DISCOVERING CHARACTERISTIC BOUNDS OF ROUGH SET THEORY FOR FUZZY SETS

KIATRISAK PALASIT

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หัวข้อวิทยานิพนธ์	การค้นหาลักษณะขอบเขตของทฤษฎีรัฟเซตสำหรับเซตวิภัชนัย
นักศึกษา	นายเกียรติศักดิ์ ปาละสิทธิ์
รหัสประจำตัว	48067402
ปริญญา	วิทยาศาสตรมหาบัณฑิต
สาขาวิชา	คณิตศาสตร์ประยุกต์
พ.ศ.	2552
อาจารย์ที่ปรึกษาวิทยานิพนธ์	ดร.พรรณทิพย์ ภัทรอินทากร

บทคัดย่อ

ในงานวิจัยนี้จะศึกษาวิเคราะห์ความสัมพันธ์ในลักษณะสหวิทยาการ ระหว่างทฤษฎีรัฟเซตและทฤษฎีเซตวิภัชนัย ซึ่งเป็นคณิตศาสตร์แขนงใหม่ที่ได้รับความสนใจจากนักวิจัยทั่วโลกในขณะนี้

โดยจะมุ่งเน้นการศึกษาคุณสมบัติขั้นสูงทางคณิตศาสตร์ที่เรียกว่า ขอบเขตรัฟเนสของเซตวิภัชนัยในเอกภพสัมพัทธ์ด้วยการนำเสนอทฤษฎีบทใหม่ พร้อมการพิสูจน์ของขอบเขตบนและล่างรัฟเนสของการดำเนินการเซตวิภัชนัยยูเนียนและอินเตอร์เซกชัน พร้อมทั้งศึกษาคุณสมบัติที่น่าสนใจต่าง ๆ ในวิทยานิพนธ์นี้

Thesis Title	Discovering Characteristic Bounds of Rough Set Theory for Fuzzy Sets
Student	Mr. Kiatrisak Palasit
Student ID	48067402
Degree	Master of Mathematics
Program	Applied Mathematics
Thesis Advisor	Dr. Puntip Pattaraintakorn

ABSTRACT

In this research, we study advanced characteristics of roughness bounds for fuzzy sets in the universe of discourse. New theorems for roughness bounds for fuzzy set operations are established and proven. Interestingly, the new upper and lower bounds reveal hidden relations between rough set theory and fuzzy set theory. The bounds of such fuzzy set operations can be determined from the roughness measure efficiently and accurately.

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CHAPTER 1

INTRODUCTION

1.1 Importance and Motivation

In 1982, rough set theory was introduced by Pawlak [38], which has emerged as another major mathematical approach for managing uncertainty arising from inexact, noisy, or incomplete information. Rough set theory has found many interesting applications [7, 9, 11, 14, 17, 24, 30, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 61, 66, 67, 70, 71, 76]. The rough sets approach seems to be of fundamental importance to logic, AI (Artificial Intelligence) and cognitive sciences, especially in the areas of machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, inductive reasoning and pattern recognition [10, 18, 20, 27, 36, 50, 54, 61, 62].

In 1965, the theory of fuzzy sets was introduced by Zadeh [74]. It is a generalization of classical sets by allowing partial membership. It provides a more realistic framework for modeling the ill-defined boundary of a class. Its applications are in pattern recognition, industrial control, medical diagnosis, image processing, market decision-making, and data mining [2, 4, 6, 12, 13, 14, 19, 23, 26, 29, 31, 33, 34, 35, 51, 52, 56, 57, 58, 59, 73, 75]. However, a fundamental difficulty with fuzzy set theory is the semantical interpretations of the degrees of membership.

Both theories represent two different approaches to handling vagueness. Fuzzy sets express the gradualness of knowledge by fuzzy membership whereas rough sets express granularity of knowledge by the indiscernibility relation. Extensive applications of rough set theory and fuzzy set theory are found in various fields [3, 8, 11, 12, 15, 21, 22, 31, 32, 67, 68, 69, 72, 77, 79]. Many researches have studied differences and connections of rough set theory and fuzzy set theory. A crucial integration is called the *roughness measure of fuzzy sets* where some notions of rough sets and fuzzy sets are integrated [3].

Nevertheless, in our view, roughness bounds of fuzzy set operations require more investigation. For this reason, we derive the bounds of roughness measures for fuzzy set operations, namely, union and intersection. Moreover, we study lower and upper bounds on the roughness measure of intersections.

1.2 Objectives

The objectives of the research are as follows.

1. To derive lower and upper bounds on the roughness measure of the fuzzy set operations union and intersection for two sets.
2. Generalize the results from 1. to operations involving more than two sets.

1.3 Scope of the Study

We study the properties of approximation and roughness of rough set theory and fuzzy set theory. The scope of this research is as follows.

1. We are interested particularly in union and intersection of fuzzy set operations.
2. We consider lower and upper approximations of fuzzy sets which depend on parameters α, β, where $0 < \beta \le \alpha \le 1$, and $\alpha-$cut $\underline{\mathscr{A}}_{\alpha}$, $\beta-$cut $\overline{\mathscr{A}}_{\beta}$ of the fuzzy sets $\underline{\mathscr{A}}$, $\overline{\mathscr{A}}$, respectively.

1.4 Expected Results

The benefits of the research are as follows.

1. New lower and upper bounds of roughness measure of the fuzzy sets operations union and intersection for two sets.
2. New lower and upper bounds of roughness measure of the fuzzy sets operations union and intersection in more general forms.

1.5 Research Methodology

Process of the Study

1.5.1 Study and review literature on theoretical concepts of rough set theory and fuzzy set theory.

1.5.2 Study the roughness measure of fuzzy sets and related work.

1.5.3 Derive roughness lower and upper bounds for fuzzy set operations.

1.5.4 Propose new theorems and their proofs on roughness lower and upper bounds of the fuzzy set operations union and intersection for two sets.

1.5.5 Propose new theorems and their proofs on roughness lower and upper bounds of the fuzzy set operations union and intersection in greater generality.

1.5.6 Refine the study and conclude, make the suggestions for further work and write the complete thesis.

CHAPTER 2

DEFINITIONS AND THEOREMS

We introduce in this chapter 2 some definitions and notations that will be used in the present research. In Section 2.1 and Section 2.2, we describe rudiments of rough set and fuzzy set theories, respectively. Section 2.3 presents a brief introduction to rough fuzzy sets. In Section 2.4, we present the roughness measure of a fuzzy set and give some properties which will be used in this thesis. Finally, we describe two operators: the certain increment operator and the uncertain decrement operator which will contribute to finding roughness bounds of the fuzzy set operations in Chapter 3.

2.1 Rough Set Theory

The theory of rough sets, proposed by Pawlak [38], provides a formal tool to deal with imprecise or incomplete information. We begin with an introduction to basic definitions of rough set theory taken from [1].

Let $\boldsymbol{U}$ be a finite non-empty set called *universe* and $\boldsymbol{R}$ be an equivalence relation on $\boldsymbol{U}$, i.e., $\boldsymbol{R}$ is reflexive, symmetric, and transitive. The pair $<\boldsymbol{U}, \boldsymbol{R}>$ is called an *approximation space* [1]. We use $\boldsymbol{U/R} = \{X_1, X_2, \ldots, X_m\}$ to denote quotient set which is the family of all equivalence classes of $\boldsymbol{R}$ where X_i is an equivalence class of $\boldsymbol{R}$, $i = 1,2,3,\ldots,m$. We use $\boldsymbol{R}(x)$ to denote an equivalence class in $\boldsymbol{R}$ containing an element $x \in \boldsymbol{U}$. This relation $\boldsymbol{R}$ decomposes set $\boldsymbol{U}$ into disjoint classes in which two elements x and y are in the same class if and only if $(x, y) \in \boldsymbol{R}$. If two elements x and y in $\boldsymbol{U}$ belong to the same equivalence class $X_i \in \boldsymbol{U/R}$, $i = 1,2,3,\ldots,m$, we say that x and y are indistinguishable.

In rough set based data analysis, the equivalence relation in an approximation space is commonly interpreted via the notion of information system. An *information system* is defined by $<\boldsymbol{U}, \boldsymbol{A}>$ where $\boldsymbol{U}$ is the universe and $\boldsymbol{A}$ is a set of attributes. Let V_a be the set of attribute values and a: $\boldsymbol{U} \rightarrow V_a$. Each subset $\boldsymbol{B}$ of the attribute set $\boldsymbol{A}$ induces an equivalence relation $\boldsymbol{I}(\boldsymbol{B})$ is called indiscernibility relation which is defined as follows.

Definition 2.1.1 (*Indiscernibility Relation* [43]) Given an information system <***U, A***> and a subset $\boldsymbol{B} \subseteq \boldsymbol{A}$, an *indiscernibility relation*, ***I*(*B*)**, is defined as

$$I(B) = \{(x, y) \in U \times U \mid a(x) = a(y) \text{ for all } a \in A\}$$

where $a(x)$ denotes the value of attribute a for element x.

$I(B)$ is called the B-indiscernibility relation. If $(x, y) \in I(B)$, then objects x and y are indiscernible from each other by attributes from B. The equivalence classes of the B-indiscernibility relation are denoted by $B(x)$. Next, we give some formal definitions of two rough set approximations and boundary region.

Definition 2.1.2 (*Lower Approximation*) Let <*U*, *A*> be an information system. For any subset $X \subseteq U$ and $B \subseteq A$, ***B-lower approximation*** of X, $\underline{B}(X)$, is defined as

$$\underline{B}(X) = \bigcup_{x \in U} \{B(x) \mid B(x) \subseteq X\}.$$

Definition 2.1.3 (*Upper Approximation*) Let <*U*, *A*> be an information system. For any subset $X \subseteq U$ and $B \subseteq A$, ***B-upper approximation*** of X, $\overline{B}(X)$, is defined as

$$\underline{B}(X) = \bigcup_{x \in U} \{B(x) \mid B(x) \cap X \neq \varnothing\}.$$

Definition 2.1.4 (*Boundary Region*) Let <*U*, *A*> be an information system. For any subset $X \subseteq U$ and $B \subseteq A$, ***B-boundary region*** of X, $BN_B(X)$, is defined as

$$BN_B(X) = \overline{B}(X) - \underline{B}(X).$$

Remark 2.1.1 As we can see from Definitions 2.1.2 – 2.1.4, they are expressed in terms of granules (small pieces) of knowledge. The lower approximation is the union of all granules which are entirely included in the set. The upper approximation is the union of all granules which have non-empty intersection with the set. The boundary region is the difference between upper approximation and lower approximation.

More precisely, the objects in $\underline{B}(X)$ can be classified with certainty as members of X on the basis of knowledge in B, while the objects in $\overline{B}(X)$ can be only classified as possible members of X on the basis of knowledge in B. The set $BN_B(X)$ consists of those objects that we cannot decisively classify in X on the basis of knowledge in B.

Definition 2.1.5 (*Crisp*) Let <*U*, *A*> be an information system. For any subset $X \subseteq U$ and $B \subseteq A$, set X is *crisp*, if the boundary region of X is empty.

Definition 2.1.6 (*Rough*) Let <*U*, *A*> be an information system. For any subset $X \subseteq U$ and $B \subseteq A$, set X is *rough*, if the boundary region of X is non-empty.

In [43], Pawlak discussed two numerical characterizations of the imprecision of a subset X as provided below.

Definition 2.1.7 (*Accuracy of Approximation*) Let <*U*, *A*> be an information system. For any subset $X \subseteq U$ and $B \subseteq A$, the *accuracy of approximation*, $\alpha_B(X)$, is defined as

$$\alpha_B(X) = \frac{|\underline{B}(X)|}{|\overline{B}(X)|}$$

where $X \neq \varnothing$, $|\underline{B}(X)|$ and $|\overline{B}(X)|$ are the cardinalities of $\underline{B}(X)$ and $\overline{B}(X)$, respectively.

Definition 2.1.8 (*Roughness of Approximation*) Let <*U*, *A*> be an information system. For any subset $X \subseteq U$ and $B \subseteq A$, the *roughness of approximation*, $\rho_B(X)$, is defined as:

$$\rho_B(X) = \frac{|BN_B(X)|}{|\overline{B}(X)|} = \frac{|\overline{B}(X)| - |\underline{B}(X)|}{|\overline{B}(X)|}$$

where $X \neq \varnothing$, $|\underline{B}(X)|$, $|\overline{B}(X)|$ and $|BN_B(X)|$ are the cardinalities of $\underline{B}(X)$, $\overline{B}(X)$ and $BN_B(X)$, respectively.

Remark 2.1.2

2.1.2.1 As $\underline{B}(X) \subseteq X \subseteq \overline{B}(X)$, we have $0 \leq \rho_R(X) \leq 1$.

2.1.2.2 By convention, when $X = \varnothing$, $\underline{B}(X) = \overline{B}(X) = \varnothing$, $\rho_B(X) = 0$.

2.1.2.3 $\rho_B(X) = 0$ if and only if set X is crisp, otherwise set X is rough in $<U, A>$.

2.1.2.4 The relationship between the roughness of approximation and accuracy of approximation is

$$\rho_B(X) = 1 - \alpha_B(X).$$

2.2 Fuzzy Set Theory

Lofti Zadeh proposed a new mathematical approach to handle vagueness called *fuzzy set theory* in 1965 [74]. It provides a better mechanism to describe behavior of data which is too ill-defined to admit precise mathematical analysis by classical approaches. In fuzzy set theory, an element belongs to a set to some degree k ($0 \leq k \leq 1$). Contrarily, classical set theory permits the membership of elements in relation to a set with precise condition: an element either belongs or does not belong to the given set. For example, in classical set theory one can be definitely sick or healthy, whereas in fuzzy set theory someone is in healthy with 70 percent (i.e., degree 0.7). Thus, fuzzy sets provide a convenient tool for representing vague concepts by employing the fuzzy membership function. Next, we provide required background of fuzzy sets taken from [3].

Let U be a finite non-empty set called the *universe*. The fuzzy membership function of a *fuzzy set* F of U is a mapping from U into the interval $[0,1]$:

$$\mu_F(x) : U \to [0,1]$$

where for each $x \in U$, we call $\mu_F(x)$ the *membership degree* of x in F.

