FIXED POINT THEOREM FOR A FINITE FAMILY OF STRICTLY PSEUDO CONTRACTIVE MAPPINGS AND VARIATIONAL INEQUALITY PROBLEMS BY EQUILIBRIUM PROBLEM CONCEPT

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หัวข้อวิทยานิพนธ์ ทฤษฎีบทจุดตรึงสำหรับวงศ์จำกัดของการส่งแบบหดเทียมโดยแท้ และปัญหาอสมการแปรผันโดยใช้แนวคิดจากปัญหาดุลยภาพ
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บทคัดย่อ

งานวิจัยนี้นำเสนอวิธีการทำซ้ำ โดยใช้แนวคิดจาก Piri[1] และ Zegeye[7] เพื่อไปพิสูจน์ทฤษฎีการลู่เข้าแบบเข้มสำหรับหาผลเฉลยของสมาชิกร่วมของเซตของระบบอสมการแปรผันสำหรับการส่งแบบผกผันทางเดียวชนิดเข้ม และ เซตของจุดตรึงร่วมของวงศ์จำกัดของการส่งแบบหดเทียมโดยแท้

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Abstract

This research is present an iterative scheme modified from work of Piri [1] and Zegeye [7] to prove the strong convergence theorem for finding a common element of the set of solutions of systems of variational inequalities for two inverse-strongly monotone mappings and the set of common fixed points of a finite family of strictly pseudo-contractive mappings.

Keywords : Variational Inequality, Inverse strongly monotone, Fixed Points, Strictly pseudo-contractive mapping.

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Mr. Sirawit Premjitpraphan

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Chapter 1

Introduction

1.1 Background

Nowadays, there are many problems can be solved by fixed point problems such as equilibrium problems, variational inequality problems etc. Fixed point problem is profitable for economics, engineering and numerous problems in physics. In this research, we suppose H be a real Hilbert space and C be a nonempty closed convex subset of H with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C. \tag{1.1}$$

We denote the set of all fixed points of T by $F(T)$, i.e., $F(T) = \{x \in C : Tx = x\}$.

The mapping B of C into itself is called *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \text{for all } x, y \in C. \tag{1.2}$$

The mapping B of C into itself is called β*-inverse strongly monotone* if there exists a positive real number β such that

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2, \quad \text{for all } x, y \in C. \tag{1.3}$$

The mapping T is said to be κ*-strictly pseudo-contractive* if there exists a constant $\kappa \in [0,1)$ such that

$$\langle x - y, Tx - Ty \rangle \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \quad \text{for all } x, y \in C. \tag{1.4}$$

We observe that T is κ-strictly pseudo-contractive implies that $I - T$ is $\frac{1-\kappa}{2}$-inverse strongly monotone. Moreover, also that $I - T$ is a continuous monotone mapping.

Note that the class of κ-strict pseudo-contraction includes the class of nonexpansive mappings, that is T is nonexpansive if and only if T is 0-strict pseudo-contractive.

Let A be a mapping of C into H. *The variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \text{for all } v \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $VI(C, A)$, that is,

$$VI(C, A) = \{x \in C : \langle Au, v - u \rangle \geq 0, \ \forall v \in C\}.$$

The following results are the inspiration for this research.

Lemma 1.1.1 [7] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H. Then, for $r>0$ and $x \in H$, there exists $z \in C$ such that

$$\langle y-z, Az\rangle + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \forall y \in C.$$

Lemma 1.1.2 [7] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H. For $r>0$ and $x \in H$, define a mapping F_r of H into C as follows :

$$F_r x := \left\{ z \in C : \langle y-z, Az\rangle + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) F_r is single-valued ;

(2) F_r is a firmly nonexpansive type mapping. i.e., for all $x, y \in H$

$$\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x-y\rangle;$$

(3) $F(F_r) = VI(C, A)$;

(4) $VI(C, A)$ is closed and convex.

Let $T_1, T_2, \ldots$ be an infinite family of nonexpansive mappings of C into itself and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \leq \lambda_i < 1$ for every $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, define a mapping W_n of C into C as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n) I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k) I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2) I, \\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1) I.
\end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$; see [13].

Theorem 1.1.3 [1] Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\vartheta = \{\phi_k : k = 1, 2, \ldots, M\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$, F be a k-Lipschitzian and η-strongly monotone of C into H operator with $k>0, \eta>0$, A be a (α, β)-strongly monotone mapping of C into H such that $\beta > \alpha\mu^2$, B be a (ζ, δ)-strongly monotone mapping of C into H such that $\delta > \zeta\eta^2$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{r_{k,n}\}$ and $\{\lambda_n\}$ be sequences of real numbers such that $0<\alpha_n,\beta_n,\gamma_n<1$, $r_{k,n}>0$ and $0<\lambda_n\le b$ for some $b\in(0,1]$. Assume that,

(1) for every $k\in\{1,2,\dots,M\}$, the bifunction ϕ_k satisfies,

  (A_1) $\phi_k(x,x)=0$ for all $x\in C$,

  (A_2) ϕ_k is monotone, i.e., $\phi_k(x,y)+\phi_k(y,x)\le 0$ for all $x,y\in C$,

  (A_3) for all $x,y,z\in C$, $\liminf\limits_{t\to 0}\phi_k(tz+(1-t)x,y)\le\phi_k(x,y)$,

  (A_4) for all $x\in C$, $y\to\phi_k(x,y)$ is convex and lower semi-continuous,

  (B_1) $\Im=\bigcap_{i=1}^{\infty}Fix(T_i)\cap EP(\ell)\cap VI(C,B)\cap VI(C,A)\neq\varnothing$,

(2) the sequence $\{\alpha_n\}$ satisfies

  (C_1) $\lim\limits_{n\to\infty}\alpha_n=0$ and

  (C_2) $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(3) the sequence $\{\beta_n\}$ satisfies

  (D_1) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$,

(4) the sequence $\{\gamma_n\}$ satisfies

  (E_1) $\lim\limits_{n\to\infty}\gamma_n=0$,

(5) the sequence $\{\delta_n\}$ and $\{\zeta_n\}$ satisfy

  (F_1) $\{\zeta_n\}\subset[c,d]\subset(0,2\alpha)$,

  (F_2) $\{\delta_n\}\subset[a,b]\subset(0,2\alpha)$ and $\lim\limits_{n\to\infty}|\delta_{n+1}-\delta_n|=0$,

(6) the sequence $\{r_{k,n}\}_{k=1}^{M}$ satisfy

  (G_1) $\lim\limits_{n\to\infty}r_{k,n}=r_k>0,\ \forall k\in\{1,2,\dots,M\}$.

For every $n\in\mathbb{N}$, let W_n be the W-mapping generated by $\{T_i\}$ and $\{\lambda_i\}$. If $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences generated by $x_1\in C$ and

$$\begin{cases} z_n = J_{r_M,n}^{\phi_M}\cdots J_{r_2,n}^{\phi_2}J_{r_1,n}^{\phi_1}x_n, \\ y_n=\gamma_n P_C(z_n-\zeta_n Az_n)+(1-\gamma_n)P_C(z_n-\delta_n Bz_n), \\ x_{n+1}=\alpha_n\gamma f(W_ny_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n\mu F)W_ny_n, \end{cases} \tag{1.6}$$

then, the sequence $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{J_{r_k,n}^{\phi_k}x_n\}_{k=1}^{M}$ converge strongly to $x^*\in\Im$, which is the unique solution of the variational inequalities :

$$\begin{cases} \left\langle (\mu F-\gamma f)x^*, x^*-x \right\rangle \leq 0, & \forall x \in \Im, \\ \left\langle Bx^*, x^*-y \right\rangle \leq 0, & \forall y \in C, \\ \left\langle Ax^*, x^*-y \right\rangle \leq 0, & \forall y \in C. \end{cases}$$

Let T be κ-strictly pseudo-contractive mapping of C into H . Define the mapping F_r^T of H into C by

$$F_r^T x := \left\{ z \in C : \left\langle y-z, (I-T)z \right\rangle + \frac{1}{r}\left\langle y-z, z-x \right\rangle \geq 0,\ \forall y \in C \right\}. \tag{1.7}$$

After we investigated the all Theorem above and research in the same direction, we have a question: Can we prove strong convergence theorem of finite family of strictly pseudo contractive mappings T without (1.7) ?

In this thesis, we give the answer and prove strong convergence theorems for finding a common element of the set of systems of variational inequalities for two inverse strongly monotone mappings and the set of common fixed point of a finite family of strictly pseudo-contractive mappings.

1.2 Objective

1.2.1 To propose an iterative scheme for finding a common element of the set of systems of variational inequality problems and fixed point problems.

1.2.2 To propose fixed point theorem for a finite family of strictly pseudo-contractive mappings and the set of systems of variational inequalities associate with two inverse strongly monotone mappings.

1.3 Scope of the study

1.3.1 Study of the strong convergence theorems of inverse strongly monotone mappings, strictly pseudo-contractive mappings, variational inequalities problems, equilibrium problems and fixed point problems.

1.3.2 Prove strong convergence theorems for finding a common element of the set of systems of variational inequalities for two inverse strongly monotone mappings and the set of common fixed point of a finite family of strictly pseudo-contractive mappings.

1.4 Method

1.4.1 Study research paper with many variational inequality problems.

1.4.2 Study definitions, lemmas, theorems and another property involving the problems in this thesis.

1.4.3 Prove strong convergence theorems for finding a common element of the set of systems of variational inequalities for two inverse strongly monotone mappings and the set of common fixed point of a finite family of strictly pseudo-contractive mapping.

1.5 Utilization of the study

1.5.1 Obtain new knowledge about fixed point theorems, equilibrium problems and variational inequality problems.

1.5.2 To obtain mathematical tools for κ-strictly pseudo-contractive mapping.

Our thesis is divided into the following chapter :
In chapter 1, we introduce the summary of the study.
In chapter 2, we give preliminaries to prove our main result.
In chapter 3, we prove the strong convergence theorem for finding a common element of the set of systems of variational inequalities for two inverse strongly monotone mappings and the set of common fixed points of a finite family of strictly pseudo-contractive mappings.
In chapter 4, we apply our main results.

Chapter 2

Preliminaries

The purpose of this chapter in order to explain fundamental concepts and definitions which to used throughout this thesis. Moreover, we give some lemmas, remarks and useful results used in the next chapters.

2.1 Linear spaces

Definition 2.1.1 [8] Let E be a nonempty set, and assume that each pair of elements x and y in E can be combined by a process called *addition* to yield an element z in E denoted by $x+y$. Assume also that this operation of addition satisfies the following conditions (v1) ~ (v4)

(v1) $(x+y)+z=x+(y+z)$,

(v2) $x+y=y+x$,

(v3) there exists a unique element in E, denoted by 0 and called the *zero element*, or the *origin*, such that $x+0=x$ for all $x\in E$,

(v4) to each $x\in E$ there corresponds a unique element in E, denoted by $-x$ and called the negative of x, such that $x+(-x)=0$.

We also assume that each scalar $\alpha\in\mathbb{R}$ and each element x in E can be combined by a process called *scalar multiplication* to yield an element y in E denoted by $y=\alpha x$ satisfying (v5) ~ (v8):

(v5) $\alpha(\beta x)=(\alpha\beta)x$,

(v6) $1\cdot x=x$,

(v7) $(\alpha+\beta)x=\alpha x+\beta x$,

(v8) $\alpha(x+y)=\alpha x+\alpha y$.

The algebraic system E defined by these operations and axioms is called a linear space. A linear space is often called a *vector space*, and its elements are spoken of as vectors.

Remark 2.1.2 [8] Since we admit the real numbers as scalars, a linear space is also called a real linear space.

Definition 2.1.3 [9] A set E in a vector space is called *convex* if for any $x,\, y \in E$ and $\alpha \in [0,1]$, we have $\alpha x + (1-\alpha)y \in E$.

2.2 Hilbert spaces

Definition 2.2.1 [10] *An inner product* on a vector space K over $\mathbb{F}$ is a function that assigns a scalar $\langle x,y\rangle$ for every $x,y \in K$, such that for all $x,\, y,\, z \in K$ and $\alpha \in \mathbb{F}$:

(I1) $\langle x+z,y\rangle = \langle x,y\rangle + \langle z,y\rangle$,

(I2) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$,

(I3) $\overline{\langle x,y\rangle} = \langle y,x\rangle$,

(I4) $\langle x,x\rangle > 0 \Leftrightarrow x \neq 0$,

A vector space K over $\mathbb{F}$ with a specific inner product is called *an inner product space*. If $\mathbb{F}=\mathbb{C}$, K is a complex inner product space and if $\mathbb{F}=\mathbb{R}$, K is a real inner product space.

Theorem 2.2.2 [10] For an inner product space K, $x,\, y,\, z \in K$, and $\alpha \in \mathbb{F}$:

(J1) $\langle x,y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$,

(J2) $\langle x,\alpha y\rangle = \bar{\alpha}\langle x,y\rangle$,

(J3) $\langle x,0\rangle = \langle 0,x\rangle = 0$,

(J4) $\langle x,x\rangle = 0 \Leftrightarrow x = 0$,

(J5) If $\langle x,y\rangle = \langle x,z\rangle$ for all $x \in K$, then $y = z$.

Remark 2.2.3 [8] An inner product space is called a real inner product space for the case when the scalars are the real numbers and $\langle x,y\rangle$ is a real number. For the case, (I3) means

$$\langle x,y\rangle = \langle y,x\rangle.$$

Remark 2.2.4 [8] Using (J1) and (J2), we obtain that for $x,\, y \in K$ and $\alpha,\, \beta \in \mathbb{C}$,

$$\langle x,\alpha y+\beta z\rangle = \bar{\alpha}\langle x,y\rangle + \bar{\beta}\langle x,z\rangle.$$

Remark 2.2.5 [8] Let K be an inner product space. For each x in K, we define its *norm* $\|x\|$ by

$$\|x\| = \sqrt{\langle x,x\rangle}.$$

Theorem 2.2.6 (The Schwarz inequality) [8] Let K be an inner product space and let x and y be elements in K. Then the following holds:

$$|\langle x,y\rangle| \leq \|x\|\|y\|.$$

Theorem 2.2.7 [8] The inner product in an inner product space K is jointly continuous:

$$x_n \to x \text{ and } y_n \to y \Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle .$$

Remark 2.2.8 [8] We of course obtain from Theorem 2.2.7 that if $x_n \to x$, then for a fixed $y \in K$,

$$\langle x_n, y \rangle \to \langle x, y \rangle \text{ and } \langle y, x_n \rangle \to \langle y, x \rangle .$$

Definition 2.2.9 (Strong Convergence) [9] A sequence $\{x_n\}$ of vectors in an inner product space K is called *strongly convergent* to a vector x in K if $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 2.2.10 (Weak Convergence) [9] A sequence $\{x_n\}$ of vectors in an inner product space K is called *weakly convergent* to a vector x in K if $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$, for every $y \in K$.

