OPEN FORM OF RUNGE-KUTTA METHOD

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บทคัดย่อ

เนื้อหาของวิทยานิพนธ์ฉบับนี้ จะทำการศึกษาและพัฒนาระเบียบวิธีรุงเง-กุตตา แบบเปิดแบบใหม่ เพื่อใช้เป็นเครื่องมือในการประมาณค่าหาผลเฉลยของการแก้ปัญหาเงื่อนไขค่าเริ่มต้นของสมการเชิงอนุพันธ์สามัญ ในอดีตที่ผ่านมาระเบียบวิธีรุงเง-กุตตาที่นิยมใช้กันอยู่นั้นเป็นแบบปิด แต่ระเบียบวิธีใหม่ที่เราได้สร้างขึ้นนี้จะเป็นแบบเปิด โดยที่วัตถุประสงค์ของเราคือต้องการสร้างระเบียบวิธีรุงเง-กุตตา แบบเปิดแบบใหม่ ซึ่งมีค่าความผิดพลาดอยู่ในรูปแบบ $O(h^{s+2})$ โดยที่ h คือช่วงที่เราพิจารณาและ s คือ จำนวนจุดที่ใช้

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ABSTRACT

The main objective of this research is to study and develop the open form of Runge-Kutta method to be used as a tool for finding the numerical solution of the initial value problem of ordinary differential equations[ODEs]. In the past, most of Runge-Kutta formulas are of the closed form,while in this research the open form will be the focal interest. The new open form Runge-Kutta method is presented truncation error value in $O(h^{s+2})$ form where h is the width of each point and s is number of stages.

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Ganokpan Songprasert

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CHAPTER 1

INTRODUCTION

Mathematical models with ordinary differential equations are used to explain many phenomena in many area of sciences, technology and social sciences. The numerical method most frequently used to find the numerical solution of initial value problems of ordinary differential equations in those models is the Runge-Kutta method.

1.1 Statement and Significance of the Problem

The Runge-Kutta method for finding the numerical solution of initial value problem of ordinary differential equations which is of the form

$$y' = f(x, y) \tag{1.1}$$

with the initial value

$$y(x_0) = y_0. \tag{1.2}$$

This type of the ordinary differential equations appears oftenly in many branchs of science. An s-stage explicit Runge-Kutta method is used to compute the approximated value of $y_{n+1}(x)$ in (1.1)-(1.2), when $y_n(x)$ is known, can be expressed by the following relations.

$$y_{n+1} = y_n + \textstyle\sum_{i=1}^{s} a_i k_i \tag{1.3}$$

$$k_i = hf(x_n + c_i h, y_n + h \textstyle\sum_{j=1}^{i-1} a_{ij} k_j),$$

$$i = 1, 2, ..., s.$$

Many mathematicians study this Runge-Kutta method in order to come up with the formulas that give the best numerical solution of the above problem. In this research we intend to construct a new formulas of the Runge-Kutta method of the open form.

1.2 Main Objective

Most of the mathematicians studied the Runge-Kutta method in closed form, but in this thesis we will study and develop new formulars of the open form and finding a suitable unknown variable for the new formulas to use as a tool to solve the initial value problem of ordinary differential equations and present truncation error value in $O(h^{s+2})$.

1.3 Scope of the study

To study the Runge-Kutta method and develop the new open form of Runge-Kutta method by three orders that is order 1, order 2 and order 3 for solving the initial value problem of ordinary differential equations are of the form

$$y' = f(x, y)$$

with the initial value

$$y(x_0) = y_0.$$

1.4 Benefit

After the study of the open formula of the Runge-Kutta method , we have the following benefits.

1.4.1 Obtain the new formulas of the open form of Runge-Kutta method for solving the initial value problem of ordinary differential equations.

1.4.2 We may use these three new formulas to compute the numerical solution of many the problems in engineering, science, astronomy and social sciences.

1.5 Research Procedure

Steps for doing this research are summarized as follow:

1.5.1 Study related document and researched papers of Runge-Kutta method that we found.

1.5.2 Formulate new formulas of the open form of Runge-Kutta method.

1.5.3 Compare the results obtained by our new formulas with those obtained by the classical Runge-Kutta formulas.

1.5.6 Summarize main results.

1.5.7 Write the thesis.

CHAPTER 2

PRELIMINARIES AND LITERATURE REVIEWS

In this chapter, we shall give definitions and theorems which are used to find the new formulated of Runge-Kutta method for solving the numerical solution of the initial value problem of the ordinary differential equations.

2.1 Basic Definitions and Theorems

The first part, we will give some basic definitions and theorems which are used for find the new construct of Runge-Kutta method for solving the numerical solution of the initial value problem of the ordinary differential equations.

2.1.1 The Runge-Kutta method is the admired method for approximate the numerical solution of the ordinary differential equations.

In 1895 C.Runge presented the numerical method for finding the numerical solution of the initial value problem of the ordinary differential equations. After that in 1901 W.Kutta improved the method of C.Runge which denoted Runge-Kutta method that has been used for finding the numerical solution of the ordinary differential equation is in the form

$$y' = f(x,y) \tag{2.1}$$

with the initial value $y(x_0) = y_0.$

which is the first order ordinary differential equations. We may find the value of $y(x_{m+1})$ by using the value of $y(x_m)$ with the help of the and the Runge-Kutta method which is in the form.

$$y_{m+1} = y_m + h\Phi(x_m, y_m; h); m = 1, 2, ... \tag{2.2}$$

when $\Phi(x_m, y_m; h) = \sum_{i=1}^{s} a_i k_i$

$$k_1 = f(x, y)$$

$$k_i = hf(x_m + c_i h, y_m + h\textstyle\sum_{j=1}^{i-1} b_{ij} k_i), i = 1, 2, ..., s$$

and $$c_i = \textstyle\sum_{j=1}^{s} b_{ij}, i = 1, 2, ..., s.$$

where a_i, c_i and b_{ij} are coefficient of the Runge-Kutta method and h is the step size.

2.1.2 The equation in the (2.1) in order p if p is the maximum positive number such that we give a dispersive Taylor series about x.

$$y_{m+1} - y_m - h\Phi(x_m, y_m; h) = O(h^{p+1}). \tag{2.3}$$

when $y(x)$ is the exact solution of the problem (2.1). The most frequently used formula in order of the Runge-Kutta method for solve the problem in the form (2.1) is the four points formula which has the form

$$y_{m+1} = y_m + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.4}$$

where

$$\begin{aligned} k_1 &= f(x_m, y_m) \\ k_2 &= f(x_m + \frac{h}{2}, y_m + \frac{h}{2}k_1) \\ k_3 &= f(x_m + \frac{h}{2}, y_m + \frac{h}{2}k_2) \\ k_4 &= f(x_m + h, y_m + hk_3). \end{aligned}$$

The Butcher array table 2.1.1 present the relation of the coefficient of the Runge-Kutta method.

Table 2.1.1 The Butcher-array for general Runge-Kutta method

$$\begin{array}{c|c} c & B \\ \hline & a^T \end{array} = \begin{array}{c|cccc} 0 & & & & \\ c_2 & b_{21} & & & \\ c_3 & b_{31} & b_{21} & & \\ \vdots & \vdots & \vdots & \ddots & \\ c_s & b_{s1} & b_{s2} & \dots & b_{s1} \\ \hline & a_1 & a_2 & \dots & a_{s-1} \end{array}$$

Form equation(2.3) the runge-kutta method will give the truncation error value in $O(h^{s+1})$ that is the error from the Runge-Kutta method order s varies to step side h order $s+1$. At the present, finding the solution of the ordinary differential equations by the Runge-Kutta method was done in order 4.

Subsequently, many mathematicians were studied and tried to develop a better Runge-Kutta method for solving the numerical solution of the initial value problem of the ordinary differential equations. For example the Runge-Kutta Merson which is of the form

$$y_{m+1} = y_m + \frac{1}{6}k_1 + \frac{4}{6}k_4 + \frac{1}{6}k_5 \tag{2.5}$$

where

$$\begin{aligned} k_1 &= hf(x_m, y_m) \\ k_2 &= hf(x_m + \frac{h}{3}, y_m + \frac{h}{3}k_1) \\ k_3 &= hf(x_m + \frac{h}{3}, y_m + \frac{h}{6}k_1 + \frac{h}{6}k_2) \\ k_4 &= hf(x_m + \frac{h}{2}, y_m + \frac{h}{8}k_1 + \frac{3h}{8}k_3) \\ k_5 &= hf(x_m + h, y_m + \frac{h}{2}k_1 - \frac{3h}{2}k_3 + 2hk_4). \end{aligned}$$

The Runge-Kutta Fehlberg formula is presented in 1969 by E.Fehlberg. He modified this formula from the original of the Runge-Kutta method in case $s=5$ with delete the term of k_2. It then can be express by

$$y_{m+1} = y_m + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5 \tag{2.6}$$

where

$$\begin{aligned}
k_1 &= hf(x_m, y_m)\\
k_2 &= hf(x_m + \frac{h}{4}, y_m + \frac{h}{4}k_1)\\
k_3 &= hf(x_m + \frac{3h}{8}, y_m + \frac{3h}{32}k_1 + \frac{9h}{32}k_2)\\
k_4 &= hf(x_m + \frac{12h}{13}, y_m + \frac{1932h}{2197}k_1 - \frac{7200h}{2197}k_2 + \frac{7296h}{2197}k_3)\\
k_5 &= hf(x_m + h, y_m + \frac{439h}{216}k_1 - 8hk_2 + \frac{3680h}{513}k_3 - \frac{845h}{4104}k_4).
\end{aligned}$$

2.1.3 We can use values α, β and A_k form Legendre Polynomials to develop another form of Runge-Kutta method.

The Legendre polynomials are orthogonal on the interval [-1,1] with respect to the weighting function $w(x) = 1$, that is,

$$\int_{-1}^{+1} \quad P_n(x)P_m(x)dx = 0, n \neq m, \tag{2.7}$$

$$\int_{-1}^{+1} \quad [P_n(x)]^2 dx = c(n) \neq 0. \tag{2.8}$$

The first few Legendre polynomials are:

$$P_0 = 1, \tag{2.9}$$

$$P_1 = x, \tag{2.10}$$

$$P_2 = x^2 - \frac{1}{3} \tag{2.11}$$

$$P_3 = x^3 - \frac{3x}{5} \tag{2.12}$$

$$P_4 = x^4 - \frac{6x^2}{7} + \frac{3}{35} \tag{2.13}$$

$$P_5 = x^5 - \frac{10x^3}{9} + \frac{5x}{21} \tag{2.14}$$

$$P_6 = x^6 - \frac{15x^4}{11} - \frac{5}{213} \tag{2.15}$$

$$P_7 = x^7 - \frac{21x^5}{13} + \frac{105x^3}{143} - \frac{35x}{429} \tag{2.16}$$

The general recursion relation is

$$P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x). \tag{2.17}$$

Table 2.1.2 Roots of Legendre polynomials and weight of the definite integral

n	x_k	A_k
2	-0.577350269189626	1.0
	0.577350269189626	1.0
3	-0.774596669241483	0.555555555555556
	0.0	0.888888888888889
	0.774596669241483	0.555555555555556
4	-0.861136311594053	0.347854845137454
	-0.339981043584856	0.652145154862546
	0.339981043584856	0.652145154862546
	0.861136311594053	0.347854845137454
5	-0.906179845938664	0.236926885056189
	-0.538469310105683	0.478628670499366
	0.0	0.568888888888889
	0.538469310105683	0.478628670499366
	0.906179845938664	0.236926885056189
6	-0.932409514203152	0.171324492379170
	-0.661209386466265	0.360761573048139
	-0.238619186083197	0.467913934572691
	0.238619186083197	0.467913934572691
	0.661209386466265	0.360761573048139
	0.932409514203152	0.171324492379170
7	-0.94910791234276	0.12948483453421
	-0.74153118559940	0.27970539148927
	-0.40584515137740	0.38183005050513
	0.0	0.41795918367348
	0.40584515137740	0.38183005050513
	0.74153118559940	0.27970539148927
	0.94910791234276	0.12948483453421

We use α_k form roots of Legendre polynomials by $\alpha_k = \frac{1+x_k}{2}$ and $a_k = \frac{A_k}{2}$. We will get α_k and a_k as shown in table 2.1.3 .