Zadeh [74] defined three basic operations on classical sets for fuzzy sets i.e., complement, intersection, and union respectively as follows:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

$$\mu_{A\cup B}\left(x\right)=\min\left(\mu_A\left(x\right),\mu_B\left(x\right)\right)$$
$$\mu_{A\cap B}\left(x\right)=\max\left(\mu_A\left(x\right),\mu_B\left(x\right)\right)$$

where ***A*** and ***B*** are fuzzy sets in ***U***.

Given a number $\alpha \in \left(0,1\right]$, the α *-cut*, or α *-level set*, of ***F*** is defined as [32]:

$$F_\alpha = \left\{x \in U \middle| \mu_F\left(x\right) \geq \alpha\right\},$$

which is a subset of ***U***.

2.3 Rough Fuzzy Set

In this section, we recall how rough sets and fuzzy sets are integrated, as rough fuzzy sets and allied notions from [5, 10, 78]. Let A: $U \rightarrow$ [0, 1] be a fuzzy set in ***U***. We use $A(x)$, $x \in U$, to denote the membership function that gives the degree of membership of x in A [1].

Definition 2.3.1 (*Lower and Upper Approximations*) The *lower* and *upper approximations* of fuzzy set A in ***U***, denoted $\underline{A}$ and $\overline{A}$, respectively, are defined as fuzzy sets in $U/R = \{X_1,...,X_n\}$, $\underline{A}$, $\overline{A}$: $U/R \rightarrow [0,1]$, such that

$$\underline{A}(X_i) = inf_{x \in X_i} A(x) \text{ and}$$

$$\overline{A}(X_i) = sup_{x \in X_i} A(x), \qquad i = 1,2,3,...,n$$

where *inf* denotes minimum and *sup* denotes maximum.

Definition 2.3.2 If $\underline{A}$ and $\overline{A}$ are lower and upper approximations of the fuzzy set A in ***U***. Then < $\underline{A}$, $\overline{A}$ > is called a *rough fuzzy set*.

Equivalently, one may call the triple < ***U, R, A***> a *rough fuzzy set* [10].

Remark 2.3.1 When A is a crisp set, $\underline{A}$ and $\overline{A}$ reduce to the collection of the equivalence classes constituting its lower and upper approximations in $\langle U, R\rangle$, respectively.

Definition 2.3.3 [78] Fuzzy sets $\underline{\mathscr{A}}, \overline{\mathscr{A}} : U \to [0,1]$ are defined as follows:

$$\underline{\mathscr{A}}(x) = \underline{A}(X_i) \text{ and } \overline{\mathscr{A}}(x) = \overline{A}(X_i)$$

if $x \in X_i$, $i = 1,2,3,\ldots,n$.

Remark 2.3.2 $\underline{\mathscr{A}}(x)$ and $\overline{\mathscr{A}}(x)$ are fuzzy sets with constant membership on the equivalence classes of U. For any x in U, $\underline{\mathscr{A}}(x)$ $(\overline{\mathscr{A}}(x))$ can be viewed as the degree to which x definitely (possible) belongs to the fuzzy set A.

2.4 Roughness Measure of the Fuzzy Sets

In [3], Banerjee and Pal proposed a roughness measure for fuzzy sets in a given approximation space which can be described as the following.

Definition 2.4.1 (*α-cut Set and β-cut Set*) Let U be the universe and $\mathscr{A}$: $U \to [0, 1]$ be a fuzzy set in U. We use $\mathscr{A}(x)$, $x \in U$, to denote the membership function that gives the degree of membership of x in A. We consider parameters α and β, where $0 < \beta \le \alpha \le 1$. The *α-cut set* ($\underline{\mathscr{A}}_\alpha$) and *β-cut set* ($\overline{\mathscr{A}}_\beta$) of the fuzzy set $\underline{\mathscr{A}}$, $\overline{\mathscr{A}}$ are, respectively, defined as [10]:

$$\underline{\mathscr{A}}_\alpha = \left\{ x \,\middle|\, \underline{\mathscr{A}}(x) \ge \alpha \right\} \text{ and}$$
$$\overline{\mathscr{A}}_\beta = \left\{ x \,\middle|\, \overline{\mathscr{A}}(x) \ge \beta \right\}.$$

We give some observations as follows: $\underline{\mathscr{A}}(x), \left(\overline{\mathscr{A}}(x)\right)$ is the collection of objects in U with α, (β) as the minimum degree of definite (possible) membership in the fuzzy set A. In other words, α, β act as the thresholds of definiteness and possibility in membership of the object of U to A, respectively. We call $\underline{\mathscr{A}}(x)$ as the

α *-lower approximation* and $\overline{\mathscr{A}}(x)$ as the β *-upper approximation* of the fuzzy set A in <U, R>.

Definition 2.4.2 (*Roughness Measure* [3]) A *roughness measure*, $\rho_A^{\alpha,\beta}$, of the fuzzy set A in U with respect to parameters α and β, where $0 < \beta \le \alpha \le 1$, and the approximation space <U, R>, where $|X|$ denotes cardinality of a set X is defined as

$$\rho_A^{\alpha,\beta} = 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|}.$$

Property 2.4.1 [3]

2.4.1.1 $\overline{\mathscr{A} \cup \mathscr{B}}_\beta = \overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta$.

2.4.1.2 $\underline{\mathscr{A} \cap \mathscr{B}}_\alpha = \underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha$.

2.4.1.3 $\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \subseteq \underline{\mathscr{A} \cup \mathscr{B}}_\alpha$.

2.4.1.4 $\overline{\mathscr{A} \cap \mathscr{B}}_\beta \subseteq \overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta$.

2.4.1.5 $\underline{\mathscr{A}}_\alpha \subseteq A \subseteq \overline{\mathscr{A}}_\beta$.

2.5 Two New Operators in Rough Set Theory

In [32, 78], Zhang et al. introduced two new operators which can improve Properties 2.4.1.3 and 2.4.1.4 for roughness properties of the fuzzy sets as described below.

Definition 2.5.1 (*Basic Factor of Inducing Rough and Correlation Basic Factor of Inducing Rough* [32, 78]) Let U be the universe and R be an equivalence relation on U. Let $X \subseteq U$. For any element $x \in X$, the two sets

$$h_X(x) = R(x) - X$$
$$l_X(x) = R(x) - h_X(x)$$

are called the *basic factor of inducing rough* and the *correlation basic factor of inducing rough* of X, respectively.

We can see clearly that $l_x(x)\cap h_x(x)=\varnothing$, $l_x(x)\cup h_x(x)=\boldsymbol{R}(x)$. Thus, $h_x(x)$ is the collection of those objects which are in $\boldsymbol{R}(x)$ but not in the set $\boldsymbol{X}$. $l_x(x)$ is the collection of those objects which are in both $\boldsymbol{R}(x)$ and $\boldsymbol{X}$.

Definition 2.5.2 (***R**-inducing Rough Region and **R**-inducing Rough Correlation Region* [32, 78]) Let $\boldsymbol{U}$ be the universe and $\boldsymbol{R}$ be an equivalence relation on $\boldsymbol{U}$. Let $\boldsymbol{X}\subseteq \boldsymbol{U}$, for any element $x\in \boldsymbol{X}$, the two sets

$$\boldsymbol{H}(x)=\bigcup\{h_X(x)|x\in \boldsymbol{RB}_{\boldsymbol{R}}(\boldsymbol{X})\cap \boldsymbol{X}\}$$
$$\boldsymbol{L}(x)=\bigcup\{l_X(x)|x\in \boldsymbol{RB}_{\boldsymbol{R}}(\boldsymbol{X})\cap \boldsymbol{X}\}$$

are called ***R**-inducing rough region* and ***R**-inducing rough correlation region* of $\boldsymbol{X}$, respectively.

Definition 2.5.3 (*Certain Increment Operator* [78]) Let $\boldsymbol{U}$ be the universe and $\boldsymbol{R}$ be an equivalence relation on $\boldsymbol{U}$. Let $\boldsymbol{X}$, $\boldsymbol{Y}\subseteq \boldsymbol{U}$. When $\boldsymbol{X}$ is extended by $\boldsymbol{Y}$, $\underline{Z}_{(.)}(.):\boldsymbol{U}\times\boldsymbol{U}\to\boldsymbol{U}$ defined by

$$\underline{Z}_X(\boldsymbol{Y})=\bigcup\left\{[x]_R\,\middle|\,x\in L\left(\boldsymbol{X}\right),l_X\left(x\right)\not\subset \boldsymbol{Y},h_X\left(x\right)\subseteq \boldsymbol{Y}\right\}$$

is called the *certain increment operator* of $\boldsymbol{X}$.

We can see that $\underline{Z}_X(\boldsymbol{Y})$ is the collection of those objects in which the certain information of $\boldsymbol{X}\cup\boldsymbol{Y}$ is larger than the union of the certain information of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Definition 2.5.4 (*Uncertain Decrement Operator* [78]) Let $\boldsymbol{U}$ be the universe and $\boldsymbol{R}$ be an equivalence relation on $\boldsymbol{U}$. Let $\boldsymbol{X}$, $\boldsymbol{Y}\subseteq \boldsymbol{U}$. When $\boldsymbol{X}$ is cut by $\boldsymbol{Y}$, $\overline{Z}_{(.)}(.):\boldsymbol{U}\times\boldsymbol{U}\to\boldsymbol{U}$ defined by

$$\overline{Z}_X(\boldsymbol{Y})=\bigcup\left\{[x]_R\,\middle|\,x\in L\left(\boldsymbol{X}\right),l_X\left(x\right)\cap \boldsymbol{Y}=\varnothing,h_X\left(x\right)\cap \boldsymbol{Y}\neq\varnothing\right\}$$

is called the *uncertain decrement operator* of $\boldsymbol{X}$.

$\overline{Z}_X(Y)$ is the collection of those objects in which the uncertain information of $X \cap Y$ is less than the intersection of the uncertain information of X and Y.

Property 2.5.1 [78]

2.5.1.1 $\underline{Z}_X(Y) = \underline{Z}_Y(X)$.
2.5.1.2 $\overline{Z}_X(Y) = \overline{Z}_Y(X)$.
2.5.1.3 $\underline{Z}_{\mathscr{A}_\alpha}(\mathscr{B}_\alpha) \cap \left(\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha\right) = \varnothing$.
2.5.1.4 $\overline{Z}_{\mathscr{A}_\beta}(\mathscr{B}_\beta) \subseteq \overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta$.

Next, let U be the universe and R be an equivalence relation on U. Let $\mathscr{A}, \mathscr{B}: U \rightarrow [0, 1]$ be two fuzzy sets in U with respect to parameters α and β, where $0 < \beta \leq \alpha \leq 1$. We have the following properties.

Property 2.5.2 [78]

2.5.2.1 $\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{Z}_{\mathscr{A}_\alpha}(\mathscr{B}_\alpha) = \underline{\mathscr{A} \cup \mathscr{B}}_\alpha$,
2.5.2.2 $\overline{\mathscr{A} \cap \mathscr{B}}_\beta = \overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta - \overline{Z}_{\mathscr{A}_\beta}(\mathscr{B}_\beta)$.

We can see that Properties 2.4.1.3 and 2.4.1.4 are now equalities (compare to Properties 2.5.2.1 and 2.5.2.2) when we use these two new operators: the uncertain decrement operator and the uncertain decrement operator to improve them. Properties 2.5.2.1 and 2.5.2.2 provide a basis for our roughness bounds of fuzzy set operations as presented in Chapter 3.

2.6 Literature Review

In this section, we consider both theoretical and practical literature review.

In 1990, Dubois and Prade [15] introduced the rigorous combinations of the above theories called *rough fuzzy sets* and *fuzzy rough sets*. Theoretical rough sets and fuzzy sets integrations were then established. These two notions were found many advantageous applications in various fields [3, 4, 5, 11, 13, 16, 28, 35, 37, 52, 53, 55, 56, 63, 64, 77]. In particular, basic concept of rough fuzzy sets was used to define the roughness measure of fuzzy sets [3]. Some possible applications for handling uncertainty in the field of pattern recognition were also conducted [4, 5, 13, 15, 16, 19, 21, 22, 26, 27, 32, 52, 54, 56, 57].

In 1996, Banerjee and Pal [3] proposed a significant measurement for rough fuzzy sets called *roughness measure of fuzzy sets*. Their roughness measure of fuzzy sets depends on the parameters α, β where $0 < \beta \leq \alpha \leq 1$. α and β are the minimum degree of definite and possible membership of a fuzzy set, respectively. Several properties of this measurement were listed in [3].

In 2004, Zhang et al. [78] invented two new operators for rough set and fuzzy set theories called *certain increment operator* and *uncertain decrement operator*. They used these two operators to modify two important inequalities (Properties 2.4.1.3 and 2.4.1.4) to equalities. By using such equalities, many properties in rough set theory, i.e., union and intersection operations can be redefined. They also employed these operators to prove that the operations of union and intersection in rough set approximations satisfy the Commutative, Associative and Distributive laws. In this thesis, we use these operators to derive roughness bounds of fuzzy sets.

In 2005, Huynh and Nakamori [22, 32] defined an alternative roughness measure for fuzzy sets based on the notions of the mass assignment and its α-cuts. It is a parameter-free measure of roughness of fuzzy sets that is a generalization of Pawlak's roughness measure. The advantage is that it avoids undesirable properties held by Banerjee and Pal's roughness measure as discussed in [3]. The relational database was analyzed by their roughness measure. They also discussed how their roughness measure can describe the quality of rough set approximation of an application called fuzzy classification.

In 2005, Rao and et al. [27] proposed to use the rough fuzzy sets as the model for images. Roughness measures of fuzzy sets can be used to optimize the intensity of images in object extraction. The experimental results showed that extracted objects with their approach provide higher accuracy compared to Shannon's probabilistic entropy approach.

In 2006, Yang and John [66] studied a parallel issue to our study which is roughness bounds for rough set operations. The results showed that operand's roughness can determine lower and upper bounds of different rough set operations. They reported that roughness measure is an important indicator for decision making applications.

However, in our view, roughness bounds of fuzzy set operations require more investigations. For this reason, we investigate the bounds of roughness measures for fuzzy set operations, namely, union and intersection that will be discussed and established in Chapter 3.

CHAPTER 3

RESULTS OF ANALYSIS

In this chapter 3, we propose new theorems on roughness lower and upper bounds of the fuzzy set operations as the following.

