THE MODIFICATION METHOD FOR FIXED POINT OF QUASI-NONEXPANSIVE MAPPING AND THE SYSTEM OF VARIATIONAL INEQUALITIES PROBLEM AND EQUILIBRIUM PROBLEM

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หัวข้อวิทยานิพนธ์	วิธีปรับปรุงสำหรับจุดตรึงของการส่งกึ่งไม่ขยาย และระบบของปัญหาอสมการการแปรผัน และปัญหาเชิงดุลยภาพ
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บทคัดย่อ

จุดประสงค์ของวิทยานิพนธ์นี้คือ เพื่อสร้างทฤษฎีบทการลู่เข้าแบบเข้มของจุดตรึงที่เกี่ยวข้องกับ

1. การหาสมาชิกร่วมของเซตของจุดตรึงของการส่งกึ่งไม่ขยาย และเซตของผลเฉลยของระบบปรับปรุงของอสมการการแปรผัน
2. การหาสมาชิกร่วมของเซตของจุดตรึงของการส่งกึ่งไม่ขยาย และเซตของผลเฉลยของปัญหาเชิงดุลยภาพ และเซตของผลเฉลยของระบบปรับปรุงของอสมการการแปรผัน

โดยไม่ใช้เงื่อนไขครึ่งปิดของการส่ง $I-T$ และ $T_\omega := (1-\omega)I+\omega T$ สำหรับทุกๆ $\omega \in \left(0,\frac{1}{2}\right)$ โดยที่ T คือการส่งกึ่งไม่ขยายในปริภูมิฮิลเบิร์ต นอกจากนี้ยังประยุกต์ทฤษฎีบทหลักดังกล่าวสำหรับการหาจุดตรึงร่วมของวงศ์จำกัดของการส่งไม่กระจาย และยกตัวอย่างการคำนวณสำหรับทฤษฎีบทหลักของวิทยานิพนธ์ฉบับนี้

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Thesis Title	The Modification Method for Fixed Point of Quasi-Nonexpansive Mapping and the System of Variational Inequalities Problem and Equilibrium Problem
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ABSTRACT

The purpose of this thesis is to introduce the strong convergence theorems involved with

1. finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities,
2. finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities,

without a demiclosed condition of $I-T$ mapping and $T_\omega := (1-\omega)I + \omega T$, for all $\omega \in \left(0,\frac{1}{2}\right)$, where T is a quasi-nonexpansive mapping in a framework of Hilbert space. Therefore, we apply main theorem for finding a common fixed point of a finite family of nonspreading mappings. Finally, we give the numerical examples to support our main results.

Keywords : Quasi-nonexpansive mapping, Variational inequality problem, Equilibrium problem, Nonspreading mapping, Fixed point problem

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Chapter 1

Introduction

1.1 Background

For the last decades, there are many new developments of technologies applied in different fields such as computer science, engineering, physics, economics etc. Furthermore, one of the important tools applied to create the technologies is "mathematical tools". The fixed point theory is one of the important mathematical tools applied to solve problems in many branches of science. The problems in science are usually transformed into a mathematical model. Most of the problems from various disciplines of sciences are nonlinear in nature. On the other hand, numerous problems in physics, optimization, and economics are reduced to find solution of fixed point problems. Iterative scheme of fixed point of nonlinear mappings is an important subject in the theory of nonlinear mappings, which is applied in a number of applied areas. Now, iterative scheme for approximating fixed point of quasi-nonexpansive mappings, the system of variational inequalities problem, and equilibrium problem have been increasingly studied by many mathematicians.

Fixed point theory is one of the most powerful and various tools of modern mathematics. Fixed point problems have been widely studied in the literature. First, we study these definitions and related literatures as follows:

Throughout this section, let H be a real Hilbert space and let C be a nonempty closed convex subset of H.

For a mapping T of C into itself, we denote $F(T)$ by the set of all *fixed points* of T, i.e.,

$$F(T)=\{x\in C:Tx=x\}.$$

Example 1.1.

1. If $T:\mathbb{R}\to\mathbb{R}$ and $T(x)=\dfrac{x+1}{2}$, then $F(T)=\{1\}$,

2. If $T:\mathbb{R}\to\mathbb{R}$ and $T(x)=x^2+5x+4$, then $F(T)=\{-2\}$,

3. If $T:\mathbb{R}\to\mathbb{R}$ and $T(x)=x^2$, then $F(T)=\{0,1\}$,

4. If $T:\mathbb{R}\to\mathbb{R}$ and $T(x)=x+5$, then $F(T)=\varnothing$,

5. If $T:\mathbb{R}\to\mathbb{R}$ and $T(x)=x$, then $F(T)=\mathbb{R}$.

Let F be a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $F:C\times C\to\mathbb{R}$ is to find $x\in C$ such that

$$F(x,y)\geq 0,\ \forall y\in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Equilibrium problems were introduced by Blum and Oettli [1] in 1994 and were included many well-known problems such as variational inequality problem, nonspreading mapping, nonexpansive mapping and fixed point problem; see for example [2, 3, 4].

Let A be an operator of C into H. Then, A is called *α-inverse strongly monotone* if there exists $\alpha>0$ such that

$$\langle x-y, Ax-Ay\rangle\geq\alpha\|Ax-Ay\|^2,$$

for all $x, y\in C$.

Let $B:C\to H$. The *variational inequality* is to find a point $u\in C$ such that

$$\langle Bu, v-u\rangle\geq 0, \tag{1.2}$$

for all $v\in C$. The set of solutions of (1.2) is denoted by $VI(C,B)$.

A mapping $T:C\to C$ is called *nonexpansive* if

$$\|Tx-Ty\|\leq\|x-y\|,$$

for all $x, y\in C$.

Recall that the mapping $T:C\to C$ is said to be *quasi-nonexpansive* if $F(T)\neq\varnothing$ and

$$\|Tx-p\|\leq\|x-p\|,$$

for all $x\in C$ and $p\in F(T)$.

Let $A, B: C \to H$ be two mappings. In 2008, Ceng et al. [5] introduced a problem for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{aligned} &\langle \rho_1 Az^* + x^* - z^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ &\langle \rho_2 Bx^* + z^* - x^*, x - z^* \rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{1.3}$$

which is called a system of variational inequalities where $\rho_1, \rho_2 > 0$.

In 2013, Kangtunyakarn [6] modified (1.3) for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{aligned} &\langle x^* - (I - \rho_1 A)(ax^* + (1-a)z^*), x - x^* \rangle \geq 0, \ \forall x \in C, \\ &\langle z^* - (I - \rho_2 B)x^*, x - z^* \rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{1.4}$$

which is called a modification of system of variational inequalities, for every $\rho_1, \rho_2 > 0$ and $a \in [0,1]$. If $a = 0$, (1.4) reduces to (1.3). Moreover, he introduced the relation between solutions of (1.4) and fixed point of the mapping G as follows:

Lemma 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B: C \to H$ be mappings. For every $\rho_1, \rho_2 > 0$ and $a \in [0,1]$, the following statements are equivalent:

1. $(x^*, z^*) \in C \times C$ is a solution of problem (1.4),
2. x^* is a fixed point of the mapping $G: C \to C$, i.e., $x^* \in F(G)$, defined by

$$G(x) = P_C(I - \rho_1 A)(ax + (1-a)P_C(I - \rho_2 B)x),$$

where $z^* = P_C(I - \rho_2 B)x^*$.

In 2012, Tian and Jin [7] proved the following strong convergence theorem of iterative scheme $\{x_n\}$ generated by (1.5) and let A be a bounded linear operator on H, and let T be a quasi-nonexpansive mapping on H, and f is a *contraction* with coefficient α; that is $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in H$. Assume the set $F(T)$ of fixed points of T is nonempty and they note that $F(T)$ is closed and convex. They assume that A is *strongly positive*; that is, there exist a constant $\bar{\gamma} > 0$ such that $\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2$, for all $x \in H$. Let $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$.

Theorem 1.2. Starting with an arbitary chosen $x_1 \in H$, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_\omega x_n, \tag{1.5}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Also $\omega \in \left(0, \frac{1}{2}\right)$, $T_\omega := (1-\omega) I + \omega T$ with two conditions on T:

1. $\|Tx - q\| \le \|x - q\|$ for any $x \in H$, and $q \in F(T)$; this means that T is a quasi-nonexpansive mapping,
2. T is demiclosed on H; that is: if $\{y_k\} \subset H$, $y_k \rightharpoonup z$, and $(I-T)y_k \to 0$ then $z \in F(T)$.

Then $\{x_n\}$ converges strongly to the $x^* \in F(T)$ which is the unique solution of the VIP: $\langle (\gamma f - A) x^*, x - x^* \rangle \le 0,\ \forall x \in F(T)$.

After study Theorem 1.1 and the research papers in the same direction, for instance [8, 9]. Many strong convergence theorems of quasi-nonexpansive mapping T were proved by assuming the following conditions:

(1a) $T_\omega := (1-\omega) I + \omega T$ for all $\omega \in \left(0, \frac{1}{2}\right)$,

(2a) T is demiclosed on H.

In 2012, Dong, Guo and Su [10] proved strong convergence theorem by using relaxed extragradient method as follows:

Theorem 1.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let F be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap F(G) \ne \varnothing$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1/2)$. Set $\beta_0 = 1$. For given $x_1 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be generated by

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geqslant 0, \quad \forall y\in C, \\ z_n = P_C(u_n-\rho_2 Bu_n), \\ y_n = \alpha_n f(x_n)+(1-\alpha_n)P_C(z_n-\rho_1 Az_n), \\ x_{n+1} = \beta_n x_n + \sigma_n \sum_{i=1}^{\infty}(\beta_{i-1}-\beta_i)T_i y_n + (1-\beta_n)(1-\sigma_n)P_C(z_n-\rho_1 Az_n), \ \forall n\in\mathbb{N}, \end{cases} \tag{1.6}$$

where $\rho_1 \in (0,2\alpha)$, $\rho_2 \in (0,2\beta)$, and sequence $\{\alpha_n\}\subset[0,1]$, $\{\beta_n\}\subset[0,1]$, $\{\sigma_n\}\subset[0,1]$, and $\{r_n\}\subset(r,\infty)$, $r>0$, are such that

(i) $\{\beta_n\}$ is strictly decreasing,

(ii) $0<\liminf_{n\to\infty}\beta_n<\limsup_{n\to\infty}\beta_n<1$,

(iii) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(iv) $\sigma_n>1/2(1-p)$, $\sum_{n=1}^{\infty}|\sigma_{n+1}-\sigma_n|<\infty$,

(v) $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $x^*=P_\Omega\cdot f(x^*)$, and (x^*,y^*) is a solution of the general system of variational inequalities (1.3) where $y^*=P_C(x^*-\rho_2 Bx^*)$; see for example [11, 12].

After we investigated the Theorem 1.2, Theorem 1.3, and related research papers, we have 2 questions as follows:

1) Can we prove strong convergence theorem of a quasi-nonexpansive mapping T without the conditions (1a) and (2a) in a framework of Hibert space?

2) Can we prove strong convergence theorem without relaxed extragradient method?

In this thesis, we give the answers for the above questions and prove two strong convergence theorems:

1) Finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities.

2) Finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities.

Moreover, we utilize our main results to obtain fixed point theorem involving a finite family of nonspreading mappings. In the last section, we give numerical examples to guarantee our main theorems.

1.2 Objectives

1.2.1 To propose new theorems for solving strong convergence theorems of variational inequality problem and equilibrium problem.

1.2.2 To propose some new knowledge about fixed point theorem used in quasi-nonexpansive mapping.

1.2.3 To apply our results for characterize fixed points of a finite family of nonspreading mappings.

1.2.4 To give numerical examples for supporting our main theorems.

1.3 Scope of the study

1.3.1 Give the basic knowledge of the strong convergence theorems of quasi-nonexpansive mappings, nonspreading mappings, variational inequality problems, and equilibrium problems in a framework of Hilbert space.

1.3.2 Prove strong convergence theorems for fixed point of quasi-nonexpansive mapping in a framework of Hilbert space.

1.3.3 Give numerical examples for supporting our main theorems.

1.4 Method

1.4.1 Collect and study research papers, text books, and important articles concerned with many problems such as variational inequality problems and equilibrium problems.

1.4.2 Study the property of quasi-nonexpansive mapping to use in main theorem.

1.4.3 Study definitions, theorems, and lemmas involving the problems in this thesis.

1.4.4 Create and prove new strong convergence theorems of quasi-nonexpansive mapping.

1.4.5 Apply the theorem in 1.4.3 to nonspreading mapping.

1.4.6 Give numerical examples to support our main theorems.

1.4.7 Write the thesis.

1.5 Utilization of the study

1.5.1 To obtain the new iterative scheme for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities.

1.5.2 To obtain the new iterative scheme for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities.

1.5.3 To obtain mathematical tools for quasi-nonexpansive mapping.

In chapter 1, we introduce the summary of the study.

In chapter 2, we review the background of this thesis for fixed point problem, the system of variational inequalities problem and equilibrium problem and give preliminaries to prove our main theorem in later chapters.

In chapter 3, we prove the following strong convergence theorems:

1) Strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities.

2) Strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities.

In chapter 4, we apply our main results for the following results:

1) Strong convergence theorems for a finite family of nonspreading mappings.

2) Numerical examples to support our main results.

In chapter 5, we give the conclusion of this thesis.

Chapter 2

Preliminaries

The purpose of this chapter is to explain fundamental concepts and definitions used throughout this thesis. Moreover, we give some lemmas, remarks and useful results used in the later chapters.

2.1 Linear spaces

Definition 2.1.1. [13] Let E be a nonempty set, and assume that each pair of elements x and y in E can be combined by a process called *addition* to yield an element z in E denoted by $x+y$. Assume also that this operation of addition satisfies the following conditions (v1) ~ (v4)

(v1) $(x+y)+z=x+(y+z)$,

(v2) $x+y=y+x$,

(v3) there exists a unique element in E, denoted by 0 and called the *zero element*, or the *origin*, such that $x+0=x$ for all $x\in E$,

(v4) to each $x\in E$ there corresponds a unique element in E, denoted by $-x$ and called the negative of x, such that $x+(-x)=0$.

We also assume that each scalar $\alpha\in\mathbb{R}$ and each element x in E can be combined by a process called *scalar multiplication* to yield an element y in E denoted by $y=\alpha x$ satisfying (v5) ~ (v8):

(v5) $\alpha(\beta x)=(\alpha\beta)x$,

(v6) $1\cdot x=x$,

(v7) $(\alpha+\beta)x=\alpha x+\beta x$,

(v8) $\alpha(x+y)=\alpha x+\alpha y$.

The algebraic system E defined by these operations and axioms is called a linear space. A linear space is often called a *vector space*, and its elements are spoken of as vectors.

Remark 2.1.2. [13] Since we admit the real numbers as scalars, a linear space is also called a real linear space.

Definition 2.1.3. [14] A set E in a vector space is called *convex* if for any $x,\ y \in E$ and $\alpha \in (0,1)$, we have $\alpha x + (1-\alpha) y \in E$.

2.2 Hilbert space

Definition 2.2.1. [15] An inner product on a vector space K over $\mathbb{F}$ is a function that assigns a scalar $\langle x, y \rangle$ for every $x, y \in K$, such that for all $x,\ y,\ z \in K$ and $\alpha \in \mathbb{F}$:

(I1) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$,

(I2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,

(I3) $\overline{\langle x, y \rangle} = \langle y, x \rangle$,

(I4) $\langle x, x \rangle > 0 \Leftrightarrow x \neq 0$,

A vector space K over $\mathbb{F}$ with a specific inner product is called an inner product space. If $\mathbb{F} = \mathbb{C}$, K is a complex inner product space, and if $\mathbb{F} = \mathbb{R}$, K is a real inner product space.

Theorem 2.2.2. [15] For an inner product space K, $x,\ y,\ z \in K$, and $\alpha \in \mathbb{F}$:

(J1) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$,

(J2) $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$,

(J3) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$,

(J4) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$,

(J5) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in K$, then $y = z$.

Remark 2.2.3. [13] An inner product space is called a real inner product space for the case when the scalars are the real numbers and $\langle x, y \rangle$ is a real number. For the case, (I3) means

$$\langle x, y \rangle = \langle y, x \rangle.$$

Remark 2.2.4. [13] Using (J1) and (J2), we obtain that for $x,\ y \in K$ and $\alpha,\ \beta \in \mathbb{C}$,

$$\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle.$$

Remark 2.2.5. [13] Let K be an inner product space. For each x in K, we define its *norm* $\|x\|$ by

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}.$$

Theorem 2.2.6. (The Schwarz inequality) [13] Let K be an inner product space and let x and y be elements in K. Then the following holds:

$$|\langle x, y \rangle| \le \|x\| \|y\|.$$

Theorem 2.2.7. [13] The inner product in an inner product space K is jointly continuous:

$$x_n \to x \text{ and } y_n \to y \Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle.$$

Remark 2.2.8. [13] We of course obtain from Theorem 2.2.7 that if $x_n \to x$, then for a fixed $y \in K$,

$$\langle x_n, y \rangle \to \langle x, y \rangle \text{ and } \langle y, x_n \rangle \to \langle y, x \rangle.$$

Definition 2.2.9. (Strong Convergence) [14] A sequence $\{x_n\}$ of vectors in an inner product space K is called strongly convergent to a vector x in K if $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 2.2.10. (Weak Convergence) [14] A sequence $\{x_n\}$ of vectors in an inner product space K is called weakly convergent to a vector x in K if $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$, for every $y \in K$.

Remark 2.2.11. We represent weak and strong convergence by "$\rightharpoonup$" and "$\to$", respectively.

Theorem 2.2.12. [14] A strongly convergent sequence is weakly convergent (to the same limit). i.e., $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.2.13. [13] If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.2.14. [13] Let $\{x_n\}$ be a Cauchy sequence of an inner product space K such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Definition 2.2.15. [13] A complete inner product space is called a *Hilbert space.*

Remark 2.2.16. [13] Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

(1) H is complete,

(2) each bounded sequence $\{x_n\}$ of H has a weakly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Theorem 2.2.17. (The nearest point theorem) [13] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $x \in H$. Then there exists a unique element y_0 in C such that

$$d(x,C) = d(x,y_0),$$

where $d(x,C) = \inf_{y \in C} \|x-y\|$.

Definition 2.2.18. [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then for each point x in H, there corresponds a unique point x_0 in C such that

$$\|x-x_0\| = d(x,C).$$

We call such a mapping defined by $Px = x_0$, or $P_C x = x_0$, the metric projection of H onto C.