Table 2.1.3 Values of α_k and a_k

n	α_k	a_k
2	0.21132486540519	0.5
	0.78867513459481	0.5
3	0.11270165871393	0.27777777777778
	0.5	0.44444444444444
	0.88729874776386	0.27777777777778
4	0.06943184629339	0.17392739273927
	0.33000949667616	0.32607260726073
	0.66999093381686	0.32607260726073
	0.93056807935077	0.17392739273927
5	0.04691005506833	0.11846344485750
	0.23076534478404	0.23931433659840
	0.5	0.28444444444444
	0.76923465521596	0.23931433659840
	0.95309009679821	0.11846344485750
6	0.03379524886878	0.08566221142163
	0.16939546599496	0.18038076798967
	0.38069034517259	0.23395695031181
	0.61930965482741	0.23395695031181
	0.83060453400504	0.18038076798967
	0.03379524886878	0.08566221142163
7	0.02544603685288	0.06474243335338
	0.12923441734417	0.13985278654048
	0.29707743296174	0.19091507570770
	0.5	0.20897959183673
	0.70292207792208	0.19091507570770
	0.87076537013802	0.13985278654048
	0.97455470737913	0.06474243335338

Therefore we will obtain the Runge-Kutta method with respect to the weighting function of the Legendre polynomials as shown below;

when n=2

$$y_{m+1} = y_m + \frac{h}{2}(k_1 + k_2) \tag{2.18}$$

where

$$\begin{gathered} k_1 = f(x_m + \tfrac{780}{3691}h, y_m + \tfrac{780}{3691}hf) \\ k_2 = f(x_m + \tfrac{780}{989}h, y_m + \tfrac{780}{989}hk_1). \end{gathered}$$

when n=3

$$y_{m+1} = y_m + \frac{h}{18}(5k_1 + 8k_2 + 5k_3) \tag{2.19}$$

where

$$\begin{gathered} k_1 = f(x_m + \tfrac{496}{4401}h, y_m + \tfrac{496}{4401}hf) \\ k_2 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hk_1) \\ k_3 = f(x_m + \tfrac{496}{559}h, y_m + \tfrac{496}{559}hk_2). \end{gathered}$$

when n=4

$$y_{m+1} = y_m + \frac{h}{3030}(527k_1 + 988k_2 + 988k_3 + k_4) \tag{2.20}$$

where

$$\begin{gathered} k_1 = f(x_m + \tfrac{842}{12127}h, y_m + \tfrac{842}{12127}hf) \\ k_2 = f(x_m + \tfrac{695}{2106}h, y_m + \tfrac{695}{2106}hk_1) \\ k_3 = f(x_m + \tfrac{739}{1103}h, y_m + \tfrac{739}{1103}hk_2) \\ k_4 = f(x_m + \tfrac{1032}{1109h}, y_m + \tfrac{1032}{1109}hk_3). \end{gathered}$$

when n=5

$$y_{m+1} = y_m + h(\frac{478}{4035}k_1 + \frac{1075}{4492}k_2 + \frac{64}{225}k_3 + \frac{1075}{4492}k_4 + \frac{478}{4035}k_5) \tag{2.21}$$

where

$$\begin{gathered} k_1 = f(x_m + \tfrac{230}{4903}h, y_m + \tfrac{230}{4903}hf) \\ k_2 = f(x_m + \tfrac{4568}{19795}h, y_m + \tfrac{4568}{19795}hk_1) \\ k_3 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hk_2) \\ k_4 = f(x_m + \tfrac{15227}{19795}h, y_m + \tfrac{15227}{19795}hk_3) \\ k_5 = f(x_m + \tfrac{1280}{1343}h, y_m + \tfrac{1280}{1343}hk4). \end{gathered}$$

when n=6

$$y_{m+1} = y_m + h(\frac{141}{1646}k_1 + \frac{559}{3099}k_2 + \frac{1163}{4971}k_3 + \frac{1163}{4971}k_4 + \frac{559}{3099}k_5 + \frac{141}{1646}k_6) \quad (2.22)$$

where

$$\begin{aligned}
k_1 &= f(x_m + \tfrac{239}{7072}h, y_m + \tfrac{239}{7072}hf) \\
k_2 &= f(x_m + \tfrac{269}{1588}h, y_m + \tfrac{269}{1588}hk_1) \\
k_3 &= f(x_m + \tfrac{761}{1999}h, y_m + \tfrac{761}{1999}hk_2) \\
k_4 &= f(x_m + \tfrac{1238}{1999h}, y_m + \tfrac{1238}{1999}hk_3) \\
k_5 &= f(x_m + \tfrac{1319}{1588}h, y_m + \tfrac{1319}{1588}hk4) \\
k_6 &= f(x_m + \tfrac{239}{7072}h, y_m + \tfrac{239}{7072}hk_5).
\end{aligned}$$

when n=7

$$\begin{aligned}
y_{m+1} = y_m + h(\frac{323}{4989}k_1 + \frac{133}{951}k_2 + \frac{290}{1519}k_3 + \frac{256}{1225}k_4 \\
+\frac{290}{1519}k_5 + \frac{133}{951}k_6 + \frac{323}{4989}k_7)
\end{aligned} \quad (2.23)$$

where

$$\begin{aligned}
k_1 &= f(x_m + \tfrac{87}{3419}h, y_m + \tfrac{87}{3419}hf) \\
k_2 &= f(x_m + \tfrac{763}{5904}h, y_m + \tfrac{763}{5904}hk_1) \\
k_3 &= f(x_m + \tfrac{986}{3319}h, y_m + \tfrac{986}{3319}hk_2) \\
k_4 &= f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hk_3) \\
k_5 &= f(x_m + \tfrac{433}{616}h, y_m + \tfrac{433}{616}hk4) \\
k_6 &= f(x_m + \tfrac{694}{797}h, y_m + \tfrac{694}{797}hk_5) \\
k_7 &= f(x_m + \tfrac{383}{393}h, y_m + \tfrac{383}{393}hk_6).
\end{aligned}$$

2.2 Literature Reviews

In 2004 Z.A.Anatassi and T.E.Simos presented[3] "A dispersive-fitted and dissipative-fitted explicit Runge-Kutta method for the numberical solution of orbital problem". They presented a new explicit Runge-Kutta method of fourth algebraic order with minimun error of the fifth algebraic order(whose limit is zero, when the step-length tends to zero) and infinite order of dispersion and dissipation. The numerical results of a wide range of methods these are applied to well-known periodic orbital problems which shown the efficiency of the new constructed method. So they have an almost fifth-order method. The new method is shown in table 2.2

Table 2.2 The Butcher-array for fifth-algebraic order of the new constructed Runge-Kutta method

0						
$\frac{321}{1000000}$	$\frac{321}{1000000}$					
$\frac{13}{20}$	$-\frac{4220827}{6420}$	$\frac{211250}{321}$				
$\frac{1}{6}$	a_{41}	a_{42}	a_{43}			
$\frac{1}{6}$	$\frac{168842389}{450684}$	$-\frac{3250000}{8667}$	$\frac{35}{351}$	$\frac{5}{12}$		
1	$\frac{168837574}{95979}$	$-\frac{13000000}{7383}$	$\frac{11060}{8671}$	a_{64}	a_{65}	
1	$\frac{1}{78}$	0	$\frac{4000}{7917}$	0	$\frac{54}{145}$	$\frac{23}{210}$

$$
\begin{aligned}
a_{41} &= -\frac{249037}{5778} + \frac{649679}{321} a_{43}, \\
a_{42} &= \frac{125000}{2889} - \frac{650000}{321} a_{43}, \\
a_{64} &= \frac{945}{667} - a_{65}, \\
a_{65} &= \frac{1890}{667} \frac{4563 a_{43} v^5 - 11600 v^3 + 69600 v - 69600 sin(v)}{v^5(-580 + 4563 a_{43})} \\
A &= -3v^{10} + 100v^8 + 72cos(v)v^6 - 1128v^6 + 480sin(v)v^5 + 5508v^4 \\
&\quad -1152cos(v)v^4 - 3960sin(v)v^3 - 8604v^2 - 900(cos(v))^2v^2 + 9504cos(v)v^2 \\
&\quad -4320sin(v)cos(v)v + 4320sin(v)v + 5184(cos(v))^2 - 10368cos(v) + 5184, \\
a_{43} &= \frac{1160}{4563} \frac{(8v^4 - 66v^2 + 30sin(v)v + 72 - 72cos(v) - \sqrt{A})}{v^6}.
\end{aligned}
$$

We can write in the following form

$$y_{m+1} = y_m + h(\frac{1}{78}k_1 + \frac{4000}{7917}k_3 + \frac{54}{145}k_5 + \frac{23}{210}k_6) \qquad (2.24)$$

where

$$\begin{aligned}
k_1 &= f(x_m, y_m) \\
k_2 &= f(x_m + \frac{321h}{1000000}, y_m + \frac{321h}{1000000}k_1) \\
k_3 &= f(x_m + \frac{13h}{20}, y_m - \frac{4220827h}{6420}k_1 + \frac{211250h}{321}k_2) \\
k_4 &= f(x_m + \frac{h}{6}, y_m + a_{41}k_1 + a_{42}k_2 + a_{43}k_3) \\
k_5 &= f(x_m + \frac{h}{6}, y_m + \frac{168842389h}{450684}k_1 - \frac{3250000h}{8667}k_2 + \frac{35h}{351}k_3 + \frac{5h}{12}k_4) \\
k_6 &= f(x_m + h, y_m + \frac{168837574h}{95979}k_1 - \frac{13000000h}{7383}k_2 + \frac{11060h}{8671}k_3 + a_{64}k_4 + a_{65}k_5).
\end{aligned}$$

In 2003 M.E.A.El-Mikkawy and M.M.M.Eisa present[4] "A general four-parameter non-FSAL embedded Runge-Kutta algorithm of orders 6 and 4 in seven stages". They shown by using seven stages per step a general four-parameter that non-FSAL embedded RK algoritham having orders 6 and 4 may be designed. A special algorithm, called RK 6(4)7 new is obtained by using suitable choices for the free parameters. This new algorithm together with the RK 5(4)7 FM in [6] and [7] are applied to some test problems, which have known exact solutions. It is found that the new algorithm is competitive conparing with the algorithms in [6] and [7].

They defined the local truncation error which is usually used as a measure of the accuracy of the process by

$$\underline{t}_{n+1} = \textstyle\sum_{i=0}^{q-1} h_n^{i+1}\underline{\Phi}_i(y(x_n)) + O(h_n^{q+1})$$

when

$$\underline{\Phi}_i(y(x)) = \tfrac{1}{i!}\tfrac{\partial^i\Phi}{\partial h_n^i} - \tfrac{1}{(i+1)!}\underline{y}^{i+1}(x), i = 0(1)q-1$$

the function $\underline{\Phi}_i$ are called the error functions. For any ERK algorithm of order p we have $\underline{\Phi}_i = 0, i = 0(1)p-1$ and $\underline{\Phi}_p \neq 0$.

To derive the RK 6(4) algorithm we shall consider $s = 7$ where s is the number of stages. In this case 45 equations for order 6 and 8 equations for order 4. A solution of these equations may be considered by assumption some coefficient and use algebraic.

The derivation gives a family of seven-stage(6,4) pairs with c_3, c_5, c_6 and b_6 as free parameters. A suitable choice for the free parameters c_3, c_5, c_6 and b_6 is found to be $(c_3, c_5, c_6, b_6) = (\frac{1}{5}, \frac{1}{2}, \frac{41}{49}, 0)$. By considering a number of test problems it is found that new RK 6(4)7 algorithm presented in Eisa [5] is competitive with the RK 5(4) 7FM algorithm in [2] and the RK 5(4)7 in [4].

In 2003 Moawwad El-Mikkawy and El-Desouky Rahmo presented[5]"A new optimized non-FSAL embedded Runge-Kutta-Nystrom algorithm of orders 6 and 4 in six stages". They presented a new 4-parameter Runge-Kutta-Nystrom (RKN) family of the non-FSAL type is constructed. The strategy used for the construction is based on the criteria listed by [8]. By an appropriate choice of the free parameters we obtained an optimized non-FSAL RKN 6(4)6 embedded algorithm. The coefficients of this algorithm are obtained by writing a program in MACSYMA(MACSYMA Reference Manual). Test results indicate the superiority of the new algorithm relative to the RKN 6(4)6FM algorithm of the FSAL type of Dormand et al.

In summary, they obtain a 4-parameters family of non-FSAL RKN 6(4) embedded algorithms for which c_3, c_4, c_5, and d_6 are free parameter. Following the criteria presented in this paper a suitable choice for the free parameters is found to be $(c_3, c_4, c_5, d_6) = (\frac{1}{3}, \frac{2}{3}, \frac{13}{15}, 0)$ and the corresponding algorithm will be referred to as RKN 6(4) 6new.

In 2004 Sirirat Kantidilokwongsa presented "Runge Kutta Method with Gauss-Legendre Quadrature Formulas". The Gauss-Legendre Quadrature Formulas is used to approximated the definite integral of the complicated functions by replacing the integral by the sum of the porduct of value of the function at the nodal points and its weights. She used the points and weights form the Gauss-Legendre Quadrature Formulas to be the points and weights in the Runge Kutta Formulas. Then the local truncation errors are almost of $O(h^{s+2})$ for the new Runge Kutta Formulas of order s.