3.1 Roughness Lower and Upper Bounds for the Union

Theorem 1 (*Lower Bound of Union*) A roughness lower bound for the union of two fuzzy sets ***A***, ***B*** in ***U***, $\rho_{A\cup B}^{\alpha,\beta}$ with respect to α,β is

$$\rho_{A\cup B}^{\alpha,\beta} \geq \rho_{A}^{\alpha,\beta} + \rho_{B}^{\alpha,\beta} - 1 - Z_*$$

where $0 < \beta \leq \alpha \leq 1$ and $Z_* = \dfrac{\left|\underline{Z}_{\mathscr{A}_\alpha}(\mathscr{B}_\alpha)\right|}{max\left\{\left|\overline{\mathscr{A}}_\beta\right|,\left|\overline{\mathscr{B}}_\beta\right|\right\}}$.

Proof. By Definition 2.4.2 and Property 2.4.1.1, we have

$$\rho_{A\cup B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}\cup\mathscr{B}}_\beta\right|} = 1 - \frac{\left|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}}_\beta\right|}.$$

By Property 2.5.2.1, we can write

$$\rho_{A\cup B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}}_\beta\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_\alpha\cup\underline{\mathscr{B}}_\alpha\cup\underline{Z}_{\mathscr{A}_\alpha}\left(\mathscr{B}_\alpha\right)\right|}{\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}}_\beta\right|}.$$

For any finite sets ***X***, ***Y***, we have $|X\cup Y| \leq |X| + |Y|$ and $|X\cup Y| \geq max\{|X|,|Y|\}$, therefore

$$\rho_{A\cup B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cup \underline{\mathscr{B}}_{\alpha} \cup \underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{A}}_{\beta} \cup \overline{\mathscr{B}}_{\beta}\right|}$$

$$\geq 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right| + \left|\underline{\mathscr{B}}_{\alpha}\right| + \left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{max\left\{\left|\overline{\mathscr{A}}_{\beta}\right|, \left|\overline{\mathscr{B}}_{\beta}\right|\right\}}.$$

If $\left|\overline{\mathscr{A}}_{\beta}\right| \geq \left|\overline{\mathscr{B}}_{\beta}\right|$, then

$$\rho_{A\cup B}^{\alpha,\beta} \geq 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right| + \left|\underline{\mathscr{B}}_{\alpha}\right| + \left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{max\left\{\left|\overline{\mathscr{A}}_{\beta}\right|, \left|\overline{\mathscr{B}}_{\beta}\right|\right\}}$$

$$= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right| + \left|\underline{\mathscr{B}}_{\alpha}\right| + \left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$$

$$= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|} - \frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|} - \frac{\left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}.$$

If $\left|\overline{\mathscr{A}}_{\beta}\right| \geq \left|\overline{\mathscr{B}}_{\beta}\right|$, then $\frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|} \leq \frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}$. We also have $\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|} = 1 - \rho_{A}^{\alpha,\beta}$, thus

$$\rho_{A\cup B}^{\alpha,\beta} \geq 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|} - \frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|} - \frac{\left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$$

$$= 1 - \left(1 - \rho_{A}^{\alpha,\beta}\right) - \left(1 - \rho_{B}^{\alpha,\beta}\right) - \frac{\left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$$

$$= \rho_{A}^{\alpha,\beta} + \rho_{B}^{\alpha,\beta} - 1 - \frac{\left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}. \qquad (3.1)$$

Similarly, if $\left|\overline{\mathscr{B}}_{\beta}\right| \geq \left|\overline{\mathscr{A}}_{\beta}\right|$, then

$$\rho_{A\cup B}^{\alpha,\beta} \geq 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right| + \left|\underline{\mathscr{B}}_{\alpha}\right| + \left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{max\left\{\left|\overline{\mathscr{A}}_{\beta}\right|, \left|\overline{\mathscr{B}}_{\beta}\right|\right\}}$$

$$= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right| + \left|\underline{\mathscr{B}}_{\alpha}\right| + \left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}$$

$$= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|} - \frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|} - \frac{\left|\underline{Z}_{\mathscr{A}_{\alpha}}\left(\mathscr{B}_{\alpha}\right)\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}.$$

If $\left|\overline{\mathscr{B}}_\beta\right| \geq \left|\overline{\mathscr{A}}_\beta\right|$, then $\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{B}}_\beta\right|} \leq \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$. We also have $\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} = 1 - \rho_A^{\alpha,\beta}$, thus

$$\rho_{A\cup B}^{\alpha,\beta} \geq 1 - \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{B}}_\beta\right|} - \frac{\left|\underline{Z}_{\mathscr{A}_\alpha}\left(\mathscr{B}_\alpha\right)\right|}{\left|\overline{\mathscr{B}}_\beta\right|}$$

$$= 1 - \left(1 - \rho_A^{\alpha,\beta}\right) - \left(1 - \rho_B^{\alpha,\beta}\right) - \frac{\left|\underline{Z}_{\mathscr{A}_\alpha}\left(\mathscr{B}_\alpha\right)\right|}{\left|\overline{\mathscr{B}}_\beta\right|}$$

$$= \rho_A^{\alpha,\beta} + \rho_B^{\alpha,\beta} - 1 - \frac{\left|\underline{Z}_{\mathscr{A}_\alpha}\left(\mathscr{B}_\alpha\right)\right|}{\left|\overline{\mathscr{B}}_\beta\right|}. \tag{3.2}$$

From (3.1) and (3.2), we get

$$\rho_{A\cup B}^{\alpha,\beta} \geq \rho_A^{\alpha,\beta} + \rho_B^{\alpha,\beta} - 1 - \frac{\left|\underline{Z}_{\mathscr{A}_\alpha}\left(\mathscr{B}_\alpha\right)\right|}{max\left\{\left|\overline{\mathscr{A}}_\beta\right|, \left|\overline{\mathscr{B}}_\beta\right|\right\}}.$$

Let us define $Z_* = \frac{\left|\underline{Z}_{\mathscr{A}_\alpha}\left(\mathscr{B}_\alpha\right)\right|}{max\left\{\left|\overline{\mathscr{A}}_\beta\right|, \left|\overline{\mathscr{B}}_\beta\right|\right\}}$, therefore a roughness lower bound for the union of two fuzzy sets is

$$\rho_{A\cup B}^{\alpha,\beta} \geq \rho_A^{\alpha,\beta} + \rho_B^{\alpha,\beta} - 1 - Z_*. \qquad \square$$

Theorem 2 (*Upper Bound of Union*) A roughness upper bound for the union of two fuzzy sets *A*, ***B*** in ***U***, $\rho_{A\cup B}^{\alpha,\beta}$ with respect to α, β is

$$\rho_{A\cup B}^{\alpha,\beta} \leq \frac{1 - \rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2 - \left(\rho_A^{\alpha,\beta} + \rho_B^{\alpha,\beta}\right)}$$

where $0 < \beta \leq \alpha \leq 1$.

Proof By Definition 2.4.2, Properties 2.4.1.1 and 2.4.1.3, we have

$$\rho_{A\cup B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}\cup\mathscr{B}}_\beta\right|} = 1 - \frac{\left|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta\right|} \leq 1 - \frac{\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta\right|}.$$

For any two crisp sets X and Y, we have $|X \cup Y| \le |X| + |Y|$ and $|X \cup Y| \ge max\{|X|, |Y|\}$,

therefore

$$\rho_{A\cup B}^{\alpha,\beta} \le 1 - \frac{|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta|} \le 1 - \frac{max\{|\underline{\mathscr{A}}_\alpha|, |\underline{\mathscr{B}}_\alpha|\}}{|\overline{\mathscr{A}}_\beta| + |\overline{\mathscr{B}}_\beta|}.$$

If $|\underline{\mathscr{A}}_\alpha| \ge |\underline{\mathscr{B}}_\alpha|$, then

$$\rho_{A\cup B}^{\alpha,\beta} \le 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta| + |\overline{\mathscr{B}}_\beta|} = 1 - \frac{1}{\frac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{A}}_\alpha|} + \frac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{A}}_\alpha|}}.$$

If $|\underline{\mathscr{A}}_\alpha| \ge |\underline{\mathscr{B}}_\alpha|$, then $\frac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{A}}_\alpha|} \le \frac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{B}}_\alpha|}$. Thus, we have

$$\rho_{A\cup B}^{\alpha,\beta} \le 1 - \frac{1}{\frac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{A}}_\alpha|} + \frac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{B}}_\alpha|}}.$$

Since we have $\rho_A^{\alpha,\beta} = 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$ and $1 - \rho_A^{\alpha,\beta} = \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$, thus

$$\begin{aligned}
\rho_{A\cup B}^{\alpha,\beta} &\le 1 - \frac{1}{\frac{1}{1-\rho_A^{\alpha,\beta}} + \frac{1}{1-\rho_B^{\alpha,\beta}}} \\
&= 1 - \frac{\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}{\left(1-\rho_B^{\alpha,\beta}\right)+\left(1-\rho_A^{\alpha,\beta}\right)} \\
&= \frac{\left(1-\rho_B^{\alpha,\beta}\right)+\left(1-\rho_A^{\alpha,\beta}\right)-\left(1-\rho_B^{\alpha,\beta}-\rho_A^{\alpha,\beta}+\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}\right)}{\left(1-\rho_B^{\alpha,\beta}\right)+\left(1-\rho_A^{\alpha,\beta}\right)} \\
&= \frac{1-\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2-\left(\rho_A^{\alpha,\beta}+\rho_B^{\alpha,\beta}\right)}. \qquad (3.3)
\end{aligned}$$

Similarly, if $|\underline{\mathscr{B}}_\alpha| \ge |\underline{\mathscr{A}}_\alpha|$, then

$$\rho_{A\cup B}^{\alpha,\beta} \le 1 - \frac{|\underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}}_\beta| + |\overline{\mathscr{B}}_\beta|} = 1 - \frac{1}{\frac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{B}}_\alpha|} + \frac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{B}}_\alpha|}}.$$

If $|\underline{\mathscr{B}}_\alpha| \geq |\underline{\mathscr{A}}_\alpha|$, then $\dfrac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{B}}_\alpha|} \leq \dfrac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{A}}_\alpha|}$. Thus, we have

$$\rho_{A\cup B}^{\alpha,\beta} \leq 1 - \frac{1}{\dfrac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{A}}_\alpha|} + \dfrac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{B}}_\alpha|}}.$$

Since we have $\rho_A^{\alpha,\beta} = 1 - \dfrac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$ and $1 - \rho_A^{\alpha,\beta} = \dfrac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$, thus

$$\begin{aligned}
\rho_{A\cup B}^{\alpha,\beta} &\leq 1 - \frac{1}{\dfrac{1}{1-\rho_A^{\alpha,\beta}} + \dfrac{1}{1-\rho_B^{\alpha,\beta}}} \\
&= 1 - \frac{\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}{\left(1-\rho_B^{\alpha,\beta}\right)+\left(1-\rho_A^{\alpha,\beta}\right)} \\
&= \frac{\left(1-\rho_B^{\alpha,\beta}\right)+\left(1-\rho_A^{\alpha,\beta}\right)-\left(1-\rho_B^{\alpha,\beta}-\rho_A^{\alpha,\beta}+\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}\right)}{\left(1-\rho_B^{\alpha,\beta}\right)+\left(1-\rho_A^{\alpha,\beta}\right)} \\
&= \frac{1-\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2-\left(\rho_A^{\alpha,\beta}+\rho_B^{\alpha,\beta}\right)}. \qquad (3.4)
\end{aligned}$$

From (3.3) and (3.4), we get a roughness upper bound for the union of two fuzzy sets

$$\rho_{A\cup B}^{\alpha,\beta} \leq \frac{1-\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2-\left(\rho_A^{\alpha,\beta}+\rho_B^{\alpha,\beta}\right)}. \qquad \square$$

3.2 Roughness Lower and Upper Bounds for the Intersection

Theorem 3 (*Lower Bound of Intersection*) A roughness lower bound for the intersection of two fuzzy sets A, B in U, $\rho_{A\cap B}^{\alpha,\beta}$, with respect to α, β is

$$\rho_{A\cap B}^{\alpha,\beta} \geq 1 - \frac{1-\rho_A^{\alpha,\beta}-\rho_B^{\alpha,\beta}+\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2-\rho_A^{\alpha,\beta}-\rho_A^{\alpha,\beta}-M_*\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}$$

where $0 < \beta \leq \alpha \leq 1$ and $M_* = \dfrac{|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta| + |\overline{Z}_{\mathscr{A}_\alpha}(\mathscr{B}_\beta)|}{min\{|\underline{\mathscr{A}}_\alpha|, |\underline{\mathscr{B}}_\alpha|\}}$.