Definition 2.2.11 A sequence $\{x_n\}$ of vectors in a normed space is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a number M such that $|x_m - x_n| < \varepsilon$ for all $m, n > N$.

Lemma 2.2.12 [8] Let $\{x_n\}$ be a Cauchy sequence of an inner product space K such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Definition 2.2.13 [8] A complete inner product space is called a *Hilbert space*.

Theorem 2.2.14 [9] In Hilbert space. A strongly convergent sequence is weakly convergent (to the same limit), i.e., $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.2.15 [8] In Hilbert space. If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Remark 2.2.16 [8] Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

(1) H is complete,

(2) each bounded sequence $\{x_n\}$ of H has a weakly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 2.2.17 [8] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then for each point x in H, there corresponds a unique point x_0 in C such that $\|x - x_0\| = d(x, C)$,
where $d(x, C) = \inf_{y \in C} \|x - y\|$. We call such a mapping defined by $Px = x_0$, or $P_C x = x_0$, the metric projection of H onto C.

Lemma 2.2.18 [8] Let C be a nonempty convex subset of a Hilbert space H. Then for $x \in H$ and $y \in C$, $\|x-y\| = d(x,C)$ if and only if

$$\langle x-y, y-z\rangle \geq 0, \text{ for all } z \in C.$$

Theorem 2.2.19 (Properties of metric projection) [8] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then the metric projection P_C of H onto C has the following properties

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x-y\rangle \text{ for all } x, y \in H.$$

Theorem 2.2.20 [8] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Definition 2.2.21 [8] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty,\infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\}$$

is closed. f is also called *convex function* if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.2.22 [8] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty,\infty]$. Let $\{x_n\}$ be a bounded sequence of C such that $x_n \rightharpoonup x_0$. Then $f(x_0) \leq \liminf\limits_{n\to\infty} f(x_n)$.

Theorem 2.2.23 [9] Weakly convergent sequences in a Hilbert space H are bounded, i.e., if $\{x_n\}$ is a weakly convergent sequence, then there exists a number M such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

2.3 Fixed point theory

We study about existence and properties of fixed points are known as fixed point theorem.

2.3.1 Fixed point of nonexpansive mapping

Theorem 2.3.1.1 [8] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.3.1.2 [8] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping C into itself. Then $F(T)$ is closed and convex.

2.3.2 Fixed point of κ-strictly pseudo-contractive mapping

Lemma 2.3.2.1 [8] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T:C\to C$ be a κ-strictly pseudo-contractive mapping with $F(T)\neq\varnothing$. Then, $F(T)$ is closed and convex.

2.4 Some useful lemmas and theorems

To prove our main result, the following tools are need.

Lemma 2.4.1 [3] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u\in C$,then for $\lambda>0$,

$$u=P_C(I-\lambda A)u \Leftrightarrow u\in VI(C,A).$$

Lemma 2.4.2 [4] Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}\subset H$ with $x_n \rightharpoonup x$,, the inequality

$$\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|,$$

holds for every $y\in H$ with $y\neq x$.

Lemma 2.4.3 Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The following identities hold :

$$(i)\ \|x\pm y\|^2=\|x\|^2\pm 2\langle x,y\rangle+\|y\|^2,\ \forall x,y\in H;$$

$$(ii)\ \|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle,\ \forall x,y\in H;$$

Lemma 2.4.4 [5] Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For every $x,y,z\in H$ and $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma=1$, we have

$$\|\alpha x+\beta y+\gamma z\|^2=\alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.4.5 [6] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1}\le(1-\alpha_n)s_n+\delta_n,\forall n\ge1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1. $\sum_{n=1}^{\infty}\alpha_n=\infty$,

2. $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}\le 0$ or $\sum_{n=1}^{\infty}|\delta_n|<\infty.$

Then $\lim_{n\to\infty}s_n=0.$

We modified the work of Zegeye ([7]) as below.

Lemma 2.4.6 Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a κ-strictly pseudo-contractive mapping of C into H with $F(T)\ne\varnothing$. For $r>0$ and $x\in H$, defined a mapping F_r^T of H into C as follows :

$$F_r^T x:=\left\{z\in C:\langle y-z,(I-T)z\rangle+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\ \forall y\in C\right\},$$

for all $x\in H$. Then the following hold :

(1) F_r^T is single-valued ;

(2) F_r^T is a firmly nonexpansive type mapping.

(3) $F\left(F_r^T\right)=VI(C,I-T)$;

(4) $VI(C,I-T)$ is closed and convex.

Proof (3) To show $F\left(F_r^T\right)=VI(C,I-T)$

Let $p\in F\left(F_r^T\right)$ and $F_r^T x:=\left\{z\in C:\langle y-z,(I-T)z\rangle+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\ \forall y\in C\right\}$.

Then, we have $p=F_r^T p$.

Therefore, $\langle y-p,(I-T)p\rangle+\frac{1}{r}\langle y-p,p-p\rangle\ge 0,\ \forall y\in C.$

It implies that $p\in VI(C,I-T)$, from which it follows that $F\left(F_r^T\right)\subseteq VI(C,I-T)$.

Next, we show that $VI(C,I-T)\subseteq F\left(F_r^T\right)$.

Let $q\in VI(C,I-T)$, we have $\langle (I-T)q,y-q\rangle\ge 0,\ \forall y\in C.$

Thus, $\langle (I-T)q,y-q\rangle+\frac{1}{r}\langle y-q,q-q\rangle\ge 0,\ \forall y\in C.$

It implies that $q\in F\left(F_r^T\right)$.

Hence $F\left(F_r^T\right)=VI(C,I-T)$.

Remark If T is a κ-strictly pseudo contractive mapping with $F(T)\ne\varnothing$, then $VI(C,I-T)=F(T)$.

Proof Let $q\in F(T)$. Then $q=Fq$,

since $\langle (I-T)q,y-q\rangle=0,\ \forall y\in C$, we have $q\in VI(C,I-T)$,

from which it follows that $F(T)\subseteq VI(C,I-T)$.

Next, we show that $VI(C,I-T)\subseteq F(T)$.

Let $q\in VI(C,I-T)$, we have

$$\langle (I-T)q, y-q\rangle \geq 0, \qquad \forall y\in C. \tag{2.1}$$

Let $\bar{q}\in F(T)$, we have

$$\begin{aligned}\|Tq-\bar{q}\|^2 &= \|q-\bar{q}-(I-T)q\|^2\\ &= \|q-\bar{q}\|^2 - 2\langle q-\bar{q},(I-T)q\rangle + \|(I-T)q\|^2.\end{aligned} \tag{2.2}$$

Since T is κ-strictly pseudo contractive and $\bar{q}\in F(T)$, we have

$$\begin{aligned}\|Tq-T\bar{q}\|^2 &\leq \|q-\bar{q}\|^2 + \kappa\|(I-T)q-(I-T)\bar{q}\|^2\\ &= \|q-\bar{q}\|^2 + \kappa\|(I-T)q\|^2.\end{aligned} \tag{2.3}$$

From (2.1), (2.2) and (2.3), we get

$$\frac{(1-k)}{2}\|(I-T)q\|^2 \leq \langle q-\bar{q},(I-T)q\rangle \leq 0,$$

which yield that $q\in F(T)$. Therefore $VI(C,I-T)\subseteq F(T)$.

Chapter 3

Convergence Theorems in Hilbert Spaces

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mappings of C into itself and let A, B be α,β - inverse strongly monotone mappings of C into H, respectively, which $\mathcal{F} = VI(C,A)\cap VI(C,B)\cap\bigcap_{i=1}^N F(T_i) \neq \emptyset$.

Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and

$$\begin{cases} z_n = F^{T_N}_{r_n^N}\cdots F^{T_2}_{r_n^2}F^{T_1}_{r_n^1}x_n \\ x_{n+1} = \alpha_n u + \beta_n P_C\left(I-\lambda_n^A A\right)x_n + \gamma_n P_C\left(I-\lambda_n^B B\right)x_n + \delta_n z_n, \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\} \subseteq [0,1]$ and $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$.

Assume the condition i)-v) holds :

i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

ii) $0 < c \leq \beta_n,\gamma_n,\delta_n \leq d < 1$ for some $c,d > 0$ and $\forall n \geq 1$,

iii) $\{\lambda_n^A\} \subseteq [e,f] \subset (0,2\alpha)$ and $\{\lambda_n^B\} \subseteq [g,h] \subset (0,2\beta)$,

iv) $\{r_n^i\}_{i=1}^N \subseteq \left[\theta,\bar{\theta}\right]$ $\exists\theta,\bar{\theta} > 0$,

v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$,
$\sum_{n=1}^{\infty}|\lambda_{n+1}^A-\lambda_n^A| < \infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}^B-\lambda_n^B| < \infty$.

Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}u$.

Proof. To show $\left(I-\lambda_n^A A\right)$ and $\left(I-\lambda_n^B B\right)$ are nonexpansive mappings.

Since A be α - inverse strongly monotone and $\{\lambda_n^A\} \subset (0,2\alpha)$, we have

$$\begin{aligned} \left\|\left(I-\lambda_n^A A\right)x-\left(I-\lambda_n^A A\right)y\right\|^2 &= \|x-y\|^2 - 2\lambda_n^A\langle x-y, Ax-Ay\rangle + \left(\lambda_n^A\right)^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - 2\alpha\lambda_n^A\|Ax-Ay\|^2 + \left(\lambda_n^A\right)^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + \lambda_n^A\left(\lambda_n^A-2\alpha\right)\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2. \end{aligned} \tag{3.1}$$

Therefore, $\left(I-\lambda_n^A A\right)$ is a nonexpansive mapping. Similarly, $\left(I-\lambda_n^B B\right)$ is a nonexpansive mapping.

Step 1. We show that $\{x_n\}$ is bounded.

Let $p \in \mathcal{F}$, $M = \max\{\|u-p\|, \|x_1-p\|\}$ and $\ell_n^N = F^{T_N}_{r_n^N}\cdots F^{T_2}_{r_n^2}F^{T_1}_{r_n^1}$, $\ell_n^0 = I$.

Since $F(T_i)=VI(C,I-T_i)=F\left(F_{r_n^i}^{T_i}\right)$, for all $i=1,2,\ldots,N$, we have

$$\begin{aligned}\|z_n-p\| &= \|\ell_n^N x_n-p\| \\ &= \|\ell_n^N x_n-\ell_n^N p\| \\ &= \left\|F_{r_n^N}^{T_N}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n-F_{r_n^N}^{T_N}x_n\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}\,p\right\| \\ &\le \left\|F_{r_n^{N-1}}^{T_{N-1}}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n-F_{r_n^{N-1}}^{T_{N-1}}x_n\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}\,p\right\| \\ &\;\vdots \\ &\le \left\|F_{r_n^1}^{T_1}x_n-F_{r_n^1}^{T_1}\,p\right\| \\ &\le \|x_n-p\|. \end{aligned} \tag{3.2}$$

By $p\in\mathcal{F}$ and $\alpha_n+\beta_n+\gamma_n+\delta_n=1$, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \left\|\alpha_n u+\beta_n P_C\left(I-\lambda_n^A A\right)x_n+\gamma_n P_C\left(I-\lambda_n^B B\right)x_n+\delta_n z_n-p\right\| \\ &= \left\|\alpha_n(u-p)+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_n^A A\right)p\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_n^B B\right)p\right)+\delta_n(z_n-p)\right\| \\ &\le \alpha_n\|u-p\|+\beta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_n^A A\right)p\right\|+\delta_n\|z_n-p\|+\gamma_n\left\|P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_n^B B\right)p\right\| \\ &\le \alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\|. \end{aligned} \tag{3.3}$$

By induction, we have $\|x_n-p\|\le M$, for all $n\in\mathbb{N}$. This implies that the sequence $\{x_n\}$ is bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \Big\|\alpha_n u+\beta_n P_C\left(I-\lambda_n^A A\right)x_n+\gamma_n P_C\left(I-\lambda_n^B B\right)x_n+\delta_n z_n \\ &\quad -\left[\alpha_{n-1}u+\beta_{n-1}P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}+\gamma_{n-1}P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}+\delta_{n-1}z_{n-1}\right]\Big\| \\ &= \Big\|(\alpha_n-\alpha_{n-1})u+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_n\right)+\beta_n\left(P_C\left(I-\lambda_{n-1}^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right) \\ &\quad +(\beta_n-\beta_{n-1})\left(P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_n\right) \\ &\quad +\gamma_n\left(P_C\left(I-\lambda_{n-1}^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right)+(\gamma_n-\gamma_{n-1})\left(P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right)+\delta_n(z_n-z_{n-1})+(\delta_n-\delta_{n-1})z_{n-1}\Big\| \end{aligned}$$

$$\begin{aligned}
&\leq \left|\alpha_n-\alpha_{n-1}\right|\|u\|+\beta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_n\right\| \\
&\quad+\beta_n\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\| \\
&\quad+\gamma_n\left\|P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_n\right\|+\gamma_n\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\| \\
&\quad+\left|\gamma_n-\gamma_{n-1}\right|\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\|+\delta_n\left\|z_n-z_{n-1}\right\|+\left|\delta_n-\delta_{n-1}\right|\left\|z_{n-1}\right\| \\
&\leq \left|\alpha_n-\alpha_{n-1}\right|\|u\|+\beta_n\left\|\left(\lambda_{n-1}^A-\lambda_n^A\right)Ax_n\right\|+\beta_n\left\|x_n-x_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\| \\
&\quad+\gamma_n\left\|\left(\lambda_{n-1}^B-\lambda_n^B\right)Bx_n\right\|+\gamma_n\left\|x_n-x_{n-1}\right\|+\left|\gamma_n-\gamma_{n-1}\right|\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\| \\
&\quad+\delta_n\left\|z_n-z_{n-1}\right\|+\left|\delta_n-\delta_{n-1}\right|\left\|z_{n-1}\right\| \\
&= \left|\alpha_n-\alpha_{n-1}\right|\|u\|+\beta_n\left|\lambda_{n-1}^A-\lambda_n^A\right|\left\|Ax_n\right\|+\beta_n\left\|x_n-x_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\| \\
&\quad+\gamma_n\left|\lambda_{n-1}^B-\lambda_n^B\right|\left\|Bx_n\right\|+\gamma_n\left\|x_n-x_{n-1}\right\|+\left|\gamma_n-\gamma_{n-1}\right|\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\| \\
&\quad+\delta_n\left\|z_n-z_{n-1}\right\|+\left|\delta_n-\delta_{n-1}\right|\left\|z_{n-1}\right\|. \qquad (3.4)
\end{aligned}$$