In summary she obtained a new formula of Runge Kutta order 4 which can be expressed by following relations:

$$y_{m+1} = y_m + \frac{h}{3030}(527k_1 + 988k_2 + 988k_3 + 527k_4) \tag{2.25}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \frac{77h}{1109}, y_m + \frac{77h}{1109}) \\
k_2 &= f(x_m + \frac{364h}{1103}, y_m + \frac{364h}{1103}k_1) \\
k_3 &= f(x_m + \frac{739h}{1103}, y_m - \frac{362h}{1175}k_1 + \frac{803h}{821}k_2) \\
k_4 &= f(x_m + \frac{1032h}{1109}, y_m + \frac{1379h}{1602}k_1 - \frac{1249h}{1742}k_2 + \frac{1664h}{2115}k_3).
\end{aligned}$$

CHAPTER 3

RESEARCH METHODOLOGY

In this chapter we shall presented the research methodology and show how to construct the open form of Runge-Kutta formula.

3.1 Runge-Kutta method

In this section we shall show how construct to the Runge-Kutta formula used for improving and implementing its.

3.2 Steps of Study

We can summarize the steps of doing this thesis as follows :

3.2.1 Find related documents and researches about Runge-Kutta method. We will look for them in the library and internet.

3.2.2 Study related document and researches about Runge-Kutta method that we have found.

3.2.3 Make a new construction of the open form of Runge-Kutta method.

First, we study the Runge-Kutta method for solving the numerical solution of the initial value problem of the ordinary differential equations. It is in the form

$$y_{m+1} = y_m + h\Phi(x_m, y_m; h); m = 1, 2, ... \tag{3.1}$$

when $\Phi(x_m, y_m; h) = \sum_{i=1}^{s} a_i k_i$

$$k_1 = f(x, y)$$

$$k_i = hf(x_m + c_i h, y_m + h\sum_{j=1}^{i-1} b_{ij} k_i), i = 1, 2, ..., s$$

and $c_i = \sum_{j=1}^{s} b_{ij}, i = 1, 2, ..., s.$

where a_i, c_i and b_{ij} are coefficient of the Runge-Kutta method and h is the width of the interval in each order.

Then we make a new construction as follows.
The Runge-Kutta method of first algebraic order form this research

$$y_{m+1} = y_m + ha_1k_1 \tag{3.2}$$

where $$k_1 = f(x_m + \alpha_1 h, y_m + \beta_1 hf).$$

The Runge-Kutta method of second algebraic order form this research, can be expressed by the following relation.

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2) \tag{3.3}$$

where

$$\begin{aligned} k_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf) \\ k_2 &= f(x_m + \alpha_2 h, y_m + \beta_2 hk_1). \end{aligned}$$

The Runge-Kutta method of third algebraic order form this research, can be expressed by the following relation.

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2 + a_3k_3) \tag{3.4}$$

where

$$\begin{aligned} k_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf) \\ I_1 &= f(x_m + \alpha_2 h, y_m + \beta_2 hf + \beta_3 hk_1) \\ k_2 &= f(x_m + \alpha_3 h, y_m + \beta_4 hf + \beta_5 hk_1 + \beta_6 hI_1) \\ I_2 &= f(x_m + \alpha_4 h, y_m + \beta_7 hk_1 + \beta_8 hI_1 + \beta_9 hk_2) \\ k_3 &= f(x_m + \alpha_5 h, y_m + \beta_{10} hI_1 + \beta_{11} hk_2 + \beta_{12} hI_2). \end{aligned}$$

3.2.4 Find the unknown coefficient α and β of the new construction

$$\alpha_i = \textstyle\sum_{j=1}^{s} \beta_{ij}, i = 1, 2, ..., s.$$

3.2.5 Compare the new Runge-Kutta method with the original Runge-Kutta method.
when

$$\begin{aligned} y_{m+1} &= y_m + hy'_m + \frac{h^2}{2!}y''_m + \frac{h^3}{3!}y'''_m + ... \qquad (3.5) \\ &= y_m + hf + \frac{h^2}{2}[f_x + ff_y] \qquad (3.6) \\ &\quad + \frac{h^3}{6}[f_{xx} + 2ff_{xy} + f_xf_y + f^2f_{yy} + ff_y^2] + O(h^4). \end{aligned}$$

3.2.6 Summarize main results.

We can summarize what we obtained from our study of the Open Form of The Runge-Kutta Method for solving the initial value problem of ordinary differential equation.

3.2.7 Write the thesis.

We use PCTeX32 to write this thesis. This thesis compose of 5 chapters. The first chapter is introduction. It is about statement and significance, objective, scope of the study, and benefits of this thesis. Chapter 2 is preliminaries and literature reviews. In this chapter contains basic definitions and theorems and researches which are related with the open form of the Runge-Kutta method for solving the initial value problem of ordinary differential equations. The third chapter is research methodology that is about how to process this thesis. Chapter 4 is the main results. It is about the open form of the Runge-Kutta method first order, second order and third order and use there formulas to obtain the numerical solution of ordinary differential equations and compare the results with some classical formulas of some initial value problem of Runge-Kutta method.

3.3 Research schedule

Work's plan	Month											
	1	2	3	4	5	6	7	8	9	10	11	12
Step 1	←	→										
Step 2		←	→									
Step 3			←	—	→							
Step 4					←	—	—	→				
Step 5								←	—	→		
Step 6										←	→	
Step 7											←	→

Details for each step, we can see in 3.2.

CHAPTER 4

MAIN RESULTS

In this chapter, we shall give the new constructed of the Runge-Kutta formulas for finding the numerical solution of the initial value problem of the ordinary differential equations. And finally we shall give some examples of this kind of problem.

The first order iterative ordinary differential equation is in the form

$$y'(x) = f(x, y(x)) \tag{4.1}$$

with the initial condition

$$y(0) = c. \tag{4.2}$$

where c is a positive real number.

The solution of the problem (4.1) with the initial condition (4.2), can be approximated by using the Runge-Kutta method. We obtain the new constructed formulas are as follow;
One point formula is of the form

$$y_{m+1} = y_m + ha_1k_1 \tag{4.3}$$

where $$k_1 = f(x_m + \alpha_1 h, y_m + \beta_1 hf).$$

when we disperse Taylor's Series of k_1 so we have

$$\begin{aligned} k_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf) \\ &= f + h[\alpha_1 f_x + \beta_1 f f_y] + O(h^2). \end{aligned}$$

Replace k_1 in equation (4.3) we will have

$$y_{m+1} = y_m + haf + h^2[a_1\alpha_1 f_x + a_1\beta_1 f f_y] + O(h^3) \tag{4.4}$$

when

$$y_{m+1} = y_m + hy'_m + \frac{h^2}{2!}y''_m + \frac{h^3}{3!}y'''_m + \ldots \tag{4.5}$$

$$= y_m + hf + \frac{h^2}{2}[f_x + f f_y] + O(h^3). \tag{4.6}$$

Hence from (4.4)and (4.6) we can compare coefficient in term h then we have

$$a_1 = 1, \alpha_1 = \beta_1 = \tfrac{1}{2}.$$

then we get the new construct of the Runge-Kutta method of first algebraic order form this research, can be express by the following relations;

$$y_{m+1} = y_m + hk_1 \tag{4.7}$$

where

$$k_1 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hf).$$

The new constructed formula of the Runge-Kutta method of second algebraic order can be expressed by the following relation.

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2) \tag{4.8}$$

where

$$\begin{aligned} k_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf) \\ k_2 &= f(x_m + \alpha_2 h, y_m + \beta_2 hk_1). \end{aligned}$$

when we disperse Taylor's Series of k_1, k_2 so we have

$$\begin{aligned} k_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf) \\ &= f + h[\alpha_1 f_x + \beta_1 f f_y] + \frac{h^2}{2}[\alpha_1^2 f_{xx} + 2\alpha_1\beta_1 f f_{xy} + \beta_1^2 f^2 f_{yy}] + O(h^3) \\ k_2 &= f(x_m + \alpha_2 h, y_m + \beta_2 hk_1) \\ &= f + h[\alpha_2 f_x + \beta_2 f f_y] + \frac{h^2}{2}[\alpha_2^2 f_{xx} + 2\alpha_2\beta_2 f f_{xy} + \beta_2^2 f^2 f_{yy}] + O(h^3). \end{aligned}$$

Replace k_1, k_2 in equation (4.8) we will have

$$\begin{aligned} y_{m+1} &= y_m + hf(a_1 + a_2) + \frac{h^2}{2}[(2a_1\alpha_1 + 2a_2\alpha_2)f_x + (2a_1\beta_1 + 2a_2\beta_2)f f_y] \\ &\quad + \frac{h^3}{6}[(3a_1\alpha_1^2 + 3a_2\alpha_2^2)f_{xx} + (6a_1\alpha_1\beta_1 + 6a_2\alpha_2\beta_2)f f_{xy} \qquad (4.9) \\ &\quad + (3a_1\beta_1^2 + 3a_2\beta_2^2)f^2 f_{yy}] + O(h^4). \end{aligned}$$

when

$$\begin{aligned} y_{m+1} &= y_m + hy'_m + \frac{h^2}{2!}y''_m + \frac{h^3}{3!}y'''_m + \ldots && (4.10) \\ &= y_m + hf + \frac{h^2}{2}[f_x + f f_y] + \frac{h^3}{6}[f_{xx} + 2f f_{xy} + f_x f_y + f^2 f_{yy} && (4.11) \\ &\quad + f f_y^2] + O(h^4). \end{aligned}$$

Hence from (4.9)and (4.11) we can compare coefficient in term h then we have

$$a_1 + a_2 = 1 \tag{4.12}$$

$$2a_1\alpha_1 + 2a_2\alpha_2 = 1 \tag{4.13}$$

$$2a_1\beta_1 + 2a_2\beta_2 = 1 \tag{4.14}$$

$$3a_1\alpha_1^2 + 3a_2\alpha_2^2 = 1 \tag{4.15}$$

$$6a_1\alpha_1\beta_1 + 6a_2\alpha_2\beta_2 = 2 \tag{4.16}$$

$$3a_1\beta_1^2 + 3a_2\beta_2^2 = 1 \tag{4.17}$$

and we known that $\alpha_1 = \beta_1, \alpha_2 = \beta_2$. when we solve these equation above, we will get these result.

$$a_1 = \tfrac{1}{2}, a_2 = \tfrac{1}{2}, \alpha_1 = \beta_1 = \tfrac{1}{2}, \alpha_2 = \beta_2 = \tfrac{1}{2}$$

then we get the new construct of the Runge-Kutta method of second algebraic order form this research, can be express by the following relations;

$$y_{m+1} = y_m + \frac{h}{2}(k_1 + k_2) \tag{4.18}$$

where

$$k_1 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hf)$$
$$k_2 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hk_1).$$

The new constructed formula of the Runge-Kutta method of third algebraic order can be expressed by the following relation.

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2 + a_3k_3) \tag{4.19}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf)\\
I_1 &= f(x_m + \alpha_2 h, y_m + \beta_2 hf + \beta_3 hk_1)\\
k_2 &= f(x_m + \alpha_3 h, y_m + \beta_4 hf + \beta_5 hk_1 + \beta_6 hI_1)\\
I_2 &= f(x_m + \alpha_4 h, y_m + \beta_7 hk_1 + \beta_8 hI_1 + \beta_9 hk_2)\\
k_3 &= f(x_m + \alpha_5 h, y_m + \beta_{10} hI_1 + \beta_{11} hk_2 + \beta_{12} hI_2).
\end{aligned}$$

we have $\alpha_1 = \frac{496}{4401}, \alpha_2 = \frac{4400}{14457}, \alpha_3 = \frac{1}{2}, \alpha_4 = \frac{557}{803}, \alpha_5 = \frac{496}{559}$