Proof. By Definition 2.4.2 and Property 2.4.1.2, we have

$$\rho_{A\cap B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A} \cap \mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_{\beta}\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_{\beta}\right|}.$$

If $X \subseteq Y$ then $|Y - X| = |Y| - |X|$, from Property 2.5.2.2, we get

$$\rho_{A\cap B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_{\beta}\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta} \cap \overline{\mathscr{B}}_{\beta} - \overline{Z}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta} \cap \overline{\mathscr{B}}_{\beta}\right| - \left|\overline{Z}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}.$$

For any finite sets X, Y we have $|X \cap Y| = |X| + |Y| - |X \cup Y|$ and, $X \cap Y \subseteq X$, $X \cap Y \subseteq Y$ then $|X \cap Y| \le |X|, |X \cap Y| \le |Y|$, therefore

$$\begin{aligned} \rho_{A\cap B}^{\alpha,\beta} &= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta} \cap \overline{\mathscr{B}}_{\beta}\right| - \left|\overline{Z}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|} \\ &\ge 1 - \frac{min\left\{\left|\underline{\mathscr{A}}_{\alpha}\right|, \left|\underline{\mathscr{B}}_{\alpha}\right|\right\}}{\left|\overline{\mathscr{A}}_{\beta}\right| + \left|\overline{\mathscr{B}}_{\beta}\right| - \left|\overline{\mathscr{A}}_{\beta} \cup \overline{\mathscr{B}}_{\beta}\right| - \left|\overline{Z}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}. \end{aligned}$$

If $\left|\underline{\mathscr{A}}_{\alpha}\right| \le \left|\underline{\mathscr{B}}_{\alpha}\right|$, then

$$\begin{aligned} \rho_{A\cap B}^{\alpha,\beta} &\ge 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right| + \left|\overline{\mathscr{B}}_{\beta}\right| - \left|\overline{\mathscr{A}}_{\beta} \cup \overline{\mathscr{B}}_{\beta}\right| - \left|\overline{Z}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|} \\ &= 1 - \frac{1}{\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|} + \frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|} - \frac{\left|\overline{\mathscr{A}}_{\beta} \cup \overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|} - \frac{\left|\overline{Z}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}}. \end{aligned}$$

If $\left|\underline{\mathscr{A}}_{\alpha}\right| \le \left|\underline{\mathscr{B}}_{\alpha}\right|$, then $\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|} \ge \frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}$. We also have $\rho_{A}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$ or $\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|} = \frac{1}{1 - \rho_{A}^{\alpha,\beta}}$, thus

$$\rho_{A\cap B}^{\alpha,\beta} \geq 1-\frac{1}{\frac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{A}}_\alpha|}+\frac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{A}}_\alpha|}-\frac{|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{A}}_\alpha|}-\frac{|\overline{Z}_{\mathscr{A}_\beta}(\mathscr{B}_\beta)|}{|\underline{\mathscr{A}}_\alpha|}}$$

$$\geq 1-\frac{1}{\frac{|\overline{\mathscr{A}}_\beta|}{|\underline{\mathscr{A}}_\alpha|}+\frac{|\overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{B}}_\alpha|}-\frac{|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta|}{|\underline{\mathscr{A}}_\alpha|}-\frac{|\overline{Z}_{\mathscr{A}_\beta}(\mathscr{B}_\beta)|}{|\underline{\mathscr{A}}_\alpha|}}$$

$$= 1-\frac{1}{\frac{1}{(1-\rho_A^{\alpha,\beta})}+\frac{1}{(1-\rho_B^{\alpha,\beta})}-\frac{|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta|+|\overline{Z}_{\mathscr{A}_\beta}(\mathscr{B}_\beta)|}{|\underline{\mathscr{A}}_\alpha|}}.$$

Let we define $M_{|\underline{\mathscr{A}}_\alpha|}=\frac{|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta|+|\overline{Z}_{\mathscr{A}_\beta}(\mathscr{B}_\beta)|}{|\underline{\mathscr{A}}_\alpha|}$, therefore

$$\rho_{A\cap B}^{\alpha,\beta} \geq 1-\frac{1}{\frac{1}{(1-\rho_A^{\alpha,\beta})}+\frac{1}{(1-\rho_B^{\alpha,\beta})}-\frac{|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta|+|\overline{Z}_{\mathscr{A}_\beta}(\mathscr{B}_\beta)|}{|\underline{\mathscr{A}}_\alpha|}}$$

$$= 1-\frac{1}{\frac{1}{(1-\rho_A^{\alpha,\beta})}+\frac{1}{(1-\rho_B^{\alpha,\beta})}-M_{|\underline{\mathscr{A}}_\alpha|}}$$

$$= 1-\frac{1}{\frac{(1-\rho_A^{\alpha,\beta})+(1-\rho_B^{\alpha,\beta})-M_{|\underline{\mathscr{A}}_\alpha|}(1-\rho_A^{\alpha,\beta})(1-\rho_B^{\alpha,\beta})}{(1-\rho_A^{\alpha,\beta})(1-\rho_B^{\alpha,\beta})}}$$

$$= 1-\frac{(1-\rho_A^{\alpha,\beta})(1-\rho_B^{\alpha,\beta})}{(1-\rho_A^{\alpha,\beta})+(1-\rho_B^{\alpha,\beta})-M_{|\underline{\mathscr{A}}_\alpha|}(1-\rho_A^{\alpha,\beta})(1-\rho_B^{\alpha,\beta})}$$

$$= 1-\frac{1-\rho_A^{\alpha,\beta}-\rho_B^{\alpha,\beta}+\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2-\rho_A^{\alpha,\beta}-\rho_A^{\alpha,\beta}-M_{|\underline{\mathscr{A}}_\alpha|}(1-\rho_A^{\alpha,\beta})(1-\rho_B^{\alpha,\beta})}. \qquad (3.5)$$

Similarly, if $|\underline{\mathscr{B}}_\alpha| \leq |\underline{\mathscr{A}}_\alpha|$, then

$$\rho_{A\cap B}^{\alpha,\beta} \geq 1-\frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|+\left|\overline{\mathscr{B}}_{\beta}\right|-\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}\right|-\left|\overline{\mathrm{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}$$

$$=1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}-\frac{\left|\overline{\mathrm{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}}.$$

If $\left|\underline{\mathscr{B}}_{\alpha}\right| \leq \left|\underline{\mathscr{A}}_{\alpha}\right|$, then $\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|} \geq \frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}$. We also have $\rho_{A}^{\alpha,\beta}=1-\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$ or $\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}=\frac{1}{1-\rho_{A}^{\alpha,\beta}}$, thus

$$\rho_{A\cap B}^{\alpha,\beta} \geq 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}-\frac{\left|\overline{\mathrm{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}}$$

$$\geq 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}-\frac{\left|\overline{\mathrm{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}}$$

$$=1-\frac{1}{\frac{1}{\left(1-\rho_{A}^{\alpha,\beta}\right)}+\frac{1}{\left(1-\rho_{B}^{\alpha,\beta}\right)}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}\right|+\left|\overline{\mathrm{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}}.$$

Let us define $M_{\left|\underline{\mathscr{B}}_{\alpha}\right|}=\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}\right|+\left|\overline{\mathrm{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}$, therefore

$$\rho_{A\cap B}^{\alpha,\beta} \geq 1-\frac{1}{\frac{1}{\left(1-\rho_{A}^{\alpha,\beta}\right)}+\frac{1}{\left(1-\rho_{B}^{\alpha,\beta}\right)}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}\right|+\left|\overline{\mathrm{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\right)\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}}$$

$$=1-\frac{1}{\frac{1}{\left(1-\rho_{A}^{\alpha,\beta}\right)}+\frac{1}{\left(1-\rho_{B}^{\alpha,\beta}\right)}-M_{\left|\underline{\mathscr{B}}_{\alpha}\right|}}$$

$$\rho_{A\cap B}^{\alpha,\beta} \ge 1 - \frac{1}{\dfrac{\left(1-\rho_A^{\alpha,\beta}\right)+\left(1-\rho_B^{\alpha,\beta}\right)-M_{|\underline{\mathscr{Z}}_\alpha|}\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}{\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}}$$

$$= 1 - \frac{\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}{\left(1-\rho_A^{\alpha,\beta}\right)+\left(1-\rho_B^{\alpha,\beta}\right)-M_{|\underline{\mathscr{Z}}_\alpha|}\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}$$

$$= 1 - \frac{1-\rho_A^{\alpha,\beta}-\rho_B^{\alpha,\beta}+\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2-\rho_A^{\alpha,\beta}-\rho_A^{\alpha,\beta}-M_{|\underline{\mathscr{Z}}_\alpha|}\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}. \tag{3.6}$$

From (3.5) and (3.6), hence

$$\rho_{A\cap B}^{\alpha,\beta} \ge 1 - \frac{1-\rho_A^{\alpha,\beta}-\rho_B^{\alpha,\beta}+\rho_A^{\alpha,\beta}\rho_B^{\alpha,\beta}}{2-\rho_A^{\alpha,\beta}-\rho_A^{\alpha,\beta}-M_*\left(1-\rho_A^{\alpha,\beta}\right)\left(1-\rho_B^{\alpha,\beta}\right)}$$

where $M_* = \dfrac{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta\right| + \left|\bar{Z}_{\mathscr{A}_\alpha}\left(\mathscr{B}_\beta\right)\right|}{min\left\{\left|\underline{\mathscr{A}}_\alpha\right|, \left|\underline{\mathscr{B}}_\alpha\right|\right\}}$. □

Theorem 4 (*Upper Bound of Intersection*) A roughness upper bound for the intersection of two fuzzy sets $\boldsymbol{A}, \boldsymbol{B}$ in $\boldsymbol{U}$, $\rho_{A\cap B}^{\alpha,\beta}$, with respect to α, β is

$$\rho_{A\cap B}^{\alpha,\beta} \le \rho_A^{\alpha,\beta} + \rho_B^{\alpha,\beta} - 1 + S_{*2}$$

where $0 < \beta \le \alpha \le 1$ and $S_{*2} = \dfrac{\left|\underline{\mathscr{A} \cup \mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_\beta\right|}$.

Proof. By Definition 2.4.2 and Property 2.4.1.2, we have

$$\rho_{A\cap B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A} \cap \mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_\beta\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_\beta\right|}.$$

For any two finite sets $\boldsymbol{X}$ and $\boldsymbol{Y}$, we have $|X \cap Y| = |X| + |Y| - |X \cup Y|$, therefore,

$$\rho_{A\cap B}^{\alpha,\beta} = 1 - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_\beta\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_\alpha\right| + \left|\underline{\mathscr{B}}_\alpha\right| - \left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B}}_\beta\right|}.$$

If we have $\overline{\mathscr{A}\cap\mathscr{B}}_\beta \subseteq \overline{\mathscr{A}}_\beta$ or $\overline{\mathscr{A}\cap\mathscr{B}}_\beta \subseteq \overline{\mathscr{B}}_\beta$, then $\left|\overline{\mathscr{A}\cap\mathscr{B}}_\beta\right| \le \left|\overline{\mathscr{A}}_\beta\right|$ or $\left|\overline{\mathscr{A}\cap\mathscr{B}}_\beta\right| \le \left|\overline{\mathscr{B}}_\beta\right|$.

Hence $\frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|} \le \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|}$ and $\frac{|\underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{B}}_\beta|} \le \frac{|\underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|}$, and thus we obtain

$$\begin{aligned}
\rho_{A\cap B}^{\alpha,\beta} &= 1 - \frac{|\underline{\mathscr{A}}_\alpha| + |\underline{\mathscr{B}}_\alpha| - |\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|} \\
&= 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|} - \frac{|\underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|} + \frac{|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|} \\
&\le 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|} - \frac{|\underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{B}}_\beta|} + \frac{|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|}.
\end{aligned}$$

From, $\rho_A^{\alpha,\beta} = 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$ and $\frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|} = 1 - \rho_A^{\alpha,\beta}$, and Property 2.4.1.3, therefore

$$\begin{aligned}
\rho_{A\cap B}^{\alpha,\beta} &\le 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|} - \frac{|\underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}}_\beta|} + \frac{|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|} \\
&= 1 - \left(1 - \rho_A^{\alpha,\beta}\right) - \left(1 - \rho_B^{\alpha,\beta}\right) + \frac{|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|} \\
&= \rho_A^{\alpha,\beta} + \rho_B^{\alpha,\beta} - 1 + \frac{|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|}.
\end{aligned}$$

Let us define $S_{*_2} = \frac{|\underline{\mathscr{A}\cup\mathscr{B}}_\alpha|}{|\overline{\mathscr{A}\cap\mathscr{B}}_\beta|}$, therefore

$$\rho_{A\cap B}^{\alpha,\beta} \le \rho_A^{\alpha,\beta} + \rho_B^{\alpha,\beta} - 1 + S_{*_2}. \qquad \square$$

Theorem 5 (*Lower Bound for* $\rho^{\alpha,\beta}_{A\cap B\cap C}$) A lower bound of the roughness measure $\rho^{\alpha,\beta}_{A\cap B\cap C}$ of A, B, and C in U with respect to α, β is given by

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \geq 1 - \frac{1}{\frac{1}{\left(1-\rho^{\alpha,\beta}_{A}\right)} + \frac{1}{\left(1-\rho^{\alpha,\beta}_{B}\right)} + \frac{1}{\left(1-\rho^{\alpha,\beta}_{C}\right)} - S_{*} - Z_{*}}$$

where $S_{*} = \dfrac{\left|\overline{\mathscr{A}}_{\beta} \cup \overline{\mathscr{B} \cap \mathscr{C}}_{\beta}\right|}{\max\left\{\left|\underline{\mathscr{A}}_{\alpha}\right|, \left|\underline{\mathscr{B}}_{\alpha}\right|, \left|\underline{\mathscr{C}}_{\alpha}\right|\right\}}$ and $Z_{*} = \dfrac{\left|\overline{\boldsymbol{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right| + \left|\overline{\boldsymbol{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta} \cap \mathscr{C}_{\beta}\right)\right|}{\max\left\{\left|\underline{\mathscr{A}}_{\alpha}\right|, \left|\underline{\mathscr{B}}_{\alpha}\right|, \left|\underline{\mathscr{C}}_{\alpha}\right|\right\}}$.