Lemma 2.2.19. [13] Let C be a nonempty convex subset of a Hilbert space H. Then for $x \in H$ and $y \in C$, $\|x-y\| = d(x,C)$ if and only if

$$\langle x-y, y-z \rangle \geq 0,$$

for all $z \in C$.

Theorem 2.2.20. (Properties of metric projection) [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then the metric projection P of H onto C has the following properties

$$\|P_C x - P_C y\| \leq \|x-y\|, \text{ for all } x, y \in H.$$

Theorem 2.2.21. [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Theorem 2.2.22. (Opial's theorem) [13] Let H be a Hilbert space and suppose $x_n \rightharpoonup x$. Then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for any $y \in H$ with $x \neq y$.

Definition 2.2.23. [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty,\infty]$, where $(-\infty,\infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called lower semicontinuous if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\},$$

is closed. f is also called convex if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.2.24. [13] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty,\infty]$. Let $\{x_n\}$ be a bounded sequence of C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Theorem 2.2.25. [14] Weakly convergent sequences in a Hilbert space H are bounded, i.e., if $\{x_n\}$ is a weakly convergent sequence, then there exists a number M such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Example 2.2.26. [16] $\mathbb{R}^n$, l_2 and $L_2(a,b)$ are Hilbert spaces.

Example 2.2.27. [16] $C[a,b]$ and $P[a,b]$ are inner product spaces but not Hilbert spaces.

2.3 Fixed point theory

We study about existence and properties of fixed points are known as fixed point theorem.

2.3.1 Fixed point of nonexpansive mapping and quasi-nonexpansive mapping

Theorem 2.3.1. [13] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.3.2. [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Theorem 2.3.3. [17] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a quasi-nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

The concept of quasi-nonexpansiveness was essentially introduced (along with some related ideas) by Diaz and Metcalf [18]. One notes that a quasi-nonexpansive mapping is nonexpansive but a nonexpansive mapping $T:C\to C$ with at least one fixed point in C is quasi-nonexpansive.

Strong convergence theorems for nonexpansive mappings. The first theorem was proved by Browder [13] as follows:

Theorem 2.3.4. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C in to itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$ and let $\{\alpha_n\}\subset(0,1)$.

(i) Fix $x_0\in C$. Then for any $n\in\mathbb{N}$, a mapping T_n of C into itself defined by

$$T_n x=(1-\alpha_n)Tx+\alpha_n x_0,$$

for all $x\in C$ has a unique fixed point u_n of T_n in C,

(ii) if $\alpha_n\to 0$, then $\{u_n\}$ converges strongly to $P_C x_0\in F(T)$.

2.3.2 Fixed point of α-inverse strongly monotone mapping and variational inequality problem

Theorem 2.3.5. [13] Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\alpha > 0$ and let $A: C \to H$ be α-inverse strongly monotone. Then $VI(C,A) \neq \emptyset$.

Lemma 2.3.6. [19] Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\rho > 0$,

$$u = P_C(I - \rho A)u \Leftrightarrow u \in VI(C,A),$$

where P_C is the metric projection of H onto C.

Lemma 2.3.7. [6] Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B: C \to H$ be α, β-inverse strongly monotone mappings, respectively, which $VI(C,A) \cap VI(C,B) \neq \emptyset$. Define a mapping $G: C \to C$ by

$$G(x) = P_C(I - \rho_1 A)(ax + (1-a)P_C(I - \rho_2 B)x),$$

for every $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and $a \in (0,1)$. Then $F(G) = VI(C,A) \cap VI(C,B)$.

2.3.3 Fixed point of nonspreading mapping

A mapping $T: C \to C$ is called *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall x, y \in C. \tag{2.1}$$

The such mapping is defined by Kohsaka and Takahashi [20]. In 2009, Iemoto and Takahashi [21] proved that (2.1) is equivalent to (2.2).

Lemma 2.3.8. Let C be a nonempty closed convex subset of a real Hilbert space H. Then the mapping $T: C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{2.2}$$

for all $x, y \in C$.

Lemma 2.3.9. [20] Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let T be a nonspreading mapping of C into itself. Then $F(T)$ is closed and convex.

Remark 2.3.10. A nonspreading mapping T with $F(T) \neq \varnothing$ is quasi-nonexpansive mapping. But the converse is not true.

2.4 Some useful Lemmas and Theorems

Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\|\cdot\|$. In this thesis, we represent weak and strong convergence by "$\rightharpoonup$" and "$\rightarrow$", respectively.

Remark 2.4.1. It is well-known that metric projection P_C has the following properties:

1. P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

2. For each $x \in H$,

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4.2. Let H be a real Hilbert space. Then there holds the following well-known results:

1. $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2,$
2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$

for all $x, y \in H$.

Lemma 2.4.3. [22] Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

For solving the equilibrium problem, we assume that the bifunction $F: C \times C \to \mathbb{R}$ satisfy the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$,

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$,

(A3) For each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x,y),$$

(A4) For each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.4.4. [1] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4.5. [23] Assume that $F: C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\},$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued,

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle,$$

(3) $F(T_r) = EP(F)$,

(4) $EP(F)$ is closed and convex.

Lemma 2.4.6. [24] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1-\alpha_n)s_n + \delta_n,\ \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

The next result is an important tool to prove our main result.

Lemma 2.4.7. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Then $VI(C, I-T) = F(T)$.

Proof. It is easy to see that $F(T) \subseteq VI(C, I-T)$.

Let $u \in VI(C, I-T)$, then we have

$$\langle v-u, (I-T)u\rangle \ge 0, \quad \forall v \in C. \tag{2.3}$$

Let $v^* \in F(T)$, we have

$$\begin{aligned} \|Tu - v^*\|^2 &= \|(u-v^*) - (I-T)u\|^2, \\ &= \|u-v^*\|^2 - 2\langle u-v^*, (I-T)u\rangle + \|(I-T)u\|^2, \\ &\le \|u-v^*\|^2. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), it follows that

$$\|(I-T)u\|^2 \le 2\langle u-v^*, (I-T)u\rangle \le 0.$$

It implies that $u \in F(T)$. Hence $VI(C, I-T) \subseteq F(T)$. □

Remark 2.4.8. From Lemma 2.3.6 and Lemma 2.4.7, we have

$$F(T) = VI(C, I-T) = F\big(P_C\left(I - \lambda(I-T)\right)\big),$$

for all $\lambda > 0$.

Chapter 3

Convergence theorems in Hilbert space

3.1 Strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities

In this section, we shall prove the strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T:C\to C$ be a quasi-nonexpansive mapping. Let $A, B:C\to H$ be α, β -inverse strongly monotone mappings, respectively. Define the mapping $G:C\to C$ by
$Gx = P_C(I-\rho_1 A)(ax+(1-a)P_C(I-\rho_2 B)x)$, for all $x\in C$ and $a\in(0,1)$. Assume that $\mathcal{F} = VI(C,A)\cap VI(C,B)\cap F(T)\neq\varnothing$. Suppose that $x_1, u\in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C(I-\lambda_n(I-T))x_n + \gamma_n G(x_n), \quad \forall n\in\mathbb{N}, \tag{3.1.1}$$

where $\rho_1\in(0,2\alpha)$, $\rho_2\in(0,2\beta)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n+\beta_n+\gamma_n = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $0<c\le\beta_n, \gamma_n\le d<1$ for all $n\in\mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n\le 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$.

Proof. We divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

Let $x, y \in C$. Since A is α-inverse strongly monotone and $\rho_1 \in (0,2\alpha)$, we have

$$\begin{aligned}\|(I-\rho_1 A)x-(I-\rho_1 A)y\|^2 &= \|x-y\|^2 - 2\rho_1\langle x-y, Ax-Ay\rangle + \rho_1^2\|Ax-Ay\|^2,\\ &\leq \|x-y\|^2 - 2\alpha\rho_1\|Ax-Ay\|^2 + \rho_1^2\|Ax-Ay\|^2,\\ &= \|x-y\|^2 + \rho_1(\rho_1-2\alpha)\|Ax-Ay\|^2,\\ &\leq \|x-y\|^2.\end{aligned}$$

Therefore $(I-\rho_1 A)$ is a nonexpansive mapping. Similarly, $(I-\rho_2 B)$ is a nonexpansive mapping. Hence $P_C(I-\rho_1 A)$ and $P_C(I-\rho_2 B)$ are nonexpansive mappings. From the definition of the mapping G, we have G is a nonexpansive mapping.

Let $x^* \in \mathcal{F}$. From Remark 2.4.8, we have

$$x^* \in F\left(P_C\left(I-\lambda_n(I-T)\right)\right). \tag{3.1.2}$$

By Lemma 2.3.7, we have

$$x^* = G(x^*) = P_C(I-\rho_1 A)\left(ax^* + (1-a)P_C(I-\rho_2 B)x^*\right). \tag{3.1.3}$$

Observe that

$$\begin{aligned}\|Tx_n - Tx^*\|^2 &= \|(x_n-x^*)-(I-T)x_n\|^2,\\ &= \|x_n-x^*\|^2 - 2\langle x_n-x^*,(I-T)x_n\rangle + \|(I-T)x_n\|^2.\end{aligned}$$

Since T is a quasi-nonexpansive mapping, we have

$$\|(I-T)x_n\|^2 \leq 2\langle x_n-x^*,(I-T)x_n\rangle. \tag{3.1.4}$$

From the nonexpansiveness of P_C and (3.1.4), we have

$$\begin{aligned}\|P_C(I-\lambda_n(I-T))x_n - x^*\|^2 &= \|P_C(I-\lambda_n(I-T))x_n - P_C(I-\lambda_n(I-T))x^*\|^2,\\ &\leq \|(I-\lambda_n(I-T))x_n - (I-\lambda_n(I-T))x^*\|^2,\\ &= \|(x_n-x^*)-\lambda_n((I-T)x_n-(I-T)x^*)\|^2,\\ &= \|x_n-x^*\|^2 - 2\lambda_n\langle x_n-x^*,(I-T)x_n\rangle + \lambda_n^2\|(I-T)x_n\|^2,\\ &\leq \|x_n-x^*\|^2 - \lambda_n\|(I-T)x_n\|^2 + \lambda_n^2\|(I-T)x_n\|^2,\\ &\leq \|x_n-x^*\|^2.\end{aligned} \tag{3.1.5}$$

Put $M_n = ax_n + (1-a)P_C(I-\rho_2 B)x_n$ and $W_n = P_C(I-\rho_1 A)M_n$. From (3.1.1) and the definition of the mapping G, we have

$$x_{n+1} = \alpha_n u + \beta_n P_C(I-\lambda_n(I-T))x_n + \gamma_n W_n.$$

From the definition of x_n, (3.1.5) and nonexpansiveness of G, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\| \\
&= \|\alpha_n(u-x^*) + \beta_n(P_C(I-\lambda_n(I-T))x_n - x^*) + \gamma_n(W_n - x^*)\|, \\
&\le \alpha_n\|u-x^*\| + \beta_n\|P_C(I-\lambda_n(I-T))x_n - x^*\| + \gamma_n\|W_n - x^*\|, \\
&\le \alpha_n\|u-x^*\| + \beta_n\|x_n - x^*\| \\
&\quad + \gamma_n\|P_C(I-\rho_1 A)(ax_n + (1-a)P_C(I-\rho_2 B)x_n) - P_C(I-\rho_1 A)(ax^* + (1-a)P_C(I-\rho_2 B)x^*)\|, \\
&= \alpha_n\|u-x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|G(x_n) - G(x^*)\|, \\
&\le \alpha_n\|u-x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\|, \\
&= \alpha_n\|u-x^*\| + (1-\alpha_n)\|x_n - x^*\|.
\end{aligned}$$

We will prove by induction that,

$$\|x_n - x^*\| \le \max\{\|u-x^*\|, \|x_1 - x^*\|\}, \text{ for all } n \in \mathbb{N}. \tag{3.1.6}$$

Let $M = \max\{\|u-x^*\|, \|x_1 - x^*\|\}$. When $n=1$, it is easy to see that (3.1.6) is true. Suppose (3.1.6) is true for $n=k$. Then

$$\begin{aligned}
\|x_{k+1}-x^*\| &\le \alpha_k\|u-x^*\| + (1-\alpha_k)\|x_k - x^*\|, \\
&\le \alpha_k M + (1-\alpha_k)M, \\
&= M.
\end{aligned}$$

Thus, (3.1.6) holds for $n=k+1$. Therefore, we can conclude that

$$\|x_n - x^*\| \le \max\{\|u-x^*\|, \|x_1 - x^*\|\},$$

for all $n \in \mathbb{N}$. This implies that the sequence $\{x_n\}$ is bounded and so is $\{(I-T)x_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

From the definition of x_n and nonexpansiveness of G, we have

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \|(\alpha_n - \alpha_{n-1})u + (\beta_n - \beta_{n-1})P_C(I-\lambda_{n-1}(I-T))x_{n-1} \\
&\quad + \beta_n(P_C(I-\lambda_n(I-T))x_n - P_C(I-\lambda_{n-1}(I-T))x_{n-1}) + \gamma_n(W_n - W_{n-1}) + (\gamma_n - \gamma_{n-1})W_{n-1}\|,
\end{aligned}$$

$$\leq|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}(I-T)\right)x_{n-1}\right\|$$
$$+\beta_n\left\|P_C\left(I-\lambda_n(I-T)\right)x_n-P_C\left(I-\lambda_{n-1}(I-T)\right)x_{n-1}\right\|+\gamma_n\|W_n-W_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|,$$
$$\leq|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}(I-T)\right)x_{n-1}\right\|$$
$$+\beta_n\left\|(x_n-x_{n-1})-\lambda_n(I-T)x_n+\lambda_{n-1}(I-T)x_{n-1}\right\|$$
$$+\gamma_n\left\|P_C(I-\rho_1A)\left(ax_n+(1-a)P_C(I-\rho_2B)x_n\right)\right.$$
$$\left.-P_C(I-\rho_1A)\left(ax_{n-1}+(1-a)P_C(I-\rho_2B)x_{n-1}\right)\right\|+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|,$$
$$=|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}(I-T)\right)x_{n-1}\right\|$$
$$+\beta_n\left\|(x_n-x_{n-1})-\lambda_n\left((I-T)x_n-(I-T)x_{n-1}\right)-(\lambda_n-\lambda_{n-1})(I-T)x_{n-1}\right\|$$
$$+\gamma_n\|G(x_n)-G(x_{n-1})\|+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|,$$
$$\leq|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}(I-T)\right)x_{n-1}\right\|+\beta_n\|x_n-x_{n-1}\|$$
$$+\lambda_n\|(I-T)x_n-(I-T)x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|(I-T)x_{n-1}\|+\gamma_n\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|,$$
$$=(1-\alpha_n)\|x_n-x_{n-1}\|+\lambda_n\|(I-T)x_n-(I-T)x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u\|$$
$$+|\beta_n-\beta_{n-1}|\left\|P_C\left(I-\lambda_{n-1}(I-T)\right)x_{n-1}\right\|+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|(I-T)x_{n-1}\|,$$
$$\leq(1-\alpha_n)\|x_n-x_{n-1}\|+\lambda_nM+|\alpha_n-\alpha_{n-1}|M+|\beta_n-\beta_{n-1}|M+|\gamma_n-\gamma_{n-1}|M+|\lambda_n-\lambda_{n-1}|M,$$

where $M:=\max_{n\in\mathbf{N}}\left\{\|(I-T)x_{n+1}-(I-T)x_n\|,\ \|u\|,\ \left\|P_C\left(I-\lambda_n(I-T)\right)x_n\right\|,\ \|W_n\|,\ \|(I-T)x_n\|\right\}.$

From the condition (ii), (iv), (v) and Lemma 2.4.6, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.1.7}$$

Step 3. We show that $\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n(I-T)\right)x_n-x_n\right\|=0.$

Put $M^*=ax^*+(1-a)P_C(I-\rho_2B)x^*$ in (3.1.3), we have $x^*=P_C(I-\rho_1A)M^*.$

Since $x^*\in VI(C,B),$ we obtain

$$\begin{aligned}M^*-x^*&=(1-a)\left(P_C(I-\rho_2B)x^*-x^*\right),\\&=(1-a)\left(P_C(I-\rho_2B)x^*-P_C(I-\rho_2B)x^*\right),\\&=0.\end{aligned} \tag{3.1.8}$$

From the definition of M_n and M^*, we have

$$\begin{aligned}\|M_n-M^*\|&=\left\|a(x_n-x^*)+(1-a)\left(P_C(I-\rho_2B)x_n-P_C(I-\rho_2B)x^*\right)\right\|,\\&\leq a\|x_n-x^*\|+(1-a)\|P_C(I-\rho_2B)x_n-P_C(I-\rho_2B)x^*\|,\\&\leq\|x_n-x^*\|,\end{aligned} \tag{3.1.9}$$

From the definition of W_n and (3.1.8), we have

$$\|W_n - x^*\|^2$$

$$= \|P_C(I-\rho_1 A)M_n - P_C(I-\rho_1 A)M^*\|^2,$$

$$\leq \langle (I-\rho_1 A)M_n - (I-\rho_1 A)M^*, W_n - x^* \rangle,$$

$$= \frac{1}{2}\left(\|(I-\rho_1 A)M_n - (I-\rho_1 A)M^*\|^2 + \|W_n - x^*\|^2 - \|(I-\rho_1 A)M_n - (I-\rho_1 A)M^* - W_n + x^*\|^2\right),$$

$$\leq \frac{1}{2}\left(\|M_n - M^*\|^2 + \|W_n - x^*\|^2 - \|(M_n - W_n) - \rho_1(AM_n - AM^*)\|^2\right),$$

which implies that

$$\|W_n - x^*\|^2$$

$$\leq \|M_n - M^*\|^2 - \|(M_n - W_n) - \rho_1(AM_n - AM^*)\|^2,$$

$$= \|M_n - M^*\|^2 - \|M_n - W_n\|^2 + 2\rho_1\langle M_n - W_n, AM_n - AM^*\rangle - \rho_1^2\|AM_n - AM^*\|^2. \qquad (3.1.10)$$