and $a_1 = \frac{5}{18}, a_2 = \frac{8}{18}, a_3 = \frac{5}{18}.$

When we disperse Taylor's Series of k_1, k_2, k_3 so we have

$$
\begin{aligned}
k_1 &= f(x_m+\alpha_1 h, y_m+\beta_1 hf) \\
&= f+h[\alpha_1 f_x+\beta_1 ff_y]+\frac{h^2}{2}[\alpha_1^2 f_{xx}+2\alpha_1\beta_1 ff_{xy}+\beta_1^2 f^2 f_{yy}]+h^3[\frac{\alpha_1^3}{6}f_{xxx} \\
&\quad +\frac{\alpha_1^2\beta_1}{2}f_{xxy}+\frac{\alpha_1\beta_1^2}{2}f_{xyy}+\frac{\beta_1^3}{6}f_{yyy}]+O(h^4) \\
I_1 &= f(x_m+\alpha_2 h, y_m+\beta_2 hf+\beta_3 hk_1) \\
&= f+h[\alpha_2 f_x+(\beta_2+\beta_3)ff_y]+h^2[\alpha_1\beta_3 f_x f_y+\beta_1\beta_3 ff_y^2+\frac{\alpha_2^2}{2}f_{xx} \\
&\quad +\alpha_2(\beta_2+\beta_3)ff_{xy}+\frac{(\beta_2+\beta_3)^2}{2}f^2 f_{yy}]+O(h^3) \\
k_2 &= f(x_m+\alpha_3 h, y_m+\beta_4 hf+\beta_5 hk_1+\beta_6 hI_1) \\
&= f+h[\alpha_3 f_x+(\beta_4+\beta_5+\beta_6)ff_y]+h^2[\alpha_1\beta_5 f_x f_y+\beta_1\beta_5 ff_y^2+\alpha_2\beta_6 f_x f_y \\
&\quad +(\beta_2+\beta_3)\beta_6 ff_y^2+\frac{\alpha_3^2}{2}f_{xx}\alpha_3(\beta_4+\beta_5+\beta_6)ff_{xy}+\frac{(\beta_4+\beta_5+\beta_6)^2}{2}f_{yy}] \\
&\quad +h^3[\frac{\alpha_1^2}{2}\beta_5 f_y f_{xx}+\alpha_1\beta_1\beta_5 ff_y f_{xy}+\frac{\beta_1^2}{2}\beta_5 f^2 f_y f_{yy}+\alpha_1\beta_3\beta_6 f_x f_y^2+\beta_1\beta_3\beta_6 ff_y^3 \\
&\quad +\frac{\alpha_2^2}{2}\beta_6 f_y f_{xx}+\alpha_2(\beta_2+\beta_3)\beta_6 ff_y f_{xy}+\frac{(\beta_2+\beta_3)^2}{2}\beta_6 f^2 f_y f_{yy}+\alpha_1\alpha_3\beta_5 f_x f_{xy} \\
&\quad +\alpha_3\beta_1\beta_5 ff_y f_{xy}+\alpha_2\alpha_3\beta_6 f_x f_{xy}+\alpha_3(\beta_2+\beta_3)\beta_6 ff_y f_{xy}+\alpha_1\beta_5^2 ff^x f_{yy} \\
&\quad +\beta_1\beta_5^2 f^2 f_y f_{yy}+\alpha_2\beta_6^2 ff_x f_{yy}+(\beta_2+\beta_3)\beta_6^2 f^2 f_y f_{yy}+\alpha_1\beta_4\beta_5 ff_x f_{yy} \\
&\quad +\beta_1\beta_4\beta_5 f^2 f_y f_{yy}+\alpha_2\beta_4\beta_6 ff_x f_{yy}+(\beta_2+\beta_3)\beta_4\beta_6 f^2 f_y f_{yy}+\alpha_1\beta_5\beta_6 ff_x f_{yy} \\
&\quad +\beta_1\beta_5\beta_6 f^2 f_y f_{yy}+\alpha_2\beta_5\beta_6 ff_x f_{yy}+(\beta_2+\beta_3)\beta_5\beta_6 f^2 f_y f_{yy}+\frac{\alpha_3^3}{6}f_{xxx} \\
&\quad +\frac{\alpha_3^2}{2}(\beta_4+\beta_5+\beta_6)ff_{xxy}+\frac{\alpha_3}{2}(\beta_4+\beta_5+\beta_6)^2 f_{xyy}+\frac{(\beta_4+\beta_5+\beta_6)^3}{6}f^3 f_{yyy}] \\
&\quad +O(h^4) \\
I_2 &= f(x_m+\alpha_4 h, y_m+\beta_7 hk_1+\beta_8 hI_1+\beta_9 hk_2) \\
&= f+h[\alpha_4 f_x+(\beta_7+\beta_8+\beta_9)ff_y]+h^2[\alpha_1\beta_7 f_x f_y+\beta_1\beta_7 ff_y^2+\alpha_2\beta_8 f_x f_y \\
&\quad +(\beta_2+\beta_3)\beta_8 ff_y^2+\alpha_3\beta_9 f_x f_y+(\beta_4+\beta_5+\beta_6)\beta_9 ff_y^2+\frac{\alpha_4^2}{2}f_{xx} \\
&\quad +\alpha_4(\beta_7+\beta_8+\beta_9)ff_{xy}+\frac{(\beta_7+\beta_8+\beta_9)^2}{2}f^2 f_{yy}]+O(h^3) \\
k_3 &= f(x_m+\alpha_5 h, y_m+\beta_{10}hI_1+\beta_{11}hk_2+\beta_{12}hI_2) \\
&= f+h[\alpha_5 f_x+(\beta_{10}+\beta_{11}+\beta_{12})ff_y]+h^2[\alpha_2\beta_{10}f_x f_y+(\beta_2+\beta_3)\beta_{10}ff_y^2 \\
&\quad +\alpha_3\beta_{11}f_x f_y+(\beta_4+\beta_5+\beta_6)\beta_{11}ff_y^2+\alpha_4\beta_{12}f_x f_y+(\beta_7+\beta_8+\beta_9)\beta_{12}ff_y^2 \\
&\quad +\frac{\alpha_5^2}{2}f_{xx}+\alpha_5(\beta_{10}+\beta_{11}+\beta_{12})ff_{xy}+\frac{(\beta_{10}+\beta_{11}+\beta_{12})^2}{2}f^2 f_{yy}] \\
&\quad +h^3[\alpha_1\beta_3\beta_{10}f_x f_y^2+\beta_1\beta_3\beta_{10}ff_y^3+\frac{\alpha_2^2}{2}\beta_{10}f_y f_{xx}+\alpha_2(\beta_2+\beta_3)\beta_{10}ff_y f_{xy}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\beta_2+\beta_3)^2}{2}\beta_{10}f^2f_yf_{yy}+\alpha_1\beta_5\beta_{11}f_xf_y^2+\beta_1\beta_5\beta_{11}ff_y^3+\alpha_2\beta_6\beta_{11}f_xf_y^2\\
&+(\beta_2+\beta_3)\beta_6\beta_{11}ff_y^3+\frac{\alpha_3^2}{2}\beta_{11}f_yf_{xx}+\alpha_3(\beta_4+\beta_5+\beta_6)\beta_{11}ff_yf_{xy}\\
&+\frac{(\beta_4+\beta_5+\beta_6)^2}{2}\beta_{11}f^2f_yf_{yy}+\alpha_1\beta_7\beta_{12}f_xf_y^2+\beta_1\beta_7\beta_{12}ff_y^3+\alpha_2\beta_8\beta_{12}f_xf_y^2\\
&+(\beta_2+\beta_3)\beta_8\beta_{12}ff_y^3+\alpha_3\beta_9\beta_{12}f_xf_y^2+(\beta_4+\beta_5+\beta_6)\beta_8\beta_{12}ff_y^3+\frac{\alpha_4^2}{2}\beta_{12}f_yf_{xx}\\
&+\alpha_4(\beta_7+\beta_8+\beta_9)\beta_{12}ff_yf_{xy}+\frac{(\beta_7+\beta_8+\beta_9)^2}{2}\beta_{12}f^2f_yf_{yy}+\alpha_2\alpha_5\beta_{10}f_xf_{xy}\\
&+\alpha_5(\beta_2+\beta_3)\beta_{10}ff_yf_{xy}+\alpha_3\alpha_5\beta_{11}f_xf_{xy}+\alpha_5(\beta_4+\beta_5+\beta_6)\beta_{11}ff_yf_{xy}\\
&+\alpha_4\alpha_5\beta_{12}f_xf_{xy}+\alpha_5(\beta_7+\beta_8+\beta_9)\beta_{12}ff_yf_{xy}+\alpha_2\beta_{10}^2ff_xf_{yy}\\
&+(\beta_2+\beta_3)\beta_{10}^2f^2f_yf_{yy}+\alpha_3\beta_{11}^2ff_xf_{yy}+(\beta_4+\beta_5+\beta_6)\beta_{11}^2f^2f_yf_{yy}\\
&+\alpha_4\beta_{12}^2ff_xf_{yy}+(\beta_7+\beta_8+\beta_9)\beta_{12}f^2f_yf_{yy}+\alpha_2\beta_{10}\beta_{11}ff_xf_{yy}\\
&+(\beta_2+\beta_3)\beta_{10}\beta_{11}f^2f_yf_{yy}+\alpha_3\beta_{10}\beta_{11}ff_xf_{yy}+(\beta_4+\beta_5+\beta_6)\beta_{10}\beta_{11}ff_yf_{yy}\\
&+\alpha_2\beta_{10}\beta_{12}ff_xf_{yy}+(\beta_2+\beta_3)\beta_{10}\beta_{12}f^2f_yf_{yy}+\alpha_4\beta_{10}\beta_{12}ff_xf_{yy}\\
&+(\beta_7+\beta_8+\beta_9)\beta_{10}\beta_{12}f^2f_yf_{yy}+\alpha_3\beta_{11}\beta_{12}ff_xf_{yy}+(\beta_4+\beta_5+\beta_6)\beta_{11}\beta_{12}f^2f_yf_{yy}\\
&+\alpha_4\beta_{11}\beta_{12}ff_xf_{yy}+(\beta_7+\beta_8+\beta_9)\beta_{11}\beta_{12}f^2f_yf_{yy}+\frac{\alpha_5^3}{6}f_{xxx}\\
&+\frac{\alpha_5^2}{2}(\beta_{10}+\beta_{11}+\beta_{12})ff_{xxy}+\frac{\alpha_5}{2}(\beta_{10}+\beta_{11}+\beta_{12})^2f^2f_{xyy}+\\
&\frac{(\beta_{10}+\beta_{11}+\beta_{12})^3}{6}f^3f_{yyy}]+O(h^4).
\end{aligned}
$$

Replace k_1, k_2, k_3 in equation (4.19) we will have

$$
\begin{aligned}
y_{m+1} \;=\;& y_m+h[\frac{5}{18}+\frac{8}{18}+\frac{5}{18}]f+h^2[(\frac{5}{18}\alpha_1+\frac{8}{18}\alpha_3+\frac{5}{18}\alpha_5)f_x\\
&+(\frac{5}{18}\beta_1+\frac{8}{18}(\beta_4+\beta_5+\beta_6)+\frac{5}{18}(\beta_{10}+\beta_{11}+\beta_{12}))ff_y]\\
&+h^3[(\frac{5}{36}\alpha_1^2+\frac{8}{36}\alpha_3^2+\frac{5}{36}\alpha_5^2)f_{xx}+(\frac{5}{18}\alpha_1\beta_1+\frac{8}{18}\alpha_3(\beta_4+\beta_5+\beta_6)\\
&+\frac{5}{18}\alpha_5(\beta_{10}+\beta_{11}+\beta_{12}))ff_{xy}+(\frac{5}{18}\alpha_1\beta_3+\frac{8}{18}(\alpha_1\beta_5+\alpha_2\beta_6)\\
&+\frac{5}{18}(\alpha_2\beta_{10}+\alpha_3\beta_{11}+\alpha_4\beta_{12}))f_xf_y+(\frac{5}{36}(\beta_1^2)+\frac{8}{36}(\beta_4+\beta_5+\beta_6)^2\\
&+\frac{5}{36}(\beta_{10}+\beta_{11}+\beta_{12})^2)f^2f_{yy}+(\frac{8}{18}\beta_6(\beta_2+\beta_3)+\frac{5}{18}(\beta_{10}(\beta_2+\beta_3)\\
&+\beta_{11}(\beta_4+\beta_5+\beta_6)+\beta_{12}(\beta_7+\beta_8+\beta_9)))ff_y^2]+O(h^4) \qquad (4.20)
\end{aligned}
$$

when

$$\begin{aligned} y_{m+1} &= y_m + hy'_m + \frac{h^2}{2!}y''_m + \frac{h^3}{3!}y'''_m + \ldots && (4.21) \\ &= y_m + hf + \frac{h^2}{2}[f_x + ff_y] && (4.22) \\ &\quad + \frac{h^3}{6}[f_{xx} + 2ff_{xy} + f_xf_y + f^2f_{yy} + ff_y^2] + \ldots. \end{aligned}$$

Hence from (4.20)and (4.22) we can compare coefficient in term h then we have

$$\frac{5}{18}\alpha_1 + \frac{8}{18}\alpha_3 + \frac{5}{18}\alpha_5 = \frac{1}{2} \quad (4.23)$$

$$\frac{5}{18}\beta_1 + \frac{8}{18}(\beta_4 + \beta_5 + \beta_6) + \frac{5}{18}(\beta_{10} + \beta_{11} + \beta_{12}) = \frac{1}{2} \quad (4.24)$$

$$\frac{5}{36}\alpha_1^2 + \frac{8}{36}\alpha_3^2 + \frac{5}{36}\alpha_5^2 = \frac{1}{6} \quad (4.25)$$

$$\frac{5}{18}\alpha_1\beta_1 + \frac{8}{18}\alpha_3(\beta_4 + \beta_5 + \beta_6) + \frac{5}{18}\alpha_5(\beta_{10} + \beta_{11} + \beta_{12}) = \frac{1}{3} \quad (4.26)$$

$$\frac{5}{18}\alpha_1\beta_3 + \frac{8}{18}(\alpha_1\beta_5 + \alpha_2\beta_6) + \frac{5}{18}(\alpha_2\beta_{10} + \alpha_3\beta_{11} + \alpha_4\beta_{12}) = \frac{1}{6} \quad (4.27)$$

$$\frac{5}{36}(\beta_1^2) + \frac{8}{36}(\beta_4 + \beta_5 + \beta_6)^2 + \frac{5}{36}(\beta_{10} + \beta_{11} + \beta_{12})^2 = \frac{1}{6} \quad (4.28)$$

$$\begin{aligned} &\frac{8}{18}\beta_6(\beta_2 + \beta_3) + \frac{5}{18}(\beta_{10}(\beta_2 + \beta_3) + \beta_{11}(\beta_4 + \beta_5 + \beta_6) \\ &+\beta_{12}(\beta_7 + \beta_8 + \beta_9)) = \frac{1}{6} \quad (4.29) \end{aligned}$$

and we known that

$$\begin{aligned} \alpha_1 &= \beta_1 && (4.30) \\ \alpha_2 &= \beta_2 + \beta_3 && (4.31) \\ \alpha_3 &= \beta_4 + \beta_5 + \beta_6 && (4.32) \\ \alpha_4 &= \beta_7 + \beta_8 + \beta_9 && (4.33) \\ \alpha_5 &= \beta_{10} + \beta_{11} + \beta_{12} && (4.34) \end{aligned}$$

from the above equations we can find $\beta_1 - \beta_{12}$ follow as

$$
\begin{aligned}
&\beta_1 = \tfrac{496}{4401}\\
&\beta_2 = -\tfrac{563}{803}\\
&\beta_3 = \tfrac{538}{915}\\
&\beta_4 = \tfrac{265}{367}\\
&\beta_5 = -\tfrac{2657}{2534}\\
&\beta_6 = \tfrac{281}{340}\\
&\beta_7 = \tfrac{557}{803}\\
&\beta_8 = 0\\
&\beta_9 = 0\\
&\beta_{10} = \tfrac{1193}{2032}\\
&\beta_{11} = 0\\
&\beta_{12} = \tfrac{782}{2605}.
\end{aligned}
$$