Proof By Definition 2.4.2 and Property 2.4.1.2, we have

$$\rho^{\alpha,\beta}_{A\cap B\cap C} = 1 - \frac{\left|\underline{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}}_{\beta}\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha} \cap \underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}}_{\beta}\right|}.$$

By Property 2.5.1.2 and Property 2.5.2.2, considering that if $X \subseteq Y$ then $|Y - X| = |Y| - |X|$, we have

$$\rho^{\alpha,\beta}_{A\cap B\cap C} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha} \cap \underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}}_{\beta}\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha} \cap \underline{\mathscr{C}}_{\alpha}\right|}{\left|\left(\overline{\mathscr{A}}_{\beta} \cap \overline{\mathscr{B} \cap \mathscr{C}}_{\beta}\right) - \overline{\boldsymbol{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta} \cap \mathscr{C}_{\beta}\right)\right|}$$
$$= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha} \cap \underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta} \cap \overline{\mathscr{B} \cap \mathscr{C}}_{\beta}\right| - \left|\overline{\boldsymbol{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta} \cap \mathscr{C}_{\beta}\right)\right|}.$$

As for any finite set, we have $|X \cup Y| = |X| + |Y| - |X \cap Y|$ and by Property 2.5.2.2,

$$\rho^{\alpha,\beta}_{A\cap B\cap C} = 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha} \cap \underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta} \cap \overline{\mathscr{B} \cap \mathscr{C}}_{\beta}\right| - \left|\overline{\boldsymbol{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta} \cap \mathscr{C}_{\beta}\right)\right|}$$
$$= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha} \cap \underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right| + \left|\overline{\mathscr{B} \cap \mathscr{C}}_{\beta}\right| - \left|\overline{\mathscr{A}}_{\beta} \cup \overline{\mathscr{B} \cap \mathscr{C}}_{\beta}\right| - \left|\overline{\boldsymbol{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta} \cap \mathscr{C}_{\beta}\right)\right|}$$
$$= 1 - \frac{\left|\underline{\mathscr{A}}_{\alpha} \cap \underline{\mathscr{B}}_{\alpha} \cap \underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right| + \left|\overline{\mathscr{B}}_{\beta}\right| + \left|\overline{\mathscr{C}}_{\beta}\right| - \left|\overline{\boldsymbol{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right| - \left|\overline{\mathscr{A}}_{\beta} \cup \overline{\mathscr{B} \cap \mathscr{C}}_{\beta}\right| - \left|\overline{\boldsymbol{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta} \cap \mathscr{C}_{\beta}\right)\right|}.$$

Next, Consider that $X\cap Y\cap Z\subseteq X$, $X\cap Y\cap Z\subseteq Y$ and $X\cap Y\cap Z\subseteq Z$ then $|X\cap Y\cap Z|\le|X|$, $|X\cap Y\cap Z|\le|Y|$ and $|X\cap Y\cap Z|\le|Z|$, therefore

$$\begin{aligned}\rho^{\alpha,\beta}_{A\cap B\cap C} &= 1-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|+\left|\overline{\mathscr{B}}_{\beta}\right|+\left|\overline{\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right|-\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}\cap\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\cap\mathscr{C}_{\beta}\right)\right|}\\ &\ge 1-\frac{\max\left\{\left|\underline{\mathscr{A}}_{\alpha}\right|,\left|\underline{\mathscr{B}}_{\alpha}\right|,\left|\underline{\mathscr{C}}_{\alpha}\right|\right\}}{\left|\overline{\mathscr{A}}_{\beta}\right|+\left|\overline{\mathscr{B}}_{\beta}\right|+\left|\overline{\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right|-\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}\cap\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\cap\mathscr{C}_{\beta}\right)\right|}.\end{aligned}$$

If we assume that $\left|\underline{\mathscr{A}}_{\alpha}\right|\ge\left|\underline{\mathscr{B}}_{\alpha}\right|$ and $\left|\underline{\mathscr{A}}_{\alpha}\right|\ge\left|\underline{\mathscr{C}}_{\alpha}\right|$, then

$$\begin{aligned}\rho^{\alpha,\beta}_{A\cap B\cap C} &\ge 1-\frac{\max\left\{\left|\underline{\mathscr{A}}_{\alpha}\right|,\left|\underline{\mathscr{B}}_{\alpha}\right|,\left|\underline{\mathscr{C}}_{\alpha}\right|\right\}}{\left|\overline{\mathscr{A}}_{\beta}\right|+\left|\overline{\mathscr{B}}_{\beta}\right|+\left|\overline{\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right|-\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}\cap\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\cap\mathscr{C}_{\beta}\right)\right|}\\ &= 1-\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|+\left|\overline{\mathscr{B}}_{\beta}\right|+\left|\overline{\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right|-\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}\cap\mathscr{C}}_{\beta}\right|-\left|\overline{\mathbf{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\cap\mathscr{C}_{\beta}\right)\right|}\\ &= 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}\cap\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\cap\mathscr{C}_{\beta}\right)\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}}.\end{aligned}$$

Because of $\left|\underline{\mathscr{A}}_{\alpha}\right|\ge\left|\underline{\mathscr{B}}_{\alpha}\right|$ and $\left|\underline{\mathscr{A}}_{\alpha}\right|\ge\left|\underline{\mathscr{C}}_{\alpha}\right|$ then $\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}\ge\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}$, $\frac{\left|\overline{\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{C}}_{\alpha}\right|}\ge\frac{\left|\overline{\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}$, thus

$$\rho^{\alpha,\beta}_{A\cap B\cap C}\ge 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{C}}_{\alpha}\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}\cap\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\cap\mathscr{C}_{\beta}\right)\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}}.$$

By Definition 2.4.2 and, we see that $\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}=\frac{1}{1-\rho^{\alpha,\beta}_{A}}$, thus we have

$$\rho^{\alpha,\beta}_{A\cap B\cap C}\ge 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{B}}_{\beta}\right|}{\left|\underline{\mathscr{B}}_{\alpha}\right|}+\frac{\left|\overline{\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{C}}_{\alpha}\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{B}_{\beta}}\left(\mathscr{C}_{\beta}\right)\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}-\frac{\left|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}\cap\mathscr{C}}_{\beta}\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{A}_{\beta}}\left(\mathscr{B}_{\beta}\cap\mathscr{C}_{\beta}\right)\right|}{\left|\underline{\mathscr{A}}_{\alpha}\right|}}$$

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \geq 1 - \frac{1}{\frac{1}{\left(1-\rho^{\alpha,\beta}_{A}\right)} + \frac{1}{\left(1-\rho^{\alpha,\beta}_{B}\right)} + \frac{1}{\left(1-\rho^{\alpha,\beta}_{C}\right)} - \frac{\left|\overline{Z}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{A}}_\alpha\right|} - \frac{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta \cap \mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{A}}_\alpha\right|}}. \tag{3.7}$$

If we assume that $\left|\underline{\mathscr{B}}_\alpha\right| \geq \left|\underline{\mathscr{A}}_\alpha\right|$ and $\left|\underline{\mathscr{B}}_\alpha\right| \geq \left|\underline{\mathscr{C}}_\alpha\right|$, then

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \geq 1 - \frac{\max\left\{\left|\underline{\mathscr{A}}_\alpha\right|, \left|\underline{\mathscr{B}}_\alpha\right|, \left|\underline{\mathscr{C}}_\alpha\right|\right\}}{\left|\overline{\mathscr{A}}_\beta\right| + \left|\overline{\mathscr{B}}_\beta\right| + \left|\overline{\mathscr{C}}_\beta\right| - \left|\overline{Z}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right| - \left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right| - \left|\overline{Z}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta \cap \mathscr{C}_\beta\right)\right|}$$

$$= 1 - \frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right| + \left|\overline{\mathscr{B}}_\beta\right| + \left|\overline{\mathscr{C}}_\beta\right| - \left|\overline{Z}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right| - \left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right| - \left|\overline{Z}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta \cap \mathscr{C}_\beta\right)\right|}$$

$$= 1 - \frac{1}{\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} + \frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} + \frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta \cap \mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}}.$$

Because of $\left|\underline{\mathscr{B}}_\alpha\right| \geq \left|\underline{\mathscr{A}}_\alpha\right|$ and $\left|\underline{\mathscr{B}}_\alpha\right| \geq \left|\underline{\mathscr{C}}_\alpha\right|$ then $\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|} \geq \frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}$, $\frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|} \geq \frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}$, thus

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \geq 1 - \frac{1}{\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|} + \frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} + \frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta \cap \mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}}.$$

By Definition 2.4.2 and, we see that $\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|} = \frac{1}{1-\rho^{\alpha,\beta}_{A}}$, thus we have

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \geq 1 - \frac{1}{\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|} + \frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} + \frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{A}}\left(\mathscr{B} \cap \mathscr{C}\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}}$$

$$= 1 - \frac{1}{\frac{1}{\left(1-\rho^{\alpha,\beta}_{A}\right)} + \frac{1}{\left(1-\rho^{\alpha,\beta}_{B}\right)} + \frac{1}{\left(1-\rho^{\alpha,\beta}_{C}\right)} - \frac{\left|\overline{Z}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} - \frac{\left|\overline{Z}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta \cap \mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}}. \tag{3.8}$$

If we assume that $\left|\underline{\mathscr{C}}_\alpha\right| \geq \left|\underline{\mathscr{A}}_\alpha\right|$ and $\left|\underline{\mathscr{C}}_\alpha\right| \geq \left|\underline{\mathscr{B}}_\alpha\right|$, then

$$
\begin{aligned}
\rho_{A\cap B\cap C}^{\alpha,\beta} &\geq 1-\frac{\max\left\{\left|\underline{\mathscr{A}}_\alpha\right|,\left|\underline{\mathscr{B}}_\alpha\right|,\left|\underline{\mathscr{C}}_\alpha\right|\right\}}{\left|\overline{\mathscr{A}}_\beta\right|+\left|\overline{\mathscr{B}}_\beta\right|+\left|\overline{\mathscr{C}}_\beta\right|-\left|\overline{\mathbf{Z}}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|-\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}\cap\mathscr{C}}_\beta\right|-\left|\overline{\mathbf{Z}}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta\cap\mathscr{C}_\beta\right)\right|} \\
&= 1-\frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|+\left|\overline{\mathscr{B}}_\beta\right|+\left|\overline{\mathscr{C}}_\beta\right|-\left|\overline{\mathbf{Z}}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|-\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}\cap\mathscr{C}}_\beta\right|-\left|\overline{\mathbf{Z}}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta\cap\mathscr{C}_\beta\right)\right|} \\
&= 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}+\frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}+\frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}\cap\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta\cap\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}}.
\end{aligned}
$$

Because of $\left|\underline{\mathscr{C}}_\alpha\right| \geq \left|\underline{\mathscr{A}}_\alpha\right|$ and $\left|\underline{\mathscr{C}}_\alpha\right| \geq \left|\underline{\mathscr{B}}_\alpha\right|$ then $\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|} \geq \frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}$, $\frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|} \geq \frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}$, thus

$$
\rho_{A\cap B\cap C}^{\alpha,\beta} \geq 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|}+\frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}+\frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}\cap\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta\cap\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}}.
$$

By Definition 2.4.2 and, we see that $\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|}=\frac{1}{1-\rho_A^{\alpha,\beta}}$, thus we have

$$
\begin{aligned}
\rho_{A\cap B\cap C}^{\alpha,\beta} &\geq 1-\frac{1}{\frac{\left|\overline{\mathscr{A}}_\beta\right|}{\left|\underline{\mathscr{A}}_\alpha\right|}+\frac{\left|\overline{\mathscr{B}}_\beta\right|}{\left|\underline{\mathscr{B}}_\alpha\right|}+\frac{\left|\overline{\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}\cap\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta\cap\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}} \\
&= 1-\frac{1}{\frac{1}{\left(1-\rho_A^{\alpha,\beta}\right)}+\frac{1}{\left(1-\rho_B^{\alpha,\beta}\right)}+\frac{1}{\left(1-\rho_C^{\alpha,\beta}\right)}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathscr{A}}_\beta\cup\overline{\mathscr{B}\cap\mathscr{C}}_\beta\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}-\frac{\left|\overline{\mathbf{Z}}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta\cap\mathscr{C}_\beta\right)\right|}{\left|\underline{\mathscr{C}}_\alpha\right|}}.
\end{aligned}
\tag{3.9}
$$

From (3.7), (3.8) and (3.9), we get

$$
\rho_{A\cap B\cap C}^{\alpha,\beta} \geq 1-\frac{1}{\frac{1}{\left(1-\rho_A^{\alpha,\beta}\right)}+\frac{1}{\left(1-\rho_B^{\alpha,\beta}\right)}+\frac{1}{\left(1-\rho_C^{\alpha,\beta}\right)}-S_*-Z_*}
$$

$$\text{where} \quad S_* = \frac{\left|\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B} \cap \mathscr{C}}_\beta\right|}{\max\left\{\left|\underline{\mathscr{A}}_\alpha\right|, \left|\underline{\mathscr{B}}_\alpha\right|, \left|\underline{\mathscr{C}}_\alpha\right|\right\}} \text{ and } Z_* = \frac{\left|\overline{\boldsymbol{Z}}_{\mathscr{B}_\beta}\left(\mathscr{C}_\beta\right)\right| + \left|\overline{\boldsymbol{Z}}_{\mathscr{A}_\beta}\left(\mathscr{B}_\beta \cap \mathscr{C}_\beta\right)\right|}{\max\left\{\left|\underline{\mathscr{A}}_\alpha\right|, \left|\underline{\mathscr{B}}_\alpha\right|, \left|\underline{\mathscr{C}}_\alpha\right|\right\}}. \qquad \square$$

Theorem 6 (*Upper Bound for* $\rho^{\alpha,\beta}_{A\cap B\cap C}$) An upper bound of the roughness measure $\rho^{\alpha,\beta}_{A\cap B\cap C}$ of the fuzzy sets $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ in $\boldsymbol{U}$ with respect to α, β is given by

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \leq 4 - \rho^{\alpha,\beta}_{A} - \rho^{\alpha,\beta}_{B} - \rho^{\alpha,\beta}_{C} - S_*$$

$$\text{where } S_* = \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\max\left\{\left|\overline{\mathscr{A}}_\beta\right|, \left|\overline{\mathscr{B}}_\beta\right|, \left|\overline{\mathscr{C}}_\beta\right|\right\}}, \text{ and}$$

$0 < \beta \leq \alpha < 1$.

Proof By Definition 2.4.2 and Properties 2.4.1.2 and 2.4.1.4, we have

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \equiv 1 - \frac{\left|\underline{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}}_\beta\right|} = 1 - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}}_\beta\right|} \leq 1 - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta \cap \overline{\mathscr{C}}_\beta\right|}.$$

For any three crisp sets $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$, we have

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|$$

and

$$|X \cap Y \cap Z| = -|X| - |Y| - |Z| + |X \cap Y| + |X \cap Z| + |Y \cap Z| + |X \cup Y \cup Z|.$$

Therefore,

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \leq 1 - \frac{-\left|\underline{\mathscr{A}}_\alpha\right| - \left|\underline{\mathscr{B}}_\alpha\right| - \left|\underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta \cap \overline{\mathscr{C}}_\beta\right|}$$
$$+ \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta \cap \overline{\mathscr{C}}_\beta\right|}.$$

Because of $X \cap Y \cap Z \subseteq X$, $X \cap Y \cap Z \subseteq Y$ and $X \cap Y \cap Z \subseteq Z$, then $|X \cap Y \cap Z| \leq |X|$, $|X \cap Y \cap Z| \leq |Y|$ and $|X \cap Y \cap Z| \leq |Z|$.