Observing that $z_n=\ell_n^N x_n=F_{r_n^N}^{T_N}\ell_n^{N-1}x_n$ and $z_{n-1}=\ell_{n-1}^N x_{n-1}=F_{r_{n-1}^N}^{T_N}\ell_{n-1}^{N-1}x_{n-1}$,

$$\left\langle y-\ell_n^N x_n,\left(I-T_i\right)\ell_n^N x_n\right\rangle+\frac{1}{r_n^N}\left\langle y-\ell_n^N x_n,\ell_n^N x_n-\ell_n^{N-1}x_n\right\rangle\geq 0 \quad ,\forall y\in C \qquad (3.5)$$

and $$\left\langle y-\ell_{n-1}^N x_{n-1},\left(I-T_i\right)\ell_{n-1}^N x_{n-1}\right\rangle+\frac{1}{r_{n-1}^N}\left\langle y-\ell_{n-1}^N x_{n-1},\ell_{n-1}^N x_{n-1}-\ell_{n-1}^{N-1}x_{n-1}\right\rangle\geq 0 \quad ,\forall y\in C. \qquad (3.6)$$

Substituting y by $\ell_{n-1}^N x_{n-1}$ in (3.5) and y by $\ell_n^N x_n$ in (3.6), we have

$$\left\langle \ell_{n-1}^N x_{n-1}-\ell_n^N x_n,\left(I-T_i\right)\ell_n^N x_n\right\rangle+\frac{1}{r_n^N}\left\langle \ell_{n-1}^N x_{n-1}-\ell_n^N x_n,\ell_n^N x_n-\ell_n^{N-1}x_n\right\rangle\geq 0, \qquad (3.7)$$

and $$\left\langle \ell_n^N x_n-\ell_{n-1}^N x_{n-1},\left(I-T_i\right)\ell_{n-1}^N x_{n-1}\right\rangle+\frac{1}{r_{n-1}^N}\left\langle \ell_n^N x_n-\ell_{n-1}^N x_{n-1},\ell_{n-1}^N x_{n-1}-\ell_{n-1}^{N-1}x_{n-1}\right\rangle\geq 0. \qquad (3.8)$$

It follow that,

$$\left\langle \ell_n^N x_n-\ell_{n-1}^N x_{n-1},\left(I-T_i\right)\ell_{n-1}^N x_{n-1}-\left(I-T_i\right)\ell_n^N x_n+\frac{1}{r_{n-1}^N}\left(\ell_{n-1}^N x_{n-1}-\ell_{n-1}^{N-1}x_{n-1}\right)-\frac{1}{r_n^N}\left(\ell_n^N x_n-\ell_n^{N-1}x_n\right)\right\rangle\geq 0. \qquad (3.9)$$

Then,

$$\left\langle \ell_n^N x_n-\ell_{n-1}^N x_{n-1},\frac{1}{r_{n-1}^N}\left(\ell_{n-1}^N x_{n-1}-\ell_{n-1}^{N-1}x_{n-1}\right)-\frac{1}{r_n^N}\left(\ell_n^N x_n-\ell_n^{N-1}x_n\right)\right\rangle\geq\left\langle \ell_n^N x_n-\ell_{n-1}^N x_{n-1},\left(I-T_i\right)\ell_n^N x_n-\left(I-T_i\right)\ell_{n-1}^N x_{n-1}\right\rangle. \qquad (3.10)$$

From monotonicity of $I-T_i$, we have

$$\left\langle \ell_n^N x_n-\ell_{n-1}^N x_{n-1},\frac{1}{r_{n-1}^N}\left(\ell_{n-1}^N x_{n-1}-\ell_{n-1}^{N-1}x_{n-1}\right)-\frac{1}{r_n^N}\left(\ell_n^N x_n-\ell_n^{N-1}x_n\right)\right\rangle\geq 0. \qquad (3.11)$$

Thus, we have

$$\left\langle \ell_n^N x_n-\ell_{n-1}^N x_{n-1},\left(\ell_{n-1}^N x_{n-1}-\ell_{n-1}^{N-1}x_{n-1}\right)-\frac{r_{n-1}^N}{r_n^N}\left(\ell_n^N x_n-\ell_n^{N-1}x_n\right)\right\rangle\geq 0, \qquad (3.12)$$

and hence,

$$\left\langle \ell_n^N x_n - \ell_{n-1}^N x_{n-1}, \ell_{n-1}^N x_{n-1} - \ell_{n-1}^{N-1} x_{n-1} - \ell_n^N x_n + \ell_n^{N-1} x_n + \left(1-\frac{r_{n-1}^N}{r_n^N}\right)\left(\ell_n^N x_n - \ell_n^{N-1} x_n\right)\right\rangle \ge 0. \tag{3.13}$$

Thus, we have

$$\begin{aligned}\left\|\ell_n^N x_n - \ell_{n-1}^N x_{n-1}\right\|^2 &= \left\langle \ell_n^N x_n - \ell_{n-1}^N x_{n-1}, \ell_n^N x_n - \ell_{n-1}^N x_{n-1}\right\rangle \\ &\le \left\langle \ell_n^N x_n - \ell_{n-1}^N x_{n-1}, \ell_n^{N-1} x_n - \ell_{n-1}^{N-1} x_{n-1} + \left(1-\frac{r_{n-1}^N}{r_n^N}\right)\left(\ell_n^{N-1} x_n - \ell_n^{N-1} x_n\right)\right\rangle \\ &\le \left\|\ell_n^N x_n - \ell_{n-1}^N x_{n-1}\right\| \left\|\ell_n^{N-1} x_n - \ell_{n-1}^{N-1} x_{n-1} + \left(1-\frac{r_{n-1}^N}{r_n^N}\right)\left(\ell_n^N x_n - \ell_n^{N-1} x_n\right)\right\|. \end{aligned} \tag{3.14}$$

Putting $\overline{M} = \max\left\{\left\|\ell_n^{i+1} x_n - \ell_n^i x_n\right\|, \left\|F_{r_n^1}^{T_1} x_n - x_n\right\|\right\}$, for all $i = 1, 2, \ldots, N-1$.

It implies that

$$\begin{aligned}\left\|\ell_n^N x_n - \ell_{n-1}^N x_{n-1}\right\| &\le \left\|\ell_n^{N-1} x_n - \ell_{n-1}^{N-1} x_{n-1}\right\| + \left|1-\frac{r_{n-1}^N}{r_n^N}\right| \left\|\ell_n^N x_n - \ell_n^{N-1} x_n\right\| \\ &= \left\|\ell_n^{N-1} x_n - \ell_{n-1}^{N-1} x_{n-1}\right\| + \frac{1}{r_n^N}\left|r_n^N - r_{n-1}^N\right| \left\|\ell_n^N x_n - \ell_n^{N-1} x_n\right\| \\ &\le \left\|\ell_n^{N-1} x_n - \ell_{n-1}^{N-1} x_{n-1}\right\| + \frac{1}{\theta}\overline{M}\left|r_n^N - r_{n-1}^N\right| \\ &= \left\|\ell_n^{N-1} x_n - \ell_{n-1}^{N-1} x_{n-1}\right\| + \frac{1}{\theta}\overline{M}\sum_{i=N}^{N}\left|r_n^i - r_{n-1}^i\right| \\ &\le \left\|\ell_n^{N-2} x_n - \ell_{n-1}^{N-2} x_{n-1}\right\| + \frac{1}{\theta}\overline{M}\sum_{i=N-1}^{N}\left|r_n^i - r_{n-1}^i\right| \\ &\le \left\|\ell_n^{N-3} x_n - \ell_{n-1}^{N-3} x_{n-1}\right\| + \frac{1}{\theta}\overline{M}\sum_{i=N-2}^{N}\left|r_n^i - r_{n-1}^i\right| \\ &\ \ \vdots \\ &\le \left\|\ell_n^2 x_n - \ell_{n-1}^2 x_{n-1}\right\| + \frac{1}{\theta}\overline{M}\sum_{i=3}^{N}\left|r_n^i - r_{n-1}^i\right| \\ &\le \left\|\ell_n^1 x_n - \ell_{n-1}^1 x_{n-1}\right\| + \frac{1}{\theta}\overline{M}\sum_{i=2}^{N}\left|r_n^i - r_{n-1}^i\right| \\ &= \left\|F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}\right\| + \frac{1}{\theta}\overline{M}\sum_{i=2}^{N}\left|r_n^i - r_{n-1}^i\right|. \end{aligned} \tag{3.15}$$

Consider that

$$\left\langle y - F_{r_n^1}^{T_1} x_n, (I-T_i) F_{r_n^1}^{T_1} x_n\right\rangle + \frac{1}{r_n^1}\left\langle y - F_{r_n^1}^{T_1} x_n, F_{r_n^1}^{T_1} x_n - x_n\right\rangle \ge 0, \qquad \forall y \in C \tag{3.16}$$

and $$\left\langle y - F_{r_{n-1}^1}^{T_1} x_{n-1}, (I-T_i) F_{r_{n-1}^1}^{T_1} x_{n-1}\right\rangle + \frac{1}{r_{n-1}^1}\left\langle y - F_{r_{n-1}^1}^{T_1} x_{n-1}, F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1}\right\rangle \ge 0, \qquad \forall y \in C. \tag{3.17}$$

Substituting y by $F_{r_{n-1}^1}^{T_1} x_{n-1}$ in (3.16) and y by $F_{r_n^1}^{T_1} x_n$ in (3.17), we have

$$\left\langle F_{r_{n-1}^1}^{T_1} x_{n-1} - F_{r_n^1}^{T_1} x_n, (I - T_i) F_{r_n^1}^{T_1} x_n \right\rangle + \frac{1}{r_n^1} \left\langle F_{r_{n-1}^1}^{T_1} x_{n-1} - F_{r_n^1}^{T_1} x_n, F_{r_n^1}^{T_1} x_n - x_n \right\rangle \ge 0,$$

and $$\left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, (I - T_i) F_{r_{n-1}^1}^{T_1} x_{n-1} \right\rangle + \frac{1}{r_{n-1}^1} \left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1} \right\rangle \ge 0.$$

These imply that

$$\left\langle F_{r_{n-1}^1}^{T_1} x_{n-1} - F_{r_n^1}^{T_1} x_n, (I - T_i) F_{r_n^1}^{T_1} x_n + \frac{1}{r_n^1} \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\rangle \ge 0,$$

and $$\left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, (I - T_i) F_{r_{n-1}^1}^{T_1} x_{n-1} + \frac{1}{r_{n-1}^1} \left(F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1} \right) \right\rangle \ge 0.$$

It follows that

$$\left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, (I - T_i) F_{r_{n-1}^1}^{T_1} x_{n-1} - (I - T_i) F_{r_n^1}^{T_1} x_n + \frac{1}{r_{n-1}^1} \left(F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1} \right) - \frac{1}{r_n^1} \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\rangle \ge 0.$$

Then

$$\left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, \frac{1}{r_{n-1}^1} \left(F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1} \right) - \frac{1}{r_n^1} \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\rangle \ge \left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, (I - T_i) F_{r_n^1}^{T_1} x_n - (I - T_i) F_{r_{n-1}^1}^{T_1} x_{n-1} \right\rangle.$$

From monotonicity of $I - T_i$, we have

$$\left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, \frac{1}{r_{n-1}^1} \left(F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1} \right) - \frac{1}{r_n^1} \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\rangle \ge 0.$$

Thus, we have

$$\left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1} - \frac{r_{n-1}^1}{r_n^1} \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\rangle \ge 0.$$

Hence

$$\left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, F_{r_{n-1}^1}^{T_1} x_{n-1} - x_{n-1} - F_{r_n^1}^{T_1} x_n + x_n + \left(1 - \frac{r_{n-1}^1}{r_n^1} \right) \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\rangle \ge 0. \tag{3.18}$$

Thus, we have

$$\begin{aligned} \left\| F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1} \right\|^2 &= \left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1} \right\rangle \le \left\langle F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}^1}{r_n^1} \right) \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\rangle \\ &\le \left\| F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1} \right\| \left\| x_n - x_{n-1} + \left(1 - \frac{r_{n-1}^1}{r_n^1} \right) \left(F_{r_n^1}^{T_1} x_n - x_n \right) \right\|. \end{aligned} \tag{3.19}$$

It implies that,

$$\begin{aligned} \left\| F_{r_n^1}^{T_1} x_n - F_{r_{n-1}^1}^{T_1} x_{n-1} \right\| &\le \| x_n - x_{n-1} \| + \left| 1 - \frac{r_{n-1}^1}{r_n^1} \right| \left\| F_{r_n^1}^{T_1} x_n - x_n \right\| \\ &= \| x_n - x_{n-1} \| + \frac{1}{r_n^1} \left| r_n^1 - r_{n-1}^1 \right| \left\| F_{r_n^1}^{T_1} x_n - x_n \right\| \\ &\le \| x_n - x_{n-1} \| + \frac{1}{\theta} \overline{M} \left| r_n^1 - r_{n-1}^1 \right|. \end{aligned} \tag{3.20}$$

From (3.15) and (3.20), we have

$$\|z_n - z_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{1}{\theta}\overline{M}\sum_{i=1}^{N}\left|r_n^i - r_{n-1}^i\right|. \tag{3.21}$$

From (3.4) and (3.21) , we have

$$\begin{aligned}
\|x_{n+1} - x_n\| \le{}& |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\left|\lambda_{n-1}^A - \lambda_n^A\right|\|Ax_n\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^A A\right)x_{n-1}\right\| \\
&+ \gamma_n\left|\lambda_{n-1}^B - \lambda_n^B\right|\|Bx_n\| + \gamma_n\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^B B\right)x_{n-1}\right\| \\
&+ \delta_n\left(\|x_n - x_{n-1}\| + \frac{1}{\theta}\overline{M}|r_n - r_{n-1}|\right) + |\delta_n - \delta_{n-1}|\|z_{n-1}\| \\
={}& (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\left|\lambda_{n-1}^A - \lambda_n^A\right|\|Ax_n\| + |\beta_n - \beta_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^A A\right)x_{n-1}\right\| \\
&+ \gamma_n\left|\lambda_{n-1}^B - \lambda_n^B\right|\|Bx_n\| + |\gamma_n - \gamma_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^B B\right)x_{n-1}\right\| + \delta_n\frac{1}{\theta}\overline{M}|r_n - r_{n-1}| + |\delta_n - \delta_{n-1}|\|z_{n-1}\| \\
\le{}& (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|Q + \left|\lambda_{n-1}^A - \lambda_n^A\right|Q + |\beta_n - \beta_{n-1}|Q \\
&+ \left|\lambda_{n-1}^B - \lambda_n^B\right|Q + |\gamma_n - \gamma_{n-1}|Q + \frac{1}{\theta}\overline{M}|r_n - r_{n-1}| + |\delta_n - \delta_{n-1}|Q.
\end{aligned} \tag{3.22}$$

where $Q = \max_{n\in\mathbb{N}}\left\{\|u\|, \|Ax_n\|, \left\|P_C\left(I - \lambda_{n-1}^A A\right)x_{n-1}\right\|, \|Bx_n\|, \left\|P_C\left(I - \lambda_{n-1}^B B\right)x_{n-1}\right\|, \|z_{n-1}\|\right\}$.