From the definition of W_n, we have

$$\|W_n - x^*\|^2 = \|P_C(I-\rho_1 A)M_n - P_C(I-\rho_1 A)M^*\|^2,$$

$$\leq \|(I-\rho_1 A)M_n - (I-\rho_1 A)M^*\|^2,$$

$$= \|(M_n - M^*) - \rho_1(AM_n - AM^*)\|^2,$$

$$= \|M_n - M^*\|^2 - 2\rho_1\langle M_n - M^*, AM_n - AM^*\rangle + \rho_1^2\|AM_n - AM^*\|^2,$$

$$\leq \|M_n - M^*\|^2 - 2\rho_1\alpha\|AM_n - AM^*\|^2 + \rho_1^2\|AM_n - AM^*\|^2,$$

$$= \|M_n - M^*\|^2 - \rho_1(2\alpha - \rho_1)\|AM_n - AM^*\|^2. \qquad (3.1.11)$$

From the definition of x_n, (3.1.5), (3.1.9) and (3.1.11), we have

$$\|x_{n+1} - x^*\|^2 \leq \alpha_n\|u - x^*\|^2 + \beta_n\|P_C(I - \lambda_n(I-T))x_n - x^*\|^2 + \gamma_n\|W_n - x^*\|^2,$$

$$\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\left(\|M_n - M^*\|^2 - \rho_1(2\alpha - \rho_1)\|AM_n - AM^*\|^2\right),$$

$$\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 - \gamma_n\rho_1(2\alpha - \rho_1)\|AM_n - AM^*\|^2,$$

$$= \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \gamma_n\rho_1(2\alpha - \rho_1)\|AM_n - AM^*\|^2.$$

It implies that

$$\gamma_n\rho_1(2\alpha - \rho_1)\|AM_n - AM^*\|^2 \leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

$$\leq \alpha_n\|u - x^*\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right).$$

From the condition (ii), (iii) and (3.1.7), we derive

$$\lim_{n\to\infty}\left\|AM_n - AM^*\right\| = 0. \tag{3.1.12}$$

From the definitions of x_n and the mapping G, (3.1.5), (3.1.9) and (3.1.10), we have

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|P_C\left(I - \lambda_n\left(I - T\right)\right)x_n - x^*\right\|^2 + \gamma_n\left\|W_n - x^*\right\|^2, \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 \\
&\quad + \gamma_n\left(\left\|M_n - M^*\right\|^2 - \left\|M_n - W_n\right\|^2 + 2\rho_1\left\langle M_n - W_n, AM_n - AM^*\right\rangle - \rho_1^2\left\|AM_n - AM^*\right\|^2\right), \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \gamma_n\left\|x_n - x^*\right\|^2 - \gamma_n\left\|M_n - W_n\right\|^2 + 2\rho_1\left\|M_n - W_n\right\|\left\|AM_n - AM^*\right\|, \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \gamma_n\left\|M_n - W_n\right\|^2 + 2\rho_1\left\|M_n - W_n\right\|\left\|AM_n - AM^*\right\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\gamma_n\left\|M_n - W_n\right\|^2 &\le \alpha_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2\rho_1\left\|M_n - W_n\right\|\left\|AM_n - AM^*\right\|, \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \left\|x_n - x_{n+1}\right\|\left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) \\
&\quad + 2\rho_1\left\|M_n - W_n\right\|\left\|AM_n - AM^*\right\|.
\end{aligned}$$

From the condition (ii), (iii), (3.1.7) and (3.1.12), we derive

$$\lim_{n\to\infty}\left\|M_n - W_n\right\| = 0. \tag{3.1.13}$$

From the properties of P_C, we have

$$\begin{aligned}
&\left\|P_C\left(I - \rho_2 B\right)x_n - x^*\right\|^2 \\
&= \left\|P_C\left(I - \rho_2 B\right)x_n - P_C\left(I - \rho_2 B\right)x^*\right\|^2, \\
&\le \left\langle \left(I - \rho_2 B\right)x_n - \left(I - \rho_2 B\right)x^*, P_C\left(I - \rho_2 B\right)x_n - x^*\right\rangle, \\
&= \frac{1}{2}\left(\left\|\left(I - \rho_2 B\right)x_n - \left(I - \rho_2 B\right)x^*\right\|^2 + \left\|P_C\left(I - \rho_2 B\right)x_n - x^*\right\|^2\right. \\
&\quad \left. - \left\|\left(I - \rho_2 B\right)x_n - \left(I - \rho_2 B\right)x^* - P_C\left(I - \rho_2 B\right)x_n + x^*\right\|^2\right), \\
&\le \frac{1}{2}\left(\left\|x_n - x^*\right\|^2 + \left\|P_C\left(I - \rho_2 B\right)x_n - x^*\right\|^2 - \left\|\left(x_n - P_C\left(I - \rho_2 B\right)x_n\right) - \rho_2\left(Bx_n - Bx^*\right)\right\|^2\right).
\end{aligned}$$

This implies that

$$\left\|P_C\left(I-\rho_2 B\right)x_n-x^*\right\|^2 \le \left\|x_n-x^*\right\|^2-\left\|\left(x_n-P_C\left(I-\rho_2 B\right)x_n\right)-\rho_2\left(Bx_n-Bx^*\right)\right\|^2,$$

$$=\left\|x_n-x^*\right\|^2-\left\|x_n-P_C\left(I-\rho_2 B\right)x_n\right\|^2$$

$$+2\rho_2\left\langle x_n-P_C\left(I-\rho_2 B\right)x_n, Bx_n-Bx^*\right\rangle-\rho_2^2\left\|Bx_n-Bx^*\right\|^2. \quad (3.1.14)$$

By using the same method as (3.1.11), we have

$$\left\|P_C\left(I-\rho_2 B\right)x_n-x^*\right\|^2 \le \left\|x_n-x^*\right\|^2-\rho_2\left(2\beta-\rho_2\right)\left\|Bx_n-Bx^*\right\|^2. \quad (3.1.15)$$

Since $x^* \in VI(C,A)$, we have

$$\left\|W_n-x^*\right\|^2 = \left\|P_C\left(I-\rho_1 A\right)M_n-P_C\left(I-\rho_1 A\right)x^*\right\|^2,$$

$$\le \left\| ax_n+(1-a)P_C\left(I-\rho_2 B\right)x_n-x^*\right\|^2,$$

$$=\left\| a\left(x_n-x^*\right)+(1-a)\left(P_C\left(I-\rho_2 B\right)x_n-x^*\right)\right\|^2,$$

$$\le a\left\|x_n-x^*\right\|^2+(1-a)\left\|P_C\left(I-\rho_2 B\right)x_n-x^*\right\|^2. \quad (3.1.16)$$

From the definition of x_n, (3.1.5), (3.1.15) and (3.1.16), we have

$$\left\|x_{n+1}-x^*\right\|^2$$

$$\le \alpha_n\left\|u-x^*\right\|^2+\beta_n\left\|P_C\left(I-\lambda_n\left(I-T\right)\right)x_n-x^*\right\|^2+\gamma_n\left\|W_n-x^*\right\|^2,$$

$$\le \alpha_n\left\|u-x^*\right\|^2+\beta_n\left\|P_C\left(I-\lambda_n\left(I-T\right)\right)x_n-x^*\right\|^2$$

$$+\gamma_n\left(a\left\|x_n-x^*\right\|^2+(1-a)\left\|P_C\left(I-\rho_2 B\right)x_n-x^*\right\|^2\right),$$

$$\le \alpha_n\left\|u-x^*\right\|^2+\beta_n\left\|P_C\left(I-\lambda_n\left(I-T\right)\right)x_n-x^*\right\|^2$$

$$+\gamma_n\left(a\left\|x_n-x^*\right\|^2+(1-a)\left(\left\|x_n-x^*\right\|^2-\rho_2\left(2\beta-\rho_2\right)\left\|Bx_n-Bx^*\right\|^2\right)\right),$$

$$\le \alpha_n\left\|u-x^*\right\|^2+\beta_n\left\|x_n-x^*\right\|^2$$

$$+\gamma_n\left(a\left\|x_n-x^*\right\|^2+(1-a)\left\|x_n-x^*\right\|^2-(1-a)\rho_2\left(2\beta-\rho_2\right)\left\|Bx_n-Bx^*\right\|^2\right),$$

$$=\alpha_n\left\|u-x^*\right\|^2+\left(1-\alpha_n\right)\left\|x_n-x^*\right\|^2-(1-a)\rho_2\gamma_n\left(2\beta-\rho_2\right)\left\|Bx_n-Bx^*\right\|^2.$$

This implies that

$$(1-a)\rho_2\gamma_n\left(2\beta-\rho_2\right)\left\|Bx_n-Bx^*\right\|^2 \le \alpha_n\left\|u-x^*\right\|^2+\left\|x_n-x^*\right\|^2-\left\|x_{n+1}-x^*\right\|^2,$$

$$\le \alpha_n\left\|u-x^*\right\|^2+\left\|x_n-x_{n+1}\right\|\left(\left\|x_n-x^*\right\|+\left\|x_{n+1}-x^*\right\|\right).$$

From the condition (ii), (iii) and (3.1.7), we have

$$\lim_{n\to\infty}\left\|Bx_n - Bx^*\right\| = 0. \tag{3.1.17}$$

From the definitions of x_n and the mapping G, (3.1.5) and (3.1.14), we have

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|P_C\left(I - \lambda_n(I-T)\right)x_n - x^*\right\|^2 \\
&\quad + \gamma_n\left(a\left\|x_n - x^*\right\|^2 + (1-a)\left\|P_C\left(I-\rho_2 B\right)x_n - x^*\right\|^2\right), \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \gamma_n\left(a\left\|x_n - x^*\right\|^2 + (1-a)\left(\left\|x_n - x^*\right\|^2 - \left\|x_n - P_C\left(I-\rho_2 B\right)x_n\right\|^2\right.\right. \\
&\quad \left.\left. + 2\rho_2\left\langle x_n - P_C\left(I-\rho_2 B\right)x_n, Bx_n - Bx^*\right\rangle - \rho_2^2\left\|Bx_n - Bx^*\right\|^2\right)\right), \\
&\le \alpha_n\left\|u - x^*\right\|^2 + (1-\alpha_n)\left\|x_n - x^*\right\|^2 - \gamma_n(1-a)\left\|x_n - P_C\left(I-\rho_2 B\right)x_n\right\|^2 \\
&\quad + 2\rho_2\gamma_n(1-a)\left\langle x_n - P_C\left(I-\rho_2 B\right)x_n, Bx_n - Bx^*\right\rangle, \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \gamma_n(1-a)\left\|x_n - P_C(I-\rho_2 B)x_n\right\|^2 \\
&\quad + 2\rho_2\gamma_n(1-a)\left\|x_n - P_C\left(I-\rho_2 B\right)x_n\right\|\left\|Bx_n - Bx^*\right\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\gamma_n(1-a)\left\|x_n - P_C\left(I-\rho_2 B\right)x_n\right\|^2 \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2\rho_2\gamma_n(1-a)\left\|x_n - P_C\left(I-\rho_2 B\right)x_n\right\|\left\|Bx_n - Bx^*\right\|, \\
&\le \alpha_n\left\|u - x^*\right\|^2 + \left\|x_n - x_{n+1}\right\|\left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) \\
&\quad + 2\rho_2\gamma_n(1-a)\left\|x_n - P_C\left(I-\rho_2 B\right)x_n\right\|\left\|Bx_n - Bx^*\right\|.
\end{aligned}$$

From the condition (ii), (iii), (3.1.7) and (3.1.17), we derive

$$\lim_{n\to\infty}\left\|x_n - P_C\left(I-\rho_2 B\right)x_n\right\| = 0. \tag{3.1.18}$$

Since

$$\begin{aligned}
\left\|M_n - x_n\right\| &= \left\| a x_n + (1-a)P_C\left(I-\rho_2 B\right)x_n - x_n \right\|, \\
&= (1-a)\left\|P_C\left(I-\rho_2 B\right)x_n - x_n\right\|,
\end{aligned}$$

from (3.1.18), we have

$$\lim_{n\to\infty}\left\|M_n - x_n\right\| = 0. \tag{3.1.19}$$

From (3.1.13) and (3.1.19), we have

$$\lim_{n\to\infty}\left\|W_n - x_n\right\| = 0. \tag{3.1.20}$$

Since

$$x_{n+1}-x_n=\alpha_n\left(u-x_n\right)+\beta_n\left(P_C\left(I-\lambda_n\left(I-T\right)\right)x_n-x_n\right)+\gamma_n\left(W_n-x_n\right),$$

from the condition (ii), (iii), (3.1.7) and (3.1.20), we have

$$\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n\left(I-T\right)\right)x_n-x_n\right\|=0. \tag{3.1.21}$$

Step 4. We show that $\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle\le 0$, where $z_0=P_{\mathcal{F}}u$.

To show this inequality, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle=\lim_{j\to\infty}\left\langle u-z_0,x_{n_j}-z_0\right\rangle.$$

Without loss of generality, we may assume that $x_{n_j}\rightharpoonup\omega$ as $j\to\infty$, where $\omega\in C$. First, we show that $\omega\in F(T)$. From Remark 2.4.8, we have $F(T)=F\left(P_C\left(I-\lambda_{n_j}\left(I-T\right)\right)\right)$. Assume that $\omega\notin F(T)$, that is, $\omega\ne P_C\left(I-\lambda_{n_j}\left(I-T\right)\right)\omega$. By $x_{n_j}\rightharpoonup\omega$ as $j\to\infty$, (3.1.21) and Opial's property, we have

$$\begin{aligned}
&\liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\|\\
&\quad<\liminf_{j\to\infty}\left\|x_{n_j}-P_C\left(I-\lambda_{n_j}\left(I-T\right)\right)\omega\right\|,\\
&\quad\le\liminf_{j\to\infty}\left(\left\|x_{n_j}-P_C\left(I-\lambda_{n_j}\left(I-T\right)\right)x_{n_j}\right\|+\left\|P_C\left(I-\lambda_{n_j}\left(I-T\right)\right)x_{n_j}-P_C\left(I-\lambda_{n_j}\left(I-T\right)\right)\omega\right\|\right),\\
&\quad\le\liminf_{j\to\infty}\left(\left\|x_{n_j}-P_C\left(I-\lambda_{n_j}\left(I-T\right)\right)x_{n_j}\right\|+\left\|x_{n_j}-\omega\right\|+\lambda_{n_j}\left\|\left(I-T\right)x_{n_j}-\left(I-T\right)\omega\right\|\right),\\
&\quad=\liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\|.
\end{aligned}$$

It is a contradiction. So, we have

$$\omega\in F(T). \tag{3.1.22}$$

Next, we show that $\omega\in VI(C,A)\cap VI(C,B)$. From Lemma 2.3.7, we have $VI(C,A)\cap VI(C,B)=F(G)$. Assume that $\omega\notin VI(C,A)\cap VI(C,B)$, we have $\omega\ne G(\omega)$. From (3.1.20), we have $W_{n_j}\rightharpoonup\omega$ as $j\to\infty$. From (3.1.20), nonexpansiveness of G and Opial's property, we have

$$\begin{aligned}
&\liminf_{j\to\infty}\left\|W_{n_j}-\omega\right\|\\
&\quad<\liminf_{j\to\infty}\left\|W_{n_j}-G(\omega)\right\|,\\
&\quad\le\liminf_{j\to\infty}\left(\left\|P_C\left(I-\rho_1A\right)\left(ax_{n_j}+(1-a)P_C\left(I-\rho_2B\right)x_{n_j}\right)-G\left(W_{n_j}\right)\right\|+\left\|G\left(W_{n_j}\right)-G(\omega)\right\|\right),
\end{aligned}$$

$$\le \liminf_{j\to\infty}\left(\left\|G(x_{n_j})-G(W_{n_j})\right\|+\left\|W_{n_j}-\omega\right\|\right),$$
$$\le \liminf_{j\to\infty}\left(\left\|x_{n_j}-W_{n_j}\right\|+\left\|W_{n_j}-\omega\right\|\right),$$
$$= \liminf_{j\to\infty}\left\|W_{n_j}-\omega\right\|.$$

It is a contradiction. So, we have

$$\omega \in VI(C,A)\cap VI(C,B). \tag{3.1.23}$$

From (3.1.22) and (3.1.23), we have $\omega\in\mathcal{F}$. Since $\langle u-z_0, x_n-z_0\rangle$ is bounded and $x_{n_j}\rightharpoonup\omega$ as $j\to\infty$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle &= \lim_{j\to\infty}\langle u-z_0,x_{n_j}-z_0\rangle,\\ &=\langle u-z_0,\omega-z_0\rangle\le 0.\end{aligned} \tag{3.1.24}$$

Step 5. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathcal{F}}u$.

From the definition of x_n and $z_0=P_{\mathcal{F}}u$, we have

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &= \left\|\alpha_n(u-z_0)+\beta_n\left(P_C\left(I-\lambda_n(I-T)\right)x_n-z_0\right)+\gamma_n(W_n-z_0)\right\|^2,\\ &\le \left\|\beta_n\left(P_C\left(I-\lambda_n(I-T)\right)x_n-z_0\right)+\gamma_n(W_n-z_0)\right\|^2+2\alpha_n\langle u-z_0,x_{n+1}-z_0\rangle,\\ &\le \beta_n\left\|P_C\left(I-\lambda_n(I-T)\right)x_n-z_0\right\|^2+\gamma_n\|W_n-z_0\|^2+2\alpha_n\langle u-z_0,x_{n+1}-z_0\rangle,\\ &\le \beta_n\|x_n-z_0\|^2+\gamma_n\|x_n-z_0\|^2+2\alpha_n\langle u-z_0,x_{n+1}-z_0\rangle,\\ &=(1-\alpha_n)\|x_n-z_0\|^2+2\alpha_n\langle u-z_0,x_{n+1}-z_0\rangle.\end{aligned}$$

From the condition (ii), (3.1.24) and Lemma 2.4.6, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathcal{F}}u$. This completes the proof. □

By using the same method as Theorem 3.1 and using Lemma 1.1 and Lemma 2.3.7, we can obtain the following corollary:

Corollary 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T:C\to C$ be a quasi-nonexpansive mapping. Let $A,B:C\to H$ be α,β-inverse strongly monotone mappings, respectively. Define the mapping $G:C\to C$ by

$Gx=P_C(I-\rho_1A)(ax+(1-a)P_C(I-\rho_2B)x)$ for all $x\in C$ and $a\in[0,1]$. Assume

$\mathcal{F}=F(G)\cap F(T)\neq\emptyset$. Suppose that $x_1,u\in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C\left(I - \lambda_n\left(I - T\right)\right)x_n + \gamma_n G\left(x_n\right), \quad \forall n \in \mathbb{N},$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < c \le \beta_n, \gamma_n \le d < 1$ for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \le 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$ and (z_0, y_0) is a solution of (1.4) where $y_0 = P_C(I - \rho_2 B) z_0$.