So that we have the new constructed formula of Runge-Kutta third algebraic as in the form of

$$y_{m+1} = y_m + \frac{h}{18}(5k_1 + 8k_2 + 5k_3) \tag{4.35}$$

where

$$
\begin{aligned}
k_1 &= f(x_m + \frac{496}{4401}h, y_m + \frac{496}{4401}hf)\\
I_1 &= f(x_m + \frac{4400}{14457}h, y_m - \frac{563}{1985}hf + \frac{538}{915}hk_1)\\
k_2 &= f(x_m + \frac{1}{2}h, y_m + \frac{265}{367}hf - \frac{2657}{2534}hk_1 + \frac{281}{340}hI_1)\\
I_2 &= f(x_m + \frac{557}{803}h, y_m + \frac{557}{803}hk_1 + 0hI_1 + 0hk_2)\\
k_3 &= f(x_m + \frac{496}{559}h, y_m + \frac{1193}{2032}hI_1 + 0hk_2 + \frac{782}{2605}hI_2).
\end{aligned}
$$

The new constructed formula of the Runge-Kutta method of forth algebraic order form this research, can be express by the following relations;

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4) \tag{4.36}$$

where

$$\begin{aligned}
I_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf) \\
k_1 &= f(x_m + \alpha_2 h, y_m + \beta_2 hf + \beta_3 hI_1) \\
I_2 &= f(x_m + \alpha_3 h, y_m + \beta_4 hf + \beta_5 hI_1 + \beta_6 hk_1) \\
k_2 &= f(x_m + \alpha_4 h, y_m + \beta_7 hf + \beta_8 hI_1 + \beta_9 hk_1 + \beta_{10} hI_2) \\
I_3 &= f(x_m + \alpha_5 h, y_m + \beta_{11} hf + \beta_{12} hI_1 + \beta_{13} hk_1 + \beta_{14} hI_2 + \beta_{15} hk_2) \\
k_3 &= f(x_m + \alpha_6 h, y_m + \beta_{16} hI_1 + \beta_{17} hk_1 + \beta_{18} hI_2 + \beta_{19} hk_2 + \beta_{20} hI_3) \\
I_4 &= f(x_m + \alpha_7 h, y_m + \beta_{21} hI_1 + \beta_{22} hk_1 + \beta_{23} hI_2 + \beta_{24} hk_2 + \beta_{25} hI_3 + \beta_{26} hk_3) \\
k_4 &= f(x_m + \alpha_8 h, y_m + \beta_{27} hI + \beta_{28} hk_1 + \beta_{29} hI_2 + \beta_{30} hk_2 + \beta_{31} hI_3 + \beta_{32} hk_3 + \beta_{33} hI_4).
\end{aligned}$$

When we disperse Taylor's Series of k_1, k_2, k_3, k_4 so we have the very large polynomial and difficult to find the value of unknown variables. So that we construct another form of the Runge-Kutta and we will get it in the form

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4) \tag{4.37}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \alpha_1 h, y_m + \alpha_1 hf) \\
k_2 &= f(x_m + \alpha_2 h, y_m + \beta_1 hf + \beta_2 hk_1) \\
k_3 &= f(x_m + \alpha_3 h, y_m + \beta_3 hf + \beta_4 hk_1 + \beta_5 hk_2) \\
k_4 &= f(x_m + \alpha_4 h, y_m + \beta_6 hf + \beta_7 hk_1 + \beta_8 hk_2 + \beta_9 hk_3).
\end{aligned}$$

after we disperse Taylor's Series of k_1, k_2, k_3, k_4 so we have

$$\begin{aligned}
k_1 &= f(x_m + \alpha_1 h, y_m + \alpha_1 hf) \\
&= f + h[\alpha_1 f_x + \alpha_1 ff_y] + \frac{h^2}{2}[\alpha_1^2 f_{xx} + 2\alpha_1^2 ff_{xy} + \alpha_1^2 f^2 f_{yy}] + h^3[\frac{\alpha_1^3}{6} f_{xxx} \\
&\quad + \frac{\alpha_1^3}{2} f_{xxy} + \frac{\alpha_1^3}{2} f_{xyy} + \frac{\alpha_1^3}{6} f_{yyy}] + O(h^4) \\
k_2 &= f(x_m + \alpha_2 h, y_m + \beta_1 hf + \beta_2 hk_1) \\
&= f + h[\alpha_2 f_x + \beta_1 ff_y + \beta_2 ff_y] + h^2[\alpha_1\beta_1 f_x f_y + \alpha_1\beta_2 ff_y^2 + \frac{\alpha_1^2\beta_2}{2} f_y f_{xx} \\
&\quad + \alpha_1^2\beta_2 ff_y f_{xy} + \frac{\alpha_1^2\beta_2}{2} f^2 f_y f_{yy} + \frac{\alpha_2^2}{2} f_{xx} + \alpha_2\beta_1 ff_{xy} + \alpha_2\beta_2 ff_{xy} + \frac{\beta_2^2}{2} f^2 f_{yy}
\end{aligned}$$

