	0.32827

**indicates significance at 1% level and *indicates significance at 5% level

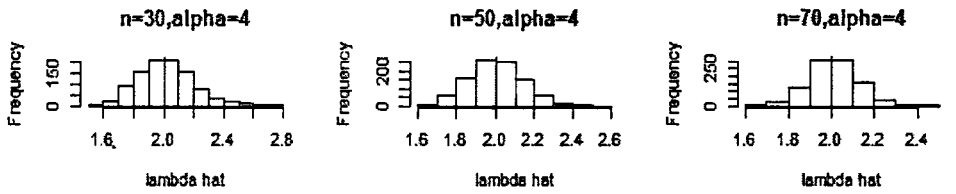

Figure 3: Histograms of estimated parameters λ by ML method when $\alpha = 4, \lambda = 2$

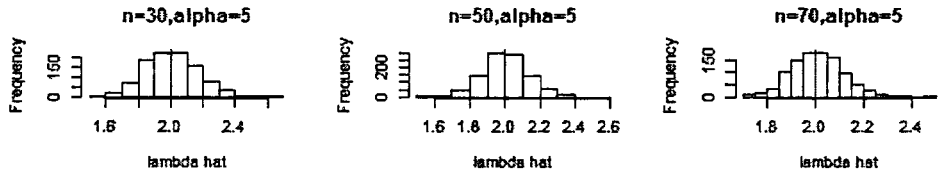

Figure 4: Histograms of estimated parameters λ by ML method when $\alpha = 5, \lambda = 2$

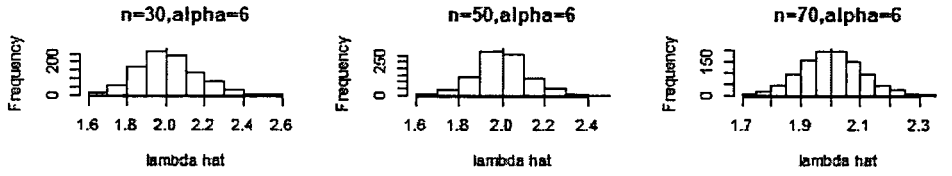

Figure 5: Histograms of estimated parameters λ by ML method when $\alpha = 6, \lambda = 2$

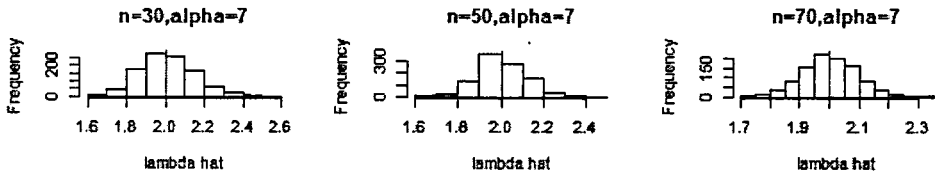

Figure 6: Histograms of estimated parameters λ by ML method when $\alpha = 7, \lambda = 2$

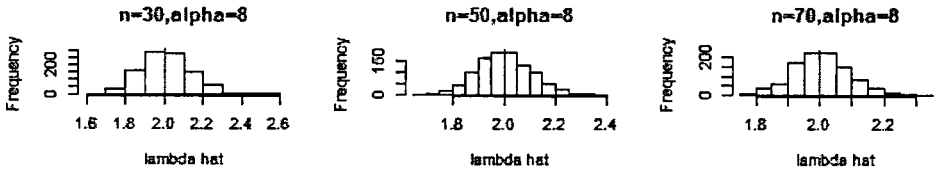

Figure 7: Histograms of estimated parameters λ by ML method when $\alpha = 8, \lambda = 2$

Table 2: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α), mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by the Bayes' method with the prior distribution is gamma distribution.