3.2 Strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities

In this section, we shall prove the strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities.

Theorem 3.3. Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \rho_1 A)\left(ax + (1-a) P_C(I - \rho_2 B)x\right)$ for all $x \in C$ and $a \in [0,1]$. Assume $\mathcal{F} = EP(F_1) \cap EP(F_2) \cap F(G) \cap F(T) \ne \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geqslant 0, \quad \forall y\in C, \\ F_2(v_n,y)+\frac{1}{s_n}\langle y-v_n,v_n-x_n\rangle \geqslant 0, \quad \forall y\in C, \\ x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n P_C\left(I-\lambda_n(I-T)\right)u_n+\delta_n G(v_n), \quad \forall n\in\mathbb{N}, \end{cases} \tag{3.3.1}$$

where $\rho_1\in(0,2\alpha)$, $\rho_2\in(0,2\beta)$ and the sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}\subseteq[0,1]$ with $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for all $n\in\mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $0<c\le\beta_n,\gamma_n,\delta_n\le d<1$ for some $c,d>0$ and for all $n\in\mathbb{N}$,

(iii) $0<e\le r_n,s_n$ for some $e>0$ and for all $n\in\mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n\le 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty, \sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty, \sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty.$

Then $\{x_n\},\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0=P_{\mathcal{F}}u$ and (x_0,z_0) is a solution of (1.4) where $z_0=P_C(I-\rho_2 B)x_0$.

Proof. In the beginning, we show that G is a nonexpansive mapping. Let $x, y\in C$. Since A, B are α,β-inverse strongly monotone, $\rho_1\in(0,2\alpha)$ and $\rho_2\in(0,2\beta)$, we have

$$\begin{aligned} \|(I-\rho_1 A)x-(I-\rho_1 A)y\|^2 &= \|x-y\|^2-2\rho_1\langle x-y,Ax-Ay\rangle+\rho_1^2\|Ax-Ay\|^2, \\ &\le \|x-y\|^2-2\alpha\rho_1\|Ax-Ay\|^2+\rho_1^2\|Ax-Ay\|^2, \\ &= \|x-y\|^2+\rho_1(\rho_1-2\alpha)\|Ax-Ay\|^2, \\ &\le \|x-y\|^2. \end{aligned}$$

Then $(I-\rho_1 A)$ is a nonexpansive mapping. Similarly, $(I-\rho_2 B)$ is a nonexpansive mapping. Then G is a nonexpansive mapping.

Next, we show $\{x_n\}$ is bounded. Let $z\in\mathcal{F}$, then $u_n=T_{r_n}x_n$ and $v_n=T_{s_n}x_n$. It is clear that $\|u_n-z\|\le\|x_n-z\|$ and $\|v_n-z\|\le\|x_n-z\|$. By Remark 2.4.8, we have

$$z\in F\left(P_C\left(I-\lambda_n(I-T)\right)\right). \tag{3.3.2}$$

Observe that

$$\begin{aligned}\|Tu_n - z\|^2 &= \|(u_n - z) - (I-T)u_n\|^2,\\ &= \|u_n - z\|^2 - 2\langle u_n - z, (I-T)u_n\rangle + \|(I-T)u_n\|^2,\\ &\le \|u_n - z\|^2.\end{aligned}$$

It implies that

$$\|(I-T)u_n\|^2 \le 2\langle u_n - z, (I-T)u_n\rangle. \tag{3.3.3}$$

From (3.3.2) and (3.3.3), we have

$$\begin{aligned}\|P_C(I-\lambda_n(I-T))u_n - z\|^2 &= \|P_C(I-\lambda_n(I-T))u_n - P_C(I-\lambda_n(I-T))z\|^2,\\ &\le \|(u_n - z) - \lambda_n((I-T)u_n - (I-T)z)\|^2,\\ &= \|u_n - z\|^2 - 2\lambda_n\langle u_n - z, (I-T)u_n\rangle + \lambda_n^2\|(I-T)u_n\|^2,\\ &\le \|u_n - z\|^2 + \lambda_n(\lambda_n - 1)\|(I-T)u_n\|^2,\\ &\le \|u_n - z\|^2.\end{aligned} \tag{3.3.4}$$

From the definition of x_n and (3.3.4), we have

$$\begin{aligned}\|x_{n+1} - z\| &= \|\alpha_n(u-z) + \beta_n(x_n - z) + \gamma_n(P_C(I-\lambda_n(I-T))u_n - z) + \delta_n(G(v_n) - z)\|,\\ &\le \alpha_n\|u-z\| + \beta_n\|x_n - z\| + \gamma_n\|P_C(I-\lambda_n(I-T))u_n - z\| + \delta_n\|G(v_n) - z\|,\\ &\le \alpha_n\|u-z\| + \beta_n\|x_n - z\| + \gamma_n\|u_n - z\| + \delta_n\|v_n - z\|,\\ &\le \alpha_n\|u-z\| + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\| + \delta_n\|x_n - z\|,\\ &= \alpha_n\|u-z\| + (1-\alpha_n)\|x_n - z\|.\end{aligned}$$

We will prove by induction that,

$$\|x_n - z\| \le \max\{\|u-z\|, \|x_1 - z\|\}, \text{ for all } n \in \mathbb{N}. \tag{3.3.5}$$

Let $M = \max\{\|u-z\|, \|x_1 - z\|\}$. When $n = 1$, it is easy to see that (3.3.5) is true. Suppose (3.3.5) is true for $n = k$. Then

$$\begin{aligned}\|x_{k+1} - z\| &\le \alpha_k\|u-z\| + (1-\alpha_k)\|x_k - z\|,\\ &\le \alpha_k M + (1-\alpha_k)M,\\ &= M.\end{aligned}$$

Thus, (3.3.5) holds for $n = k+1$. Therefore, we can conclude that

$$\|x_n - z\| \le \max\{\|u-z\|, \|x_1 - z\|\},$$

for all $n \in \mathbb{N}$. This implies that the sequence $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{v_n\}$, $\{(I-T)u_n\}$ and $\{P_C(I-\lambda_n(I-T))u_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From the definition of x_n and nonexpansiveness of G, we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad =\|(\alpha_n-\alpha_{n-1})u+\beta_n(x_n-x_{n-1})+(\beta_n-\beta_{n-1})x_{n-1} \\
&\qquad +\gamma_n\left(P_C(I-\lambda_n(I-T))u_n-P_C(I-\lambda_{n-1}(I-T))u_{n-1}\right)+(\gamma_n-\gamma_{n-1})P_C(I-\lambda_{n-1}(I-T))u_{n-1} \\
&\qquad +\delta_n(G(v_n)-G(v_{n-1}))+(\delta_n-\delta_{n-1})G(v_{n-1})\|, \\
&\quad \le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\qquad +\gamma_n\|P_C(I-\lambda_n(I-T))u_n-P_C(I-\lambda_{n-1}(I-T))u_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))u_{n-1}\| \\
&\qquad +\delta_n\|G(v_n)-G(v_{n-1})\|+|\delta_n-\delta_{n-1}|\|G(v_{n-1})\|, \\
&\quad \le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\qquad +\gamma_n\|(u_n-u_{n-1})-(\lambda_n(I-T)u_n-\lambda_n(I-T)u_{n-1})-(\lambda_n(I-T)u_{n-1}-\lambda_{n-1}(I-T)u_{n-1})\| \\
&\qquad +|\gamma_n-\gamma_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))u_{n-1}\|+\delta_n\|v_n-v_{n-1}\|+|\delta_n-\delta_{n-1}|\|G(v_{n-1})\|, \\
&\quad \le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\qquad +\gamma_n\|u_n-u_{n-1}\|+\lambda_n\|(I-T)u_n-(I-T)u_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|(I-T)u_{n-1}\| \\
&\qquad +|\gamma_n-\gamma_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))u_{n-1}\|+\delta_n\|v_n-v_{n-1}\|+|\delta_n-\delta_{n-1}|\|G(v_{n-1})\|. \qquad (3.3.6)
\end{aligned}
$$

On the other hand, from $u_n=T_{r_n}x_n$ and $u_{n+1}=T_{r_{n+1}}x_{n+1}$, we have

$$F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C, \qquad (3.3.7)$$

and

$$F_1(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0,\quad \forall y\in C. \qquad (3.3.8)$$

Putting $y=u_{n+1}$ in (3.3.7) and $y=u_n$ in (3.3.8), we have

$$F_1(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\ge 0,$$

and

$$F_1(u_{n+1},u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0.$$

From (A2), we have

$$\left\langle u_{n+1}-u_n, \frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle \geq 0.$$

So

$$\left\langle u_{n+1}-u_n, u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}\left(u_{n+1}-x_{n+1}\right)\right\rangle \geq 0.$$

Then

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\leq \left\langle u_{n+1}-u_n, x_{n+1}-x_n+u_{n+1}-x_{n+1}-\frac{r_n}{r_{n+1}}\left(u_{n+1}-x_{n+1}\right)\right\rangle,\\ &= \left\langle u_{n+1}-u_n, x_{n+1}-x_n+\left(1-\frac{r_n}{r_{n+1}}\right)\left(u_{n+1}-x_{n+1}\right)\right\rangle,\\ &\leq \|u_{n+1}-u_n\|\left(\|x_{n+1}-x_n\|+\left|1-\frac{r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|\right),\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq \|x_{n+1}-x_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|,\\ &\leq \|x_{n+1}-x_n\|+\frac{1}{e}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|. \end{aligned} \tag{3.3.9}$$

From $v_n=T_{s_n}x_n$ and $v_{n+1}=T_{s_{n+1}}x_{n+1}$. By using the same method as (3.3.9), we have

$$\|v_{n+1}-v_n\| \leq \|x_{n+1}-x_n\|+\frac{1}{e}|s_{n+1}-s_n|\|v_{n+1}-x_{n+1}\|. \tag{3.3.10}$$

From (3.3.6), (3.3.9) and (3.3.10), we have

$$\begin{aligned}&\|x_{n+1}-x_n\|\\ &\leq |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\gamma_n\left(\|x_n-x_{n-1}\|+\frac{1}{e}|r_n-r_{n-1}|\|u_n-x_n\|\right)\\ &+\lambda_n\|(I-T)u_n-(I-T)u_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|(I-T)u_{n-1}\|\\ &+|\gamma_n-\gamma_{n-1}|\|P_C\left(I-\lambda_{n-1}(I-T)\right)u_{n-1}\|+\delta_n\left(\|x_n-x_{n-1}\|+\frac{1}{e}|s_n-s_{n-1}|\|v_n-x_n\|\right)\\ &+|\delta_n-\delta_{n-1}|\|G(v_{n-1})\|,\end{aligned}$$

$$\le|\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\gamma_n\|x_n-x_{n-1}\|+\frac{1}{e}|r_n-r_{n-1}|\|u_n-x_n\|$$
$$+\lambda_n\|(I-T)u_n-(I-T)u_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|(I-T)u_{n-1}\|$$
$$+|\gamma_n-\gamma_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))u_{n-1}\|+\delta_n\|x_n-x_{n-1}\|$$
$$+\frac{1}{e}|s_n-s_{n-1}|\|v_n-x_n\|+|\delta_n-\delta_{n-1}|\|G(v_{n-1})\|,$$
$$\le(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M+|\beta_n-\beta_{n-1}|M+|\gamma_n-\gamma_{n-1}|M+|\delta_n-\delta_{n-1}|M$$
$$+|\lambda_n-\lambda_{n-1}|M+\lambda_nM+\frac{1}{e}|r_n-r_{n-1}|M+\frac{1}{e}|s_n-s_{n-1}|M,$$

where $M:=\max_{n\in\mathbb{N}}\{\ \|u\|,\ \|x_n\|,\ \|P_C(I-\lambda_n(I-T))u_n\|,\ \|G(v_n)\|,\ \|(I-T)u_n\|,$
$\|(I-T)u_{n+1}-(I-T)u_n\|,\ \|u_n-x_n\|,\ \|v_n-x_n\|\ \}.$

From the condition (i), (iv), (v) and Lemma 2.4.6, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3.11}$$

Since T_{r_n} is a firmly nonexpansive mapping, we obtain

$$\|u_n-z\|^2=\|T_{r_n}x_n-T_{r_n}z\|^2,$$
$$\le\langle T_{r_n}x_n-T_{r_n}z,x_n-z\rangle,$$
$$=\langle u_n-z,x_n-z\rangle,$$
$$=\frac{1}{2}\left(\|u_n-z\|^2+\|x_n-z\|^2-\|u_n-x_n\|^2\right).$$

It implies that

$$\|u_n-z\|^2\le\|x_n-z\|^2-\|u_n-x_n\|^2. \tag{3.3.12}$$

By using the same method as (3.3.12), we have

$$\|v_n-z\|^2\le\|x_n-z\|^2-\|v_n-x_n\|^2. \tag{3.3.13}$$

From the definition of x_n, (3.3.4), (3.3.12) and (3.3.13), we have

$$\|x_{n+1}-z\|^2$$
$$=\left\|\alpha_n(u-z)+\beta_n(x_n-z)+\gamma_n\left(P_C(I-\lambda_n(I-T))u_n-z\right)+\delta_n(G(v_n)-z)\right\|^2,$$
$$\le\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\|P_C(I-\lambda_n(I-T))u_n-z\|^2+\delta_n\|G(v_n)-z\|^2$$
$$-\beta_n\gamma_n\|P_C(I-\lambda_n(I-T))u_n-x_n\|^2-\beta_n\delta_n\|G(v_n)-x_n\|^2,$$

$$\leq \alpha_n \|u-z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|u_n - z\|^2 + \delta_n \|v_n - z\|^2 - \beta_n\gamma_n \left\|P_C\left(I - \lambda_n(I-T)\right)u_n - x_n\right\|^2$$
$$-\beta_n\delta_n \|G(v_n) - x_n\|^2,$$
$$\leq \alpha_n \|u-z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \left(\|x_n - z\|^2 - \|u_n - x_n\|^2\right) + \delta_n \left(\|x_n - z\|^2 - \|v_n - x_n\|^2\right)$$
$$-\beta_n\gamma_n \left\|P_C\left(I - \lambda_n(I-T)\right)u_n - x_n\right\|^2 - \beta_n\delta_n \|G(v_n) - x_n\|^2,$$
$$= \alpha_n \|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \gamma_n \|u_n - x_n\|^2 - \delta_n \|v_n - x_n\|^2$$
$$-\beta_n\gamma_n \left\|P_C\left(I - \lambda_n(I-T)\right)u_n - x_n\right\|^2 - \beta_n\delta_n \|G(v_n) - x_n\|^2,$$
$$\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \gamma_n \|u_n - x_n\|^2 - \delta_n \|v_n - x_n\|^2 - \beta_n\gamma_n \left\|P_C\left(I - \lambda_n(I-T)\right)u_n - x_n\right\|^2$$
$$-\beta_n\delta_n \|G(v_n) - x_n\|^2,$$

which implies that

$$\gamma_n \|u_n - x_n\|^2 \leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2,$$
$$\leq \alpha_n \|u-z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right).$$

From the conditions (i), (ii) and (3.3.11), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.3.14}$$

By using the same method as (3.3.14), we can imply that

$$\lim_{n\to\infty} \|v_n - x_n\| = \lim_{n\to\infty} \left\|P_C\left(I - \lambda_n(I-T)\right)u_n - x_n\right\| = \lim_{n\to\infty} \|G(v_n) - x_n\| = 0. \tag{3.3.15}$$

From (3.3.14) and (3.3.15), we have

$$\lim_{n\to\infty} \|u_n - v_n\| = 0. \tag{3.3.16}$$

Afterwards, we show that $\limsup_{n\to\infty} \langle u - x_0, x_n - x_0 \rangle \leq 0$, where $x_0 = P_{\mathcal{F}} u$. Without loss of generality, we may assume that $u_{n_j} \rightharpoonup \omega$ as $j \to \infty$, where $\omega \in C$. First, we will show that $\omega \in EP(F_1)$ and $\omega \in EP(F_2)$. In fact, since $u_n = T_{r_n} x_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

By condition (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq F_1(y, u_n), \quad \forall y \in C.$$

In particular, it follows that

$$\left\langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \right\rangle \geq F_1\left(y, u_{n_j}\right), \quad \forall y \in C.$$

Since $\frac{u_{n_j} - x_{n_j}}{r_{n_j}} \to 0$ and $u_{n_j} \rightharpoonup \omega$ as $j \to \infty$. From (A4), we have

$$F_1(y, \omega) \le 0, \ \ \forall y \in C.$$

For any $t \in (0,1]$ and $y \in C$, let $y_t := ty + (1-t)\omega$. Since $y \in C$ and $\omega \in C$, we have $y_t \in C$. Hence $F_1(y_t, \omega) \le 0$. By using (A1) and (A4), we have

$$\begin{aligned} 0 &= F_1(y_t, y_t), \\ &= F_1(y_t, ty + (1-t)\omega), \\ &= tF_1(y_t, y) + (1-t)F_1(y_t, \omega), \\ &\le tF_1(y_t, y), \end{aligned}$$

and hence

$$F_1(y_t, y) \ge 0.$$

From (A3), we have

$$F_1(\omega, y) \ge 0, \ \ \forall y \in C.$$

Hence

$$\omega \in EP(F_1). \tag{3.3.17}$$

From (3.3.16), we have $v_{n_j} \rightharpoonup \omega$ as $j \to \infty$. By using the same method as (3.3.17), we have

$$\omega \in EP(F_2). \tag{3.3.18}$$

Continually, we show that $\omega \in F(T)$. From Remark 2.4.8, we have $F(T) = F\left(P_C\left(I - \lambda_{n_j}(I-T)\right)\right)$. Assume that $\omega \notin F(T)$, we have $\omega \ne P_C\left(I - \lambda_{n_j}(I-T)\right)\omega$. From (3.3.14), we have $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$. By (3.13), (3.14), the condition (iv) and Opial's property, we have

$$\begin{aligned} &\liminf_{j\to\infty} \left\| x_{n_j} - \omega \right\| \\ &\quad < \liminf_{j\to\infty} \left\| x_{n_j} - P_C\left(I - \lambda_{n_j}(I-T)\right)\omega \right\|, \\ &\quad \le \liminf_{j\to\infty} \Big(\left\| x_{n_j} - P_C\left(I - \lambda_{n_j}(I-T)\right)u_{n_j} \right\| + \left\| P_C\left(I - \lambda_{n_j}(I-T)\right)u_{n_j} - P_C\left(I - \lambda_{n_j}(I-T)\right)x_{n_j} \right\| \\ &\qquad + \left\| P_C\left(I - \lambda_{n_j}(I-T)\right)x_{n_j} - P_C\left(I - \lambda_{n_j}(I-T)\right)\omega \right\| \Big), \\ &\quad \le \liminf_{j\to\infty} \Big(\left\| u_{n_j} - x_{n_j} \right\| + \lambda_{n_j} \left\| (I-T)u_{n_j} - (I-T)x_{n_j} \right\| + \left\| x_{n_j} - \omega \right\| + \lambda_{n_j} \left\| (I-T)x_{n_j} - (I-T)\omega \right\| \Big), \\ &\quad = \liminf_{j\to\infty} \left\| x_{n_j} - \omega \right\|. \end{aligned}$$