If we assume that $\left|\overline{\mathscr{A}}_\beta\right| \geq \left|\overline{\mathscr{B}}_\beta\right|$ and $\left|\overline{\mathscr{A}}_\beta\right| \geq \left|\overline{\mathscr{C}}_\beta\right|$ such that

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \leq 1 - \frac{-\left|\underline{\mathscr{A}}_\alpha\right| - \left|\underline{\mathscr{B}}_\alpha\right| - \left|\underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta \cap \overline{\mathscr{C}}_\beta\right|} + \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta \cap \overline{\mathscr{C}}_\beta\right|}$$

$$\leq 1 - \frac{-\left|\underline{\mathscr{A}}_\alpha\right| - \left|\underline{\mathscr{B}}_\alpha\right| - \left|\underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} + \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right| + \left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$$

$$= 1 + \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} + \frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} + \frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}.$$

Because of $\left|\overline{\mathscr{A}}_\beta\right| \geq \left|\overline{\mathscr{B}}_\beta\right|$ and $\left|\overline{\mathscr{A}}_\beta\right| \geq \left|\overline{\mathscr{C}}_\beta\right|$ then $\frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} \leq \frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{B}}_\beta\right|}$ and $\frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} \leq \frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}$,

therefore,

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \leq 1 + \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} + \frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} + \frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$$

$$\leq 1 + \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} + \frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{B}}_\beta\right|} + \frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|} - \frac{\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}.$$

Since $\rho^{\alpha,\beta}_{A} = 1 - \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$, therefore

$$\rho_{A\cap B\cap C}^{\alpha,\beta} \le 1+\frac{|\mathscr{A}_\alpha|}{|\overline{\mathscr{A}}_\beta|}+\frac{|\mathscr{B}_\alpha|}{|\overline{\mathscr{B}}_\beta|}+\frac{|\mathscr{C}_\alpha|}{|\overline{\mathscr{C}}_\beta|}-\frac{|\mathscr{A}_\alpha\cap\mathscr{B}_\alpha|}{|\overline{\mathscr{A}}_\beta|}-\frac{|\mathscr{A}_\alpha\cap\mathscr{C}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$$
$$-\frac{|\mathscr{B}_\alpha\cap\mathscr{C}_\alpha|}{|\overline{\mathscr{A}}_\beta|}-\frac{|\mathscr{A}_\alpha\cup\mathscr{B}_\alpha\cup\mathscr{C}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$$
$$=4-\rho_A^{\alpha,\beta}-\rho_B^{\alpha,\beta}-\rho_C^{\alpha,\beta}$$
$$-\frac{|\mathscr{A}_\alpha\cap\mathscr{B}_\alpha|+|\mathscr{A}_\alpha\cap\mathscr{C}_\alpha|+|\mathscr{B}_\alpha\cap\mathscr{C}_\alpha|+|\mathscr{A}_\alpha\cup\mathscr{B}_\alpha\cup\mathscr{C}_\alpha|}{|\overline{\mathscr{A}}_\beta|}. \quad (3.10)$$

If we assume that $|\overline{\mathscr{B}}_\beta|\ge|\overline{\mathscr{A}}_\beta|$ and $|\overline{\mathscr{B}}_\beta|\ge|\overline{\mathscr{C}}_\beta|$, then

$$\rho_{A\cap B\cap C}^{\alpha,\beta} \le 1-\frac{-|\mathscr{A}_\alpha|-|\mathscr{B}_\alpha|-|\mathscr{C}_\alpha|+|\mathscr{A}_\alpha\cap\mathscr{B}_\alpha|}{|\overline{\mathscr{A}}_\beta\cap\overline{\mathscr{B}}_\beta\cap\overline{\mathscr{C}}_\beta|}$$
$$+\frac{|\mathscr{A}_\alpha\cap\mathscr{C}_\alpha|+|\mathscr{B}_\alpha\cap\mathscr{C}_\alpha|+|\mathscr{A}_\alpha\cup\mathscr{B}_\alpha\cup\mathscr{C}_\alpha|}{|\overline{\mathscr{A}}_\beta\cap\overline{\mathscr{B}}_\beta\cap\overline{\mathscr{C}}_\beta|}$$
$$\le 1-\frac{-|\mathscr{A}_\alpha|-|\mathscr{B}_\alpha|-|\mathscr{C}_\alpha|+|\mathscr{A}_\alpha\cap\mathscr{B}_\alpha|}{|\overline{\mathscr{B}}_\beta|}$$
$$+\frac{|\mathscr{A}_\alpha\cap\mathscr{C}_\alpha|+|\mathscr{B}_\alpha\cap\mathscr{C}_\alpha|+|\mathscr{A}_\alpha\cup\mathscr{B}_\alpha\cup\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}$$
$$=1+\frac{|\mathscr{A}_\alpha|}{|\overline{\mathscr{B}}_\beta|}+\frac{|\mathscr{B}_\alpha|}{|\overline{\mathscr{B}}_\beta|}+\frac{|\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}-\frac{|\mathscr{A}_\alpha\cap\mathscr{B}_\alpha|}{|\overline{\mathscr{B}}_\beta|}-\frac{|\mathscr{A}_\alpha\cap\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}$$
$$-\frac{|\mathscr{B}_\alpha\cap\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}-\frac{|\mathscr{A}_\alpha\cup\mathscr{B}_\alpha\cup\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}.$$

Because of $|\overline{\mathscr{B}}_\beta|\ge|\overline{\mathscr{A}}_\beta|$ and $|\overline{\mathscr{B}}_\beta|\ge|\overline{\mathscr{C}}_\beta|$ then $\frac{|\mathscr{A}_\alpha|}{|\overline{\mathscr{B}}_\beta|}\le\frac{|\mathscr{A}_\alpha|}{|\overline{\mathscr{A}}_\beta|}$ and $\frac{|\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}\le\frac{|\mathscr{C}_\alpha|}{|\overline{\mathscr{C}}_\beta|}$,

therefore

$$\rho_{A\cap B\cap C}^{\alpha,\beta} \le 1+\frac{|\mathscr{A}_\alpha|}{|\overline{\mathscr{B}}_\beta|}+\frac{|\mathscr{B}_\alpha|}{|\overline{\mathscr{B}}_\beta|}+\frac{|\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}-\frac{|\mathscr{A}_\alpha\cap\mathscr{B}_\alpha|}{|\overline{\mathscr{B}}_\beta|}-\frac{|\mathscr{A}_\alpha\cap\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}$$
$$-\frac{|\mathscr{B}_\alpha\cap\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}-\frac{|\mathscr{A}_\alpha\cup\mathscr{B}_\alpha\cup\mathscr{C}_\alpha|}{|\overline{\mathscr{B}}_\beta|}$$

$$\rho_{A\cap B\cap C}^{\alpha,\beta} \le 1+\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}+\frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}+\frac{\left|\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}$$
$$-\frac{\left|\underline{\mathscr{B}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}\cup\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}.$$

Since $\rho_{A}^{\alpha,\beta}=1-\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$, therefore

$$\rho_{A\cap B\cap C}^{\alpha,\beta} \le 1+\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}+\frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}+\frac{\left|\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}$$
$$-\frac{\left|\underline{\mathscr{B}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}\cup\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}$$
$$=4-\rho_{A}^{\alpha,\beta}-\rho_{B}^{\alpha,\beta}-\rho_{C}^{\alpha,\beta}$$
$$-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}\right|+\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{B}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}\cup\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{B}}_{\beta}\right|}. \qquad (3.11)$$

If we assume that $\left|\overline{\mathscr{C}}_{\beta}\right|\ge\left|\overline{\mathscr{A}}_{\beta}\right|$ and $\left|\overline{\mathscr{C}}_{\beta}\right|\ge\left|\overline{\mathscr{B}}_{\beta}\right|$, then

$$\rho_{A\cap B\cap C}^{\alpha,\beta} \le 1-\frac{-\left|\underline{\mathscr{A}}_{\alpha}\right|-\left|\underline{\mathscr{B}}_{\alpha}\right|-\left|\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\cap\overline{\mathscr{B}}_{\beta}\cap\overline{\mathscr{C}}_{\beta}\right|}$$
$$+\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{B}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}\cup\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\cap\overline{\mathscr{B}}_{\beta}\cap\overline{\mathscr{C}}_{\beta}\right|}$$
$$\le 1-\frac{-\left|\underline{\mathscr{A}}_{\alpha}\right|-\left|\underline{\mathscr{B}}_{\alpha}\right|-\left|\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}$$
$$+\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{B}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|+\left|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}\cup\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}$$
$$=1+\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}+\frac{\left|\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}+\frac{\left|\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}$$
$$-\frac{\left|\underline{\mathscr{B}}_{\alpha}\cap\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}\cup\underline{\mathscr{C}}_{\alpha}\right|}{\left|\overline{\mathscr{C}}_{\beta}\right|}.$$

Because of $\left|\overline{\mathscr{C}}_\beta\right| \geq \left|\overline{\mathscr{A}}_\beta\right|$ and $\left|\overline{\mathscr{C}}_\beta\right| \geq \left|\overline{\mathscr{B}}_\beta\right|$ then $\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} \leq \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$ and $\frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} \leq \frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{B}}_\beta\right|}$,

therefore

$$\begin{aligned}
\rho^{\alpha,\beta}_{A\cap B\cap C} &\leq 1+\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}+\frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}+\frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} \\
&\quad -\frac{\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} \\
&\leq 1+\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}+\frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{B}}_\beta\right|}+\frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} \\
&\quad -\frac{\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}.
\end{aligned}$$

Since $\rho^{\alpha,\beta}_{A} = 1-\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$, therefore

$$\begin{aligned}
\rho^{\alpha,\beta}_{A\cap B\cap C} &\leq 1+\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}+\frac{\left|\underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{B}}_\beta\right|}+\frac{\left|\underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} \\
&\quad -\frac{\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}-\frac{\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|} \\
&= 4-\rho^{\alpha,\beta}_{A}-\rho^{\alpha,\beta}_{B}-\rho^{\alpha,\beta}_{C} \\
&\quad -\frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|+\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|+\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|+\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\left|\overline{\mathscr{C}}_\beta\right|}. \qquad (3.12)
\end{aligned}$$

From (3.10), (3.11) and (3.12), we get

$$\rho^{\alpha,\beta}_{A\cap B\cap C} \leq 4-\rho^{\alpha,\beta}_{A}-\rho^{\alpha,\beta}_{B}-\rho^{\alpha,\beta}_{C}-S_*$$

where $S_* = \frac{\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha\right|+\left|\underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|+\left|\underline{\mathscr{B}}_\alpha \cap \underline{\mathscr{C}}_\alpha\right|+\left|\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \cup \underline{\mathscr{C}}_\alpha\right|}{\max\left\{\left|\overline{\mathscr{A}}_\beta\right|,\left|\overline{\mathscr{B}}_\beta\right|,\left|\overline{\mathscr{C}}_\beta\right|\right\}}$. □

CHAPTER 4

BOUNDS IN GENERAL FORMS

In this chapter 4, we propose new corollaries and their proofs on roughness lower and upper bounds of the fuzzy set operations union and intersection in greater generality as the following.

4.1 Roughness Upper Bounds for the Union

Corollary 1 (*Upper Bound for* $\rho^{\alpha,\beta}_{A_1 \cup A_2 \cup \ldots \cup A_n}$) An upper bound of the roughness measure $\rho^{\alpha,\beta}_{A_1 \cup A_2 \cup \ldots \cup A_n}$ of fuzzy sets A_1, A_2, A_3, …, A_n in U, with respect to α, β is given by

$$\rho^{\alpha,\beta}_{A_1 \cup A_2 \cup A_3 \ldots \cup A_n} \leq 1 - \frac{\prod_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)}{\sum_{i=1}^{n}\left(\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(1-\rho^{\alpha,\beta}_{A_j}\right)\right)}$$

where $0 < \beta \leq \alpha \leq 1$.

Proof By Definition 2.4.2, Properties 2.4.1.1 and 2.4.1.3, we have

$$\rho^{\alpha,\beta}_{A_1 \cup A_2 \cup A_3 \ldots \cup A_n} = 1 - \frac{\left|\underline{\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \ldots \cup \mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \ldots \cup \mathscr{A}_n}_{\beta}\right|} = 1 - \frac{\left|\underline{\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \ldots \cup \mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1}_{\beta} \cup \overline{\mathscr{A}_2}_{\beta} \cup \overline{\mathscr{A}_3}_{\beta} \ldots \cup \overline{\mathscr{A}_n}_{\beta}\right|}$$

$$\leq 1 - \frac{\left|\underline{\mathscr{A}_1}_{\alpha} \cup \underline{\mathscr{A}_2}_{\alpha} \cup \underline{\mathscr{A}_3}_{\alpha} \ldots \cup \underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1}_{\beta} \cup \overline{\mathscr{A}_2}_{\beta} \cup \overline{\mathscr{A}_3}_{\beta} \ldots \cup \overline{\mathscr{A}_n}_{\beta}\right|}.$$

For any crisp sets $X_1, X_2, \ldots, X_n$, we have $|X_1 \cup X_2 \cup \ldots \cup X_n| \leq |X_1| + |X_2| + \ldots + |X_n|$ and $|X_1 \cup X_2 \cup \ldots \cup X_n| \geq max|X_1|, |X_2|, \ldots, |X_n|$, therefore

$$\rho^{\alpha,\beta}_{A_1\cup A_2\cup A_3\ldots\cup A_n} \le 1-\frac{\left|\underline{\mathscr{A}_1}_\alpha \cup \underline{\mathscr{A}_2}_\alpha \cup \underline{\mathscr{A}_3}_\alpha \ldots \cup \underline{\mathscr{A}_n}_\alpha\right|}{\left|\overline{\mathscr{A}_1}_\beta \cup \overline{\mathscr{A}_2}_\beta \cup \overline{\mathscr{A}_3}_\beta \ldots \cup \overline{\mathscr{A}_n}_\beta\right|} \le 1-\frac{max\left\{\left|\underline{\mathscr{A}_1}_\alpha\right|,\left|\underline{\mathscr{A}_2}_\alpha\right|,\left|\underline{\mathscr{A}_3}_\alpha\right|,\ldots,\left|\underline{\mathscr{A}_n}_\alpha\right|\right\}}{\left|\overline{\mathscr{A}_1}_\beta\right|+\left|\overline{\mathscr{A}_2}_\beta\right|+\left|\overline{\mathscr{A}_3}_\beta\right|+\ldots+\left|\overline{\mathscr{A}_n}_\beta\right|}.$$