From (3.22), conditions (ii), (v) and Lemma 2.4.5, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.23}$$

Step 3. We show that $\lim_{n\to\infty}\left\|x_n - F_{r_n^k}^{T_k}x_n\right\| = 0$ for every $k \in \{1,2,\ldots,N\}$.

First we claim that

$$\lim_{n\to\infty}\left\|\ell_n^k x_n - \ell_n^{k-1}x_n\right\| = 0, \quad \forall k = 1,2,\ldots,N. \tag{3.24}$$

Let $p \in \mathcal{F}$ and $k \in \{1,2,\ldots,N\}$. Since $F_{r_n^k}^{T_k}$ is firmly nonexpansive, we obtain

$$\begin{aligned}
\left\|\ell_n^k x_n - p\right\|^2 &= \left\|F_{r_n^k}^{T_k}\ell_n^{k-1}x_n - F_{r_n^k}^{T_k}p\right\|^2 \le \left\langle F_{r_n^k}^{T_k}\ell_n^{k-1}x_n - p, \ell_n^{k-1}x_n - p\right\rangle \\
&= \frac{1}{2}\left[\left\|F_{r_n^k}^{T_k}\ell_n^{k-1}x_n - p\right\|^2 + \left\|\ell_n^{k-1}x_n - p\right\|^2 - \left\|F_{r_n^k}^{T_k}\ell_n^{k-1}x_n - \ell_n^{k-1}x_n\right\|^2\right].
\end{aligned}$$

It follows that

$$\left\|\ell_n^k x_n - p\right\|^2 \le \|x_n - p\|^2 - \left\|\ell_n^k x_n - \ell_n^{k-1}x_n\right\|^2. \tag{3.25}$$

From (3.25), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\alpha_n(u-p)+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-p\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-p\right)+\delta_n(z_n-p)\right\|^2 \\
&\le \left\|\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-p\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-p\right)+\delta_n(z_n-p)\right\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\
&\le \beta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-p\right\|^2+\gamma_n\left\|P_C\left(I-\lambda_n^B B\right)x_n-p\right\|^2+\delta_n\|z_n-p\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\
&\le (\beta_n+\gamma_n)\|x_n-p\|^2+\delta_n\left\|F_{r_n^N}^{T_N}\cdots F_{r_n^{k+1}}^{T_{k+1}}\ell_n^k x_n-p\right\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\
&\le (\beta_n+\gamma_n)\|x_n-p\|^2+\delta_n\left\|\ell_n^k x_n-p\right\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\
&\le (\beta_n+\gamma_n)\|x_n-p\|^2+\delta_n\left[\|x_n-p\|^2-\left\|\ell_n^k x_n-\ell_n^{k-1}x_n\right\|^2\right]+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\
&\le \|x_n-p\|^2-\delta_n\left\|\ell_n^k x_n-\ell_n^{k-1}x_n\right\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle .
\end{aligned}$$

Then we have

$$\begin{aligned}
\delta_n\left\|\ell_n^k x_n-\ell_n^{k-1}x_n\right\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\
&\le \left[\|x_n-p\|+\|x_{n+1}-p\|\right]\|x_{n+1}-x_n\|+2\alpha_n\langle u-p,x_{n+1}-p\rangle .
\end{aligned}$$

Using condition i) , ii) and Step 2, we have

$$\lim_{n\to\infty}\left\|\ell_n^k x_n-\ell_n^{k-1}x_n\right\|=0, \quad \forall k\in\{1,2,\ldots,N\}.$$

Hence, from (3.24), we obtain

$$\lim_{n\to\infty}\left\|x_n-F_{r_n^1}^{T_1}x_n\right\|=0.$$

By Induction, we assume that $\lim_{n\to\infty}\left\|x_n-F_{r_n^k}^{T_k}x_n\right\|=0, \ \forall k\in\{1,2,\ldots,N\}$.

We must to show that $\lim_{n\to\infty}\left\|x_n-F_{r_n^{k+1}}^{T_{k+1}}x_n\right\|=0, \ \forall k\in\{1,2,\ldots,N\}$.

$$\begin{aligned}
\left\|x_n-F_{r_n^{k+1}}^{T_{k+1}}x_n\right\| &\le \left\|F_{r_n^{k+1}}^{T_{k+1}}x_n-F_{r_n^{k+1}}^{T_{k+1}}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n\right\|+\left\|F_{r_n^{k+1}}^{T_{k+1}}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n-F_{r_n^k}^{T_k}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n\right\| \\
&\quad+\left\|F_{r_n^k}^{T_k}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n-F_{r_n^{k-1}}^{T_{k-1}}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n\right\|+\ldots+\left\|F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n-F_{r_n^1}^{T_1}x_n\right\|+\left\|F_{r_n^1}^{T_1}x_n-x_n\right\| \\
&= \left\|F_{r_n^{k+1}}^{T_{k+1}}x_n-\ell_n^{k+1}x_n\right\|+\left\|\ell_n^{k+1}x_n-\ell_n^k x_n\right\|+\ldots+\left\|\ell_n^2 x_n-\ell_n^1 x_n\right\|+\left\|\ell_n^1 x_n-x_n\right\| \\
&= \left\|F_{r_n^{k+1}}^{T_{k+1}}x_n-\ell_n^{k+1}x_n\right\|+\sum_{i=0}^{k}\left\|\ell_n^{i+1}x_n-\ell_n^i x_n\right\|. \qquad (3.26)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\|F_{r_n^{k+1}}^{T_{k+1}} x_n - \ell_n^{k+1} x_n\right\| &= \left\|F_{r_n^{k+1}}^{T_{k+1}} x_n - F_{r_n^{k+1}}^{T_{k+1}} \cdots F_{r_n^2}^{T_2} F_{r_n^1}^{T_1} x_n\right\| \\
&\le \left\|x_n - F_{r_n^k}^{T_k} \cdots F_{r_n^2}^{T_2} F_{r_n^1}^{T_1} x_n\right\| \\
&\le \left\|x_n - F_{r_n^k}^{T_k} x_n\right\| + \left\|F_{r_n^k}^{T_k} x_n - F_{r_n^k}^{T_k} \cdots F_{r_n^2}^{T_2} F_{r_n^1}^{T_1} x_n\right\| \\
&= \left\|x_n - F_{r_n^k}^{T_k} x_n\right\| + \left\|F_{r_n^k}^{T_k} x_n - \ell_n^k x_n\right\| \\
&\le \left\|x_n - F_{r_n^k}^{T_k} x_n\right\| + \left\|x_n - F_{r_n^{k-1}}^{T_{k-1}} x_n\right\| + \left\|F_{r_n^{k-1}}^{T_{k-1}} x_n - \ell_n^{k-1} x_n\right\| \\
&= \sum_{i=k-1}^{k} \left\|x_n - F_{r_n^i}^{T_i} x_n\right\| + \left\|F_{r_n^{k-1}}^{T_{k-1}} x_n - \ell_n^{k-1} x_n\right\| \\
&\;\vdots \\
&= \sum_{i=2}^{k} \left\|x_n - F_{r_n^i}^{T_i} x_n\right\| + \left\|F_{r_n^2}^{T_2} x_n - \ell_n^2 x_n\right\| \\
&\le \sum_{i=1}^{k} \left\|x_n - F_{r_n^i}^{T_i} x_n\right\| + \left\|F_{r_n^1}^{T_1} x_n - \ell_n^1 x_n\right\| \\
&= \sum_{i=1}^{k} \left\|x_n - F_{r_n^i}^{T_i} x_n\right\|.
\end{aligned} \tag{3.27}$$

Therefore, from (3.26) and (3.27), we have

$$\left\|x_n - F_{r_n^{k+1}}^{T_{k+1}} x_n\right\| \le \sum_{i=1}^{k} \left\|x_n - F_{r_n^i}^{T_i} x_n\right\| + \sum_{i=0}^{k} \left\|\ell_n^{i+1} x_n - \ell_n^i x_n\right\|.$$

By assumption and (3.24), we get

$$\lim_{n\to\infty} \left\|x_n - F_{r_n^{k+1}}^{T_{k+1}} x_n\right\| = 0, \quad \forall k \in \{1,2,\ldots,N\}.$$

Therefore $\lim_{n\to\infty} \left\|x_n - F_{r_n^j}^{T_j} x_n\right\| = 0, \quad \forall j \in \{1,2,\ldots,N\}$ (3.28)

Step 4. We show that $\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\left\|P_C\left(I - \lambda_n^A A\right)x_n - x_n\right\| = \lim_{n\to\infty}\left\|P_C\left(I - \lambda_n^B B\right)x_n - x_n\right\| = 0.$

Since $z_n = \ell_n^k x_n = F_{r_n^N}^{T_N} \cdots F_{r_n^2}^{T_2} F_{r_n^1}^{T_1} x_n$, we have

$$\begin{aligned}
\|x_n - z_n\| &= \left\|x_n - F_{r_n^N}^{T_N} \cdots F_{r_n^2}^{T_2} F_{r_n^1}^{T_1} x_n\right\| \\
&\le \left\|x_n - F_{r_n^N}^{T_N} x_n\right\| + \left\|F_{r_n^N}^{T_N} x_n - F_{r_n^N}^{T_N} \cdots F_{r_n^2}^{T_2} F_{r_n^1}^{T_1} x_n\right\| \\
&\le \left\|x_n - F_{r_n^N}^{T_N} x_n\right\| + \left\|x_n - F_{r_n^{N-1}}^{T_{N-1}} \cdots F_{r_n^2}^{T_2} F_{r_n^1}^{T_1} x_n\right\| \\
&\le \left\|x_n - F_{r_n^N}^{T_N} x_n\right\| + \left\|x_n - F_{r_n^{N-1}}^{T_{N-1}} x_n\right\| + \ldots + \left\|x_n - F_{r_n^2}^{T_2} x_n\right\| + \left\|x_n - F_{r_n^1}^{T_1} x_n\right\| \\
&\le \sum_{k=1}^{N} \left\|x_n - F_{r_n^k}^{T_k} x_n\right\|.
\end{aligned} \tag{3.29}$$

It follows from Step 3 that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.30}$$

From definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\alpha_n u + \beta_n P_C\left(I - \lambda_n^A A\right)x_n + \gamma_n P_C\left(I - \lambda_n^B B\right)x_n + \delta_n z_n - p\right\|^2 \\
&= \left\|\alpha_n u + \beta_n P_C\left(I - \lambda_n^A A\right)x_n + \gamma_n P_C\left(I - \lambda_n^B B\right)x_n + \delta_n z_n - (\alpha_n + \beta_n + \gamma_n + \delta_n)p\right\|^2 \\
&= \left\|\alpha_n(u-p) + \beta_n\left(P_C\left(I - \lambda_n^A A\right)x_n - p\right) + \gamma_n\left(P_C\left(I - \lambda_n^B B\right)x_n - p\right) + \delta_n(z_n - p)\right\|^2 \\
&= \left\|\beta_n\left(P_C\left(I - \lambda_n^A A\right)x_n - p\right) + \delta_n(z_n - p) + (\alpha_n + \gamma_n)\left(\frac{\alpha_n(u-p)}{\alpha_n + \gamma_n} + \frac{\gamma_n\left(P_C\left(I - \lambda_n^B B\right)x_n - p\right)}{\alpha_n + \gamma_n}\right)\right\|^2 \\
&= \left\|\beta_n\left(P_C\left(I - \lambda_n^A A\right)x_n - p\right) + \delta_n(z_n - p) + c_n G_n\right\|^2, \qquad (3.31)
\end{aligned}$$

where $c_n = \alpha_n + \gamma_n$ and $G_n = \dfrac{\alpha_n(u-p)}{\alpha_n + \gamma_n} + \dfrac{\gamma_n\left(P_C\left(I - \lambda_n^B B\right)x_n - p\right)}{\alpha_n + \gamma_n}$.

From (3.31), Lemma 2.4.4 and $p \in \mathcal{F}$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - p\right\|^2 + \delta_n\|z_n - p\|^2 + c_n\|G_n\|^2 - \beta_n\delta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\|^2 \\
&\le \beta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - p\right\|^2 + \delta_n\|x_n - p\|^2 + c_n\|G_n\|^2 - \beta_n\delta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\|^2 \\
&\le (\delta_n + \beta_n)\|x_n - p\|^2 - \beta_n\delta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\|^2 + c_n\left(\frac{\alpha_n\|u-p\|^2}{\alpha_n + \gamma_n} + \frac{\gamma_n\left\|P_C\left(I - \lambda_n^B B\right)x_n - p\right\|^2}{\alpha_n + \gamma_n}\right) \\
&= (\delta_n + \beta_n)\|x_n - p\|^2 - \beta_n\delta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\|^2 + \alpha_n\|u-p\|^2 + \gamma_n\left\|P_C\left(I - \lambda_n^B B\right)x_n - p\right\|^2 \\
&\le \|x_n - p\|^2 - \beta_n\delta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\|^2 + \alpha_n\|u-p\|^2. \qquad (3.32)
\end{aligned}$$

It implies that

$$\begin{aligned}
\beta_n\delta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|u-p\|^2 \\
&\le \left(\|x_n - p\| + \|x_{n+1} - p\|\right)\|x_n - x_{n+1}\| + \alpha_n\|u-p\|^2. \qquad (3.33)
\end{aligned}$$

By condition i) and Step 2, we have

$$\lim_{n\to\infty}\left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\| = 0. \tag{3.34}$$

Since

$$\left\|P_C\left(I - \lambda_n^A A\right)x_n - x_n\right\| \le \left\|P_C\left(I - \lambda_n^A A\right)x_n - z_n\right\| + \|z_n - x_n\|, \tag{3.35}$$

it follows by (3.30) and (3.34) that

$$\lim_{n\to\infty}\left\|P_C\left(I - \lambda_n^A A\right)x_n - x_n\right\| = 0. \tag{3.36}$$

Consider

$$x_{n+1}-x_n = \alpha_n\left(x_n-u\right)+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-x_n\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-x_n\right)+\delta_n\left(x_n-z_n\right). \quad (3.37)$$

From condition i), (3.30), (3.36), and Step 2, we have

$$\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n^B B\right)x_n-x_n\right\|=0. \quad (3.38)$$

Step 5. We show that $\limsup_{n\to\infty}\langle u-p,x_n-p\rangle\le 0$, where $p=P_{\mathcal{F}}u$.