It is a contradiction. So, we have

$$\omega \in F(T). \tag{3.3.19}$$

After that, we show that $\omega \in F(G)$. Assume that $\omega \notin F(G)$, then $\omega \neq G(\omega)$. From (3.3.14), we have $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, (3.14), the condition (iv) and Opial's property, we have

$$\begin{aligned}
\liminf_{j\to\infty} \left\| x_{n_j} - \omega \right\| &< \liminf_{j\to\infty} \left\| x_{n_j} - G(\omega) \right\|, \\
&\leq \liminf_{j\to\infty} \left(\left\| x_{n_j} - G(v_{n_j}) \right\| + \left\| G(v_{n_j}) - G(x_{n_j}) \right\| + \left\| G(x_{n_j}) - G(\omega) \right\| \right), \\
&\leq \liminf_{j\to\infty} \left(\left\| v_{n_j} - x_{n_j} \right\| + \left\| x_{n_j} - \omega \right\| \right), \\
&= \liminf_{j\to\infty} \left\| x_{n_j} - \omega \right\|.
\end{aligned}$$

It is a contradiction. So, we have

$$\omega \in F(G). \tag{3.3.20}$$

Therefore, $\omega \in \mathcal{F}$. Since $\langle u - x_0, x_n - x_0 \rangle$ is bounded and $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - x_0, x_n - x_0 \rangle &= \lim_{j\to\infty} \left\langle u - x_0, x_{n_j} - x_0 \right\rangle, \\
&= \langle u - x_0, \omega - x_0 \rangle \leq 0.
\end{aligned} \tag{3.3.21}$$

Finally, we show that the sequence $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0 = P_{\mathcal{F}}u$. From the definition of x_n, (3.3.4) and $x_0 = P_{\mathcal{F}}u$, we have

$$\begin{aligned}
\left\| x_{n+1} - x_0 \right\|^2 &= \left\| \alpha_n (u - x_0) + \beta_n (x_n - x_0) + \gamma_n \left(P_C \left(I - \lambda_n (I - T) \right) u_n - x_0 \right) + \delta_n \left(G(v_n) - x_0 \right) \right\|^2, \\
&\leq \left\| \beta_n (x_n - x_0) + \gamma_n \left(P_C \left(I - \lambda_n (I - T) \right) u_n - x_0 \right) + \delta_n \left(G(v_n) - x_0 \right) \right\|^2 \\
&\quad + 2\alpha_n \langle u - x_0, x_{n+1} - x_0 \rangle, \\
&\leq (1 - \alpha_n) \left\| x_n - x_0 \right\|^2 + 2\alpha_n \langle u - x_0, x_{n+1} - x_0 \rangle.
\end{aligned}$$

From the condition (i), (3.3.21) and Lemma 2.4.6, we can conclude that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathcal{F}}u$. Consequently, we can obtain that $\{u_n\}$ and $\{v_n\}$ also converge strongly to $x_0 = P_{\mathcal{F}}u$. This completes the proof. □

From Theorem 3.3, if we take $a = 0$, we have the following corollary:

Corollary 3.4. Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \rho_1 A)(P_C(I - \rho_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = EP(F_1) \cap EP(F_2) \cap F(G) \cap F(T) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ F_2(v_n, y) + \dfrac{1}{s_n}\langle y - v_n, v_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - T))u_n + \delta_n G(v_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ for some $c, d > 0$ and for all $n \in \mathbb{N}$,

(iii) $0 < e \leq r_n, s_n$ for some $e > 0$ and for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \leq 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$

$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty.$

Then $\{x_n\}, \{u_n\}$ and $\{v_n\}$ converge strongly to $x_0 = P_{\mathcal{F}} u$ and (x_0, z_0) is a solution of (1.3) where $z_0 = P_C(I - \rho_2 B)x_0$.

By using the same method as Theorem 3.3 and using Lemma 2.3.7, we can obtain the following corollary:

Corollary 3.5. Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \rho_1 A)(P_C(I - \rho_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = EP(F_1) \cap EP(F_2) \cap F(T) \cap VI(C,A) \cap VI(C,B) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ F_2(v_n, y) + \dfrac{1}{s_n}\langle y - v_n, v_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - T))u_n \\ \qquad + \delta_n P_C(I - \rho_1 A)(a x_n + (1-a) P_C(I - \rho_2 B)x_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, for all $n \in \mathbb{N}$, and $a \in (0,1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c, d > 0$ and for all $n \in \mathbb{N}$,

(iii) $0 < e \le r_n, s_n$ for some $e > 0$ and for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \le 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$

$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty.$

Then $\{x_n\}, \{u_n\}$ and $\{v_n\}$ converge strongly to $x_0 = P_{\mathcal{F}} u$ and (x_0, z_0) is a solution of (1.4) where $z_0 = P_C(I - \rho_2 B)x_0$.

Chapter 4

Applications

4.1 Strong convergence theorems for a finite family of nonspreading mappings

In this section, we use our main results to prove Theorem 4.1.3, Theorem 4.1.5, Corollary 4.1.4 and Corollary 4.1.6. Before proving these theorems, we need the following definitions and lemmas.

In 2009, Kangtunyakarn and Suantai [26] introduced the S-mapping generated by $T_1,T_2,...,T_N$ and $\lambda_1,\lambda_2,...,\lambda_N$ as follows:

Definition 4.1.1. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j=1,2,...,N$, let $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I$, where $I=[0,1]$ and $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$. Define the mapping $S:C\to C$ as follows:

$$
\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1T_1U_0+\alpha_2^1U_0+\alpha_3^1I,\\
U_2 &= \alpha_1^2T_2U_1+\alpha_2^2U_1+\alpha_3^2I,\\
U_3 &= \alpha_1^3T_3U_2+\alpha_2^3U_2+\alpha_3^3I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1}T_{N-1}U_{N-2}+\alpha_2^{N-1}U_{N-2}+\alpha_3^{N-1}I,\\
S &= U_N=\alpha_1^NT_NU_{N-1}+\alpha_2^NU_{N-1}+\alpha_3^NI.
\end{aligned}
$$

This mapping is called the *S-mapping* generated by $T_1,T_2,...,T_N$ and $\alpha_1,\alpha_2,...,\alpha_N$.

For every $i=1,2,...,N$. Put $\alpha_3^i=0$ in Definition 4.1.1, then the S-mapping is reduced to the K-mapping defined by Kangtunyakarn and Suantai [27] as follows:

Definition 4.1.2. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of mappings of C into itself, and let $\lambda_1,\lambda_2,\ldots,\lambda_N$ be real numbers such that $0\le\lambda_i\le 1$ for every $i=1,2,\ldots,N.$ Define a mapping $K:C\to C$ as follows:

$$
\begin{aligned}
U_0 &= I,\\
U_1 &= \lambda_1 T_1 + (1-\lambda_1)I,\\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1,\\
U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3)U_2,\\
&\vdots\\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2},\\
K &= U_N = \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}
$$

The such mapping K is called the K*-mapping* generated by $T_1,T_2,\ldots,T_N$ and $\lambda_1,\lambda_2,\ldots,\lambda_N$.

Lemma 4.1.1. [28] Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i)\neq\emptyset$, and let $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I,\quad j=1,2,\ldots,N,$ where $I=[0,1]$, $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$, $\alpha_1^j,\alpha_3^j\in(0,1)$ for all $j=1,2,\ldots,N-1$ and $\alpha_1^N\in(0,1]$, $\alpha_3^N\in[0,1)$, $\alpha_2^j\in[0,1)$ for all $j=1,2,\ldots,N.$ Let S be the mapping generated by $T_1,T_2,\ldots,T_N$ and $\alpha_1,\alpha_2,\ldots,\alpha_N$. Then $F(S)=\bigcap_{i=1}^{N} F(T_i)$ and S is a quasi-nonexpansive mapping.

Lemma 4.1.2. [29] Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i)\neq\emptyset$, and let $\lambda_1,\lambda_2,\ldots,\lambda_N$ be real numbers such that $0<\lambda_i<1$ for every $i=1,2,\ldots,N-1$ and $0<\lambda_N\le 1$. Let K be the K-mapping generated by $T_1,T_2,\ldots,T_N$ and $\lambda_1,\lambda_2,\ldots,\lambda_N$. Then $F(K)=\bigcap_{i=1}^{N} F(T_i)$ and K is quasi-nonexpansive mapping.

By using these results, we obtain the following theorem.

Theorem 4.1.3. Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \varnothing$, and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \rho_1 A)(ax + (1-a)P_C(I - \rho_2 B)x)$ for all $x \in C$ and $a \in (0,1)$. Assume $\mathcal{F} = VI(C,A) \cap VI(C,B) \cap \bigcap_{i=1}^N F(T_i) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - K))x_n + \gamma_n G(x_n), \quad \forall n \in \mathbb{N},$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \leq 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

Proof. By using Theorem 3.1 and Lemma 4.1.2, we obtain the conclusion.

The following result is direct proof from Theorem 4.1.3. Therefore, we omit the proof.

Corollary 4.1.4. Let C be a nonempty closed convex subset of a real Hilbert space. Let T be a nonspreading mappings of C into itself with $F(T) \neq \varnothing$. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by

$Gx = P_C(I-\rho_1 A)(ax+(1-a)P_C(I-\rho_2 B)x)$ for all $x \in C$ and $a \in (0,1)$. Assume $\mathcal{F} = VI(C,A) \cap VI(C,B) \cap F(T) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C(I-\lambda_n(I-T))x_n + \gamma_n G(x_n), \quad \forall n \in \mathbb{N},$$

where $\rho_1 \in (0,2\alpha)$, $\rho_2 \in (0,2\beta)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < c \le \beta_n, \gamma_n \le d < 1$ for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \le 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

Theorem 4.1.5 Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,\ldots,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1,2,\ldots,N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1,2,\ldots,N$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by

$G(x) = P_C(I-\rho_1 A)(ax+(1-a)P_C(I-\rho_2 B)x)$ for all $x \in C$ and $a \in [0,1]$. Assume $\mathcal{F} = EP(F_1) \cap EP(F_2) \cap F(G) \cap \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geqslant 0, & \forall y\in C,\\ F_2(v_n,y)+\frac{1}{s_n}\langle y-v_n,v_n-x_n\rangle\geqslant 0, & \forall y\in C,\\ x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n P_C\left(I-\lambda_n(I-S)\right)u_n+\delta_n G(v_n), & \forall n\in\mathbb{N}, \end{cases}$$

where $\rho_1\in(0,2\alpha)$, $\rho_2\in(0,2\beta)$ and the sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}\subseteq[0,1]$ with $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for all $n\in\mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $0<c\leq\beta_n,\gamma_n,\delta_n\leq d<1$ for some $c,d>0$ and for all $n\in\mathbb{N}$,

(iii) $0<e\leq r_n,s_n$ for some $e>0$ and for all $n\in\mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n\leq 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty,\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty,\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty,\sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty.$

Then $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0=P_{\mathcal{F}}u$ and (x_0,z_0) is a solution of (1.4) where $z_0=P_C(I-\rho_2 B)x_0$.

Proof. By using Theorem 3.3 and Lemma 4.1.1, we obtain the conclusion.

The following result is direct proof from Theorem 4.1.5. Therefore, we omit the proof.

Corollary 4.1.6. Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1,F_2:C\times C\to\mathbb{R}$ be bifunctions satisfying (A1)-(A4). Let T be a nonspreading mappings of C into itself with $F(T)\neq\varnothing$. Let $A,B:C\to H$ be α,β -inverse strongly monotone mappings, respectively. Define the mapping $G:C\to C$ by

$G(x)=P_C(I-\rho_1 A)\left(ax+(1-a)P_C(I-\rho_2 B)x\right)$ for all $x\in C$ and $a\in[0,1]$. Assume

$\mathcal{F}=EP(F_1)\cap EP(F_2)\cap F(G)\cap F(T)\neq\varnothing$. Suppose that $x_1,u\in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geqslant 0, & \forall y\in C,\\ F_2(v_n,y)+\frac{1}{s_n}\langle y-v_n,v_n-x_n\rangle\geqslant 0, & \forall y\in C,\\ x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n P_C\left(I-\lambda_n(I-T)\right)u_n+\delta_n G(v_n), & \forall n\in\mathbb{N}, \end{cases}$$

where $\rho_1\in(0,2\alpha)$, $\rho_2\in(0,2\beta)$ and the sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}\subseteq[0,1]$ with $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for all $n\in\mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $0<c\leq\beta_n,\gamma_n,\delta_n\leq d<1$ for some $c,d>0$ and for all $n\in\mathbb{N}$,

(iii) $0<e\leq r_n, s_n$ for some $e>0$ and for all $n\in\mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n\leq 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty, \sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty, \sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty.$

Then $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0=P_{\mathcal{F}}u$ and (x_0,z_0) is a solution of (1.4) where $z_0=P_C(I-\rho_2 B)x_0$.

4.2 Example and Numerical Results

In this section, Example 4.2.1 and Example 4.2.2 are given for supporting Theorem 3.1 and Theorem 3.3, respectively.

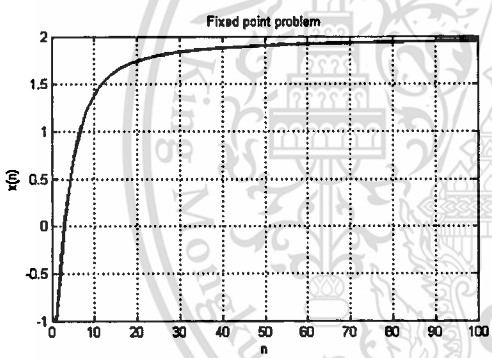
Example 4.2.1. Let $\mathbb{R}$ be the set of real numbers and let the mappings $A, B:\mathbb{R}\to\mathbb{R}$ be defined by $Ax=\frac{x-2}{3}$ and $Bx=\frac{x-2}{5}$, $\forall x\in\mathbb{R}$, respectively. Define the mapping $G:\mathbb{R}\to\mathbb{R}$ by $Gx=(I-\rho_1 A)\left(ax+(1-a)(I-\rho_2 B)x\right)$ for all $x\in\mathbb{R}$. Define the mapping $T:\mathbb{R}\to\mathbb{R}$ by $Tx=\frac{x+2}{2}$, $\forall x\in\mathbb{R}$. Let $x_1,u\in\mathbb{R}$ and $\{x_n\}$ be generated by

$$x_{n+1}=\alpha_n u+\beta_n P_{\mathbb{R}}\left(I-\lambda_n(I-T)\right)x_n+\gamma_n G(x_n),\quad \forall n\in\mathbb{N}, \tag{4.2.1}$$

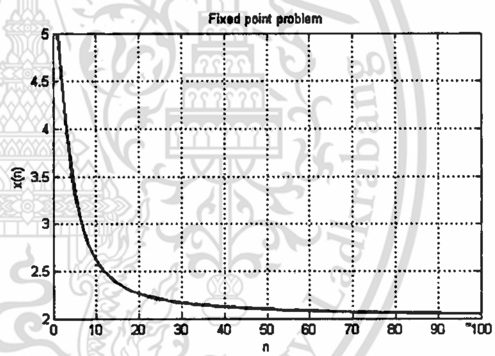
where $a \in (0,1)$, $\rho_1 \in (0,2)$, $\rho_2 \in (0,2)$, $\alpha_n = \frac{1}{2n}, \beta_n = \frac{3n-1}{16n}, \gamma_n = \frac{13n-7}{16n}$ and $\lambda_n = \frac{1}{2n^2}$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to 2. It is easy to see that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the all conditions in the Theorem 3.1, A and B are 1-inverse strongly monotone mappings and $VI(C,A) \cap VI(C,B) \cap F(T) = \{2\}$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ converge strongly to 2. For numerical results, choose $a = 0.5$, $\rho_1 = 1$ and $\rho_2 = 1$ for all $n \in \mathbb{N}$. We can rewrite (4.2.1) as follows:

$$\begin{aligned} x_{n+1} &= \left(\frac{1}{2n}\right)u + \left(\frac{3n-1}{16n}\right)\left(I - \frac{1}{2n^2}(I-T)\right)x_n \\ &+ \left(\frac{13n-7}{16n}\right)(I-A)\left(\frac{1}{2}x_n + \frac{1}{2}(I-B)x_n\right), \quad \forall n \in \mathbb{N}\ , \end{aligned} \tag{4.2.2}$$

The following figures show the values of sequences $\{x_n\}$ where $u = x_1 = -1$ and $u = x_1 = 5$ and $n = 100$.



Figures 4.1 : The values of sequence $\{x_n\}$, where $u = x_1 = -1$ and $n = 100$.



Figures 4.2 : The values of sequence $\{x_n\}$, where $u = x_1 = 5$ and $n = 100$.

Table 1 in Appendix B shows the values of sequence $\{x_n\}$ where $u = x_1 = -1$ and $u = x_1 = 5$ and $n = 100$ of Example 4.2.1.