If $\left|\underline{\mathscr{A}_k}_\alpha\right| \ge \left|\underline{\mathscr{A}_l}_\alpha\right|$ $\forall l$ where $1 \le k, l \le n$ and $k \ne l$ then

$$\rho^{\alpha,\beta}_{A_1\cup A_2\cup A_3\ldots\cup A_n} \le 1-\frac{\left|\underline{\mathscr{A}_k}_\alpha\right|}{\left|\overline{\mathscr{A}_1}_\beta\right|+\left|\overline{\mathscr{A}_2}_\beta\right|+\left|\overline{\mathscr{A}_3}_\beta\right|+\ldots+\left|\overline{\mathscr{A}_n}_\beta\right|}$$

$$=1-\frac{1}{\frac{\left|\overline{\mathscr{A}_1}_\beta\right|}{\left|\underline{\mathscr{A}_k}_\alpha\right|}+\frac{\left|\overline{\mathscr{A}_2}_\beta\right|}{\left|\underline{\mathscr{A}_k}_\alpha\right|}+\frac{\left|\overline{\mathscr{A}_3}_\beta\right|}{\left|\underline{\mathscr{A}_k}_\alpha\right|}+\ldots+\frac{\left|\overline{\mathscr{A}_n}_\beta\right|}{\left|\underline{\mathscr{A}_k}_\alpha\right|}}.$$

If $\left|\underline{\mathscr{A}_k}_\alpha\right| \ge \left|\underline{\mathscr{A}_l}_\alpha\right|$, then $\frac{\left|\overline{\mathscr{A}_l}_\beta\right|}{\left|\underline{\mathscr{A}_k}_\alpha\right|} \le \frac{\left|\overline{\mathscr{A}_l}_\beta\right|}{\left|\underline{\mathscr{A}_l}_\alpha\right|}$. Thus, we have

$$\rho^{\alpha,\beta}_{A_1\cup A_2\cup A_3\ldots\cup A_n} \le 1-\frac{1}{\frac{\left|\overline{\mathscr{A}_1}_\beta\right|}{\left|\underline{\mathscr{A}_1}_\alpha\right|}+\frac{\left|\overline{\mathscr{A}_2}_\beta\right|}{\left|\underline{\mathscr{A}_2}_\alpha\right|}+\frac{\left|\overline{\mathscr{A}_3}_\beta\right|}{\left|\underline{\mathscr{A}_3}_\alpha\right|}+\ldots+\frac{\left|\overline{\mathscr{A}_n}_\beta\right|}{\left|\underline{\mathscr{A}_n}_\alpha\right|}}.$$

Since we have $\rho^{\alpha,\beta}_A = 1-\frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$ and $1-\rho^{\alpha,\beta}_A = \frac{\left|\underline{\mathscr{A}}_\alpha\right|}{\left|\overline{\mathscr{A}}_\beta\right|}$, thus

$$\rho^{\alpha,\beta}_{A_1\cup A_2\cup A_3\ldots\cup A_n} \le 1-\frac{1}{\frac{1}{1-\rho^{\alpha,\beta}_{A_1}}+\frac{1}{1-\rho^{\alpha,\beta}_{A_2}}+\frac{1}{1-\rho^{\alpha,\beta}_{A_3}}+\ldots+\frac{1}{1-\rho^{\alpha,\beta}_{A_n}}}$$

$$=1-\frac{\left(1-\rho^{\alpha,\beta}_{A_1}\right)\left(1-\rho^{\alpha,\beta}_{A_2}\right)\ldots\left(1-\rho^{\alpha,\beta}_{A_n}\right)}{\begin{array}{l}\left(1-\rho^{\alpha,\beta}_{A_2}\right)\left(1-\rho^{\alpha,\beta}_{A_3}\right)\ldots\left(1-\rho^{\alpha,\beta}_{A_n}\right)+\left(1-\rho^{\alpha,\beta}_{A_1}\right)\left(1-\rho^{\alpha,\beta}_{A_3}\right)\ldots\left(1-\rho^{\alpha,\beta}_{A_n}\right)\\+\ldots+\left(1-\rho^{\alpha,\beta}_{A_1}\right)\left(1-\rho^{\alpha,\beta}_{A_2}\right)\ldots\left(1-\rho^{\alpha,\beta}_{A_{n-1}}\right)\end{array}}$$

$$=1-\frac{\prod_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)}{\sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\ne i}}^{n}\left(1-\rho^{\alpha,\beta}_{A_j}\right)\right)}.$$

Therefore, we get

$$\rho^{\alpha,\beta}_{A_1\cup A_2\cup A_3\ldots\cup A_n} \le 1-\frac{\prod_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)}{\sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\ne i}}^{n}\left(1-\rho^{\alpha,\beta}_{A_j}\right)\right)}. \qquad \square$$

4.2 Roughness Upper Bounds for the Intersection

Corollary 2 (*Upper Bound for* $\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n}$, *n is even*) Let $A_1, A_2, A_3, \ldots, A_n \in \boldsymbol{U}$. An upper bound of the roughness measure $\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n}$ of fuzzy sets $A_1, A_2, A_3, \ldots, A_n$ where n is even with respect to α, β is given by

$$\begin{aligned}\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n} \le{}& 1-\sum_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)-\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}}\right)\\ &-\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}}\right)\\ &-\ldots-\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_{n-1}}}\right)+F_*^{\cap\text{even}}+S_{*n},\end{aligned}$$

where

$$F_*^{\cap\text{even}}=\frac{\begin{aligned}&\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\right|\right)+\sum_{i_1=1}^{n-3}\sum_{i_2=i_1+1}^{n-2}\sum_{i_3=i_2+1}^{n-1}\sum_{i_4=i_3+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\right|\right)\\&+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-2}}}_{\alpha}\right|\right)\end{aligned}}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|},$$

$$S_{*n}=\frac{\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|},\ 0<\beta\le\alpha\le 1 \text{ and } n \text{ is even.}$$

.

Proof By Definition 2.4.2 and Property 2.4.1.2, we have

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n} = 1-\frac{\left|\underline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|} = 1-\frac{\left|\underline{\mathscr{A}_1}_{\alpha}\cap\underline{\mathscr{A}_2}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}.$$

For any finite set $X_1, X_2, X_3, \ldots, X_n$, where n is even, we have

$|X_1\cup X_2\cup\ldots\cup X_n| = \sum_{i=1}^{n}|X_i| - X^{\cap\text{even}} + X^{\cap\text{odd}} - |X_1\cap X_2\cap\ldots\cap X_n|$, thus

$|X_1\cap X_2\cap\ldots\cap X_n| = \sum_{i=1}^{n}|X_i| - X^{\cap\text{even}} + X^{\cap\text{odd}} - |X_1\cup X_2\cup X_3\cup\ldots\cup X_n|$, where

$$X^{\cap\text{even}} = \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}\left(\left|X_{i_1}\cap X_{i_2}\right|\right) + \sum_{i_1=1}^{n-3}\sum_{i_2=i_1+1}^{n-2}\sum_{i_3=i_2+1}^{n-1}\sum_{i_4=i_3+1}^{n}\left(\left|X_{i_1}\cap X_{i_2}\cap X_{i_3}\cap X_{i_4}\right|\right) + \ldots + \sum_{i_1=1}^{3}\sum_{i_2=i_1+1}^{4}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(\left|X_{i_1}\cap X_{i_2}\cap\ldots\cap X_{i_{n-2}}\right|\right)$$

$$X^{\cap\text{odd}} = \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(\left|X_{i_1}\cap X_{i_2}\cap X_{i_3}\right|\right) + \sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(\left|X_{i_1}\cap X_{i_2}\cap X_{i_3}\cap X_{i_4}\cap X_{i_5}\right|\right) + \ldots + \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(\left|X_{i_1}\cap X_{i_2}\cap\ldots\cap X_{i_{n-1}}\right|\right).$$

Therefore,

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap A_3\cap\ldots A_n} = 1-\frac{\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right| - F^{\cap\text{even}} + F^{\cap\text{odd}} - \left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\underline{\mathscr{A}_3}_{\alpha}\cup\ldots\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}$$

where

$$F^{\cap\text{even}} = \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\right|\right) + \sum_{i_1=1}^{n-3}\sum_{i_2=i_1+1}^{n-2}\sum_{i_3=i_2+1}^{n-1}\sum_{i_4=i_3+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\right|\right) + \ldots + \sum_{i_1=1}^{3}\sum_{i_2=i_1+1}^{4}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-2}}}_{\alpha}\right|\right)$$

and,

$$F^{\cap \text{odd}} = \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)$$
$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha} \cap \underline{\mathscr{A}_{i_4}}_{\alpha} \cap \underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)$$
$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \ldots \cap \underline{\mathscr{A}_{i_{n-1}}}_{\alpha}\right|\right).$$

If we have $\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta} \subseteq \overline{\mathscr{A}_i}_{\beta}$, where $i = 1, 2, 3, \ldots n$ and n is even, then $\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right| \le \left|\overline{\mathscr{A}_i}_{\beta}\right|$.

Hence $\sum_{i=1}^{n}\dfrac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_i}_{\beta}\right|} \le \dfrac{\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}$, and thus we obtain

$$\rho^{\alpha,\beta}_{A_1 \cap A_2 \cap \ldots \cap A_n} = 1 - \frac{\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right| - F^{\cap \text{even}} + F^{\cap \text{odd}} - \left|\underline{\mathscr{A}_1}_{\alpha} \cup \underline{\mathscr{A}_2}_{\alpha} \cup \ldots \cup \underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}$$
$$= 1 - \frac{\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|} + \frac{F^{\cap \text{even}} - F^{\cap \text{odd}} + \left|\underline{\mathscr{A}_1}_{\alpha} \cup \underline{\mathscr{A}_2}_{\alpha} \cup \ldots \cup \underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}$$
$$\le 1 - \sum_{i=1}^{n}\frac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_i}_{\beta}\right|} + \frac{F^{\cap \text{even}} - F^{\cap \text{odd}} + \left|\underline{\mathscr{A}_1}_{\alpha} \cup \underline{\mathscr{A}_2}_{\alpha} \cup \ldots \cup \underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}.$$

Consider

$$\frac{F^{\cap \text{odd}}}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|} = \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}$$
$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha} \cap \underline{\mathscr{A}_{i_4}}_{\alpha} \cap \underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}$$
$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \ldots \cap \underline{\mathscr{A}_{i_{n-1}}}_{\alpha}\right|\right)}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}.$$

$$\geq \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}}_{\beta}\right|\right)}$$

$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\cap\underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}\cap\mathscr{A}_{i_4}\cap\mathscr{A}_{i_5}}_{\beta}\right|\right)}$$

$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-1}}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\ldots\cap\mathscr{A}_{i_{n-1}}}_{\beta}\right|\right)}.$$

Thus, we have

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n}\leq 1-\sum_{i=1}^{n}\frac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_i}_{\beta}\right|}+\frac{F^{\cap\text{even}}-F^{\cap\text{odd}}+\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}$$

$$\leq 1-\sum_{i=1}^{n}\frac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_i}_{\beta}\right|}-\frac{F^{\cap\text{odd}}}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}+\frac{F^{\cap\text{even}}+\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}$$

$$\leq 1-\sum_{i=1}^{n}\frac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_i}_{\beta}\right|}-\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}}_{\beta}\right|\right)}$$

$$-\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\cap\underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}\cap\mathscr{A}_{i_4}\cap\mathscr{A}_{i_5}}_{\beta}\right|\right)}$$

$$-\ldots-\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-1}}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\ldots\cap\mathscr{A}_{i_{n-1}}}_{\beta}\right|\right)}$$

$$+\frac{F^{\cap\text{even}}+\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}.$$

From Definition 2.4.2, $\rho^{\alpha,\beta}_{A}=1-\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$ or $\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}=1-\rho^{\alpha,\beta}_{A}$, and from Property 2.4.1.2,

therefore

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n} \le 1-\sum_{i=1}^{n}\frac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_i}_{\beta}\right|}-\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}}_{\beta}\right|\right)}$$

$$-\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\cap\underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}\cap\mathscr{A}_{i_4}\cap\mathscr{A}_{i_5}}_{\beta}\right|\right)}$$

$$-\ldots-\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-1}}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\ldots\cap\mathscr{A}_{i_{n-1}}}_{\beta}\right|\right)}$$

$$+\frac{F^{\cap\text{even}}+\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}$$

and,

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n} \le 1-\sum_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)-\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}}\right)$$

$$-\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}}\right)$$

$$-\ldots-\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_{n-1}}}\right)$$

$$+\frac{F^{\cap\text{even}}+\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}.$$

Let us define $F_*^{\cap\text{even}}=\dfrac{F^{\cap\text{even}}}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}$ and $S_{*_n}=\dfrac{\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}$,

therefore

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n} \le 1-\sum_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)-\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}}\right)$$

$$-\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}}\right) \qquad \square$$

$$-\ldots-\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_{n-1}}}\right)+F_*^{\cap\text{even}}+S_{*_n}.$$

Corollary 3 (*Upper Bound for* $\rho^{\alpha,\beta}_{A_1 \cap A_2 \cap \ldots \cap A_n}$ *, n is odd*) Let $A_1, A_2, A_3, \ldots, A_n \in U$. An upper bound of the roughness measure $\rho^{\alpha,\beta}_{A_1 \cap A_2 \cap \ldots \cap A_n}$ of fuzzy sets $A_1, A_2, A_3, \ldots, A_n$, where n is odd with respect to α, β is given by

$$\begin{aligned}\rho^{\alpha,\beta}_{A_1 \cap A_2 \cap \ldots \cap A_n} \leq 1 &+ \sum_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right) + \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}}\right) \\ &+ \sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}}\right) \\ &+ \ldots + \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\ldots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_{n-2}}}\right) - F_{\bullet}^{\cap\text{even}} - Z_{\bullet k},\end{aligned}$$

where

$$F_{\bullet}^{\cap\text{even}} = \frac{\begin{aligned}&\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha}\right|\right) + \sum_{i_1=1}^{n-3}\sum_{i_2=i_1+1}^{n-2}\sum_{i_3=i_2+1}^{n-1}\sum_{i_4=i_3+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha} \cap \underline{\mathscr{A}_{i_4}}_{\alpha}\right|\right) \\ &+ \ldots + \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\ldots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \ldots \cap \underline{\mathscr{A}_{i_{n-1}}}_{\alpha}\right|\right)\end{aligned}}{\left|\overline{\mathscr{A}_{k}}_{\beta}\right|},$$

and,

$$Z_{\bullet k} = \frac{\left|\underline{\mathscr{A}_1}_{\alpha} \cup \underline{\mathscr{A}_2}_{\alpha} \cup \underline{\mathscr{A}_3}_{\alpha} \cup \ldots \underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|},$$ if $\left|\overline{\mathscr{A}_k}_{\beta}\right| \geq \left|\overline{\mathscr{A}_i}_{\beta}\right|$ $\forall i$ where $i = 1, 2, 3, \ldots, n$ and n is odd.