$$
\begin{aligned}
& +\beta_1\beta_2 f^2 f_{yy} + \frac{\beta_1^2}{2} f^2 f_{yy}] + h^3[\alpha_1\alpha_2\beta_2 f_x f_{xy} + \alpha_1\alpha_2\beta_2 f f_y f_{xy} + \alpha_1\beta_2^2 f f_x f_{yy} \\
& +\alpha_1\beta_2^2 f^2 f_y f_{yy} + \alpha_1\beta_1\beta_2 f_x f_{yy} + \alpha_1\beta_1\beta_2 f f_y f_{yy} + \frac{\alpha_2^3}{6} f_{xxx} + \frac{\alpha_2^2\beta_1}{2} f f_{xxy} \\
& +\frac{\alpha_2^2\beta_2}{2} f f_{xxy} + \frac{\alpha_2\beta_2^2}{2} f^2 f_{xyy} + \alpha_2\beta_1\beta_2 f^2 f_{xyy} + \frac{\alpha_2\beta_1^2}{2} f^2 f_{xyy} + \frac{\beta_1\beta_2^2}{2} f^3 f_{yyy} \\
& +\frac{\beta_1^2\beta_2}{2} f^3 f_{yyy} + \frac{\beta_2^3}{6} f^3 f_{yyy} + \frac{\beta_1^3}{6} f^3 f_{yyy}] + O(h^4) \\
k_3 &= f(x_m + \alpha_3 h, y_m + \beta_3 h f + \beta_4 h k_1 + \beta_5 h k_2) \\
&= f + h[\alpha_3 f_x + \beta_3 f f_y + \beta_4 f f_y + \beta_4 f f_y] + h^2[\alpha_1\beta_4 f_x f_y + \alpha_1\beta_4 f f_y^2 + \alpha_2\beta_5 f_x f_y \\
& +\beta_1\beta_5 f f_y^2 + \beta_1\beta_2 f f_y^2 + \frac{\alpha_3^2}{2} f_{xx} + \alpha_3\beta_3 f f_{xy} + \alpha_3\beta_4 f f_{xy} + \alpha_3\beta_5 f f_{xy} \\
& +\beta_3\beta_5 f^2 f_{yy} + \beta_4\beta_5 f^2 f_{yy} + \frac{\beta_5^2}{2} f^2 f_{yy} + \frac{\beta_4^2}{2} f^2 f_{yy} + \frac{\beta_3^2}{2} f^2 f_{yy} + \beta_3\beta_4 f^2 f_{yy}] \\
& +h^3[\frac{\alpha_1^2\beta_4}{2} f_y f_{xx} + \alpha_1^2\beta_4 f f_y f_{xy} + \frac{\alpha_1^2\beta_4}{2} f^2 f_y f_{yy} + \alpha_1\beta_2\beta_5 f_x f_y^2 + \alpha_1\beta_2\beta_5 f f_y^3 \\
& +\frac{\alpha_2^2\beta_5}{2} f_y f_{xx} + \alpha_2\beta_1\beta_5 f f_y f_{xy} + \alpha_2\beta_2\beta_5 f f_y f_{xy} + \frac{\beta_2^2\beta_5}{2} f^2 f_y f_{yy} + \beta_1\beta_2\beta_5 f^2 f_y f_{yy} \\
& +\frac{\beta_1^2\beta_5}{2} f^2 f_y f_{yy} + \alpha_1\alpha_3\beta_4 f_x f_{xy} + \alpha_1\alpha_3\beta_4 f f_y f_{xy} + \alpha_2\alpha_3\beta_5 f_x f_{xy} + \alpha_3\beta_1\beta_5 f f_y f_{xy} \\
& +\alpha_3\beta_2\beta_5 f f_y f_{xy} + \alpha_2\beta_3\beta_5 f_x f_{yy} + \beta_1\beta_3\beta_5 f f_y f_{yy} + \beta_2\beta_3\beta_5 f f_y f_{yy} + \alpha_1\beta_4\beta_5 f f_x f_{yy} \\
& +\alpha_1\beta_4\beta_5 f^2 f_y f_{yy} + \alpha_2\beta_4\beta_5 f f_x f_{yy} + \beta_1\beta_4\beta_5 f^2 f_y f_{yy} + \beta_2\beta_4\beta_5 f^2 f_y f_{yy} \\
& +\alpha_2\beta_5^2 f f_x f_{yy} + \beta_1\beta_5^2 f^2 f_y f_{yy} + \beta_2\beta_5^2 f^2 f_y f_{yy} + \alpha_1\beta_4^2 f f_x f_{yy} + \alpha_1\beta_4^2 f^2 f_y f_{yy} \\
& +\alpha_1\beta_3\beta_4 f f_x f_{yy} + \alpha_1\beta_3\beta_4 f^2 f_y f_{yy} + \frac{\alpha_3^3}{6} f_{xxx} + \frac{\alpha_3^3\beta_3}{2} f f_{xxy} + \frac{\alpha_3^3\beta_4}{2} f f_{xxy} \\
& +\frac{\alpha_3^3\beta_5}{2} f f_{xxy} + \alpha_3\beta_3\beta_5 f^2 f_{xyy} + \alpha_3\beta_4\beta_5 f^2 f_{xyy} + \frac{\alpha_3\beta_5^2}{2} f^2 f_{xyy} + \frac{\alpha_3\beta_4^2}{2} f^2 f_{xyy} \\
& +\alpha_3\beta_3\beta_4 f^2 f_{xyy} + \frac{\alpha_3\beta_3^2}{2} f^2 f_{xyy} + \frac{\beta_3^3}{6} f^3 f_{yyy} + \beta_3\beta_4\beta_5 f^3 f_{yyy} + \frac{\beta_4^2\beta_5}{2} f^3 f_{yyy} \\
& +\frac{\beta_4\beta_5^2}{2} f^3 f_{yyy} + \frac{\beta_3\beta_4^2}{2} f^3 f_{yyy} + \frac{\beta_3^2\beta_4}{2} f^3 f_{yyy} + \frac{\beta_4^3}{6} f^3 f_{yyy} + \frac{\beta_5^3}{6} f^3 f_{yyy} \\
& +\frac{\beta_3^2\beta_5}{2} f^3 f_{yyy} + \frac{\beta_3\beta_5^2}{2} f^3 f_{yyy}] + O(h^4) \\
k_4 &= f(x_m + \alpha_4 h, y_m + \beta_6 h f + \beta_7 h k_1 + \beta_8 h k_2 + \beta_9 h k_3) \\
&= f + h[\alpha_4 f_x + \beta_6 f f_y + \beta_7 f f_y + \beta_8 f f_y + \beta_9 f f_y] + h^2[\alpha_1\beta_7 f_x f_y + \alpha_1\beta_7 f f_y^2 \\
& +\alpha_2\beta_8 f_x f_y + \beta_1\beta_8 f f_y^2 + \beta_2\beta_8 f f_y^2 + \alpha_3\beta_9 f f_y^2 + \beta_4\beta_9 f f_y^2 + \beta_5\beta_9 f f_y^2 + \frac{\alpha_4^2}{2} f_{xx} \\
& +\alpha_4\beta_6 f f_{xy} + \alpha_4\beta_7 f f_{xy} + \alpha_4\beta_8 f f_{xy} + \alpha_4\beta_9 f f_{xy} + \beta_6\beta_9 f^2 f_{yy} + \beta_7\beta_9 f^2 f_{yy} \\
& +\beta_8\beta_9 f^2 f_{yy} + \frac{\beta_9^2}{2} f^2 f_{yy} + \frac{\beta_8^2}{2} f^2 f_{yy} + \beta_7\beta_8 f^2 f_{yy} + \beta_6\beta_8 f^2 f_{yy} + \frac{\beta_7^2}{2} f^2 f_{yy} \\
& +\beta_6\beta_7 f^2 f_{yy} + \frac{\beta_6^2}{2} f^2 f_{yy}] + h^3[\frac{\alpha_1^2\beta_7}{2} f_y f_{xx} + \alpha_1^2\beta_7 f f_y f_{xy} + \frac{\alpha_1^2\beta_7}{2} f^2 f_y f_{yy} \\
& +\alpha_1\beta_2\beta_8 f_x f_y^2 + \alpha_1\beta_2\beta_8 f f_y^3 + \frac{\alpha_2^2\beta_8}{2} f_y f_{xx} + \alpha_2\beta_1\beta_8 f f_y f_{xy} + \alpha_2\beta_2\beta_8 f f_y f_{xy}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\beta_2^2\beta_8}{2}f^2f_yf_{yy}+\beta_1\beta_2\beta_8f^2f_yf_{yy}+\frac{\beta_1^2\beta_8}{2}f^2f_yf_{yy}+\alpha_1\beta_4\beta_9f_xf_y^2+\alpha_1\beta_4\beta_9ff_y^3\\
&+\alpha_2\beta_5\beta_9f_xf_y^2+\beta_1\beta_5\beta_9ff_y^3+\beta_2\beta_5\beta_9ff_y^3+\frac{\alpha_3^2\beta_9}{2}f_yf_{xx}+\alpha_3\beta_3\beta_9ff_yf_{xy}\\
&+\alpha_3\beta_4\beta_9ff_yf_{xy}+\alpha_3\beta_5\beta_9ff_yf_{xy}+\beta_3\beta_5\beta_9f^2f_yf_{yy}+\beta_4\beta_5\beta_9f^2f_yf_{yy}\\
&+\frac{\beta_5^2\beta_9}{2}f^2f_yf_{yy}+\frac{\beta_4^2\beta_9}{2}f^2f_yf_{yy}+\beta_3\beta_4\beta_9f^2f_yf_{yy}+\frac{\beta_3^2\beta_9}{2}f^2f_yf_{yy}\\
&+\alpha_1\alpha_4\beta_7f_xf_{xy}+\alpha_1\alpha_4\beta_7ff_yf_{xy}+\alpha_2\alpha_4\beta_8f_xf_{xy}+\alpha_4\beta_1\beta_8ff_yf_{xy}\\
&+\alpha_4\beta_2\beta_8ff_yf_{xy}+\alpha_3\alpha_4\beta_9f_xf_{xy}+\alpha_4\alpha_3\beta_9ff_yf_{xy}+\alpha_4\beta_4\beta_9ff_yf_{xy}\\
&+\alpha_4\beta_5\beta_9ff_yf_{xy}+\alpha_3\beta_6\beta_9ff_xf_{yy}+\beta_3\beta_6\beta_9f^2f_yf_{yy}+\beta_4\beta_6\beta_9f^2f_yf_{yy}\\
&+\beta_5\beta_6\beta_9f^2f_yf_{yy}+\alpha_3\beta_7\beta_9ff_xf_{yy}+\beta_3\beta_7\beta_9f^2f_yf_{yy}+\beta_4\beta_7\beta_9f^2f_yf_{yy}\\
&+\alpha_1\beta_7\beta_9ff_xf_{yy}+\alpha_1\beta_7\beta_9f^2f_yf_{yy}+\beta_2\beta_8\beta_9f^2f_yf_{yy}+\beta_1\beta_8\beta_9f^2f_yf_{yy}\\
&+\alpha_2\beta_8\beta_9ff_xf_{yy}+\alpha_3\beta_8\beta_9ff_xf_{yy}+\beta_3\beta_8\beta_9f^2f_yf_{yy}+\beta_4\beta_8\beta_9f^2f_yf_{yy}\\
&+\beta_5\beta_8\beta_9f^2f_yf_{yy}+\alpha_3\beta_9^2ff_xf_{yy}+\beta_3\beta_9^2f^2f_yf_{yy}+\beta_4\beta_9^2f^2f_yf_{yy}+\beta_5\beta_9^2f^2f_yf_{yy}\\
&+\alpha_2\beta_8^2ff_xf_{yy}+\beta_1\beta_8^2f^2f_yf_{yy}+\beta_2\beta_8^2f^2f_yf_{yy}+\alpha_2\beta_7\beta_8ff_xf_{yy}+\beta_1\beta_7\beta_8f^2f_yf_{yy}\\
&+\beta_2\beta_7\beta_8f^2f_yf_{yy}+\alpha_1\beta_7\beta_8ff_xf_{yy}+\alpha_1\beta_7\beta_8f^2f_yf_{yy}+\alpha_2\beta_6\beta_8ff_xf_{yy}\\
&+\beta_1\beta_6\beta_8f^2f_yf_{yy}+\beta_2\beta_6\beta_8f^2f_yf_{yy}+\alpha_1\beta_7^2ff_xf_{yy}+\alpha_1\beta_7^2f^2f_yf_{yy}\\
&+\alpha_1\beta_6\beta_7ff_xf_{yy}+\alpha_1\beta_6\beta_7f^2f_yf_{yy}+\frac{\alpha_4^3}{6}f_{xxx}+\frac{\alpha_4^2\beta_6}{2}ff_{xxy}+\frac{\alpha_4^2\beta_7}{2}ff_{xxy}\\
&+\frac{\alpha_4^2\beta_8}{2}ff_{xxy}+\frac{\alpha_4^2\beta_9}{2}ff_{xxy}+\alpha_4\beta_6\beta_9f^2f_{xyy}+\alpha_4\beta_7\beta_9f^2f_{xyy}+\alpha_4\beta_8\beta_9f^2f_{xyy}\\
&+\frac{\alpha_4\beta_9^2}{2}f^2f_{xyy}+\frac{\alpha_4\beta_8^2}{2}f^2f_{xyy}+\alpha_4\beta_7\beta_8f^2f_{xyy}+\alpha_4\beta_6\beta_8f^2f_{xyy}+\frac{\alpha_4\beta_9^2}{2}f^2f_{xyy}\\
&+\alpha_4\beta_6\beta_7f^2f_{xyy}+\frac{\alpha_4\beta_6^2}{2}f^2f_{xyy}+\frac{\beta_7^2\beta_9}{2}f^3f_{yyy}+\beta_6\beta_8\beta_9f^3f_{yyy}+\frac{\beta_6\beta_9^2}{2}f^3f_{yyy}\\
&+\frac{\beta_7\beta_9^2}{2}f^3f_{yyy}+\frac{\beta_8\beta_9^2}{2}f^3f_{yyy}+\frac{\beta_9^3}{6}f^3f_{yyy}+\beta_7\beta_8\beta_9f^3f_{yyy}+\frac{\beta_8^2\beta_9}{2}f^3f_{yyy}\\
&+\beta_6\beta_7\beta_9f^3f_{yyy}+\frac{\beta_6^2\beta_9}{2}f^3f_{yyy}+\frac{\beta_7\beta_8^2}{2}f^3f_{yyy}+\frac{\beta_8^3}{6}f^3f_{yyy}+\frac{\beta_6\beta_8^2}{2}f^3f_{yyy}\\
&+\frac{\beta_7^2\beta_8}{2}f^3f_{yyy}+\frac{\beta_6^2\beta_8}{2}f^3f_{yyy}+\beta_6\beta_7\beta_8f^3f_{yyy}+\frac{\beta_7^3}{6}f^3f_{yyy}+\frac{\beta_6\beta_7^2}{2}f^3f_{yyy}\\
&+\frac{\beta_6^2\beta_7}{2}f^3f_{yyy}+\frac{\beta_6^3}{6}f^3f_{yyy}]+O(h^4).
\end{aligned}
\tag{4.38}
$$

after we disperse Taylor's Series of k_1, k_2, k_3, k_4 so we have

$$
\begin{aligned}
y_{m+1} \;=\;& y_m+h[(a_1\alpha_1+a_2\alpha_2+a_3\alpha_3+a_4\alpha_4)f_x+(a_1\alpha_1+a_2(\beta_1+\beta_2)\\
&+a_3(\beta_3+\beta_4+\beta_5)+a_4(\beta_6+\beta_7+\beta_8+\beta_9))ff_y]+\frac{h^2}{2}[(a_1\alpha_1^2+a_2\alpha_2^2\\
&+a_3\alpha_3^2+a_4\alpha_4^2)f_{xx}+2(a_1\alpha_1^2+a_2\alpha_2(\beta_1+\beta_2)+a_3\alpha_3(\beta_3+\beta_4+\beta_5)\\
&+a_4\alpha_4(\beta_6+\beta_7+\beta_8+\beta_9))ff_{xy}+(a_2\alpha_1(\beta_1+\beta_2)+a_3(\alpha_1\beta_3+\alpha_2\beta_4+\alpha_3\beta_5)
\end{aligned}
$$

$$\begin{aligned}
&+a_4(\alpha_1\beta_6+\alpha_2\beta_7+\alpha_3\beta_8+\alpha_4\beta_9))f_xf_y+(a_1\alpha_1^2+a_2(\beta_1+\beta_2)^2\\
&+a_3(\beta_3+\beta_4+\beta_5)^2+a_4(\beta_6+\beta_7+\beta_8+\beta_9)^2)f^2f_{yy}+2(a_2\alpha_1(\beta_1+\beta_2)\\
&+a_3(\alpha_1\beta_3+\beta_1\beta_4+\beta_2\beta_5)+a_4(\alpha_1\beta_6+\beta_7(\beta_1+\beta_2)+\beta_8(\beta_3+\beta_4+\beta_5)))ff_y^2]\\
&+\frac{h^3}{6}[(a_1\alpha_1^3+a_2\alpha_2^3+a_3\alpha_3^3+a_4\alpha_4^3)f_{xxx}+3(a_1\alpha_1^3+a_2\alpha_2^2(\beta_1+\beta_2)\\
&+a_3\alpha_3^2(\beta_3+\beta_4+\beta_5)+a_4\alpha_4^2(\beta_6+\beta_7+\beta_8+\beta_9))ff_{xxy}+6(a_2\alpha_1^2\alpha_2\\
&+a_3\alpha_3(\alpha_1\beta_3+\alpha_2\beta_4+\alpha_3\beta_5)+a_4\alpha_4(\alpha_1\beta_6+\alpha_2\beta_7+\alpha_3\beta_8+\alpha_4\beta_9))f_xf_{xy}\\
&+6(a_3\alpha_1\beta_1(\alpha_1+\alpha_2)+a_3((\alpha_1+\alpha_3)\alpha_1\beta_3+(\alpha_2+\alpha_4)\beta_1\beta_4)+a_4(\alpha_1^2\beta_6\\
&+\alpha_2\beta_7(\beta_1+\beta_2)+\alpha_3\beta_8(\beta_3+\beta_4+\beta_5)+\alpha_4(\alpha_1\beta_6+\beta_7(\beta_1+\beta_2)\\
&+\beta_8(\beta_3+\beta_4+\beta_5))))ff_yf_{xy}+3(a_1\alpha_1^3+a_2\alpha_2(\beta_1+\beta_2)^2+a_3\alpha_3(\beta_3+\beta_4+\beta_5)^2\\
&+a_4\alpha_4(\beta_6+\beta_7+\beta_8+\beta_9)^2)f^2f_{xyy}+3(a_2\alpha_1^2(\beta_1+\beta_2)+a_3(\alpha_1^2\beta_3+\alpha_2^2\beta_4+\alpha_3^2\beta_5)\\
&+a_4(\alpha_1^2\beta_6+\alpha_2^2\beta_7+\alpha_3^2\beta_8+\alpha_4^2\beta_9))f_yf_{xx}+6(a_2\alpha_1(\beta_1+\beta_2)^2\\
&+a_3\alpha_3(\beta_3+\beta_4+\beta_5)(\alpha_1\beta_3+\alpha_2\beta_4+\alpha_3\beta_5)+a_4(\beta_6+\beta_7+\beta_8+\beta_9)(\alpha_1\beta_6\\
&+\alpha_2\beta_7+\alpha_3\beta_8+\alpha_4\beta_9))ff_xf_{yy}+6(a_2(\alpha_1^2(\beta_1+\beta_2)+(\beta_1+\beta_2)^2\alpha_1)\\
&+a_3(\alpha_1^2\beta_3+(\beta_1+\beta_2)^2\beta_4+(\beta_3+\beta_4+\beta_5)(\alpha_1\beta_3+\beta_1\beta_4+\beta_2\beta_5))\\
&+a_4(\beta_6+\beta_7+\beta_8+\beta_9)(\alpha_1\beta_6+\beta_7(\beta_1+\beta_2)+\beta_8(\beta_3+\beta_4+\beta_5)))f^2f_yf_{yy}\\
&+(a_1\alpha_1^3+a_2(\beta_1+\beta_2)^3+a_3(\beta_3+\beta_4+\beta_5)^3+a_4(\beta_6+\beta_7+\beta_8+\beta_9)^3)f^3f_{yyy}\\
&+(a_3\alpha_1\beta_4(\beta_1+\beta_2)+a_4(\alpha_1\beta_7(\beta_1+\beta_2)+\beta_8(\alpha_1\beta_3+\alpha_2\beta_4+\alpha_3\beta_5))f_xf_y^2\\
&+(a_3\alpha_1\beta_4(\beta_1+\beta_2)+a_4(\alpha_1\beta_7(\beta_1+\beta_2)+\beta_8(\alpha_1\beta_3+\beta_4(\beta_1+\beta_2)))ff_y^3]\\
&+O(h^5). \qquad (4.39)
\end{aligned}$$