Example 4.2.2. Let $\mathbb{R}$ be the set of real numbers and define the mappings $A, B:\mathbb{R}\to\mathbb{R}$ by $Ax=\dfrac{x-2}{3}$ and $Bx=\dfrac{x-2}{5}$, $\forall x\in\mathbb{R}$, respectively. Define the mapping $T:\mathbb{R}\to\mathbb{R}$ by $Tx=\dfrac{x+2}{2}$, $\forall x\in\mathbb{R}$, let $F_1, F_2:\mathbb{R}\to\mathbb{R}$ be defined by

$$F_1(x,y)=-(x-y)(-4+x+y),\ \ \forall x, y\in\mathbb{R},$$

and

$$F_2(x,y)=-2(x-2)^2+(x-2)(y-2)+(y-2)^2,\ \ \forall x,y\in\mathbb{R}.$$

By the definition of F_1, we have

$$\begin{aligned}
0 &\le F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \\
&=-(u_n-y)(-4+u_n+y)+\frac{1}{r_n}(y-u_n)(u_n-x_n) \\
&=-(u_n-y)(-4+u_n+y)+\frac{1}{r_n}(yu_n-yx_n-u_n^2+u_nx_n) \\
&\Leftrightarrow \\
0 &\le -r_n(u_n-y)(-4+u_n+y)+(yu_n-yx_n-u_n^2+u_nx_n) \\
&=4r_nu_n-u_n^2-r_nu_n^2+u_nx_n+(-4r_n+u_n-x_n)y+r_ny^2.
\end{aligned}$$

Let $G(y)=r_ny^2+(-4r_n+u_n-x_n)y+4r_nu_n-u_n^2-r_nu_n^2+u_nx_n$ which is a quadratic function of y with coefficient $a=r_n$, $b=-4r_n+u_n-x_n$, and $c=4r_nu_n-u_n^2-r_nu_n^2+u_nx_n$. Determine the discriminant Δ of G as follows

$$\begin{aligned}
\Delta &= b^2-4ac \\
&=(-4r_n+u_n-x_n)^2-4r_n\left(4r_nu_n-u_n^2-r_nu_n^2+u_nx_n\right) \\
&=16r_n^2-8r_nu_n-16r_n^2u_n+u_n^2+4r_nu_n^2+4r_n^2u_n^2+8r_nx_n-2u_nx_n-4r_nu_nx_n+x_n^2 \\
&=(-4r_n+u_n+2r_nu_n-x_n)^2.
\end{aligned}$$

We know that $G(y)\ge 0, \forall y\in\mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta\le 0$. So we obtain

$$u_n=\frac{4r_n+x_n}{1+2r_n}. \tag{4.2.3}$$

By using the same method as (4.2.3), we have

$$v_n = \frac{6s_n + x_n}{1+3s_n}. \tag{4.2.4}$$

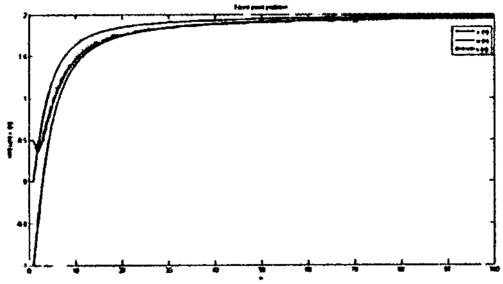
Let $x_1, u \in \mathbb{R}$ and $\{x_n\}$ generated by (3.3.1) as follows:

$$\begin{cases} F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ F_2(v_n, y) + \frac{1}{s_n}\langle y - v_n, v_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n (I - T)\right)u_n + \delta_n G(v_n), \quad \forall n \geq 1, \end{cases}$$

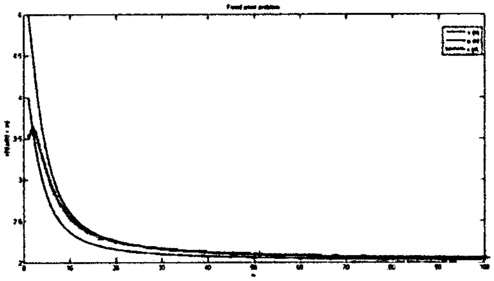
where $a = 0.5$, $\rho_1 = 1$, $\rho_2 = 1$, $r_n = \frac{n}{3n+1}$, $s_n = \frac{n}{4n+1}$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{3n-1}{16n}$, $\gamma_n = \frac{10n-3}{16n}$, $\delta_n = \frac{3n-4}{16n}$ and $\lambda_n = \frac{1}{2n^2}$ for all $n \in \mathbb{N}$. It is easy to see that all parameters satisfy all conditions of Theorem 3.3. From the definition of F_1, F_2, G and T, we have $EP(F_1) \cap EP(F_2) \cap F(G) \cap F(T) = \{2\}$. From Theorem 3.3, we can conclude that the sequence $\{x_n\}, \{u_n\}$ and $\{v_n\}$ converge strongly to 2. From (4.2.3) and (4.2.4), we can rewrite (3.3.1) as follows:

$$\begin{cases} u_n = \frac{4r_n + x_n}{1+2r_n}, \\ v_n = \frac{6s_n + x_n}{1+3s_n}, \\ x_{n+1} = \frac{1}{2n}u + \frac{3n-1}{16n}x_n + \frac{10n-3}{16n}\left(I - \frac{1}{2n^2}(I-T)\right)u_n + \frac{3n-4}{16n}G(v_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.2.5}$$

The following figures show the values of sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ where $u = x_1 = -1$ and $u = x_1 = 5$ and $n = 100$.



Figures 4.3 : The values of sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ where $u = x_1 = -1$ and $n = 100$.



Figures 4.4 : The values of sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ where $u = x_1 = 5$ and $n = 100$.

Table 2 in Appendix B shows the values of sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ where $u = x_1 = -1$ and $u = x_1 = 5$ and $n = 100$ of Example 4.2.2.

Chapter 5

Conclusions

In this chapter, we conclude all theorems and corollaries obtained in this thesis.

(1) Let C be a nonempty closed convex subset of a real Hilbert space H and let $T:C\to C$ be a quasi-nonexpansive mapping. Let $A,B:C\to H$ be α,β-inverse strongly monotone mappings, respectively. Define the mapping $G:C\to C$ by $Gx=P_C(I-\rho_1 A)(ax+(1-a)P_C(I-\rho_2 B)x)$, for all $x\in C$ and $a\in(0,1)$. Assume that $\mathcal{F}=VI(C,A)\cap VI(C,B)\cap F(T)\neq\varnothing$. Suppose that $x_1,u\in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1}=\alpha_n u+\beta_n P_C(I-\lambda_n(I-T))x_n+\gamma_n G(x_n),\quad \forall n\in\mathbb{N},$$

where $\rho_1\in(0,2\alpha)$, $\rho_2\in(0,2\beta)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n+\beta_n+\gamma_n=1$,

(ii) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(iii) $0<c\le\beta_n,\gamma_n\le d<1$ for all $n\in\mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n\le 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$.

Then $\{x_n\}$ converges strongly to $z_0=P_{\mathcal{F}}u$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H and let $T:C\to C$ be a quasi-nonexpansive mapping. Let $A,B:C\to H$ be α,β-inverse strongly monotone mappings, respectively. Define the mapping $G:C\to C$ by $Gx=P_C(I-\rho_1 A)(ax+(1-a)P_C(I-\rho_2 B)x)$ for all $x\in C$ and $a\in[0,1]$. Assume $\mathcal{F}=F(G)\cap F(T)\neq\varnothing$. Suppose that $x_1,u\in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C\left(I - \lambda_n (I - T)\right)x_n + \gamma_n G(x_n), \quad \forall n \in \mathbb{N},$$

where $\rho_1 \in (0,2\alpha)$, $\rho_2 \in (0,2\beta)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < c \le \beta_n, \gamma_n \le d < 1$ for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \le 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$ and (z_0, y_0) is a solution of (1.4) where $y_0 = P_C(I - \rho_2 B) z_0$.

(3) Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \rho_1 A)\left(ax + (1-a)P_C(I - \rho_2 B)x\right)$ for all $x \in C$ and $a \in [0,1]$. Assume $\mathcal{F} = EP(F_1) \cap EP(F_2) \cap F(G) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\},\{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geqslant 0, & \forall y \in C, \\ F_2(v_n, y) + \dfrac{1}{s_n}\langle y - v_n, v_n - x_n \rangle \geqslant 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n (I - T)\right)u_n + \delta_n G(v_n), & \forall n \in \mathbb{N}, \end{cases}$$

where $\rho_1 \in (0,2\alpha)$, $\rho_2 \in (0,2\beta)$ and the sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c, d > 0$ and for all $n \in \mathbb{N}$,

(iii) $0 < e \le r_n, s_n$ for some $e > 0$ and for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $0 < \lambda_n \le 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty}|s_{n+1} - s_n| < \infty.$

Then $\{x_n\},\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0 = P_{\mathcal{F}}u$ and (x_0, z_0) is a solution of (1.4) where $z_0 = P_C(I - \rho_2 B)x_0$.

(4) Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \rho_1 A)(P_C(I - \rho_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = EP(F_1) \cap EP(F_2) \cap F(G) \cap F(T) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in C, \\ F_2(v_n, y) + \frac{1}{s_n}\langle y - v_n, v_n - x_n\rangle \geqslant 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n(I - T)\right)u_n + \delta_n G(v_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and the sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c, d > 0$ and for all $n \in \mathbb{N}$,

(iii) $0 < e \le r_n, s_n$ for some $e > 0$ and for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $0 < \lambda_n \le 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty, \sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty, \sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty.$

Then $\{x_n\},\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0 = P_{\mathcal{F}}u$ and (x_0, z_0) is a solution of (1.3) where $z_0 = P_C(I-\rho_2 B)x_0$.

(5) Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C\times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I-\rho_1 A)(P_C(I-\rho_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = EP(F_1)\cap EP(F_2)\cap F(T)\cap VI(C,A)\cap VI(C,B)\neq\emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y\in C,\\ F_2(v_n, y) + \frac{1}{s_n}\langle y-v_n, v_n - x_n\rangle \geqslant 0, \quad \forall y\in C,\\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I-\lambda_n(I-T))u_n \\ \qquad + \delta_n P_C(I-\rho_1 A)(a x_n + (1-a)P_C(I-\rho_2 B)x_n), \quad \forall n\in\mathbb{N}, \end{cases}$$

where $\rho_1 \in (0,2\alpha)$, $\rho_2 \in (0,2\beta)$ and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$, for all $n\in\mathbb{N}$, and $a\in(0,1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < c \le \beta_n, \gamma_n, \delta_n \le d < 1$ for some $c, d > 0$ and for all $n\in\mathbb{N}$,

(iii) $0 < e \le r_n, s_n$ for some $e > 0$ and for all $n\in\mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $0<\lambda_n\le 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty, \sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty, \sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty.$

Then $\{x_n\},\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0 = P_{\mathcal{F}}u$ and (x_0, z_0) is a solution of (1.4) where $z_0 = P_C(I-\rho_2 B)x_0$.

(6) Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Let $A, B: C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G: C \to C$ by $Gx = P_C(I-\rho_1 A)(ax + (1-a)P_C(I-\rho_2 B)x)$ for all $x \in C$ and $a \in (0,1)$. Assume $\mathcal{F} = VI(C,A) \cap VI(C,B) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I-K))x_n + \gamma_n G(x_n), \quad \forall n \in \mathbb{N},$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \leq 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$.

(7) Let C be a nonempty closed convex subset of a real Hilbert space. Let T be a nonspreading mappings of C into itself with $F(T) \neq \emptyset$. Let $A, B: C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G: C \to C$ by $Gx = P_C(I-\rho_1 A)(ax + (1-a)P_C(I-\rho_2 B)x)$ for all $x \in C$ and $a \in (0,1)$. Assume

$\mathcal{F} = VI(C,A) \cap VI(C,B) \cap F(T) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C\left(I - \lambda_n (I - T)\right)x_n + \gamma_n G(x_n), \quad \forall n \in \mathbb{N},$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < c \le \beta_n, \gamma_n \le d < 1$ for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \le 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

(8) Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1, F_2 : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1, 2, \ldots, N$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by

$G(x) = P_C(I - \rho_1 A)\left(ax + (1-a)P_C(I - \rho_2 B)x\right)$ for all $x \in C$ and $a \in [0,1]$. Assume

$\mathcal{F} = EP(F_1) \cap EP(F_2) \cap F(G) \cap \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geqslant 0, & \forall y\in C, \\ F_2(v_n,y)+\frac{1}{s_n}\langle y-v_n,v_n-x_n\rangle\geqslant 0, & \forall y\in C, \\ x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n P_C\left(I-\lambda_n(I-S)\right)u_n+\delta_n G(v_n), & \forall n\in\mathbb{N}, \end{cases}$$

where $\rho_1\in(0,2\alpha)$, $\rho_2\in(0,2\beta)$ and the sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}\subseteq[0,1]$ with $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for all $n\in\mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $0<c\leq\beta_n,\gamma_n,\delta_n\leq d<1$ for some $c,d>0$ and for all $n\in\mathbb{N}$,

(iii) $0<e\leq r_n,s_n$ for some $e>0$ and for all $n\in\mathbb{N}$,

(iv) $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n\leq 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty, \sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty, \sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty.$

Then $\{x_n\},\{u_n\}$ and $\{v_n\}$ converge strongly to $x_0=P_{\mathcal{F}}u$ and (x_0,z_0) is a solution of (1.4) where $z_0=P_C(I-\rho_2 B)x_0$.

(9) Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_1,F_2:C\times C\to\mathbb{R}$ be bifunctions satisfying (A1)-(A4). Let T be a nonspreading mappings of C into itself with $F(T)\neq\varnothing$. Let $A,B:C\to H$ be α,β-inverse strongly monotone mappings, respectively. Define the mapping $G:C\to C$ by $G(x)=P_C(I-\rho_1 A)\left(ax+(1-a)P_C(I-\rho_2 B)x\right)$ for all $x\in C$ and $a\in[0,1]$. Assume $\mathcal{F}=EP(F_1)\cap EP(F_2)\cap F(G)\cap F(T)\neq\varnothing$. Suppose that $x_1,u\in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geqslant 0, & \forall y \in C, \\ F_2(v_n, y) + \frac{1}{s_n}\langle y - v_n, v_n - x_n \rangle \geqslant 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n (I - T)\right) u_n + \delta_n G(v_n), & \forall n \in \mathbb{N}, \end{cases}$$

where $\rho_1 \in (0, 2\alpha)$, $\rho_2 \in (0, 2\beta)$ and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ for some $c, d > 0$ and for all $n \in \mathbb{N}$,

(iii) $0 < e \leq r_n, s_n$ for some $e > 0$ and for all $n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n \leq 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$

$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty.$

Then $\{x_n\}, \{u_n\}$ and $\{v_n\}$ converge strongly to $x_0 = P_{\mathcal{F}} u$ and (x_0, z_0) is a solution of (1.4) where $z_0 = P_C(I - \rho_2 B) x_0$.

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APPENDIX A.

THE RESEARCH PAPERS

วิธีการใหม่สำหรับหาเซตของผลเฉลยของปัญหาอสมการการแปรผัน

The New Method for Finding the Solution Sets of Variational Inequality problems.

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บทคัดย่อ

จุดมุ่งหมายของงานวิจัยนี้ เราสร้างขั้นตอนวิธีการทำซ้ำชนิดใหม่ของลำดับ $\{x_n\}$ สำหรับหาเซตของผลเฉลยของระบบปรับปรุงของอสมการการแปรผันในปริภูมิฮิลเบิร์ต ที่เกี่ยวข้องกับการส่งแบบผกผันทางเดียวแบบ α และยังพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มของลำดับ $\{x_n\}$ สู่จุด $z_0 = P_F u$ โดย $F = VI(C,A) \cap VI(C,B) \neq \emptyset$ นอกจากนี้ เรายังได้ให้ตัวอย่างที่สร้างมาจากทฤษฎีบทหลัก เพื่อสนับสนุนทฤษฎีบทดังกล่าว

คำสำคัญ : อสมการการการแปรผัน, การส่งแบบผกผันทางเดียวแบบ α, จุดตรึง

Abstract

For the purpose of this article, we introduce the new method of iterative scheme $\{x_n\}$ for finding the set of solutions of a modified system of variational inequalities in a framework of Hilbert space involving α - inverse strongly monotone mapping. By assuming $F = VI(C,A) \cap VI(C,B) \neq \emptyset$, we prove strong convergence theorem $\{x_n\}$ converges strongly to $z_0 = P_F u$. Moreover, we give numerical example to support our main theorem.

Keywords : Variational Inequality, α - inverse strongly monotone, Fixed Point.

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1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H. A Mapping $T:C\to C$ is called nonexpansive if

$$\|Tx-Ty\|\le\|x-y\|, \text{ for all } x,y\in C. \tag{1.1}$$

A mapping $A:C\to H$ is called α - *inverse strongly monotone* if there exists a positive real number $\alpha>0$ such that

$$\langle Ax-Ay,x-y\rangle\ge\alpha\|Ax-Ay\|^2, \tag{1.2}$$

for all $x,y\in C$. Denote $F(T)$ by the set of fixed points of T; that is $F(T)=\{x\in C:Tx=x\}$. Fixed point problems are widely used in the literature such as optimization, physics, engineering and applied sciences; see [7,8].

Let $B:C\to H$. The *variational inequality* is to find a point $u\in C$ such that

$$\langle Bu,v-u\rangle\ge 0, \tag{1.3}$$

for all $v\in C$. The set of solutions of (1.3) is denoted by $VI(C,B)$. The variational inequalities were initially studied and introduced by Stampacchia see [3,4].

Let $D_1,D_2:C\to H$ be two mappings. In 2008, Ceng et al. [1] introduced a problem for finding $(x^*,z^*)\in C\times C$ such that

$$\begin{cases}\langle\lambda_1 D_1 z^*+x^*-z^*,x-x^*\rangle\ge 0,\forall x\in C,\\ \langle\lambda_2 D_2 x^*+z^*-x^*,x-z^*\rangle\ge 0,\forall x\in C,\end{cases} \tag{1.4}$$

which is called a system of variational inequalities where $\lambda_1,\lambda_2>0$. In 2013, Kangtunyakarn [2] modified (1.4) for finding $(x^*,z^*)\in C\times C$ such that

$$\begin{cases}\langle x^*-(I-\lambda_1 D_1)(ax^*+(1-a)z^*),x-x^*\rangle\ge 0,\forall x\in C,\\ \langle z^*-(I-\lambda_2 D_2)x^*,x-z^*\rangle\ge 0,\forall x\in C,\end{cases} \tag{1.5}$$

which is called a modification of system of variational inequalities, for every $\lambda_1,\lambda_2>0$ and $a\in[0,1]$. If $a=0$, (1.5) reduces to (1.4). He introduced the relation between solutions of (1.5) and fixed point of the mapping G as follows:

Lemma 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $a \in [0,1]$, the following statements are equivalent:

1. $(x^*, z^*) \in C \times C$ is a solution of problem (1.5),
2. x^* is a fixed point of the mapping $G : C \to C$, i.e., $x^* \in F(G)$, defined by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)x), \quad (1.6)$$

where $z^* = P_C(I - \lambda_2 D_2)x^*$.