Proof By Definition 2.4.2 and Property 2.4.1.2

$$\rho^{\alpha,\beta}_{A_1 \cap A_2 \cap \ldots \cap A_n} = 1 - \frac{\left|\underline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|} = 1 - \frac{\left|\underline{\mathscr{A}_1}_{\alpha} \cap \underline{\mathscr{A}_2}_{\alpha} \cap \ldots \cap \underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}$$

For any finite set $X_1, X_2, X_3, \ldots X_n$, where n are odd number.

We have $\left|X_1 \cup X_2 \cup \ldots \cup X_n\right| = \sum_{i=1}^{n}\left|X_i\right| - X^{\cap\text{even}} + X^{\cap\text{odd}} + \left|X_1 \cap X_2 \cap \ldots \cap X_n\right|$, thus

$$\left|X_1 \cap X_2 \cap \ldots \cap X_n\right| = -\sum_{i=1}^{n}\left|X_i\right| + X^{\cap \text{even}} - X^{\cap \text{odd}} + \left|X_1 \cup X_2 \cup \ldots \cup X_n\right|, \text{ where}$$

$$X^{\cap \text{even}} = \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}\left(\left|X_{i_1} \cap X_{i_2}\right|\right) + \sum_{i_1=1}^{n-3}\sum_{i_2=i_1+1}^{n-2}\sum_{i_3=i_2+1}^{n-1}\sum_{i_4=i_3+1}^{n}\left(\left|X_{i_1} \cap X_{i_2} \cap X_{i_3} \cap X_{i_4}\right|\right)$$
$$+ \ldots + \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(\left|X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_{n-1}}\right|\right)$$

and,

$$X^{\cap \text{odd}} = \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(\left|X_{i_1} \cap X_{i_2} \cap X_{i_3}\right|\right)$$
$$+ \sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(\left|X_{i_1} \cap X_{i_2} \cap X_{i_3} \cap X_{i_4} \cap X_{i_5}\right|\right)$$
$$+ \ldots + \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(\left|X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_{n-2}}\right|\right).$$

Therefore,

$$\rho^{\alpha,\beta}_{A_1 \cap A_2 \cap \ldots \cap A_n} = 1 - \frac{-\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right| - F^{\cap \text{odd}} + F^{\cap \text{even}} + \left|\underline{\mathscr{A}_1}_{\alpha} \cup \underline{\mathscr{A}_2}_{\alpha} \cup \ldots \cup \underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right|}$$

where

$$F^{\cap \text{even}} = \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha}\right|\right) + \sum_{i_1=1}^{n-3}\sum_{i_2=i_1+1}^{n-2}\sum_{i_3=i_2+1}^{n-1}\sum_{i_4=i_3+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha} \cap \underline{\mathscr{A}_{i_4}}_{\alpha}\right|\right) +$$
$$\ldots + \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \ldots \cap \underline{\mathscr{A}_{i_{n-1}}}_{\alpha}\right|\right),$$

and,

$$F^{\cap \text{odd}} = \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)$$
$$+ \sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \underline{\mathscr{A}_{i_3}}_{\alpha} \cap \underline{\mathscr{A}_{i_4}}_{\alpha} \cap \underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)$$
$$+ \ldots + \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha} \cap \underline{\mathscr{A}_{i_2}}_{\alpha} \cap \ldots \cap \underline{\mathscr{A}_{i_{n-2}}}_{\alpha}\right|\right).$$

If we have $\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta} \subseteq \overline{\mathscr{A}_i}_{\beta}$ then $\left|\overline{\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_n}_{\beta}\right| \leq \left|\overline{\mathscr{A}_i}_{\beta}\right|$.

If $\left|\overline{\mathscr{A}_k}_{\beta}\right| \geq \left|\overline{\mathscr{A}_i}_{\beta}\right|$ $\forall i$ where $i = 1, 2, 3, \ldots, n$ and n is odd, thus we obtain

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n} = 1-\frac{-\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right| - F^{\cap\mathrm{odd}} + F^{\cap\mathrm{even}} + \left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1\cap\mathscr{A}_2\cap\ldots\cap\mathscr{A}_n}_{\beta}\right|}$$

$$\leq 1-\frac{-\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right| - F^{\cap\mathrm{odd}} + F^{\cap\mathrm{even}} + \left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}$$

$$= 1+\frac{\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}+\frac{F^{\cap\mathrm{odd}}}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}-\frac{F^{\cap\mathrm{even}}}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}_1}_{\alpha}\cup\underline{\mathscr{A}_2}_{\alpha}\cup\ldots\cup\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|},$$

From

$$\frac{\sum_{i=1}^{n}\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}=\frac{\left|\underline{\mathscr{A}_1}_{\alpha}\right|+\left|\underline{\mathscr{A}_2}_{\alpha}\right|+\left|\underline{\mathscr{A}_3}_{\alpha}\right|+\ldots+\left|\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}$$

$$=\frac{\left|\underline{\mathscr{A}_1}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}+\frac{\left|\underline{\mathscr{A}_2}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}+\frac{\left|\underline{\mathscr{A}_3}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}+\ldots+\frac{\left|\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}$$

$$\leq\frac{\left|\underline{\mathscr{A}_1}_{\alpha}\right|}{\left|\overline{\mathscr{A}_1}_{\beta}\right|}+\frac{\left|\underline{\mathscr{A}_2}_{\alpha}\right|}{\left|\overline{\mathscr{A}_2}_{\beta}\right|}+\frac{\left|\underline{\mathscr{A}_3}_{\alpha}\right|}{\left|\overline{\mathscr{A}_3}_{\beta}\right|}+\ldots+\frac{\left|\underline{\mathscr{A}_n}_{\alpha}\right|}{\left|\overline{\mathscr{A}_n}_{\beta}\right|},$$

because of $\left|\overline{\mathscr{A}_k}_{\beta}\right|\geq\left|\overline{\mathscr{A}_i}_{\beta}\right|$ where $i = 1, 2, 3, \ldots, n$ and n is odd, then $\frac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}\leq\frac{\left|\underline{\mathscr{A}_i}_{\alpha}\right|}{\left|\overline{\mathscr{A}_i}_{\beta}\right|}$.

$$\frac{F^{\cap\mathrm{odd}}}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}=\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}$$

$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\cap\underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}$$

$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-2}}}_{\alpha}\right|\right)}{\left|\overline{\mathscr{A}_k}_{\beta}\right|}.$$

$$\leq \sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}}_{\beta}\right|\right)}$$

$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\cap\underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}\cap\mathscr{A}_{i_4}\cap\mathscr{A}_{i_5}}_{\beta}\right|\right)}$$

$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-2}}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\ldots\cap\mathscr{A}_{i_{n-2}}}_{\beta}\right|\right)}.$$

Thus, we have

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap A_3\cap\ldots A_n}=1+\frac{\sum_{i=1}^{n}\left|\underline{\mathscr{A}_{i}}_{\alpha}\right|}{\left|\overline{\mathscr{A}_{k}}_{\beta}\right|}+\frac{F^{\cap\text{odd}}}{\left|\overline{\mathscr{A}_{k}}_{\beta}\right|}-\frac{F^{\cap\text{even}}}{\left|\overline{\mathscr{A}_{k}}_{\beta}\right|}-\frac{\left|\underline{\mathscr{A}_{1}}_{\alpha}\cup\underline{\mathscr{A}_{2}}_{\alpha}\cup\underline{\mathscr{A}_{3}}_{\alpha}\cup\ldots\underline{\mathscr{A}_{n}}_{\alpha}\right|}{\left|\overline{\mathscr{A}_{k}}_{\beta}\right|}$$

$$\leq 1+\sum_{i=1}^{n}\frac{\left|\underline{\mathscr{A}_{i}}_{\alpha}\right|}{\left|\overline{\mathscr{A}_{i}}_{\beta}\right|}+F_{*}^{\cap\text{odd}}-\frac{F^{\cap\text{even}}+\left|\underline{\mathscr{A}_{1}}_{\alpha}\cup\underline{\mathscr{A}_{2}}_{\alpha}\cup\underline{\mathscr{A}_{3}}_{\alpha}\cup\ldots\underline{\mathscr{A}_{n}}_{\alpha}\right|}{\left|\overline{\mathscr{A}_{k}}_{\beta}\right|}$$

where let us define

$$F_{*}^{\cap\text{odd}}=\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}}_{\beta}\right|\right)}$$

$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\underline{\mathscr{A}_{i_3}}_{\alpha}\cap\underline{\mathscr{A}_{i_4}}_{\alpha}\cap\underline{\mathscr{A}_{i_5}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}\cap\mathscr{A}_{i_4}\cap\mathscr{A}_{i_5}}_{\beta}\right|\right)}$$

$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_{\alpha}\cap\underline{\mathscr{A}_{i_2}}_{\alpha}\cap\ldots\cap\underline{\mathscr{A}_{i_{n-2}}}_{\alpha}\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\ldots\cap\mathscr{A}_{i_{n-2}}}_{\beta}\right|\right)}.$$

From Definition 2.4.2, $\rho^{\alpha,\beta}_{A}=1-\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}$ or $\frac{\left|\underline{\mathscr{A}}_{\alpha}\right|}{\left|\overline{\mathscr{A}}_{\beta}\right|}=1-\rho^{\alpha,\beta}_{A}$, and from Property 2.4.1.2,

therefore

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n} \le 1+\sum_{i=1}^{n}\frac{\left|\underline{\mathscr{A}_i}_\alpha\right|}{\left|\overline{\mathscr{A}_i}_\beta\right|}+\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_\alpha\cap\underline{\mathscr{A}_{i_2}}_\alpha\cap\underline{\mathscr{A}_{i_3}}_\alpha\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}}_\beta\right|\right)}$$

$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_\alpha\cap\underline{\mathscr{A}_{i_2}}_\alpha\cap\underline{\mathscr{A}_{i_3}}_\alpha\cap\underline{\mathscr{A}_{i_4}}_\alpha\cap\underline{\mathscr{A}_{i_5}}_\alpha\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\mathscr{A}_{i_3}\cap\mathscr{A}_{i_4}\cap\mathscr{A}_{i_5}}_\beta\right|\right)}$$

$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\frac{\left(\left|\underline{\mathscr{A}_{i_1}}_\alpha\cap\underline{\mathscr{A}_{i_2}}_\alpha\cap\ldots\cap\underline{\mathscr{A}_{i_{n-2}}}_\alpha\right|\right)}{\left(\left|\overline{\mathscr{A}_{i_1}\cap\mathscr{A}_{i_2}\cap\ldots\cap\mathscr{A}_{i_{n-2}}}_\beta\right|\right)}$$

$$-\frac{\boldsymbol{F}^{\cap\text{even}}+\left|\underline{\mathscr{A}_1}_\alpha\cup\underline{\mathscr{A}_2}_\alpha\cup\underline{\mathscr{A}_3}_\alpha\cup\ldots\underline{\mathscr{A}_n}_\alpha\right|}{\left|\overline{\mathscr{A}_k}_\beta\right|}$$

$$=1+\sum_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)+\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}}\right)$$

$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}}\right)$$

$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_{n-2}}}\right)$$

$$-\frac{\boldsymbol{F}^{\cap\text{even}}+\left|\underline{\mathscr{A}_1}_\alpha\cup\underline{\mathscr{A}_2}_\alpha\cup\ldots\cup\underline{\mathscr{A}_n}_\alpha\right|}{\left|\overline{\mathscr{A}_k}_\beta\right|}.$$

Let us define $\boldsymbol{F}_*^{\cap\text{even}}=\dfrac{\boldsymbol{F}^{\cap\text{even}}}{\left|\overline{\mathscr{A}_k}_\beta\right|}$ and $Z_{*k}=\dfrac{\left|\underline{\mathscr{A}_1}_\alpha\cup\underline{\mathscr{A}_2}_\alpha\cup\ldots\cup\underline{\mathscr{A}_n}_\alpha\right|}{\left|\overline{\mathscr{A}_k}_\beta\right|}$, if $\left|\overline{\mathscr{A}_k}_\beta\right|\ge\left|\overline{\mathscr{A}_i}_\beta\right|$ $\forall i$ where $i = 1, 2, 3,\ldots, n$ and n is odd, therefore

$$\rho^{\alpha,\beta}_{A_1\cap A_2\cap\ldots\cap A_n}\le 1+\sum_{i=1}^{n}\left(1-\rho^{\alpha,\beta}_{A_i}\right)+\sum_{i_1=1}^{n-2}\sum_{i_2=i_1+1}^{n-1}\sum_{i_3=i_2+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}}\right)$$

$$+\sum_{i_1=1}^{n-4}\sum_{i_2=i_1+1}^{n-3}\sum_{i_3=i_2+1}^{n-2}\sum_{i_4=i_3+1}^{n-1}\sum_{i_5=i_4+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}}\right) \qquad \square$$

$$+\ldots+\sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}\left(1-\rho^{\alpha,\beta}_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_{n-2}}}\right)-\boldsymbol{F}_*^{\cap\text{even}}-Z_{*k}.$$

CHAPTER 5

CONCLUSION AND SUGGESTION

To this end, theoretically, rough set theory, fuzzy set theory and their connections are described and discussed. We formulate roughness lower and upper bounds of the fuzzy set under union and intersection operations and prove true. New theorems of roughness bounds for fuzzy set operations are established and prove true theoretically. Interestingly, the acquired upper and lower bounds reveal hidden relations between rough set theory and fuzzy set theory. The bounds of such fuzzy set operations can be determined from the roughness measure efficiently and accurately. The results are several useful indications of the roughness for the fuzzy operations involving two or more large fuzzy sets. This is beneficial for many applications in pattern recognition and image analysis problems.

Some open issues can be drawn here such as finding and developing a mechanism or an algorithm to identify roughness bound in real applications (e.g., pattern recognition and image analysis). More importantly are the extensions of minimum roughness upper bounds and maximum roughness lower bounds of the fuzzy set operations.

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AUTHOR BIOGRAPHY

Author	Kiatrisak Palasit
Date of Birth	December 13, 1976
Address	1 M. 4, Natakwan, Mueng, Rayong 21000, Thailand
Bachelor Degree	B.Ed. (Mathematics)
Institution	Faculty of Education, Burapha University, Chonburi, Thailand.