We can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-p,x_n-p\rangle=\lim_{j\to\infty}\langle u-p,x_{n_j}-p\rangle. \quad (3.39)$$

Since $\lambda_n^A\in[e,f]$, we may assume that $\lambda_{n_j}^A\to\lambda\in[e,f]\subseteq(0,2\alpha)$, as $j\to\infty$

Without loss of generality, we can assume that $x_{n_j}\rightharpoonup\omega$ as $j\to\infty$, where $\omega\in C$. We show that $\omega\in\mathcal{F}$.

Assume that $\omega\notin VI(C,A)$. Since $VI(C,A)=F\left(P_C\left(I-\lambda A\right)\right)$, we have $\omega\neq P_C\left(I-\lambda A\right)\omega$.

By nonexpansiveness of $P_C\left(I-\lambda A\right)$ and Opial's condition, we have

$$\begin{aligned}
\liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\| &< \liminf_{j\to\infty}\left\|x_{n_j}-P_C\left(I-\lambda A\right)\omega\right\| \\
&\le \liminf_{j\to\infty}\left\|x_{n_j}-P_C\left(I-\lambda_{n_j}^A A\right)x_{n_j}\right\|+\liminf_{j\to\infty}\left\|P_C\left(I-\lambda_{n_j}^A A\right)x_{n_j}-P_C\left(I-\lambda A\right)\omega\right\| \\
&\le \liminf_{j\to\infty}\left\|P_C\left(I-\lambda_{n_j}^A A\right)x_{n_j}-P_C\left(I-\lambda_{n_j}^A A\right)\omega\right\|+\liminf_{j\to\infty}\left\|P_C\left(I-\lambda_{n_j}^A A\right)\omega-P_C\left(I-\lambda A\right)\omega\right\| \\
&\le \liminf_{j\to\infty}\left\|P_C\left(I-\lambda_{n_j}^A A\right)x_{n_j}-P_C\left(I-\lambda_{n_j}^A A\right)\omega\right\|+\liminf_{j\to\infty}\left|\lambda-\lambda_{n_j}^A\right|\left\|A\omega\right\| \\
&\le \liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\|. \quad (3.40)
\end{aligned}$$

This is a contradiction. We obtain $\omega\in VI(C,A)$. (3.41)

By using the same above method, we have $\omega\in VI(C,B)$. (3.42)

Next, we will show that $\omega\in\bigcap_{i=1}^{N}F\left(T_i\right)$. Since $r_n^i\in\left[\theta,\overline{\theta}\right]$,

We may assume that $r_{n_j}^i\to r\in\left[\theta,\overline{\theta}\right]$ as $j\to\infty$, $\forall i=1,2,\ldots,N$

Consider

$$\left\langle y-F_{r_{n_j}^i}^{T_k}\omega,\left(I-T_k\right)F_{r_{n_j}^i}^{T_k}\omega\right\rangle+\frac{1}{r_{n_j}^i}\left\langle y-F_{r_{n_j}^i}^{T_k}\omega,F_{r_{n_j}^i}^{T_k}\omega-\omega\right\rangle\ge 0 \quad ,\forall y\in C \quad (3.43)$$

and $$\left\langle y-F_r^{T_k}\omega,\left(I-T_k\right)F_r^{T_k}\omega\right\rangle+\frac{1}{r}\left\langle y-F_r^{T_k}\omega,F_r^{T_k}\omega-\omega\right\rangle\ge 0 \quad ,\forall y\in C \quad (3.44)$$

for all $k\in\{1,2,\ldots,N\}$.

Substituting y by $F_r^{T_k}\omega$ in (3.43) and y by $F_{r_{n_j}^i}^{T_k}\omega$ in (3.44) for all $k\in\{1,2,\ldots,N\}$, we have

$$\left\langle F_r^{T_k}\omega-F_{r_{n_j}^i}^{T_k}\omega,\left(I-T_k\right)F_{r_{n_j}^i}^{T_k}\omega\right\rangle+\frac{1}{r_{n_j}^i}\left\langle F_r^{T_k}\omega-F_{r_{n_j}^i}^{T_k}\omega,F_{r_{n_j}^i}^{T_k}\omega-\omega\right\rangle\ge 0, \quad (3.45)$$

and $$\left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega,(I-T_k)F^{T_k}_{r}\omega\right\rangle+\frac{1}{r}\left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega, F^{T_k}_{r}\omega-\omega\right\rangle \geq 0. \tag{3.46}$$

It follows that

$$\left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega,(I-T_k)F^{T_k}_{r}\omega-(I-T_k)F^{T_k}_{r^i_{n_j}}\omega+\frac{1}{r}\left(F^{T_k}_{r}\omega-\omega\right)-\frac{1}{r^i_{n_j}}\left(F^{T_k}_{r^i_{n_j}}\omega-\omega\right)\right\rangle \geq 0. \tag{3.47}$$

Then,

$$\left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega,\frac{1}{r}\left(F^{T_k}_{r}\omega-\omega\right)-\frac{1}{r^i_{n_j}}\left(F^{T_k}_{r^i_{n_j}}\omega-\omega\right)\right\rangle \geq \left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega,(I-T_k)F^{T_k}_{r^i_{n_j}}\omega-(I-T_k)F^{T_k}_{r}\omega\right\rangle \tag{3.48}$$

From monotonicity of $I-T_i$, we have

$$\left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega,\frac{1}{r}\left(F^{T_k}_{r}\omega-\omega\right)-\frac{1}{r^i_{n_j}}\left(F^{T_k}_{r^i_{n_j}}\omega-\omega\right)\right\rangle \geq 0. \tag{3.49}$$

Then,

$$\left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega, F^{T_k}_{r}\omega-\omega-F^{T_k}_{r^i_{n_j}}\omega+\omega+\left(1-\frac{r}{r^i_{n_j}}\right)\left(F^{T_k}_{r^i_{n_j}}\omega-\omega\right)\right\rangle \geq 0. \tag{3.50}$$

Thus, we have

$$\begin{aligned}\left\|F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega\right\|^2 &= \left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega, F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega\right\rangle \leq \left\langle F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega,\left(1-\frac{r}{r^i_{n_j}}\right)\left(F^{T_k}_{r^i_{n_j}}\omega-\omega\right)\right\rangle \\ &\leq \left\|F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega\right\|\left|1-\frac{r}{r^i_{n_j}}\right|\left\|F^{T_k}_{r^i_{n_j}}\omega-\omega\right\|.\end{aligned} \tag{3.51}$$

From (3.40), we have

$$\left\|F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega\right\| \leq \left|1-\frac{r}{r^i_{n_j}}\right|\left\|F^{T_k}_{r^i_{n_j}}\omega-\omega\right\|, \text{ for all } k\in\{1,2,\ldots,N\}.$$

Hence $\lim\limits_{j\to\infty}\left\|F^{T_k}_{r^i_{n_j}}\omega - F^{T_k}_{r}\omega\right\| = 0$, for all $k\in\{1,2,\ldots,N\}$. (3.52)

Assume that $\omega \neq F^{T_{k_0}}_{r}\omega$, for some $k_0\in\{1,2,\ldots,N\}$. By nonexpansiveness of $F^{T_k}_{r^i_{n_j}}$, (3.28), (3.52) and Opial's condition, we have

$$\begin{aligned}\liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\| &< \liminf_{j\to\infty}\left\|x_{n_j}-F^{T_{k0}}_{r}\omega\right\| \\ &\leq \liminf_{j\to\infty}\left\|x_{n_j}-F^{T_{k0}}_{r^{k0}_{n_j}}x_{n_j}\right\| + \liminf_{j\to\infty}\left\|F^{T_{k0}}_{r^{k0}_{n_j}}x_{n_j}-F^{T_{k0}}_{r}\omega\right\| \\ &\leq \liminf_{j\to\infty}\left\|F^{T_{k0}}_{r^{k0}_{n_j}}x_{n_j}-F^{T_{k0}}_{r^{k0}_{n_j}}\omega\right\| + \liminf_{j\to\infty}\left\|F^{T_{k0}}_{r^{i}_{n_j}}\omega-F^{T_{k0}}_{r}\omega\right\|\end{aligned}$$

$$\leq \liminf_{j\to\infty} \left\| x_{n_j} - \omega \right\|. \tag{3.53}$$

This is a contradiction. Thus, $\omega \in \bigcap_{j=1}^{N} F\left(F_r^{T_j}\right)$.

Since $F\left(F_r^{T_i}\right) = VI\left(C, I - T_i\right) = F\left(T_i\right)$, we have $\omega \in \bigcap_{i=1}^{N} F\left(T_i\right)$. (3.54)

From $x_{n_j} \rightharpoonup \omega$, (3.41), (3.42), (3.54) and Lemma 2.2.18, we conclude that

$$\limsup_{n\to\infty} \langle u - p, x_n - p\rangle = \lim_{j\to\infty} \langle u - z, x_{n_j} - p\rangle = \langle u - p, \omega - p\rangle \leq 0. \tag{3.55}$$

Step 6. Finally, we will show that the sequence $\{x_n\}$ converge strongly to $p = P_{\mathcal{F}}u$.

Consider,

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &= \left\|\alpha_n\left(u - p\right) + \beta_n\left(P_C\left(I - \lambda_n^A A\right)x_n - p\right) + \gamma_n\left(P_C\left(I - \lambda_n^B B\right)x_n - p\right) + \delta_n\left(z_n - p\right)\right\|^2 \\
&\leq \left\|\beta_n\left(P_C\left(I - \lambda_n^A A\right)x_n - p\right) + \gamma_n\left(P_C\left(I - \lambda_n^B B\right)x_n - p\right) + \delta_n\left(z_n - p\right)\right\|^2 + 2\alpha_n\left\langle u - p, x_{n+1} - p\right\rangle \\
&\leq \beta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - p\right\|^2 + \gamma_n\left\|P_C\left(I - \lambda_n^B B\right)x_n - p\right\|^2 + \delta_n\left\|z_n - p\right\|^2 + 2\alpha_n\left\langle u - p, x_{n+1} - p\right\rangle \\
&\leq \left(1 - \alpha_n\right)\left\|x_n - p\right\|^2 + 2\alpha_n\left\langle u - p, x_{n+1} - p\right\rangle.
\end{aligned} \tag{3.56}$$

From (3.56), condition i) and Lemma 2.4.5, we can conclude that $\{x_n\}$ converge strongly to $p = P_{\mathcal{F}}u$. This completes the proof.

Chapter 4

Applications

In this section, we use our main result to prove Corollary 4.3 and Corollary 4.4, we used the following definitions and lemmas.

In 2009, Kangtunyakarn and Suantai [12] introduced the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ as follows.

Definition 4.1 [12] Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into itself. For each $j = 1,2,\ldots,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in J \times J \times J$ where $J = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 4.2 [11] Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in J \times J \times J$, $j = 1,2,\ldots,N$, where $J = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.

Corollary 4.3 Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into C and let $A, B : C \to H$ be α, β - inverse strongly monotone mappings, respectively, such that $\mathcal{F} = VI(C,A) \cap VI(C,B) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and

$$\begin{cases} z_n = F_{r_n}^T x_n, \\ x_{n+1} = \alpha_n u + \beta_n P_C\left(I - \lambda_n^A A\right) x_n + \gamma_n P_C\left(I - \lambda_n^B B\right) x_n + \delta_n z_n, \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $[0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$.

Assume the condition i)-v) holds :

i) $\lim\limits_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c,d > 0$ and $\forall n \ge 1$,

iii) $\{\lambda_n^A\} \subseteq [e,f] \subset (0,2\alpha)$ and $\{\lambda_n^B\} \subseteq [g,h] \subset (0,2\beta)$,

iv) $\{r_n\} \subseteq [\theta,\overline{\theta}]$ $\exists\theta,\overline{\theta} > 0$,

v) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|, \sum\limits_{n=1}^{\infty}|\beta_{n+1}-\beta_n|, \sum\limits_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|, \sum\limits_{n=1}^{\infty}|r_{n+1}-r_n|, \sum\limits_{n=1}^{\infty}|\lambda_{n+1}^A-\lambda_n^A|, \sum\limits_{n=1}^{\infty}|\lambda_{n+1}^B-\lambda_n^B| < \infty.$

Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}u$.

Proof. Since nonexpansive mapping implies 0-strictly pseudo-contractive mapping and take $N = 1$ and $T_1 = T$ in Theorem 3.1 to conclude the result.

Corollary 4.4 Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into C with $\bigcap_{i=1}^N F(T_i) \ne \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in J \times J \times J$, $j = 1,2,\ldots,N$, where $J = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $A, B : C \to H$ be α, β - inverse strongly monotone mappings, respectively, such that $\mathcal{F} = VI(C,A) \cap VI(C,B) \cap \bigcap_{i=1}^N F(T_i) \ne \emptyset$. Suppose that $u, x_1 \in C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = F_{r_n}^S x_n , \\ x_{n+1} = \alpha_n u + \beta_n P_C\left(I - \lambda_n^A A\right)x_n + \gamma_n P_C\left(I - \lambda_n^B B\right)x_n + \delta_n z_n . \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $[0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$.

Assume the condition i)-v) holds :

i) $\lim\limits_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c,d > 0$ and $\forall n \ge 1$,

iii) $\{\lambda_n^A\} \subseteq [e,f] \subset (0,2\alpha)$ and $\{\lambda_n^B\} \subseteq [g,h] \subset (0,2\beta)$,

iv) $\{r_n\} \subseteq [\theta,\overline{\theta}]$ $\exists\theta,\overline{\theta} > 0$,

v) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|, \sum\limits_{n=1}^{\infty}|\beta_{n+1}-\beta_n|, \sum\limits_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|, \sum\limits_{n=1}^{\infty}|r_{n+1}-r_n|, \sum\limits_{n=1}^{\infty}|\lambda_{n+1}^A-\lambda_n^A|, \sum\limits_{n=1}^{\infty}|\lambda_{n+1}^B-\lambda_n^B| < \infty.$

Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}u$.