Motivated by [2], we introduce the new method of iterative scheme $\{x_n\}$ for finding the set of solutions of a modified system of variational inequalities in a framework of Hilbert space involving α - inverse strongly monotone mapping. By assuming $F = VI(C,A) \cap VI(C,B) \neq \varnothing$, we prove strong convergence theorem $\{x_n\}$ converges strongly to $z_0 = P_F u$. Moreover, we show the example and numerical results of our main theorem.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. Throughout this paper, we denote weak and strong convergence by notations "$\rightharpoonup$" and "$\to$", respectively. For every $x \in H$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well-known that for each $x \in H$,

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \quad (2.1)$$

Recall that H satisfies *Opial's condition* [7], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad (2.2)$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. Let H be a real Hilbert space. Then

1. $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2$,
2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$,

for all $x, y \in H$.

Lemma 2.2. ([2]) Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2 - inverse strongly monotone mappings, respectively, which $VI(C, D_1) \cap VI(C, D_1) \neq \varnothing$. Define a mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)x),$$

for every $\lambda_1 \in (0,2d_1)$, $\lambda_2 \in (0,2d_2)$ and $a \in (0,1)$. Then $F(G) = VI(C,D_1) \cap VI(C,D_2)$.

Lemma 2.3. ([7]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1-\alpha_n)s_n + \delta_n, \forall n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. MAIN RESULT

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α, β - inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I-\lambda_1 A)(ax + (1-a)P_C(I-\lambda_2 B)x)$ for all $x \in C$ and $a \in (0,1)$. Assume $F = VI(C,A) \cap VI(C,B) \neq \varnothing$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)P_C(I-\lambda_1 A)\big(ax_n + (1-a)P_C(I-\lambda_2 B)x_n\big), \quad \forall n \ge 1, \tag{3.1}$$

where $\lambda_1 \in (0,2\alpha)$, $\lambda_2 \in (0,2\beta)$ and $\{\alpha_n\}$ are sequences in $[0,1]$. Suppose the following conditions holds:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_F u$.

Proof. We divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

Let $x, y \in C$. Since A is α - inverse strongly monotone and $\lambda_1 \in (0,2\alpha)$, we have

$$\begin{aligned} \|(I-\lambda_1 A)x - (I-\lambda_1 A)y\|^2 &= \|x-y\|^2 - 2\lambda_1\langle x-y, Ax-Ay\rangle + \lambda_1^2\|Ax-Ay\|^2 \\ &\le \|x-y\|^2 - 2\alpha\lambda_1\|Ax-Ay\|^2 + \lambda_1^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + \lambda_1(\lambda_1 - 2\alpha)\|Ax-Ay\|^2 \\ &\le \|x-y\|^2. \end{aligned} \tag{3.2}$$

Therefore $(I-\lambda_1 A)$ is a nonexpansive mapping. Similarly, $(I-\lambda_2 B)$ is a nonexpansive mapping. Hence $P_C(I-\lambda_1 A)$ and $P_C(I-\lambda_2 B)$ are nonexpansive mappings. From definition of the mapping G, we have G is a nonexpansive mapping.

Let $x^* \in F$. By Lemma 2.2, we have

$$x^* = G(x^*) = P_C(I-\lambda_1 A)\big(ax^* + (1-a)P_C(I-\lambda_2 B)x^*\big). \tag{3.3}$$

Put $M_n = ax_n + (1-a)P_C(I-\lambda_2 B)x_n$. From (3.1), we have

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C (I - \lambda_1 A) M_n. \tag{3.4}$$

From the definition of x_n and nonexpansiveness of G, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n (u - x^*) + (1-\alpha_n)(P_C (I - \lambda_1 A) M_n - x^*)\| \\
&\le \alpha_n \|u - x^*\| + (1-\alpha_n)\|P_C (I - \lambda_1 A) M_n - x^*\| \\
&= \alpha_n \|u - x^*\| + (1-\alpha_n)\|P_C (I - \lambda_1 A)(a x_n + (1-a) P_C (I - \lambda_2 B) x_n) - P_C (I - \lambda_1 A)(a x^* + (1-a) P_C (I - \lambda_2 B) x^*)\| \\
&= \alpha_n \|u - x^*\| + (1-\alpha_n)\|G(x_n) - G(x^*)\| \\
&\le \alpha_n \|u - x^*\| + (1-\alpha_n)\|x_n - x^*\|.
\end{aligned} \tag{3.5}$$

By induction, we can conclude that

$$\|x_n - x^*\| \le \max\{\|u - x^*\|, \|x_1 - x^*\|\}, \tag{3.6}$$

for all $n \ge 1$. This implies that the sequence $\{x_n\}$ is bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

From the definition of x_n and nonexpansiveness of G, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1}) u + (1-\alpha_n)(P_C (I - \lambda_1 A) M_n - P_C (I - \lambda_1 A) M_{n-1}) + (\alpha_{n-1} - \alpha_n) P_C (I - \lambda_1 A) M_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|P_C (I - \lambda_1 A)(a x_n + (1-a) P_C (I - \lambda_2 B) x_n) - P_C (I - \lambda_1 A)(a x_{n-1} + (1-a) P_C (I - \lambda_2 B) x_{n-1})\| \\
&\quad + |\alpha_{n-1} - \alpha_n|\|P_C (I - \lambda_1 A) M_{n-1}\| \\
&= |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|G(x_n) - G(x_{n-1})\| + |\alpha_{n-1} - \alpha_n|\|P_C (I - \lambda_1 A) M_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u\| + |\alpha_{n-1} - \alpha_n|\|P_C (I - \lambda_1 A) M_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M + |\alpha_{n-1} - \alpha_n| M
\end{aligned} \tag{3.7}$$

where $M := \max_{n\in N}\{\|u\|, \|P_C (I - \lambda_1 A) M_n\|\}.$

From the condition (i), (ii) and Lemma 2.3, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Step 3. We show that $\lim_{n\to\infty} \|P_C (I - \lambda_1 A) M_n - x_n\| = 0.$

From the definition of x_n, we have

$$x_{n+1} - x_n = \alpha_n (u - x_n) + (1-\alpha_n)(P_C (I - \lambda_1 A) M_n - x_n). \tag{3.9}$$

That is

$$\|P_C (I - \lambda_1 A) M_n - x_n\| \le \alpha_n \|u - x_n\| + \alpha_n \|P_C (I - \lambda_1 A) M_n - x_n\| + \|x_{n+1} - x_n\|. \tag{3.10}$$

From the condition (i) and (3.8), we derive

$$\lim_{n\to\infty}\left\|P_C(I-\lambda_1 A)M_n - x_n\right\| = 0. \tag{3.11}$$

Step 4. We show that $\limsup_{n\to\infty}\langle u-z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_{\mathfrak{F}} u$.

$$\limsup_{n\to\infty}\langle u-z_0, x_n-z_0\rangle = \lim_{j\to\infty}\langle u-z_0, x_{n_j}-z_0\rangle. \tag{3.12}$$

Without loss of generality, we may assume that $x_{n_j} \overset{w}{\to} \omega$ as $j\to\infty$, where $\omega\in C$. First, we show that $\omega\in VI(C,A)\cap VI(C,B)$. From Lemma 2.2, we have $VI(C,A)\cap VI(C,B)=F(G)$. Assume that $\omega\notin F(G)$, that $\omega\ne G(\omega)$. Since $x_{n_j}\overset{w}{\to}\omega$ as $j\to\infty$, (3.11) and Opial's property, we have

$$\begin{aligned}
\liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\| &< \liminf_{j\to\infty}\left\|x_{n_j}-G(\omega)\right\| \\
&\le \liminf_{j\to\infty}\left(\left\|x_{n_j}-P_C(I-\lambda_1 A)\left(ax_{n_j}+(1-a)P_C(I-\lambda_2 B)x_{n_j}\right)\right\|+\left\|P_C(I-\lambda_1 A)\left(ax_{n_j}+(1-a)P_C(I-\lambda_2 B)x_{n_j}\right)-G(\omega)\right\|\right) \\
&\le \liminf_{j\to\infty}\left(\left\|x_{n_j}-P_C(I-\lambda_1 A)\left(ax_{n_j}+(1-a)P_C(I-\lambda_2 B)x_{n_j}\right)\right\|+\left\|G\left(x_{n_j}\right)-G(\omega)\right\|\right) \qquad (3.13)\\
&\le \liminf_{j\to\infty}\left(\left\|x_{n_j}-P_C(I-\lambda_1 A)\left(ax_{n_j}+(1-a)P_C(I-\lambda_2 B)x_{n_j}\right)\right\|+\left\|x_{n_j}-\omega\right\|\right) \\
&\le \liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\|.
\end{aligned}$$

This is a contradiction, we have

$$\omega\in F(G)=VI(C,A)\cap VI(C,B). \tag{3.14}$$

Since $x_{n_j}\overset{w}{\to}\omega$ as $j\to\infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u-z_0, x_n-z_0\rangle &= \lim_{j\to\infty}\langle u-z_0, x_{n_j}-z_0\rangle \\
&= \langle u-z_0, \omega-z_0\rangle \le 0.
\end{aligned} \tag{3.15}$$

Step 5. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.
From the definition of x_n and Lemma 2.1, we have

$$\begin{aligned}
\left\|x_{n+1}-z_0\right\|^2 &= \left\|\alpha_n(u-z_0)+(1-\alpha_n)\left(P_C(I-\lambda_1 A)M_n-z_0\right)\right\|^2 \\
&\le \left\|(1-\alpha_n)\left(P_C(I-\lambda_1 A)M_n-z_0\right)\right\|^2+2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle \\
&\le (1-\alpha_n)\left\|P_C(I-\lambda_1 A)\left(ax_n+(1-a)P_C(I-\lambda_2 B)x_n\right)-P_C(I-\lambda_1 A)\left(az_0+(1-a)P_C(I-\lambda_2 B)z_0\right)\right\|^2+2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle \\
&= (1-\alpha_n)\left\|G(x_n)-G(z_0)\right\|^2+2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle \qquad (3.16)\\
&\le (1-\alpha_n)\left\|x_n-z_0\right\|^2+2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle.
\end{aligned}$$

From the condition (i), (3.15) and Lemma 2.3, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$. This completes the proof.

4. EXAMPLE AND NUMERICAL RESULTS

In this section, an example is given for supporting Theorem 3.1.

Example 4.1. Let □ be the set of real numbers and let the mappings $A,B:□\to□$ define by $Ax=\frac{x-1}{2}$ and $Bx=\frac{x-1}{5}$, $\forall x\in□$, respectively. Let $x_1,u\in□$ and $\{x_n\}$ be a sequence generated by

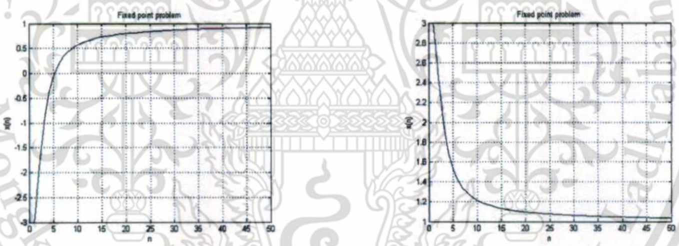
$$x_{n+1}=\alpha_n u+(1-\alpha_n)P_{□}(I-\lambda_1 A)\left(ax_n+(1-a)P_{□}(I-\lambda_2 B)x_n\right),\quad \forall n\geq 1, \tag{4.1}$$

where $\lambda_1\in(0,2)$, $\lambda_2\in(0,2)$, $a\in(0,1)$ and $\alpha_n=\frac{1}{2n}$ for all $n\geq 1$. Then $\{x_n\}$ converge strongly to 1.

Solution It is easy to see that the sequence $\{\alpha_n\}$ satisfies the conditions (i) and (ii) in the theorem 3.1, A and B are 1-inverse strongly monotone mappings and $VI(C,A)\cap VI(C,B)=\{1\}$. From the theorem 3.1, we can conclude that the sequence $\{x_n\}$ converge strongly to 1. For numerical results, choose $\lambda_1=1, \lambda_2=1, a=0.5, u=x_1=-3$ and $u=x_1=3$ in the iterative (4.1) becomes

$$x_{n+1}=\frac{1}{2n}u+(1-\frac{1}{2n})\left((0.5x_n+0.5(x_n-Bx_n))-A\left(0.5x_n+0.5(x_n-Bx_n)\right)\right),\quad \forall n\geq 1, \tag{4.2}$$

The following graphs shows the value of the sequence $\{x_n\}$ defined by (4.2) where $u=x_1=-3$ and $u=x_1=3$



(a) $u=x_1=-3$ (b) $u=x_1=3$

Figure 1: The convergence comparison with different initial value u and x_1.

The following table shows the value of the sequence $\{x_n\}$ defined by (4.2) where $u=x_1=-3$ and $u=x_1=3$

n	$u=x_1=-3$ x_n	$u=x_1=3$ x_n
1	-3.0000	3.0000
2	-1.9000	2.4500
3	-0.9788	1.9894
4	-0.4087	1.7043
5	-0.0547	1.5273
⋮	⋮	⋮
25	0.8456	1.0772

⋮	⋮	⋮
46	0.9184	1.0408
47	0.9202	1.0399
48	0.9219	1.0390
49	0.9236	1.0382
50	0.9251	1.0374

Table 1: The values of $\{x_n\}$ with $n = N = 50$.

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Approximation Method for Fixed Points of Nonlinear Mapping and Variational Inequalities with Application[1]

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Abstract : In this paper, we introduce the new method of iterative scheme $\{x_n\}$ for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without demiclose condition and $T_\omega := (1-\omega)I + \omega T$, when T is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$ in a framework of Hilbert space. Using our main result, we obtain strong convergence theorems involving a finite family of nonspreading mapping and another corollary.

Keywords : quasi-nonexpansive mapping; variational inequality; fixed point; nonspreading mapping.
2010 Mathematics Subject Classification : 46C05; 47H09; 47H10.

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote $F(T)$ by the set of all fixed points of T. Recall that the mapping $T : C \to C$ is

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said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|,$$

for all $x \in C$ and $p \in F(T)$. Fixed point problems have been investigated in the following literature; see [1–3].

A mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

Let $B : C \to H$. The *variational inequality* is to find a point $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, B)$.

The variational inequalities were initially studied and introduced by Stampacchia [4, 5]. This problem is widely used in economics, social sciences and other fields, see for example [6–8].

Let $D_1, D_2 : C \to H$ be two mappings. In 2008, Ceng et al. [9] introduced a problem for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 D_1 z^* + x^* - z^*, x - x^* \rangle \geq 0, \forall x \in C, \\ \langle \lambda_2 D_2 x^* + z^* - x^*, x - z^* \rangle \geq 0, \forall x \in C, \end{cases} \tag{1.2}$$

which is called a system of variational inequalities where $\lambda_1, \lambda_2 > 0$.

In 2013, Kangtunyakarn [10] modified (1.2) for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1 - a) z^*), x - x^* \rangle \geq 0, \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2) x^*, x - z^* \rangle \geq 0, \forall x \in C, \end{cases} \tag{1.3}$$

which is called a modification of system of variational inequalities, for every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$. If $a = 0$, (1.3) reduces to (1.2). He introduced the relation between solutions of (1.3) and fixed point of the mapping G as follows:

Lemma 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$, the following statements are equivalent:*

1. *$(x^*, z^*) \in C \times C$ is a solution of problem (1.3),*
2. *x^* is a fixed point of the mapping $G : C \to C$, i.e., $x^* \in F(G)$, defined by*

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)x),$$

where $z^* = P_C(I - \lambda_2 D_2)x^*$.

Moreover, he introduced a new iterative algorithm $\{x_n\}$ for finding a common element of the set of fixed points of a finite family of κ_i−strictly pseudo-contractive mappings and the set of solutions of problem (1.3) in Hilbert space. The sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(I - \lambda_2 D_2)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C\left(a x_n + (1-a)y_n - \lambda_1 D_1\left(a x_n + (1-a)y_n\right)\right), \forall n \geq 1, \end{cases}$$

where $D_1, D_2 : C \to H$ are d_1, d_2-inverse strongly monotone mappings, respectively, and $S : C \to C$ is S-mapping generated by a finite family of strictly pseudo-contractive mapping and finite real numbers. Under suitable conditions of the parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \lambda_1, \lambda_2, a$, he proved a strong convergence theorem of iterative scheme $\{x_n\}$.

In 2012, Tian and Jin [11] proved the following strong convergence theorem of iterative scheme $\{x_n\}$ generated by (1.4).

Theorem 1.2. *Starting with an arbitrary chosen $x_1 \in H$, let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)T_\omega x_n, \tag{1.4}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Also $\omega \in (0, \frac{1}{2}), T_\omega := (1-\omega)I + \omega T$ with two conditions on T:

1. *$\|Tx - q\| \leq \|x - q\|$ for any $x \in H$, and $q \in F(T)$; this means that T is a quasi-nonexpansive mapping;*

2. *T is demiclosed on H; that is: if $\{y_k\} \subset H, y_k \rightharpoonup z$, and $(I-T)y_k \to 0$, then $z \in F(T)$.*

Then $\{x_n\}$ converges strongly to the $x^* \in F(T)$ which is the unique solution of the VIP:

$$\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0, \forall x \in F(T).$$

Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping T by assuming the following conditions:

(1) $T_\omega := (1-\omega)I + \omega T$,

(2) T is demiclosed on H.

see for example [12] and [13].

Motivated by [10], we introduced the new method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without the conditions (1) and (2) in a framework of Hilbert space. Using our main result, we obtain strong convergence theorems involving a finite family of nonspreading mapping and another corollary.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Throughout this paper, we denote weak and strong convergence by notations "$\rightharpoonup$" and "$\rightarrow$", respectively. For every $x \in H$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the *metric projection* of H onto C.

Remark 2.1. *It is well-known that metric projection P_C has the following properties:*

1. *P_C is firmly nonexpansive, i.e.,*
$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \ \forall x, y \in H.$$
2. *For each $x \in H$,*
$$z = P_C(x) \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \ \forall y \in C.$$

Recall that H satisfies *Opial's condition* [14], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality
$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$
holds for every $y \in H$ with $y \neq x$.

Lemma 2.2. *Let H be a real Hilbert space. Then there holds the following well-known results:*

1. $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2$,
2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$,

for all $x, y \in H$.