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01481	0.18183	2.00353	2.02610	2.57585	0.01014*
	50	2.00854	0.14085	1.99980	2.01728	1.91704	0.05552
	70	2.00814	0.11764	2.00084	2.01544	2.18889	0.02884*
5	30	2.01202	0.16394	2.00184	2.02219	2.31807	0.02065*
	50	2.00617	0.12447	1.99844	2.01389	1.56638	0.11758
	70	2.00546	0.10635	1.99886	2.01206	1.62236	0.10504
6	30	2.00836	0.14972	1.99907	2.01765	1.76519	0.07784
	50	2.00292	0.11283	1.99592	2.00992	0.81784	0.41365
	70	2.00240	0.09790	1.99633	2.00848	0.77561	0.43816
7	30	2.00799	0.13555	1.99958	2.01640	1.86336	0.06270
	50	2.00296	0.10858	0.50119	0.50459	0.86222	0.38877
	70	2.00216	0.09075	1.99653	2.00779	0.75208	0.45218
8	30	2.00818	0.13019	2.00010	2.01626	1.98734	0.04716*
	50	2.00358	0.10346	1.99716	2.01000	1.09495	0.27380
	70	2.00258	0.08397	1.99737	2.00780	0.97338	0.33060

**indicates significance at 1% level and *indicates significance at 5% level

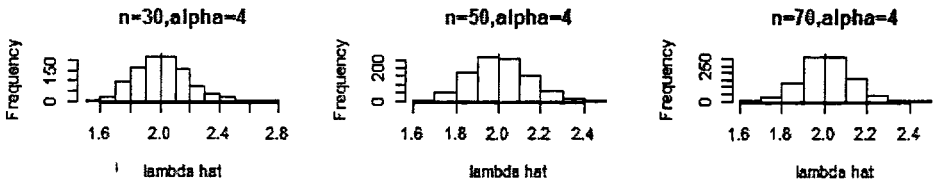

Figure 8: Histograms of estimated parameters λ by Bayes' method when $\alpha = 4, \lambda = 2$

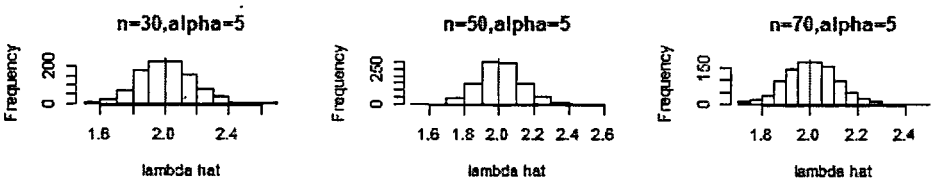

Figure 9: Histograms of estimated parameters λ by Bayes' method when $\alpha = 5, \lambda = 2$

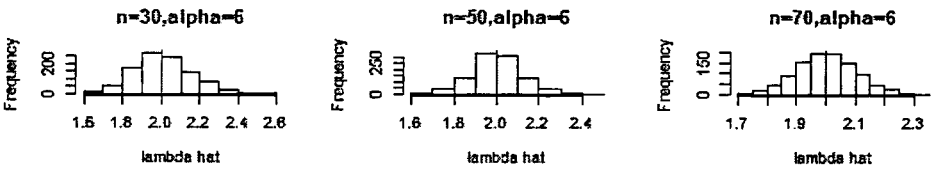

Figure 10: Histograms of estimated parameters λ by Bayes' method when $\alpha = 6, \lambda = 2$

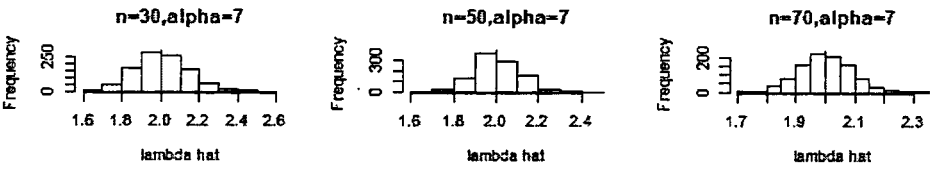

Figure 11: Histograms of estimated parameters λ by Bayes' method when $\alpha = 7, \lambda = 2$