Proof. By using Corollary 4.3 and Lemma 4.2, we obtain the conclusion.

Chapter 5

Conclusions

In this chapter, we conclude all theorems and corollaries obtained in this thesis.

1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mapping of C into itself and let $A,B:C\to H$ be α,β- inverse strongly monotone mappings, respectively, which $\mathcal{F}=VI(C,A)\cap VI(C,B)\cap\bigcap_{i=1}^N F(T_i)\neq\emptyset$.

   Let $\{x_n\}$ be a sequence generated by $u,x_1\in C$ and

   $$\begin{cases} z_n = F_{r_n^N}^{T_N}\cdots F_{r_n^2}^{T_2}F_{r_n^1}^{T_1}x_n \\ x_{n+1}=\alpha_n u+\beta_n P_C\left(I-\lambda_n^A A\right)x_n+\gamma_n P_C\left(I-\lambda_n^B B\right)x_n+\delta_n z_n, \end{cases}$$

   for all $n\geq 1$, where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}\subseteq[0,1]$ and $\alpha_n+\beta_n+\gamma_n+\delta_n=1$.

   Assume the condition i)-v) holds :

   i) $\lim\limits_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$,

   ii) $0<c\leq\beta_n,\gamma_n,\delta_n\leq d<1$ for some $c,d>0$ and $\forall n\geq 1$,

   iii) $\{\lambda_n^A\}\subseteq[e,f]\subset(0,2\alpha)$ and $\{\lambda_n^B\}\subseteq[g,h]\subset(0,2\beta)$,

   iv) $\{r_n^i\}_{i=1}^N\subseteq\left[\theta,\bar{\theta}\right]$ $\exists\theta,\bar{\theta}>0$,

   v) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum\limits_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum\limits_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$, $\sum\limits_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$,

   $\sum\limits_{n=1}^{\infty}\left|\lambda_{n+1}^A-\lambda_n^A\right|<\infty$, $\sum\limits_{n=1}^{\infty}\left|\lambda_{n+1}^B-\lambda_n^B\right|<\infty$

   Then $\{x_n\}$ converges strongly to $p=P_{\mathcal{F}}u$.

2. Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into C and let $A,B:C\to H$ be α,β- inverse strongly monotone mappings, respectively, such that $\mathcal{F}=VI(C,A)\cap VI(C,B)\cap F(T)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by $u,x_1\in C$ and

   $$\begin{cases} z_n = F_{r_n}^{T}x_n, \\ x_{n+1}=\alpha_n u+\beta_n P_C\left(I-\lambda_n^A A\right)x_n+\gamma_n P_C\left(I-\lambda_n^B B\right)x_n+\delta_n z_n, \end{cases}$$

   for all $n\geq 1$, where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}$ are sequences in $[0,1]$ and $\alpha_n+\beta_n+\gamma_n+\delta_n=1$.

Assume the condition i)-v) holds :

i) $\lim\limits_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c, d > 0$ and $\forall n \ge 1$,

iii) $\{\lambda_n^A\} \subseteq [e, f] \subset (0, 2\alpha)$ and $\{\lambda_n^B\} \subseteq [g, h] \subset (0, 2\beta)$,

iv) $\{r_n\} \subseteq [\theta, \overline{\theta}] \quad \exists \theta, \overline{\theta} > 0$,

v) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, $\sum\limits_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$, $\sum\limits_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$,

$\sum\limits_{n=1}^{\infty}|\lambda_{n+1}^A - \lambda_n^A| < \infty$, $\sum\limits_{n=1}^{\infty}|\lambda_{n+1}^B - \lambda_n^B| < \infty$

Then $\{x_n\}$ converge strongly to $p = P_{\mathcal{F}}u$.

3. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into C with $\bigcap_{i=1}^N F(T_i) \ne \varnothing$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in J \times J \times J$, $j = 1, 2, \ldots, N$, where $J = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $A, B : C \to H$ be α, β - inverse strongly monotone mappings, respectively, such that $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^N F(T_i) \ne \varnothing$. Suppose that $u, x_1 \in C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = F_{r_n}^S x_n , \\ x_{n+1} = \alpha_n u + \beta_n P_C\left(I - \lambda_n^A A\right)x_n + \gamma_n P_C\left(I - \lambda_n^B B\right)x_n + \delta_n z_n . \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $[0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$.

Assume the condition i)-v) holds :

i) $\lim\limits_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c, d > 0$ and $\forall n \ge 1$,

iii) $\{\lambda_n^A\} \subseteq [e, f] \subset (0, 2\alpha)$ and $\{\lambda_n^B\} \subseteq [g, h] \subset (0, 2\beta)$,

iv) $\{r_n\} \subseteq [\theta, \overline{\theta}] \quad \exists \theta, \overline{\theta} > 0$,

v) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, $\sum\limits_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$, $\sum\limits_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$,

$\sum\limits_{n=1}^{\infty}|\lambda_{n+1}^A - \lambda_n^A| < \infty$, $\sum\limits_{n=1}^{\infty}|\lambda_{n+1}^B - \lambda_n^B| < \infty$.

Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}u$.

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APPENDIX

ทฤษฎีบทการลู่เข้าของปัญหาจุดตรึงของการส่งแบบไม่ขยายและปัญหาอสมการการแปรผัน

Convergence Theorem Of The Fixed Point Problems Of Nonexpansive Mapping And Variational Inequality Problems

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บทคัดย่อ

ในงานวิจัยนี้นำเสนอวิธีการทำซ้ำ ซึ่งได้รับแรงดลใจและแนวคิดจากงานวิจัยของ H.Piri ([1]) และ Zegeye ([7]) และรูปแบบการนิยามการส่งค่าเซตบางประการ เราได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มเพื่อหาสมาชิกร่วมของเซตผลเฉลยของปัญหาระบบอสมการการแปรผันสำหรับสองการส่งแบบผกผันทางเดียวแบบเข้ม และเซตของจุดตรึงของการส่งแบบไม่ขยาย

คำสำคัญ : อสมการการการแปรผัน, การส่งแบบผกผันทางเดียวแบบ α, จุดตรึง

Abstract

In this paper, we propose an iterative scheme which received inspiration and ideas from the research of H. Piri ([1]) , Zegeye ([7]) and the definition of a certain set value mapping. We prove the strong convergence theorem for finding a common element of the set of solutions of systems of variational inequalities for two inverse-strongly monotone mappings and the set of fixed points of nonexpansive mapping.

Keywords : Variational Inequality, α - inverse strongly monotone, Fixed Point.

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1. INTRODUCTION

Let H be a real Hilbert space and C be nonempty closed convex subset of H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A mapping T of C into itself is called nonexpansive if

$$\|Tx-Ty\|\leq\|x-y\|,\ \text{for all } x,y\in C \tag{1.1}$$

We note the set of fixed point of T by $F(T)$. i.e., $F(T)=\{x\in C: Tx=x\}$.

Let $B:C\to H$ is called monotone if

$$\langle x-y, Bx-By\rangle\geq 0,\ \text{for all } x,y\in C. \tag{1.2}$$

Let $B:C\to H$ is called β -inverse strongly monotone if

$$\langle Bx-By, x-y\rangle\geq\beta\|Bx-By\|^2,\ \text{for all } x,y\in C. \tag{1.3}$$

Recall the mapping T is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0,1)$ such that

$$\langle x-y, Tx-Ty\rangle \le \|x-y\|^2 + \kappa\|(I-T)x-(I-T)y\|^2, \text{ for all } x,y \in C. \qquad (1.4)$$

We observe that T is if κ-strictly pseudo-contractive mapping then $I-T$ is $\frac{1-\kappa}{2}$-inverse strongly monotone.

Moreover, also this implies that $I-T$ is a continuous monotone mapping.

Let $B: C \to H$. The variational inequality problem is to find a point $u \in C$ such that

$$\langle Bu, v-u\rangle \ge 0, \text{ for all } v \in C. \qquad (1.5)$$

The set of solutions of (1.5) is denoted by $VI(C,B)$, that is, $VI(C,B) = \{x \in C : \langle Bu, v-u\rangle \ge 0, \ \forall y \in C\}$.

Let C be nonempty closed and convex subset of a real Hilbert space H. Let $A: C \to H$ be a continuous monotone mapping. For $r > 0$ and $x \in H$, define a mapping $F_r: H \to C$ as follows :

$$F_r x := \left\{ z \in C : \langle y-z, Az\rangle + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \ \forall y \in C \right\}$$

is single-valued and firmly nonexpansive that satisfies, $F(F_r) = VI(C,A)$.

In 2012, H.Piri ([1]) introduce the following general iterative methods :

$$\begin{cases} z_n = J^{\phi_M}_{r_{M,n}} \cdots J^{\phi_2}_{r_{2,n}} J^{\phi_1}_{r_{1,n}} x_n, \\ y_n = \gamma_n P_C(z_n - \zeta_n A z_n) + (1-\gamma_n) P_C(z_n - \delta_n B z_n), \\ x_{n+1} = \alpha_n \gamma f(W_n y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F) W_n y_n \end{cases} \qquad (1.6)$$

such that A is (α,β) – strongly monotone such that $\beta > \alpha\mu^2$, B is (ζ,δ) – strongly monotone such that $\delta > \zeta\eta^2$, $T_i: C \to C$ is a nonexpansive mapping for all $i = 1,2,\ldots$ and $J^{\phi}_r: H \to C$ defined by $J^{\phi}_r(x) = \left\{ z \in C : \phi(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \ \forall y \in C \right\}$ which is single-valued and firmly nonexpansive and satisfies $Fix(J^{\phi}_r) = EP(\phi)$, $\forall r > 0$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ be real sequence such that $0 < \alpha_n, \beta_n, \gamma_n < 1$, $r_{k,n} > 0$ and $0 < \lambda_n \le b$ for some $b \in (0,1]$ satisfies the conditions :

(1) for every $k \in \{1,2,\ldots,M\}$ the bifunction ϕ_k satisfies,

- (A_1) $\phi_k(x,x) = 0$ for all $x \in C$,
- (A_2) ϕ_k is monotone, $\phi_k(x,y) + \phi_k(y,x) \le 0$ for all $x,y \in C$,
- (A_3) $\liminf\limits_{t\to 0} \phi_k(tz + (1-t)x, y) \le \phi_k(x,y)$ for all $x,y,z \in C$,
- (A_4) $y \to \phi_k(x,y)$ is convex and lower semi-continuous for all $x \in C$,
- (B_1) $\Im = \bigcap_{i=1}^{\infty} Fix(T_i) \cap EP(\ell) \cap VI(C,B) \cap VI(C,A) \ne \varnothing$,

(2) the sequence $\{\alpha_n\}$ satisfies

- (C_1) $\lim\limits_{n\to\infty} \alpha_n = 0$ and
- (C_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(3) the sequence $\{\beta_n\}$ satisfies

- (D_1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(4) the sequence $\{\beta_n\}$ satisfies

- (E_1) $\lim\limits_{n\to\infty} \gamma_n = 0$,

(5) the sequence $\{\delta_n\}$ and $\{\zeta_n\}$ satisfies

- (F_1) $\{\zeta_n\} \subset [c,d] \subset (0,2\alpha)$,
- (F_2) $\{\delta_n\} \subset [a,b] \subset (0,2\alpha)$ and $\lim\limits_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$,

(6) The sequence $\{r_{k,n}\}_{k=1}^{M}$ satisfies

(G_1) $\lim_{n\to\infty} r_{k,n} = r_k > 0, \forall k \in \{1,2,\ldots,M\}$.

They show that the sequence $\{x_n\},\{y_n\},\{z_n\}$ and $\{J^{\phi_k}_{r_k,n}x_n\}_{k=1}^{M}$ converge strongly to $x^* \in \Im$, which is the unique solution of the variational inequalities :

$$\begin{cases} \langle(\mu F-\gamma f)x^*, x^*-x\rangle \le 0, & \forall x\in\Im, \\ \langle Bx^*, x^*-y\rangle \le 0, & \forall y\in C, \\ \langle Ax^*, x^*-y\rangle \le 0, & \forall y\in C. \end{cases}$$

Let $T:C\to H$ be κ-strictly pseudo-contractive mapping. From Lemma 2.6, define the mapping $F_r^T : H\to C$ by

$$F_r^T x := \left\{ z\in C: \langle y-z,(I-T)z\rangle + \frac{1}{r}\langle y-z, z-x\rangle \ge 0,\ \forall y\in C \right\}. \tag{1.7}$$

In this paper, we propose an iterative scheme which received inspiration and ideas from the research of H. Piri ([1]) , Zegeye ([7]) and the definition of a certain set value mapping. We prove the strong convergence theorem for finding a common element of the set of solutions of systems of variational inequalities for two inverse-strongly monotone mappings and the set of fixed points of nonexpansive mapping.

2. PRELIMINARIES

In this paper, let H be a real Hilbert space and C be a nonempty closed convex subset of H and the symbols "$\to$" and "$\rightharpoonup$" are denoted by strong and weak convergence, respectively. For every $x\in H$, there exists a unique nearest point $P_C x$ in C such that $\|x-P_C x\| = \min\|x-y\|$ for all $y\in C$. P_C is called metric projection of H onto C.

To prove our main result, the following tools are need.

Lemma 2.1. ([3]) For each $x\in H$ and $z\in C$, $P_C(x)=z \Leftrightarrow \langle x-z, z-y\rangle \ge 0, \quad \forall y\in C$.

It is well known that P_C is a firmly nonexpansive mapping of H onto C, i.e., $\|P_C x-P_C y\|^2 \le \langle P_C x-P_C y, x-y\rangle \quad ,\forall x,y\in H$.

Lemma 2.2. ([3]) Let H be a Hilbert space, C be a nonempty closed convex subset of H and A be a mapping of C into H. Let $u\in C$,then for $\lambda>0$,

$$u = P_C(I-\lambda A)u \Leftrightarrow u\in VI(C,A),$$

where P_C is called metric projection of H onto C.

Lemma 2.3. ([4]) For each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}\subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n-x\| < \liminf_{n\to\infty}\|x_n-y\|,$$

holds for every $y\in H$ with $y\ne x$.

Lemma 2.4. ([5]) For every $x,y,z\in H$ and $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma=1$, we have

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.5. ([6]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1-\alpha_n)s_n+\delta_n, \forall n\ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1. $\sum_{n=1}^{\infty}\alpha_n=\infty,$
2. $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}\le 0$ or $\sum_{n=1}^{\infty}|\delta_n|<\infty.$

Then $\lim_{n\to\infty} s_n = 0$.