when

$$\begin{aligned}
y_{m+1} &= y_m+hy'_m+\frac{h^2}{2!}y''_m+\frac{h^3}{3!}y'''_m+\frac{h^4}{24}y''''_m+\ldots\\
&= y_m+hf+\frac{h^2}{2}[f_x+ff_y]+\frac{h^3}{6}[f_{xx}+2ff_{xy}+f_xf_y+f^2f_{yy}+ff_y^2]\\
&\quad+\frac{h^4}{24}[f_{xxx}+3ff_{xxy}+3f_xf_{xy}+f_yf_{xx}+5ff_yf_{xy}+f^3f_{yyy}+3ff_xf_{yy}\\
&\quad+3f^2f_{xyy}+4f^2f_yf_{yy}+f_xf_y^2+ff_y^3]+O(h^5). \qquad (4.40)
\end{aligned}$$

Hence from (4.39)and (4.40) we can compare coefficient in term h and we know that

$$\alpha_2 = \beta_1+\beta_2 \qquad (4.41)$$

$$\alpha_3 = \beta_3+\beta_4+\beta_5 \qquad (4.42)$$

$$\alpha_4 = \beta_6+\beta_7+\beta_8+\beta_9. \qquad (4.43)$$

Then we can find unknown variables follow as

$A_1 = \frac{527}{3030}$
$A_2 = \frac{988}{3030}$
$A_3 = \frac{988}{3030}$
$A_4 = \frac{527}{3030}$
$\alpha_1 = \frac{842}{12127}$
$\alpha_2 = \frac{695}{2106}$
$\alpha_3 = \frac{739}{1103}$
$\alpha_4 = \frac{1132}{1109}$
$\beta_1 = \frac{1108}{847}$
$\beta_2 = -\frac{1029}{1052}$
$\beta_3 = \frac{3874}{119}$
$\beta_4 = -\frac{21851}{962}$
$\beta_5 = -\frac{3604}{393}$
$\beta_6 = -\frac{9226}{101}$
$\beta_7 = \frac{93381}{1115}$
$\beta_8 = \frac{5366}{635}$
$\beta_9 = \frac{772}{10035}.$

So we have the new constructed formula of Runge-Kutta forth algebraic in the form

$$y_{m+1} = y_m + h(\frac{527}{3030}k_1 + \frac{988}{3030}k_2 + \frac{988}{3030}k_3 + \frac{527}{3030}k_4) \tag{4.44}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \frac{842}{12127}h, y_m + \frac{842}{12127}hf) \\
k_2 &= f(x_m + \frac{695}{2106}h, y_m + \frac{1108}{847}hf - \frac{1029}{1052}hk_1) \\
k_3 &= f(x_m + \frac{739}{1103}h, y_m + \frac{3874}{119}hf - \frac{21851}{962}hk_1 - \frac{3604}{393}hk_2) \\
k_4 &= f(x_m + \frac{1032}{1109}h, y_m - \frac{9226}{101}hf + \frac{93381}{1115}hk_1 + \frac{5366}{635}hk_2 + \frac{772}{10035}hk_3).
\end{aligned}$$

The example of problem A, B, C, D, E, F and G where problem B and D are the autonomous problems.We shall use the new formulas to find the numerical solution of following problems.

Problem A: Consider solution of

$$y' = (x+2)e^{2x} \tag{4.45}$$

when $$y(0) = 0$$

and the analytical solution is $$y = \frac{x+2}{2}e^{2x} - \frac{1}{4}e^{2x} - \frac{3}{4}.$$

Table 4.1 Solution and absolute true error of problem A

	$x = 0.1$, $h = 0.1$ $y = 0.227122206$		$x = 1$, $h = 0.00001$ $y = 8.486320124$	
	calculated y	absolute true error	calculated y	absolute true error
1p	0.22712220653	$5.621832354 \times 10^{-4}$	8.4863201237	$3.9857695810 \times 10^{-8}$
2p	0.22712220653	$5.6216832354 \times 10^{-4}$	8.4863201237	$3.9857695810 \times 10^{-8}$
3p	0.22712220653	$1.8644641386 \times 10^{-11}$	8.4863201237	$3.9610313252 \times 10^{-8}$
4po	0.22712220653	$9.0949470177 \times 10^{-13}$	8.4863201237	$3.9595761336 \times 10^{-8}$
2rk	0.22824728961	$1.1250830778 \times 10^{-3}$	8.4863200845	$1.4084798750 \times 10^{-7}$
3rk	0.22712245534	$2.4881023819 \times 10^{-7}$	8.4563200840	$1.4042598195 \times 10^{-7}$
4rk	0.22712245534	$2.4881023819 \times 10^{-7}$	8.4563200840	$1.4042598195 \times 10^{-7}$
2pl	0.22712220653	$1.6585818230 \times 10^{-7}$	8.4863201237	$3.9595761336 \times 10^{-8}$
3pl	0.22712220653	$6.9821908255 \times 10^{-9}$	8.4863201237	$3.9581209421 \times 10^{-8}$
4pl	0.22712220653	$7.3366663855 \times 10^{-9}$	8.4863201237	$2.7897102569 \times 10^{-9}$
5pl	0.22712220653	$2.8355771065 \times 10^{-9}$	8.4863201237	$2.2169767180 \times 10^{-4}$
6pl	0.22712220653	$4.5110483211 \times 10^{-3}$	8.4863201237	$2.0433365717 \times 10^{-4}$
7pl	0.22712220653	$3.7182871893 \times 10^{-8}$	8.4863201237	$2.2318848642 \times 10^{-4}$

Problem B: Consider solution of

$$y' = y^3 \tag{4.46}$$

when $$y(0) = 0.1$$

and the analytical solution is $$y = \frac{1}{\sqrt{2 \times (50-x)}}.$$

Table 4.2 Solution and absolute true error of problem B

	$x = 1.0$, $h = 0.001$ $y = 0.101015254$		$x = 1$, $h = 0.00001$ $y = 0.101015254$	
	calculated y	absolute true error	calculated y	absolute true error
1p	0.10101525446	$1.7053025658 \times 10^{-12}$	0.10101525446	$9.0949470177 \times 10^{-13}$
2p	0.10101525446	$1.5916157281 \times 10^{-12}$	0.10101525446	$9.0949470177 \times 10^{-13}$
3p	0.10101525446	$1.4779288904 \times 10^{-12}$	0.10101525446	$9.0949470177 \times 10^{-13}$
4po	0.10101525446	$4.7748471843 \times 10^{-11}$	0.10101525446	$3.4106051316 \times 10^{-13}$
2rk	0.10010015015	$1.0027179087 \times 10^{-10}$	0.10101525445	$7.3896444519 \times 10^{-12}$
3rk	0.10010015025	$1.1368683772 \times 10^{-13}$	0.10101525445	$7.3896444519 \times 10^{-12}$
4rk	0.10010015025	$1.1368683772 \times 10^{-13}$	0.10101525445	$7.3896444519 \times 10^{-12}$
2pl	0.10101525446	$1.5916157281 \times 10^{-12}$	0.10101525446	$9.0949470177 \times 10^{-13}$
3pl	0.10101525446	$1.4779288904 \times 10^{-12}$	0.10101525446	$9.0949470177 \times 10^{-13}$
4pl	0.10101525446	$1.4779288904 \times 10^{-12}$	0.10101525446	$1.0306735021 \times 10^{-8}$
5pl	0.10101525446	$7.3896444519 \times 10^{-12}$	0.10101525446	$1.0314579413 \times 10^{-8}$
6pl	0.10101525446	$2.6410589271 \times 10^{-9}$	0.10101525446	$1.0139160622 \times 10^{-8}$
7pl	0.10101525446	$1.8894752429 \times 10^{-10}$	0.10101525446	$1.0497160474 \times 10^{-8}$

Problem C: Consider solution of

$$y' = \frac{y}{x} - \frac{y^2}{x} \tag{4.47}$$

when $y(1) = 0.5$

and the analytical solution is $y = \frac{x}{x+1}$.

Table 4.3 Solution and absolute true error of problem C

	$x = 1.1$, $h = 0.1$ $y = 0.523809524$		$x = 2$, $h = 0.001$ $y = 0.6666666667$	
	calculated y	absolute true error	calculated y	absolute true error
1p	0.52380952381	$1.4880952222 \times 10^{-5}$	0.66666666667	$8.3737177192 \times 10^{-9}$
2p	0.52380952381	$1.4180774087 \times 10^{-5}$	0.66666666667	$6.7229848355 \times 10^{-9}$
3p	0.52380952381	$3.5006451071 \times 10^{-9}$	0.66666666667	$8.1854523160 \times 10^{-12}$
4po	0.52380952381	$6.6499524110 \times 10^{-7}$	0.66666666667	$2.5100234780 \times 10^{-8}$
2rk	0.52383522727	$2.5703463507 \times 10^{-5}$	0.666667681	$1.0119947547 \times 10^{-8}$
3rk	0.52380990924	$3.8543112169 \times 10^{-7}$	0.66666666667	$3.1832314562 \times 10^{-11}$
4rk	0.52380952791	$4.0981831262 \times 10^{-9}$	0.66666666667	$3.1832314562 \times 10^{-11}$
2pl	0.52380952381	$6.3141669671 \times 10^{-7}$	0.66666666667	$1.1123120203 \times 10^{-9}$
3pl	0.52380952381	$3.7239260564 \times 10^{-8}$	0.66666666667	$2.7102942113 \times 10^{-10}$
4pl	0.52380952381	$2.7101941669 \times 10^{-7}$	0.66666666667	$2.5374902179 \times 10^{-10}$
5pl	0.52380952381	$4.6517652663 \times 10^{-7}$	0.66666666667	$1.7353158910 \times 10^{-9}$
6pl	0.52380952381	$1.8593541336 \times 10^{-4}$	0.66666666667	$1.0191758520 \times 10^{-5}$
7pl	0.52380952381	$6.9702946348 \times 10^{-7}$	0.66666666667	$2.9232978704 \times 10^{-8}$

Problem D: Consider solution of

$$y' = 5y - 4 \tag{4.48}$$

when $$y(-0.2) = \frac{0.8e+1}{e}$$

and the analytical solution is $$y = e^{5x} + \frac{4}{5}.$$

Table 4.4Solution and absolute true error of problem D

	$x=-0.1$, $h=0.1$ $y=1.40653066$		$x=0.8$, $h=0.001$ $y=55.39815003$	
	calculated y	absolute true error	calculated y	absolute true error
1l	1.4065306597	$8.7265678055 \times 10^{-3}$	55.398150033	$1.1331950082 \times 10^{-3}$
2p	1.4065306597	$2.9784515409 \times 10^{-3}$	55.398150033	$2.8436194407 \times 10^{-4}$
3p	1.4065306597	$1.0059288979 \times 10^{-4}$	55.398150033	$5.5978307500 \times 10^{-7}$
4po	1.4065306597	$1.0156787430 \times 10^{-3}$	55.398150033	$1.8981278408 \times 10^{-3}$
2rk	1.3978040919	$8.7265678085 \times 10^{-3}$	55.397016838	$1.1332228896 \times 10^{-3}$
3rk	1.4054682469	$1.0624127844 \times 10^{-3}$	55.398148616	$1.4448305592 \times 10^{-6}$
4rk	1.4064262663	$1.0439340622 \times 10^{-4}$	55.398150033	$5.5978307500 \times 10^{-8}$
2pl	1.4065306597	$4.8944902956 \times 10^{-3}$	55.398150033	$5.6730589131 \times 10^{-4}$
3pl	1.4065306597	$1.5885375597 \times 10^{-3}$	55.398150033	$1.2564513600 \times 10^{-4}$
4pl	1.4065306597	$8.8535064970 \times 10^{-4}$	55.398150033	$1.4505186118 \times 10^{-4}$
5pl	1.4065306597	$2.7799061627 \times 10^{-3}$	55.398150033	$3.2637163531 \times 10^{-4}$
6pl	1.4065306597	$7.2636281311 \times 10^{-3}$	55.398150033	$1.0840995784 \times 10^{-1}$
7pl	1.4065306597	$5.4784064960 \times 10^{-3}$	55.398150033	$5.9385987697 \times 10^{-4}$

Problem E: Consider solution of

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \tag{4.49}$$

when $$y(0) = 1$$

and the analytical solution is $y = \frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$.