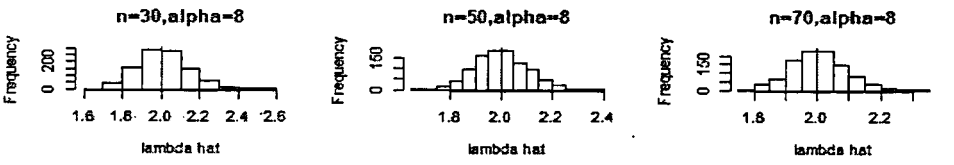

Figure 12: Histograms of estimated parameters λ by Bayes' method when $\alpha = 8, \lambda = 2$

Table 3: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α), mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by the MCMC method

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01548	0.18527	2.00398	2.02698	2.64203	0.00837**
	50	2.00863	0.14272	1.99978	2.01749	1.91279	0.05606
	70	2.00821	0.11864	2.00085	2.01557	2.18763	0.02893*
5	30	2.01263	0.16678	2.00228	2.02298	2.39519	0.01679*
	50	2.00619	0.12570	1.99839	2.01399	1.55833	0.11947
	70	2.00539	0.10700	1.99875	2.01203	1.59412	0.11123
6	30	2.00901	0.15182	1.99959	2.01843	1.87699	0.06081
	50	2.00307	0.11355	1.99602	2.01012	0.85485	0.39284
	70	2.00240	0.09845	1.99629	2.00851	0.76954	0.44175
7	30	2.00812	0.13732	1.99960	2.01664	1.87001	0.06177
	50	2.00303	0.10919	0.49905	0.50244	0.87680	0.38081
	70	2.00234	0.09129	1.99667	2.00800	0.80901	0.41870
8	30	2.00819	0.13142	1.99983	2.01614	1.97182	0.04891*
	50	2.00379	0.10439	1.99732	2.01027	1.14932	0.25070
	70	2.00250	0.08444	1.99726	2.00774	0.93714	0.34891

**indicates significance at 1% level and *indicates significance at 5% level

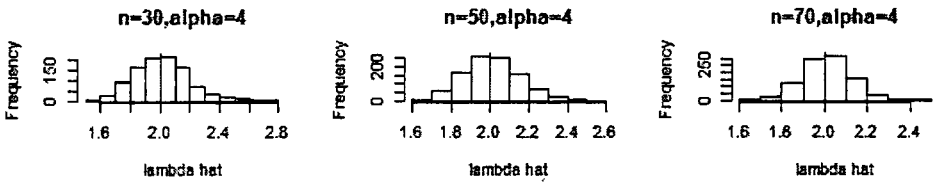

Figure 13: Histograms of estimated parameters λ by MCMC method when $\alpha = 4, \lambda = 2$

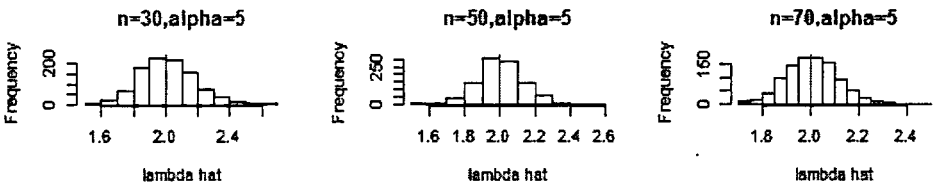

Figure 14: Histograms of estimated parameters λ by MCMC method when $\alpha = 5, \lambda = 2$