We modified the work of Zegeye ([7]) as below.

Lemma 2.6. Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T:C\to H$ be κ-strictly pseudo-contractive mapping with $F(T)\neq\varnothing$. For $r>0$ and $x\in H$. Define a mapping $F_r^T: H\to C$ as follows :

$$F_r^T x := \left\{ z\in C: \langle y-z,(I-T)z\rangle+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\right\},$$

for all $x\in H$. Then the following hold :

(1) F_r^T is single-valued ;

(2) F_r^T is a firmly nonexpansive type mapping. i.e., for all $x,y\in H$;

(3) $F(F_r^T)=VI(C,I-T)$;

(4) $VI(C,I-T)$ is closed and convex.

Remark If T is nonexpansive mapping, then $VI(C,I-T)=F(T)$.

3. MAIN RESULT

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping and let $A,B: C\to H$ be α,β - inverse strongly monotone mappings, respectively, such that $\mathcal{F}=VI(C,A)\cap VI(C,B)\cap F(T)\neq\varnothing$.

Let $\{x_n\}$ be a sequence generated by $u,x_1\in C$ and

$$\begin{cases} x_{n+1}=\alpha_n u+\beta_n P_C\left(I-\lambda_n^A A\right)x_n+\gamma_n P_C\left(I-\lambda_n^B B\right)x_n+\delta_n z_n, \\ z_n=F_{r_n}^T x_n. \end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}\subseteq[0,1]$ and $\alpha_n+\beta_n+\gamma_n+\delta_n=1$.

Assume the condition i) - v) holds :

i) $\lim\limits_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$,

ii) $0<c\leq\beta_n,\gamma_n,\delta_n\leq d<1$ for some $c,d>0$ and $\forall n\geq 1$,

iii) $\{\lambda_n^A\}\subseteq[e,f]\subset(0,2\alpha)$ and $\{\lambda_n^B\}\subseteq[g,h]\subset(0,2\beta)$,

iv) $\{r_n\}\subseteq\left[\theta,\bar{\theta}\right]$ $\exists\theta,\bar{\theta}>0$,

v) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|$, $\sum\limits_{n=1}^{\infty}|\beta_{n+1}-\beta_n|$, $\sum\limits_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|$, $\sum\limits_{n=1}^{\infty}|r_{n+1}-r_n|$, $\sum\limits_{n=1}^{\infty}|\lambda_{n+1}^A-\lambda_n^A|$, $\sum\limits_{n=1}^{\infty}|\lambda_{n+1}^B-\lambda_n^B|<\infty$.

Then $\{x_n\}$ converges strongly to $p=P_{\mathcal{F}}u$.

Proof. We will show $\left(I-\lambda_n^A A\right)$ and $\left(I-\lambda_n^B B\right)$ are nonexpansive mapping.

Since A be α - inverse strongly monotone mapping and $\{\lambda_n^A\}\subset(0,2\alpha)$, we have

$$\begin{aligned}\left\|\left(I-\lambda_n^A A\right)x-\left(I-\lambda_n^A A\right)y\right\|^2 &= \|x-y\|^2-2\lambda_n^A\langle x-y,Ax-Ay\rangle+\left(\lambda_n^A\right)^2\|Ax-Ay\|^2\\ &\leq \|x-y\|^2-2\alpha\lambda_n^A\|Ax-Ay\|^2+\left(\lambda_n^A\right)^2\|Ax-Ay\|^2\\ &= \|x-y\|^2+\lambda_n^A\left(\lambda_n^A-2\alpha\right)\|Ax-Ay\|^2\\ &\leq \|x-y\|^2. \end{aligned} \tag{3.1}$$

Therefore $\left(I-\lambda_n^A A\right)$ is nonexpansive mapping. Similarly, $\left(I-\lambda_n^B B\right)$ is nonexpansive mapping.

Step 1. The sequence $\{x_n\}$ is bounded.

Let $p\in\mathcal{F}$ and $M=max\{\|u-p\|,\|x_1-p\|\}$. Since $F(T)=VI(C,I-T)=F\left(F_{r_n}^T\right)$, we have

$$\begin{aligned}\|z_n-p\|&=\left\|F_{r_n}^T x_n-p\right\|\\ &=\left\|F_{r_n}^T x_n-F_{r_n}^T p\right\|\\ &\leq\|x_n-p\|. \end{aligned} \tag{3.2}$$

Since $p\in\mathcal{F}$, lemma 2.2 and $\alpha_n+\beta_n+\gamma_n+\delta_n=1$,

$$
\begin{aligned}
\|x_{n+1}-p\| &= \left\|\alpha_n u+\beta_n P_C\left(I-\lambda_n^A A\right)x_n+\gamma_n P_C\left(I-\lambda_n^B B\right)x_n+\delta_n z_n-p\right\| \\
&= \left\|\alpha_n(u-p)+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_n^A A\right)p\right)+\delta_n(z_n-p)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_n^B B\right)p\right)\right\| \\
&\le \alpha_n\|u-p\|+\beta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_n^A A\right)p\right\|+\delta_n\|z_n-p\|+\gamma_n\left\|P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_n^B B\right)p\right\| \\
&\le \alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\| \\
&\le M \qquad (3.3)
\end{aligned}
$$

By induction, we have $\|x_n-p\| \le M$, for all $n\in\mathbb{N}$. This implies that the sequence $\{x_n\}$ is bounded and $\{z_n\}$ is bounded too.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

From the definition of x_n , we have

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|\alpha_n u+\beta_n P_C\left(I-\lambda_n^A A\right)x_n+\gamma_n P_C\left(I-\lambda_n^B B\right)x_n+\delta_n z_n-\left[\alpha_{n-1}u+\beta_{n-1}P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}+\gamma_{n-1}P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}+\delta_n z_{n-1}\right]\right\| \\
&= \Big\|(\alpha_n-\alpha_{n-1})u+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_n\right)+\beta_n\left(P_C\left(I-\lambda_{n-1}^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right) \\
&\quad +(\beta_n-\beta_{n-1})\left(P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_n\right)+\gamma_n\left(P_C\left(I-\lambda_{n-1}^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right) \\
&\quad +(\gamma_n-\gamma_{n-1})\left(P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right)+\delta_n(z_n-z_{n-1})+(\delta_n-\delta_{n-1})z_{n-1}\Big\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_n\right\|+\beta_n\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_n-P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\| \\
&\quad +|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\|+\gamma_n\left\|P_C\left(I-\lambda_n^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_n\right\|+\gamma_n\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_n-P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\| \\
&\quad +|\gamma_n-\gamma_{n-1}|\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\|+\delta_n\|z_n-z_{n-1}\|+|\delta_n-\delta_{n-1}|\|z_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\left\|\left(\lambda_{n-1}^A-\lambda_n^A\right)Ax_n\right\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\| \\
&\quad +\gamma_n\left\|\left(\lambda_{n-1}^B-\lambda_n^B\right)Bx_n\right\|+\gamma_n\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\|+\delta_n\|z_n-z_{n-1}\|+|\delta_n-\delta_{n-1}|\|z_{n-1}\| \\
&= |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\left|\lambda_{n-1}^A-\lambda_n^A\right|\|Ax_n\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}^A A\right)x_{n-1}\right\|+\gamma_n\left|\lambda_{n-1}^B-\lambda_n^B\right|\|Bx_n\| \\
&\quad +\gamma_n\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\left\|P_C\left(I-\lambda_{n-1}^B B\right)x_{n-1}\right\|+\delta_n\|z_n-z_{n-1}\|+|\delta_n-\delta_{n-1}|\|z_{n-1}\| \qquad (3.4)
\end{aligned}
$$

Observing that $z_n=F_{r_n}^T x_n$ and $z_{n-1}=F_{r_{n-1}}^T x_{n-1}$,

$$\left\langle y-F_{r_n}^T x_n,(I-T)F_{r_n}^T x_n\right\rangle+\frac{1}{r_n}\left\langle y-F_{r_n}^T x_n,F_{r_n}^T x_n-x_n\right\rangle \ge 0 \quad ,\forall y\in C \qquad (3.5)$$

and $$\left\langle y-F_{r_{n-1}}^T x_{n-1},(I-T)F_{r_{n-1}}^T x_{n-1}\right\rangle+\frac{1}{r_{n-1}}\left\langle y-F_{r_{n-1}}^T x_{n-1},F_{r_{n-1}}^T x_{n-1}-x_{n-1}\right\rangle \ge 0 \quad ,\forall y\in C \qquad (3.6)$$

Taking $y=F_{r_{n-1}}^T x_{n-1}$ in (3.5) and $y=F_{r_n}^T x_n$ in (3.6), we have

$$\left\langle F_{r_{n-1}}^T x_{n-1}-F_{r_n}^T x_n,(I-T)F_{r_n}^T x_n\right\rangle+\frac{1}{r_n}\left\langle F_{r_{n-1}}^T x_{n-1}-F_{r_n}^T x_n,F_{r_n}^T x_n-x_n\right\rangle \ge 0\,, \qquad (3.7)$$

and $$\left\langle F_{r_n}^T x_n-F_{r_{n-1}}^T x_{n-1},(I-T)F_{r_{n-1}}^T x_{n-1}\right\rangle+\frac{1}{r_{n-1}}\left\langle F_{r_n}^T x_n-F_{r_{n-1}}^T x_{n-1},F_{r_{n-1}}^T x_{n-1}-x_{n-1}\right\rangle \ge 0\,. \qquad (3.8)$$

It follow that,

$$\left\langle F_{r_n}^T x_n-F_{r_{n-1}}^T x_{n-1},(I-T)F_{r_{n-1}}^T x_{n-1}-(I-T)F_{r_n}^T x_n+\frac{1}{r_{n-1}}\left(F_{r_{n-1}}^T x_{n-1}-x_{n-1}\right)-\frac{1}{r_n}\left(F_{r_n}^T x_n-x_n\right)\right\rangle \ge 0\,. \qquad (3.9)$$

Then, $$\left\langle F_{r_n}^T x_n-F_{r_{n-1}}^T x_{n-1},\frac{1}{r_{n-1}}\left(F_{r_{n-1}}^T x_{n-1}-x_{n-1}\right)-\frac{1}{r_n}\left(F_{r_n}^T x_n-x_n\right)\right\rangle \ge \left\langle F_{r_n}^T x_n-F_{r_{n-1}}^T x_{n-1},(I-T)F_{r_n}^T x_n-(I-T)F_{r_{n-1}}^T x_{n-1}\right\rangle\,. \qquad (3.10)$$

Since $I-T:C\to H$ be a monotone, it follow that

$$\left\langle F_{r_n}^T x_n-F_{r_{n-1}}^T x_{n-1},\left(F_{r_{n-1}}^T x_{n-1}-x_{n-1}\right)-\frac{r_{n-1}}{r_n}\left(F_{r_n}^T x_n-x_n\right)\right\rangle \ge 0\,. \qquad (3.11)$$

and hence

$$\left\langle F_{r_n}^T x_n-F_{r_{n-1}}^T x_{n-1},F_{r_{n-1}}^T x_{n-1}-x_{n-1}-F_{r_n}^T x_n+x_n+\left(1-\frac{r_{n-1}}{r_n}\right)\left(F_{r_n}^T x_n-x_n\right)\right\rangle \ge 0\,. \qquad (3.12)$$

Thus, we have

$$\left\|F_{r_n}^T x_n - F_{r_{n-1}}^T x_{n-1}\right\|^2 = \left\langle F_{r_n}^T x_n - F_{r_{n-1}}^T x_{n-1}, F_{r_n}^T x_n - F_{r_{n-1}}^T x_{n-1}\right\rangle \le \left\langle F_{r_n}^T x_n - F_{r_{n-1}}^T x_{n-1}, x_n - x_{n-1} + \left(1-\frac{r_{n-1}}{r_n}\right)\left(F_{r_n}^T x_n - x_n\right)\right\rangle$$

$$\le \left\|F_{r_n}^T x_n - F_{r_{n-1}}^T x_{n-1}\right\| \left\|x_n - x_{n-1} + \left(1-\frac{r_{n-1}}{r_n}\right)\left(F_{r_n}^T x_n - x_n\right)\right\|. \tag{3.13}$$

It implies that,

$$\|z_n - z_{n-1}\| = \left\|F_{r_n}^T x_n - F_{r_{n-1}}^T x_{n-1}\right\| \le \|x_n - x_{n-1}\| + \left|1-\frac{r_{n-1}}{r_n}\right| \left\|F_{r_n}^T x_n - x_n\right\| = \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}| \left\|F_{r_n}^T x_n - x_n\right\|$$

$$\le \|x_n - x_{n-1}\| + \frac{1}{\theta}\overline{M}|r_n - r_{n-1}|. \tag{3.14}$$

From (3.4) and (3.14), we have

$$\|x_{n+1} - x_n\| \le |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\left|\lambda_{n-1}^A - \lambda_n^A\right|\|Ax_n\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^A A\right)x_{n-1}\right\| + \gamma_n\left|\lambda_{n-1}^B - \lambda_n^B\right|\|Bx_n\|$$

$$+ \gamma_n\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^B B\right)x_{n-1}\right\| + \delta_n\left(\|x_n - x_{n-1}\| + \frac{1}{\theta}\overline{M}|r_n - r_{n-1}|\right) + |\delta_n - \delta_{n-1}|\|z_{n-1}\|$$

$$= (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\left|\lambda_{n-1}^A - \lambda_n^A\right|\|Ax_n\| + |\beta_n - \beta_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^A A\right)x_{n-1}\right\|$$

$$+ \gamma_n\left|\lambda_{n-1}^B - \lambda_n^B\right|\|Bx_n\| + |\gamma_n - \gamma_{n-1}|\left\|P_C\left(I - \lambda_{n-1}^B B\right)x_{n-1}\right\| + \delta_n\frac{1}{\theta}\overline{M}|r_n - r_{n-1}| + |\delta_n - \delta_{n-1}|\|z_{n-1}\|$$

$$\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M + \left|\lambda_{n-1}^A - \lambda_n^A\right|M + |\beta_n - \beta_{n-1}|M$$

$$+ \gamma_n\left|\lambda_{n-1}^B - \lambda_n^B\right|M + |\gamma_n - \gamma_{n-1}|M + \frac{1}{\theta}\overline{M}|r_n - r_{n-1}| + |\delta_n - \delta_{n-1}|M. \tag{3.15}$$

where $M = \max_{n\in\mathbf{N}}\left\{\|u\|, \|Ax_n\|, \left\|P_C\left(I - \lambda_{n-1}^A A\right)x_{n-1}\right\|, \|Bx_n\|, \left\|P_C\left(I - \lambda_{n-1}^B B\right)x_{n-1}\right\|, \|z_{n-1}\|\right\}$.