Table 4.5 Solution and absolute true error of problem E

	$x = 0.1$, $h = 0.1$ $y = 1.0050251887$		$x = 1.1$, $h = 0.0001$ $y = 6.1100397659$	
	calculated y	absolute true error	calculated y	absolute true error
1p	1.0050251887	$3.1426987334 \times 10^{-5}$	6.1100397659	$3.8785328798 \times 10^{-5}$
2p	1.0050251887	$1.2677008272 \times 10^{-5}$	6.1100397659	$1.3973019750 \times 10^{-5}$
3p	1.0050251887	$8.1881808001 \times 10^{-8}$	6.1100397659	$9.0723915491 \times 10^{-8}$
4po	1.0050251887	$7.4555300671 \times 10^{-6}$	6.1100397659	$1.1752735008 \times 10^{-4}$
2rk	1.0049751860	$5.0002730000 \times 10^{-5}$	6.1100181633	$4.8349300000 \times 10^{-2}$
3rk	1.0050377575	$1.2568810000 \times 10^{-5}$	6.1100398300	$6.4137570000 \times 10^{-5}$
4rk	1.005035295	$3.8697180571 \times 10^{-8}$	6.1100398412	$7.5335265137 \times 10^{-8}$
2pl	1.0050251887	$1.7947466404 \times 10^{-5}$	6.1100397659	$1.6510231944 \times 10^{-5}$
3pl	1.0050251887	$1.0360490705 \times 10^{-6}$	6.1100397659	$3.5843768273 \times 10^{-6}$
4pl	1.0050251887	$8.0720565165 \times 10^{-6}$	6.1100397659	$4.3276013457 \times 10^{-6}$
5pl	1.0050251887	$1.3902459614 \times 10^{-5}$	6.1100397659	$1.0383068002 \times 10^{-6}$
6pl	1.0050251887	$7.9716702749 \times 10^{-4}$	6.1100397659	$7.8329318276 \times 10^{-3}$
7pl	1.0050251887	$2.0876292183 \times 10^{-5}$	6.1100397659	$1.7090084002 \times 10^{-3}$

Problem F: Consider solution of

$$y' = \frac{xy}{y^2 - x^2} \tag{4.50}$$

when $$y(0) = 1$$

and the analytical solution is $$y = \sqrt{x^2 + \sqrt{x^4 + 1}}.$$

Table 4.6 Solution and absolute true error of problem F

	$x = 0.1$, $h = 0.1$ $y = 1.0050124371$		$x = 10.0$, $h = 0.0001$ $y = 14.142312344$	
	calculated y	absolute true error	calculated y	absolute true error
1p	1.0050124371	$9.4214556157 \times 10^{-6}$	14.142312395	$5.0873495638 \times 10^{-8}$
2p	1.0050124371	$6.2027374952 \times 10^{-6}$	14.142312395	$5.088804755 \times 10^{-8}$
3p	1.0050124371	$6.1118043959 \times 10^{-10}$	14.142312395	$5.0902599469 \times 10^{-8}$
4po	1.0050124371	$1.3999670045 \times 10^{-6}$	14.142312395	$2.6202178560 \times 10^{-7}$
2rk	1.0050505051	$3.8067940000 \times 10^{-5}$	14.142151819	$1.6195740000 \times 10^{-3}$
3rk	1.0050081470	$4.2900660000 \times 10^{-6}$	14.142151819	$1.6195740000 \times 10^{-3}$
4rk	1.0050124478	$1.0702933650 \times 10^{-8}$	14.142312344	$5.0902599469 \times 10^{-8}$
2pl	1.0050124371	$6.2087711676 \times 10^{-6}$	14.142312395	$5.0873495638 \times 10^{-8}$
3pl	1.0050124371	$3.0354385672 \times 10^{-7}$	14.142312395	$5.0902599469 \times 10^{-8}$
4pl	1.0050124371	$2.7187088563 \times 10^{-6}$	14.142312395	$5.0902599469 \times 10^{-8}$
5pl	1.0050124371	$4.6301356633 \times 10^{-6}$	14.142312395	$2.4709152058 \times 10^{-8}$
6pl	1.0050124371	$8.0568359954 \times 10^{-4}$	14.142312395	$5.6911085267 \times 10^{-7}$
7pl	1.0050124371	$6.8882782216 \times 10^{-6}$	14.142312395	$5.9648300521 \times 10^{-7}$

Problem G: Consider solution of

$$y'' + y = 0.001e^{ix} \tag{4.51}$$

when

$$y(0) = 1, y'(0) = 0.9995i, y \in \Im$$

or

$$\begin{aligned} u'' + u &= 0.001cos(x), u(0) = 1, u'(0) = 1 \\ v'' + v &= 0.001sin(x), v(0) = 0, v'(0) = 0.9995 \end{aligned} \tag{4.52}$$

and the analytical solution is

$$\begin{aligned} y(x) &= u(x) + iv(x), u, v \in \Re \\ u(x) &= cos(x) + 0.0005xsin(x) \\ v(x) &= sin(x) - 0.0005xcos(x). \end{aligned} \tag{4.53}$$

Table 4.7 Solution and absolute true error of problem G

	$x = 1.0$, $h = 0.1$ $y = 0.54072304136 + i0.84174113596$	
	calculated y	absolute true error
3rk	0.54069781765+i0.841167735890	$5.739545938679427 \times 10^{-4}$
3p	0.54072434784+i0.841199814250	$5.413232865924947 \times 10^{-4}$

remark

$1p$: the new formula of the open form of Runge-Kutta method order 1

$2p$: the new formula of the open form of Runge-Kutta method order 2

$3p$: the new formula of the open form of Runge-Kutta method order 3

$4po$: the other new formula of the open form of $2rk$: the Runge-Kutta method order 2

$3rk$: the Runge-Kutta method order 3

$4rk$: the Runge-Kutta method order 4

Runge-Kutta method order 4

$2pl$: the open form of Runge-Kutta method order 2 with respect to the weighting function of the Legendre polynomials

$3pl$: the open form of Runge-Kutta method order 3 with respect to the weighting function of the Legendre polynomials

$4pl$: the open form of Runge-Kutta method order 4 with respect to the weighting function of the Legendre polynomials

$5pl$: the open form of Runge-Kutta method order 5 with respect to the weighting function of the Legendre polynomials

$6pl$: the open form of Runge-Kutta method order 6 with respect to the weighting function of the Legendre polynomials

$7pl$: the open form of Runge-Kutta method order 7 with respect to the weighting function of the Legendre polynomials

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS

In this chapter, we shall give the conclusions and suggestions on the open form of Runge-Kutta method for find the numerical solution of initial value problem of ordinary differential equations.

5.1 Conclusions

In this thesis we study the open form of Runge-Kutta method for finding the numerical solution of initial value problem of ordinary differential equations which is of the form

$$y' = f(x, y) \tag{5.1}$$

with the initial value

$$y(x_0) = y_0.$$

After study the Runge-Kutta method, we develop the new open form of Runge-Kutta method first order, the second order and third order are shown below;

n=1

$$y_{m+1} = y_m + ha_1k_1 \tag{5.2}$$

where $$k_1 = f(x_m + \alpha_1 h, y_m + \beta_1 hf).$$

n=2

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2) \tag{5.3}$$

where

$$k_1 = f(x_m + \alpha_1 h, y_m + \beta_1 hf)$$
$$k_2 = f(x_m + \alpha_2 h, y_m + \beta_2 hk_1).$$

n=3

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2 + a_3k_3) \tag{5.4}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \alpha_1 h, y_m + \beta_1 hf) \\
I_1 &= f(x_m + \alpha_2 h, y_m + \beta_2 hf + \beta_3 hk_1) \\
k_2 &= f(x_m + \alpha_3 h, y_m + \beta_4 hf + \beta_5 hk_1 + \beta_6 hI_1) \\
I_2 &= f(x_m + \alpha_4 h, y_m + \beta_7 hk_1 + \beta_8 hI_1 + \beta_9 hk_2) \\
k_3 &= f(x_m + \alpha_5 h, y_m + \beta_{10} hI_1 + \beta_{11} hk_2 + \beta_{12} hI_2).
\end{aligned}$$

where a, α and β are unknown coefficient of the Runge-Kutta method and h is setp side and s is order of each method.

Input the values of unknown variable that we solve back to the new Runge-Kutta method, we will got the new completed Runge-Kutta method first order, second order and third order for this research.

n=1

$$y_{m+1} = y_m + hk_1 \tag{5.5}$$

where

$$k_1 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hf).$$

n=2

$$y_{m+1} = y_m + \frac{h}{2}(k_1 + k_2) \tag{5.6}$$

where

$$\begin{gathered}
k_1 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hf) \\
k_2 = f(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}hk_1).
\end{gathered}$$

n=3

$$y_{m+1} = y_m + \frac{h}{18}(5k_1 + 8k_2 + 5k_3) \tag{5.7}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \frac{496}{4401}h, y_m + \frac{496}{4401}hf) \\
I_1 &= f(x_m + \frac{4400}{14457}h, y_m - \frac{563}{1985}hf + \frac{538}{915}hk_1) \\
k_2 &= f(x_m + \frac{1}{2}h, y_m + \frac{265}{367}hf - \frac{2657}{2534}hk_1 + \frac{281}{340}hI_1) \\
I_2 &= f(x_m + \frac{557}{803}h, y_m + \frac{557}{803}hk_1 + 0hI_1 + 0hk_2) \\
k_3 &= f(x_m + \frac{496}{559}h, y_m + \frac{1193}{2032}hI_1 + 0hk_2 + \frac{782}{2605}hI_2).
\end{aligned}$$

The testing of the new Runge-Kutta method was done by solving problem A, B, C, D, E, F and G by step side 0.1, 0.001 and 0.00001 as shown in chapter 4 to confirm that three new Runge-Kutta methods can solve the initial value problem in equation (5.1) form.

The truncation error of three new Runge-Kutta methods will present in $O(h^{s+2})$ forms by s is order of each method and so the truncation error will be $O(h^3), O(h^4) and O(h^5)$ respectively.

For the fourth order formula, its the truncation error is in the form of $O(h^{s+1})$ and

$$y_{m+1} = y_m + h(a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4) \tag{5.8}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \alpha_1 h, y_m + \alpha_1 hf) \\
k_2 &= f(x_m + \alpha_2 h, y_m + \beta_1 hf + \beta_2 hk_1) \\
k_3 &= f(x_m + \alpha_3 h, y_m + \beta_3 hf + \beta_4 hk_1 + \beta_5 hk_2) \\
k_4 &= f(x_m + \alpha_4 h, y_m + \beta_6 hf + \beta_7 hk_1 + \beta_8 hk_2 + \beta_9 hk_3).
\end{aligned}$$

After calculating for every unknown of the formula we obtained the complete formula of the forth order as formula

$$y_{m+1} = y_m + h(\frac{527}{3030}k_1 + \frac{988}{3030}k_2 + \frac{988}{3030}k_3 + \frac{527}{3030}k_4) \tag{5.9}$$

where

$$\begin{aligned}
k_1 &= f(x_m + \frac{842}{12127}h, y_m + \frac{842}{12127}hf) \\
k_2 &= f(x_m + \frac{695}{2106}h, y_m + \frac{1108}{847}hf - \frac{1029}{1052}hk_1) \\
k_3 &= f(x_m + \frac{739}{1103}h, y_m + \frac{3874}{119}hf - \frac{21851}{962}hk_1 - \frac{3604}{393}hk_2) \\
k_4 &= f(x_m + \frac{1032}{1109}h, y_m - \frac{9226}{101}hf + \frac{93381}{1115}hk_1 + \frac{5366}{635}hk_2 + \frac{772}{10035}hk_3).
\end{aligned}$$

The truncation error of the above formula is in the form of $O(h^5)$.

5.2 suggestions

The x values in the new open form of Runge-Kutta method are in the range (x_m, x_{m+1}). we called this form "open form" but in general most research was done in closed form.

This research will be fruitful for scientists interested in the Runge-Kutta method. They can use these results as a guide line to developming other Runge-Kutta formulas, for instance the development of the open form of Runge-Kutta method of order four or five. They may use these to find the numerical solution of the initial value problem of ordinary differential equations and present truncation error value in $O(h^6)$ and $O(h^7)$ forms.

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