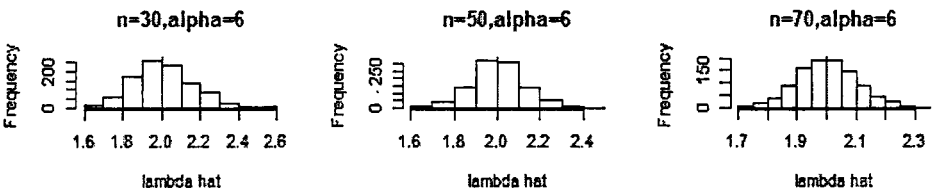

Figure 15: Histograms of estimated parameters λ by MCMC method when $\alpha = 6, \lambda = 2$

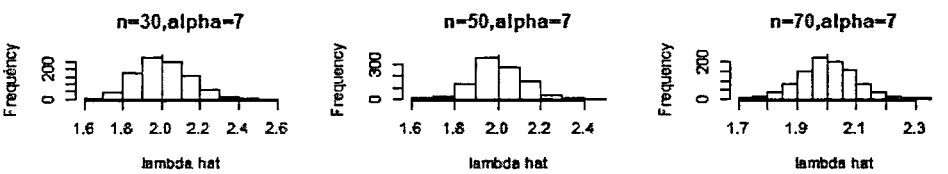

Figure 16: Histograms of estimated parameters λ by MCMC method when $\alpha = 7, \lambda = 2$

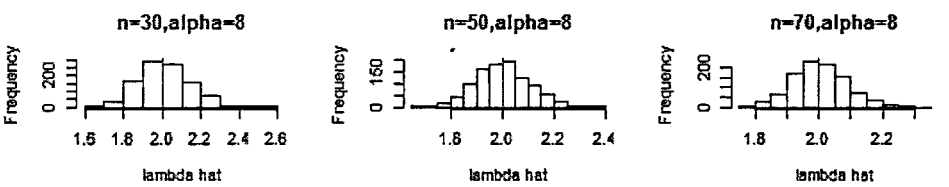

Figure 17: Histograms of estimated parameters λ by MCMC method when $\alpha = 8, \lambda = 2$

Table 4: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α), mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by applied Bayes' with MCMC method

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01502	0.18340	2.00364	2.02640	2.58934	0.00976**
	50	2.00957	0.14168	2.00078	2.01837	2.13696	0.03284*
	70	2.00723	0.11793	1.99991	2.01455	1.93772	0.05294
5	30	2.01171	0.16491	2.00148	2.02195	2.24579	0.02494*
	50	2.00659	0.12503	1.99884	2.01435	1.66796	0.09564
	70	2.00482	0.10657	1.99820	2.01143	1.42925	0.15324
6	30	2.00814	0.15058	1.99879	2.01748	1.70878	0.08780
	50	2.00298	0.11322	1.99595	2.01001	0.83206	0.40557
	70	2.00168	0.09803	1.99560	2.00776	0.54175	0.58811
7	30	2.00712	0.13605	1.99868	2.01556	1.65427	0.09839
	50	2.00306	0.10891	0.49889	0.50227	0.88747	0.37504
	70	2.00230	0.09096	1.99666	2.00794	0.79944	0.42423
8	30	2.00804	0.13068	1.99963	2.01585	1.94636	0.05189
	50	2.00351	0.10372	1.99708	2.00995	1.07095	0.28445
	70	2.00220	0.08408	1.99698	2.00742	0.82779	0.40799