From (3.15), conditions (ii), (V) and lemma 2.5 , we obtain $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$ (3.16)

Step 3. We show that $\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\left\|P_C\left(I - \lambda_n^A A\right)x_n - x_n\right\| = \lim_{n\to\infty}\left\|P_C\left(I - \lambda_n^B B\right)x_n - x_n\right\| = 0.$

Let $p \in \mathcal{F}$. Since $F_{r_n}^T$ is firmly nonexpansive, we obtain

$$\left\|F_{r_n}^T x_n - p\right\|^2 = \left\|F_{r_n}^T x_n - F_{r_n}^T p\right\|^2 \le \left\langle F_{r_n}^T x_n - p, x_n - p\right\rangle$$

$$= \frac{1}{2}\left[\left\|F_{r_n}^T x_n - p\right\|^2 + \|x_n - p\|^2 - \left\|F_{r_n}^T x_n - x_n\right\|^2\right]$$

$$\le \|x_n - p\|^2 - \left\|F_{r_n}^T x_n - x_n\right\|^2 \tag{3.17}$$

Using (3.17), we obtain

$$\|x_{n+1} - p\|^2 = \left\|\alpha_n(u-p) + \beta_n\left(P_C\left(I - \lambda_n^A A\right)x_n - p\right) + \gamma_n\left(P_C\left(I - \lambda_n^B B\right)x_n - p\right) + \delta_n(z_n - p)\right\|^2$$

$$= \left\|\beta_n\left(P_C\left(I - \lambda_n^A A\right)x_n - p\right) + \gamma_n\left(P_C\left(I - \lambda_n^B B\right)x_n - p\right) + \delta_n(z_n - p)\right\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle$$

$$\le \beta_n\left\|P_C\left(I - \lambda_n^A A\right)x_n - p\right\|^2 + \gamma_n\left\|P_C\left(I - \lambda_n^B B\right)x_n - p\right\|^2 + \delta_n\|z_n - p\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle$$

$$\le (\beta_n + \gamma_n)\|x_n - p\|^2 + \delta_n\left\|F_{r_n}^T x_n - p\right\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle$$

$$\le (\beta_n + \gamma_n)\|x_n - p\|^2 + \delta_n\left[\|x_n - p\|^2 - \left\|F_{r_n}^T x_n - x_n\right\|^2\right] + 2\alpha_n\langle u - p, x_{n+1} - p\rangle$$

$$\le \|x_n - p\|^2 - \delta_n\left\|F_{r_n}^T x_n - x_n\right\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle. \tag{3.18}$$

Then, we have

$$\delta_n\left\|F_{r_n}^T x_n - x_n\right\|^2 \le \left[\|x_n - p\| + \|x_{n+1} - p\|\right]\|x_{n+1} - x_n\| + 2\alpha_n\langle u - p, x_{n+1} - p\rangle.$$

Using condition i), ii) and step 2, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\left\|F_{r_n}^T x_n - x_n\right\| = 0. \tag{3.19}$$

From definition of x_n , we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\alpha_n(u-p)+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-p\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-p\right)+\delta_n(z_n-p)\right\|^2 \\
&= \left\|\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-p\right)+\delta_n(z_n-p)+(\alpha_n+\gamma_n)\left(\frac{\alpha_n(u-p)}{\alpha_n+\gamma_n}+\frac{\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-p\right)}{\alpha_n+\gamma_n}\right)\right\|^2 \\
&= \left\|\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-p\right)+\delta_n(z_n-p)+c_nG_n\right\|^2 . \qquad (3.20)
\end{aligned}
$$

where $c_n=\alpha_n+\gamma_n$ and $G_n=\frac{\alpha_n(u-p)}{\alpha_n+\gamma_n}+\frac{\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-p\right)}{\alpha_n+\gamma_n}$.

From lemma 2.4 and $p\in\mathcal{F}$, we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-p\right\|^2+\delta_n\|z_n-p\|^2+c_n\|G_n\|^2-\beta_n\delta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-z_n\right\|^2 \\
&\le (\delta_n+\beta_n)\|x_n-p\|^2-\beta_n\delta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-z_n\right\|^2+c_n\left(\frac{\alpha_n\|u-p\|^2}{\alpha_n+\gamma_n}+\frac{\gamma_n\left\|P_C\left(I-\lambda_n^B B\right)x_n-p\right\|^2}{\alpha_n+\gamma_n}\right) \\
&\le \|x_n-p\|^2-\beta_n\delta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-z_n\right\|^2+\alpha_n\|u-p\|^2 . \qquad (3.21)
\end{aligned}
$$

It implies that

$$\beta_n\delta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-z_n\right\|^2 \le \left(\|x_n-p\|+\|x_{n+1}-p\|\right)\|x_n-x_{n+1}\|+\alpha_n\|u-p\|^2 . \qquad (3.22)$$

Then $\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n^A A\right)x_n-z_n\right\|=0$. (3.23)

Since $\left\|P_C\left(I-\lambda_n^A A\right)x_n-x_n\right\|\le\left\|P_C\left(I-\lambda_n^A A\right)x_n-z_n\right\|+\|z_n-x_n\|$ and (3.24)

from (3.19) and (3.23), we have $\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n^A A\right)x_n-x_n\right\|=0$. (3.25)

Consider that $x_{n+1}-x_n = \alpha_n(x_n-u)+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-x_n\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-x_n\right)+\delta_n(x_n-z_n)$. (3.26)

From condition i), (3.19), (3.25), and step 2, we have $\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n^B B\right)x_n-x_n\right\|=0$. (3.27)

Step 4. We show that $\limsup_{n\to\infty}\langle u-p,x_n-p\rangle\le 0$, where $p=P_{\mathcal{F}}u$.

We can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-p,x_n-p\rangle=\lim_{k\to\infty}\langle u-p,x_{n_k}-p\rangle. \qquad (3.28)$$

Since $\lambda_n^A\in[e,f]$, we may assume that $\lambda_{n_k}^A\to\lambda\in[e,f]\subseteq(0,2\alpha)$.

Without loss of generality, we can assume that $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$, where $\omega\in C$. We will show that $\omega\in\mathcal{F}$.

Assume that $\omega\notin VI(C,A)$. Since $VI(C,A)=F\left(P_C(I-\lambda A)\right)$, we have $\omega\ne P_C(I-\lambda A)\omega$.

By nonexpansiveness of $P_C(I-\lambda A)$ and Opial's condition, we have

$$
\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\|x_{n_k}-P_C(I-\lambda A)\omega\| \\
&\le \liminf_{k\to\infty}\left\|x_{n_k}-P_C\left(I-\lambda_{n_k}^A A\right)x_{n_k}\right\| + \liminf_{k\to\infty}\left\|P_C\left(I-\lambda_{n_k}^A A\right)x_{n_k}-P_C(I-\lambda A)\omega\right\| \\
&\le \liminf_{k\to\infty}\left\|P_C\left(I-\lambda_{n_k}^A A\right)x_{n_k}-P_C\left(I-\lambda_{n_k}^A A\right)\omega\right\|+\liminf_{k\to\infty}\left\|P_C\left(I-\lambda_{n_k}^A A\right)\omega-P_C(I-\lambda A)\omega\right\| \\
&\le \liminf_{k\to\infty}\left\|P_C\left(I-\lambda_{n_k}^A A\right)x_{n_k}-P_C\left(I-\lambda_{n_k}^A A\right)\omega\right\|+\liminf_{k\to\infty}\left|\lambda-\lambda_{n_k}^A\right|\|A\omega\| \\
&\le \liminf_{k\to\infty}\|x_{n_k}-\omega\| . \qquad (3.29)
\end{aligned}
$$

This is a contradiction. We obtain $\omega\in VI(C,A)$. (3.30)

By using the same above method, we have $\omega\in VI(C,B)$. (3.31)

Next, we will show that $\omega\in F(T)$. Since $r_n\in\left[\theta,\bar{\theta}\right]$, we may assume that $r_{n_j}\to r\in\left[\theta,\bar{\theta}\right]$, $\forall j=1,2,\ldots,M$.

Consider that

$$\left\langle y-F_{r_{n_j}}^T\omega,(I-T)F_{r_{n_j}}^T\omega\right\rangle+\frac{1}{r_{n_j}}\left\langle y-F_{r_{n_j}}^T\omega,F_{r_{n_j}}^T\omega-\omega\right\rangle\ge 0 \quad ,\forall y\in C \qquad (3.32)$$

and $$\left\langle y-F_r^T\omega,(I-T)F_r^T\omega\right\rangle+\frac{1}{r}\left\langle y-F_r^T\omega,F_r^T\omega-\omega\right\rangle\geq 0 \qquad ,\forall y\in C \tag{3.33}$$

Taking $y=F_r^T\omega$ in (3.32) and $y=F_{r_{n_j}}^T\omega$ in (3.33), we have

$$\left\langle F_r^T\omega-F_{r_{n_j}}^T\omega,(I-T)F_{r_{n_j}}^T\omega\right\rangle+\frac{1}{r_{n_j}}\left\langle F_r^T\omega-F_{r_{n_j}}^T\omega,F_{r_{n_j}}^T\omega-\omega\right\rangle\geq 0\ , \tag{3.34}$$

and $$\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,(I-T)F_r^T\omega\right\rangle+\frac{1}{r}\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,F_r^T\omega-\omega\right\rangle\geq 0\ . \tag{3.35}$$

It follows that

$$\left\langle F_{r_{n_j}}^T x_n-F_r^T\omega,(I-T)F_r^T\omega-(I-T)F_{r_{n_j}}^T\omega+\frac{1}{r}\left(F_r^T\omega-\omega\right)-\frac{1}{r_{n_j}}\left(F_{r_{n_j}}^T\omega-\omega\right)\right\rangle\geq 0\ . \tag{3.36}$$

Then, $$\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,\frac{1}{r}\left(F_r^T\omega-\omega\right)-\frac{1}{r_{n_j}}\left(F_{r_{n_j}}^T\omega-\omega\right)\right\rangle\geq\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,(I-T)F_{r_{n_j}}^T\omega-(I-T)F_r^T\omega\right\rangle . \tag{3.37}$$

Since $I-T:C\rightarrow H$ be a monotone, we have

$$\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,F_r^T\omega-\omega-\frac{r}{r_{n_j}}\left(F_{r_{n_j}}^T\omega-\omega\right)\right\rangle\geq 0\ . \tag{3.38}$$

then, $$\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,F_r^T\omega-\omega-F_{r_{n_j}}^T\omega+\omega+\left(1-\frac{r}{r_{n_j}}\right)\left(F_{r_{n_j}}^T\omega-\omega\right)\right\rangle\geq 0\ . \tag{3.39}$$

Thus, we have

$$\begin{aligned}\left\|F_{r_{n_j}}^T\omega-F_r^T\omega\right\|^2&=\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,F_{r_{n_j}}^T\omega-F_r^T\omega\right\rangle\leq\left\langle F_{r_{n_j}}^T\omega-F_r^T\omega,\left(1-\frac{r}{r_{n_j}}\right)\left(F_{r_{n_j}}^T\omega-\omega\right)\right\rangle\\&\leq\left\|F_{r_{n_j}}^T\omega-F_r^T\omega\right\|\left|1-\frac{r}{r_{n_j}}\right|\left\|F_{r_{n_j}}^T\omega-\omega\right\| .\end{aligned} \tag{3.40}$$

From (3.40), we have

$$\left\|F_{r_{n_j}}^T\omega-F_r^T\omega\right\|\leq\left|1-\frac{r}{r_{n_j}}\right|\left\|F_{r_{n_j}}^T\omega-\omega\right\| .$$

Hence $\lim_{j\to\infty}\left\|F_{r_{n_j}}^T\omega-F_r^T\omega\right\|=0$. (3.41)

Assume that $\omega\neq F_r^T\omega$. By non-expansive of $F_{r_{n_j}}^T$, (3.19), (3.41) and Opial's condition, we have

$$\begin{aligned}\liminf_{k\to\infty}\left\|x_{n_k}-\omega\right\|&<\liminf_{k\to\infty}\left\|x_{n_k}-F_r^T\omega\right\|\\&\leq\liminf_{k\to\infty}\left\|x_{n_k}-F_{r_{n_j}}^T x_{n_k}\right\|+\liminf_{k\to\infty}\left\|F_{r_{n_j}}^T x_{n_k}-F_r^T\omega\right\|\\&\leq\liminf_{k\to\infty}\left\|F_{r_{n_j}}^T x_{n_k}-F_{r_{n_j}}^T\omega\right\|+\liminf_{j\to\infty}\left\|F_{r_{n_j}}^T\omega-F_r^T\omega\right\|\\&\leq\liminf_{k\to\infty}\left\|x_{n_k}-\omega\right\| .\end{aligned} \tag{3.42}$$

This is a contradiction, we obtain $\omega\in F\left(F_r^T\right)$.

Since $F\left(F_r^T\right)=VI\left(C,I-T\right)=F\left(T\right)$, $\omega\in F\left(T\right)$. (3.43)

From $x_{n_k}\rightharpoonup\omega$, (3.30), (3.31), (3.43) and lemma 2.1, we conclude that

$$\limsup_{n\to\infty}\langle u-z,x_n-z\rangle=\lim_{k\to\infty}\langle u-z,x_{n_k}-z\rangle=\langle u-z,\omega-z\rangle\leq 0\ . \tag{3.44}$$

Step 5. Finally, we will show that the sequence $\{x_n\}$ converges strongly to $p=P_{\mathcal{F}}u$.

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \left\|\alpha_n(u-p)+\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-p\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-p\right)+\delta_n(z_n-p)\right\|^2 \\ &= \left\|\beta_n\left(P_C\left(I-\lambda_n^A A\right)x_n-p\right)+\gamma_n\left(P_C\left(I-\lambda_n^B B\right)x_n-p\right)+\delta_n(z_n-p)\right\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\ &\le \beta_n\left\|P_C\left(I-\lambda_n^A A\right)x_n-p\right\|^2+\gamma_n\left\|P_C\left(I-\lambda_n^B B\right)x_n-p\right\|^2+\delta_n\|z_n-p\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle \\ &\le (1-\alpha_n)\|x_n-p\|^2+2\alpha_n\langle u-p,x_{n+1}-p\rangle . \end{aligned} \tag{3.45}$$

From (3.45), condition i) and lemma 2.5, we can conclude that $\{x_n\}$ converges strongly to $p=P_{\mathcal{F}}u$.

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