**indicates significance at 1% level and *indicates significance at 5% level

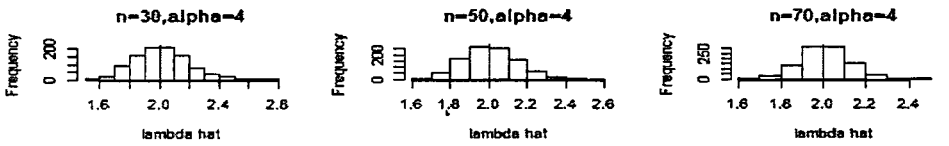

Figure 18: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 4, \lambda = 2$

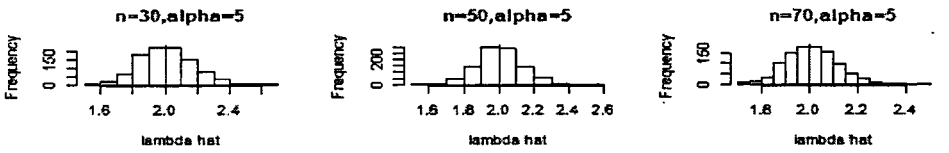

Figure 19: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 5, \lambda = 2$

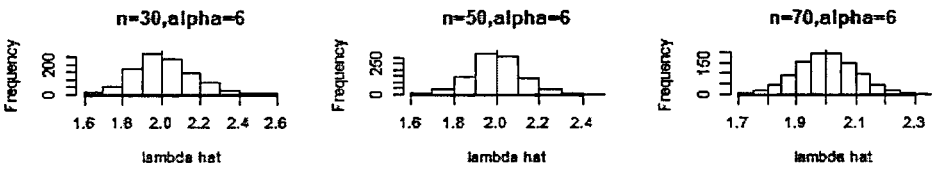

Figure 20: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 6, \lambda = 2$

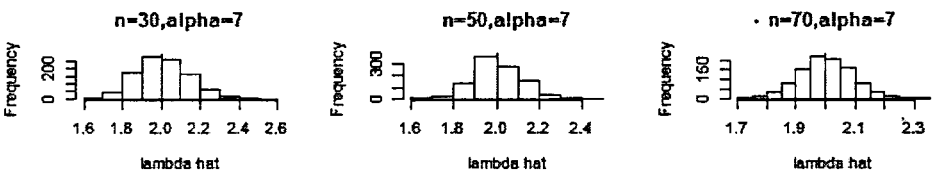

Figure 21: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 7, \lambda = 2$

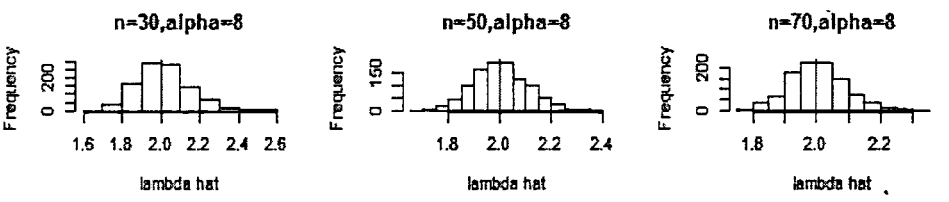

Figure 22: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 8, \lambda = 2$

From Tables 1-4, we can be summarized the estimation method that the estimated parameters are equal the true parameter shown in Table 5 and 6 based on α.

Table 5: An efficiency of parameter estimation in each situation with $\alpha = 0.01$

n	α (Shape Parameter)				
	4	5	6	7	8
30	Bayes'	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC
50	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC
70	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC

Table 6: An efficiency of parameter estimation in each situation with $\alpha = 0.05$

n	α (Shape Parameter)				
	4	5	6	7	8
30	-	-	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	Bayes'- MCMC
50	ML Bayes' MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC
70	Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC

2.6 Conclusion

In this research, we have tested the hypothesis of mean among estimated parameter (λ) from gamma distribution and mean of estimator from ML, Bayes', MCMC, and Bayes'-MCMC method. Through a simulation study, ML method performs a good method in most all cases except of moderate sample size and small shape parameter. The Bayes' method is used the parameter from gamma distribution to estimate parameter of prior distribution, so it makes the posterior estimator is outperforms than the other methods in all cases at significance level 0.01. The MCMC method is a reasonable working method as well as Bayes' and ML when the sample sizes and shape parameter are presented on moderate values. However, we apply Bayes'- MCMC that shown the good estimator more than the other method especially large shape parameter and large sample sizes. For this case, the user can

use Bayes'- MCMC method without to set the parameter of prior distribution on Bayes' method.

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