Lemma 2.3 ([15]). *Let $(E, \langle .,. \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*
$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = \ & \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ & - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.\end{aligned}$$

Lemma 2.4 ([16]). *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E and let $S : C \rightarrow C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 2.5 ([17]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*
$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 1$$
where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6 ([10]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively, which $VI(C, D_1) \cap VI(C, D_2) \neq \emptyset$. Define a mapping $G : C \to C$ by*

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a) P_C(I - \lambda_2 D_2) x),$$

for every $\lambda_1 \in (0, 2d_1), \lambda_2 \in (0, 2d_2)$ and $a \in (0,1)$. Then $F(G) = VI(C, D_1) \cap VI(C, D_2)$.

Lemma 2.7 ([18]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for* $\lambda > 0$,

$$u = P_C(I - \lambda A) u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

The next result is very important for our main result.

Lemma 2.8. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Then $VI(C, I - T) = F(T)$.*

Proof. It is easy to see that $F(T) \subseteq VI(C, I - T)$.
Let $u \in VI(C, I - T)$, then we have

$$\langle v - u, (I - T)u \rangle \ge 0, \ \forall v \in C. \tag{2.1}$$

Let $v^* \in F(T)$, then we have

$$\|Tu - v^*\|^2 \le \|u - v^*\|^2. \tag{2.2}$$

On the other hand

$$\begin{aligned} \|Tu - v^*\|^2 &= \|(u - v^*) - (I - T)u\|^2 \\ &= \|u - v^*\|^2 - 2\langle u - v^*, (I - T)u \rangle + \|(I - T)u\|^2. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we have

$$\|u - v^*\|^2 - 2\langle u - v^*, (I - T)u \rangle + \|(I - T)u\|^2 \le \|u - v^*\|^2.$$

From (2.1), we have

$$\|(I - T)u\|^2 \le 2\langle u - v^*, (I - T)u \rangle.$$

It follows that $u \in F(T)$. Hence $VI(C, I - T) \subseteq F(T)$. □

Remark 2.9. *From Lemma 2.7 and 2.8, we have*

$$F(T) = VI(C, I - T) = F(P_C(I - \lambda(I - T))),$$

for all $\lambda > 0$.

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n G x_n, \ \forall n \geq 1, \tag{3.1}$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$. Suppose the following conditions holds:

(i) $\alpha_n + \beta_n + \gamma_n = 1,$

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1,$

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1,$

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

Proof. We divide the proof into five steps.
Step 1. We show that $\{x_n\}$ is bounded.
Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda_1 \in (0, 2\alpha)$, we have

$$\begin{aligned}
\|(I - \lambda_1 A)x - (I - \lambda_1 A)y\|^2 &= \|x - y\|^2 - 2\lambda_1 \langle x - y, Ax - Ay\rangle + \lambda_1^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\alpha\lambda_1 \|Ax - Ay\|^2 + \lambda_1^2 \|Ax - Ay\|^2 \\
&= \|x - y\|^2 + \lambda_1(\lambda_1 - 2\alpha) \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 .
\end{aligned}$$

Therefore $(I - \lambda_1 A)$ is a nonexpansive mapping. Similarly, $(I - \lambda_2 B)$ is a nonexpansive mapping. Hence $P_C(I - \lambda_1 A)$ and $P_C(I - \lambda_2 B)$ are nonexpansive mappings. From definition of the mapping G, we have G is a nonexpansive mapping.
Let $x^* \in \mathcal{F}$. From Remark 2.9, we have

$$x^* \in F\left(P_C(I - \lambda_n(I - T))\right).$$

By Lemma 2.6, we have

$$x^* = G(x^*) = P_C(I - \lambda_1 A)(ax^* + (1-a)P_C(I - \lambda_2 B)x^*).$$

Observe that

$$\begin{aligned}
\|Tx_n - Tx^*\|^2 &= \|(x_n - x^*) - (I - T)x_n\|^2 \\
&= \|x_n - x^*\|^2 - 2\langle x_n - x^*, (I - T)x_n\rangle + \|(I - T)x_n\|^2 .
\end{aligned}$$

Since T is a quasi-nonexpansive mapping, we have

$$\|(I-T)x_n\|^2 \le 2\langle x_n - x^*, (I-T)x_n\rangle. \tag{3.2}$$

From the nonexpansiveness of P_C and (3.2), we have

$$\begin{aligned}
\|P_C(I-\lambda_n(I-T))x_n - x^*\|^2 &= \|P_C(I-\lambda_n(I-T))x_n - P_C(I-\lambda_n(I-T))x^*\|^2 \\
&\le \|(I-\lambda_n(I-T))x_n - (I-\lambda_n(I-T))x^*\|^2 \\
&= \|(x_n - x^*) - \lambda_n((I-T)x_n - (I-T)x^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n\langle x_n - x^*, (I-T)x_n\rangle \\
&\quad + \lambda_n^2\|(I-T)x_n\|^2 \\
&\le \|x_n - x^*\|^2 - \lambda_n\|(I-T)x_n\|^2 + \lambda_n^2\|(I-T)x_n\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned} \tag{3.3}$$

Put $M_n = ax_n + (1-a)P_C(I-\lambda_2 B)x_n$ and $W_n = P_C(I-\lambda_1 A)M_n$. From (3.1), we have

$$x_{n+1} = \alpha_n u + \beta_n P_C(I-\lambda_n(I-T))x_n + \gamma_n W_n.$$

From the definition of x_n, (3.3) and nonexpansiveness of G, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + \beta_n(P_C(I-\lambda_n(I-T))x_n - x^*) + \gamma_n(W_n - x^*)\| \\
&\le \alpha_n\|u - x^*\| + \beta_n\|P_C(I-\lambda_n(I-T))x_n - x^*\| + \gamma_n\|W_n - x^*\| \\
&\le \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| \\
&\quad + \gamma_n\|P_C(I-\lambda_1 A)(ax_n + (1-a)P_C(I-\lambda_2 B)x_n) \\
&\quad - P_C(I-\lambda_1 A)(ax^* + (1-a)P_C(I-\lambda_2 B)x^*)\| \\
&= \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|G(x_n) - G(x^*)\| \\
&\le \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\
&= \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\le \max\{\|u - x^*\|, \|x_n - x^*\|\}.
\end{aligned}$$

By induction, we can conclude that

$$\|x_n - x^*\| \le \max\{\|u - x^*\|, \|x_1 - x^*\|\},$$

for all $n \ge 1$. This implies that the sequence $\{x_n\}$ is bounded and so is $\{(I-T)x_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

From the definition of x_n and nonexpansiveness of G, we have

$$\begin{aligned}\|x_{n+1}-x_n\| =&\|(\alpha_n-\alpha_{n-1})u+(\beta_n-\beta_{n-1})P_C(I-\lambda_{n-1}(I-T))x_{n-1}\\ &+\beta_n(P_C(I-\lambda_n(I-T))x_n-P_C(I-\lambda_{n-1}(I-T))x_{n-1})\\ &+\gamma_n(W_n-W_{n-1})+(\gamma_n-\gamma_{n-1})W_{n-1}\|\\ \le&|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))x_{n-1}\|\\ &+\beta_n\|P_C(I-\lambda_n(I-T))x_n-P_C(I-\lambda_{n-1}(I-T))x_{n-1}\|\\ &+\gamma_n\|W_n-W_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|\\ \le&|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))x_{n-1}\|\\ &+\beta_n\|(x_n-x_{n-1})-\lambda_n(I-T)x_n+\lambda_{n-1}(I-T)x_{n-1}\|\\ &+\gamma_n\|P_C(I-\lambda_1A)(ax_n+(1-a)P_C(I-\lambda_2B)x_n)\\ &-P_C(I-\lambda_1A)(ax_{n-1}+(1-a)P_C(I-\lambda_2B)x_{n-1})\|\\ &+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|\\ =&|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))x_{n-1}\|\\ &+\beta_n\|(x_n-x_{n-1})-\lambda_n\left((I-T)x_n-(I-T)x_{n-1}\right)\\ &-\left(\lambda_n-\lambda_{n-1}\right)(I-T)x_{n-1}\|+\gamma_n\|G(x_n)-G(x_{n-1})\|\\ &+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|\\ \le&|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))x_{n-1}\|\\ &+\beta_n\|x_n-x_{n-1}\|+\lambda_n\|(I-T)x_n-(I-T)x_{n-1}\|\\ &+|\lambda_n-\lambda_{n-1}|\,\|(I-T)x_{n-1}\|+\gamma_n\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|\\ =&(1-\alpha_n)\|x_n-x_{n-1}\|+\lambda_n\|(I-T)x_n-(I-T)x_{n-1}\|\\ &+|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda_{n-1}(I-T))x_{n-1}\|\\ &+|\gamma_n-\gamma_{n-1}|\|W_{n-1}\|+|\lambda_n-\lambda_{n-1}|\,\|(I-T)x_{n-1}\|\\ \le&(1-\alpha_n)\|x_n-x_{n-1}\|+\lambda_nM+|\alpha_n-\alpha_{n-1}|M+|\beta_n-\beta_{n-1}|M\\ &+|\gamma_n-\gamma_{n-1}|M+|\lambda_n-\lambda_{n-1}|M,\end{aligned}$$

where $M:=\max_{n\in\mathbb{N}}\{\|(I-T)x_{n+1}-(I-T)x_n\|,\ \|u\|,\ \|P_C(I-\lambda_n(I-T))x_n\|,\ \|W_n\|,$
$\|(I-T)x_n\|\}.$

From the condition $(ii),(iv),(v)$ and Lemma 2.5, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

Step 3. We show that $\lim_{n\to\infty}\|P_C(I-\lambda_n(I-T))x_n-x_n\|=0$.
Since $x^*=P_C(I-\lambda_1A)(ax^*+(1-a)P_C(I-\lambda_2B)x^*)$ and
$M^*=ax^*+(1-a)P_C(I-\lambda_2B)x^*$, we have $x^*=P_C(I-\lambda_1A)M^*$.

Since $x^* \in VI(C,B)$, we obtain

$$\begin{aligned} M^* - x^* &= (1-a)\left(P_C(I-\lambda_2 B)x^* - x^*\right) \\ &= (1-a)\left(P_C(I-\lambda_2 B)x^* - P_C(I-\lambda_2 B)x^*\right) \\ &= 0. \end{aligned} \tag{3.5}$$

From the definition of M_n and M^*, we have

$$\begin{aligned} \|M_n - M^*\| &= \|a(x_n - x^*) + (1-a)\left(P_C(I-\lambda_2 B)x_n - P_C(I-\lambda_2 B)x^*\right)\| \\ &\le a\|x_n - x^*\| + (1-a)\|P_C(I-\lambda_2 B)x_n - P_C(I-\lambda_2 B)x^*\| \\ &\le a\|x_n - x^*\| + (1-a)\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{3.6}$$

From the definition of W_n, we have

$$\begin{aligned} \|W_n - x^*\|^2 &= \|P_C(I-\lambda_1 A)M_n - P_C(I-\lambda_1 A)M^*\|^2 \\ &\le \langle (I-\lambda_1 A)M_n - (I-\lambda_1 A)M^*, W_n - x^*\rangle \\ &= \frac{1}{2}\left(\|(I-\lambda_1 A)M_n - (I-\lambda_1 A)M^*\|^2 + \|W_n - x^*\|^2\right. \\ &\quad \left. - \|(I-\lambda_1 A)M_n - (I-\lambda_1 A)M^* - W_n + x^*\|^2\right) \\ &\le \frac{1}{2}\left(\|M_n - M^*\|^2 + \|W_n - x^*\|^2\right. \\ &\quad \left. - \|(M_n - W_n) - \lambda_1(AM_n - AM^*)\|^2\right), \end{aligned}$$

which implies that

$$\begin{aligned} \|W_n - x^*\|^2 &\le \|M_n - M^*\|^2 - \|(M_n - W_n) - \lambda_1(AM_n - AM^*)\|^2 \\ &= \|M_n - M^*\|^2 - \|M_n - W_n\|^2 + 2\lambda_1\langle M_n - W_n, AM_n - AM^*\rangle \\ &\quad - \lambda_1^2\|AM_n - AM^*\|^2. \end{aligned} \tag{3.7}$$

From the definition of W_n, we have

$$\begin{aligned} &\|W_n - x^*\|^2 \\ &= \|P_C(I-\lambda_1 A)M_n - P_C(I-\lambda_1 A)M^*\|^2 \\ &\le \|(I-\lambda_1 A)M_n - (I-\lambda_1 A)M^*\|^2 \\ &= \|(M_n - M^*) - \lambda_1(AM_n - AM^*)\|^2 \\ &= \|M_n - M^*\|^2 - 2\lambda_1\langle M_n - M^*, AM_n - AM^*\rangle + \lambda_1^2\|AM_n - AM^*\|^2 \\ &\le \|M_n - M^*\|^2 - 2\lambda_1\alpha\|AM_n - AM^*\|^2 + \lambda_1^2\|AM_n - AM^*\|^2 \\ &= \|M_n - M^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|AM_n - AM^*\|^2. \end{aligned} \tag{3.8}$$

From the definition of x_n, (3.3), (3.6) and (3.8), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \le\ & \alpha_n\|u-x^*\|^2+\beta_n\|P_C(I-\lambda_n(I-T))x_n-x^*\|^2 \\
& +\gamma_n\|W_n-x^*\|^2 \\
\le\ & \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2 \\
& +\gamma_n\left(\|M_n-M^*\|^2-\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2\right) \\
\le\ & \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2 \\
& -\gamma_n\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2 \\
=\ & \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2 \\
& -\gamma_n\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\gamma_n\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2 \\
&\quad\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\quad\le \alpha_n\|u-x^*\|^2+\|x_n-x_{n+1}\|\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right). \qquad (3.9)
\end{aligned}$$

From the condition (ii) and (3.4), we derive

$$\lim_{n\to\infty}\|AM_n-AM^*\|=0. \qquad (3.10)$$

From the definition of x_n, (3.3), (3.6) and (3.7), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|u-x^*\|^2+\beta_n\|P_C(I-\lambda_n(I-T))x_n-x^*\|^2+\gamma_n\|W_n-x^*\|^2 \\
&\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2 \\
&\quad+\gamma_n(\|M_n-M^*\|^2-\|M_n-W_n\|^2+2\lambda_1\langle M_n-W_n,AM_n-AM^*\rangle \\
&\quad-\lambda_1^2\|AM_n-AM^*\|^2) \\
&\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2-\gamma_n\|M_n-W_n\|^2 \\
&\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\| \\
&= \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\gamma_n\|M_n-W_n\|^2 \\
&\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\gamma_n\|M_n-W_n\|^2 &\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\| \\
&\le \alpha_n\|u-x^*\|^2+\|x_n-x_{n+1}\|\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right) \\
&\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\|. \qquad (3.11)
\end{aligned}$$

From the condition (ii), (3.4) and (3.10), we derive

$$\lim_{n\to\infty} \|M_n - W_n\| = 0. \tag{3.12}$$

From the property of P_C, we have

$$\begin{aligned}
\|P_C(I-\lambda_2 B)x_n - x^*\|^2 &= \|P_C(I-\lambda_2 B)x_n - P_C(I-\lambda_2 B)x^*\|^2 \\
&\le \langle (I-\lambda_2 B)x_n - (I-\lambda_2 B)x^*, P_C(I-\lambda_2 B)x_n - x^*\rangle \\
&= \frac{1}{2}\ (\|(I-\lambda_2 B)x_n - (I-\lambda_2 B)x^*\|^2 \\
&\quad + \|P_C(I-\lambda_2 B)x_n - x^*\|^2 \\
&\quad - \|(I-\lambda_2 B)x_n - (I-\lambda_2 B)x^* - P_C(I-\lambda_2 B)x_n + x^*\|^2) \\
&\le \frac{1}{2}\ (\|x_n - x^*\|^2 + \|P_C(I-\lambda_2 B)x_n - x^*\|^2 \\
&\quad - \|(x_n - P_C(I-\lambda_2 B)x_n) - \lambda_2(Bx_n - Bx^*)\|^2).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|P_C(I-\lambda_2 B)x_n - x^*\|^2 &\le \|x_n - x^*\|^2 \\
&\quad - \|(x_n - P_C(I-\lambda_2 B)x_n) - \lambda_2(Bx_n - Bx^*)\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - P_C(I-\lambda_2 B)x_n\|^2 \\
&\quad + 2\lambda_2\langle x_n - P_C(I-\lambda_2 B)x_n, Bx_n - Bx^*\rangle \\
&\quad - \lambda_2^2 \|Bx_n - Bx^*\|^2.
\end{aligned} \tag{3.13}$$

By using the same method as (3.8), we have

$$\|P_C(I-\lambda_2 B)x_n - x^*\|^2 \le \|x_n - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|Bx_n - Bx^*\|^2. \tag{3.14}$$

Since $x^* \in VI(C, A)$, we have

$$\begin{aligned}
\|W_n - x^*\|^2 &= \|P_C(I-\lambda_1 A)M_n - P_C(I-\lambda_1 A)x^*\|^2 \\
&\le \|a x_n + (1-a)P_C(I-\lambda_2 B)x_n - x^*\|^2 \\
&= \|a(x_n - x^*) + (1-a)(P_C(I-\lambda_2 B)x_n - x^*)\|^2 \\
&\le a\|x_n - x^*\|^2 + (1-a)\|P_C(I-\lambda_2 B)x_n - x^*\|^2.
\end{aligned} \tag{3.15}$$

From the definition of x_n, (3.3), (3.14) and (3.15), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|P_C(I-\lambda_n(I-T))x_n - x^*\|^2 + \gamma_n\|W_n - x^*\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|P_C(I-\lambda_n(I-T))x_n - x^*\|^2 \\
&\quad + \gamma_n\left(a\|x_n - x^*\|^2 + (1-a)\|P_C(I-\lambda_2 B)x_n - x^*\|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|P_C(I - \lambda_n(I - T))x_n - x^*\|^2 \\
&\quad + \gamma_n \left(a \|x_n - x^*\|^2 + (1-a)\left(\|x_n - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|Bx_n - Bx^*\|^2\right)\right) \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n(a\|x_n - x^*\|^2 + (1-a)\|x_n - x^*\|^2 \\
&\quad - (1-a)\lambda_2(2\beta - \lambda_2)\|Bx_n - Bx^*\|^2) \\
&= \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - (1-a)\lambda_2\gamma_n(2\beta - \lambda_2)\|Bx_n - Bx^*\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1-a)&\lambda_2\gamma_n(2\beta - \lambda_2)\|Bx_n - Bx^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|). \qquad (3.16)
\end{aligned}$$

From the condition (ii) and (3.4), we have

$$\lim_{n\to\infty} \|Bx_n - Bx^*\| = 0. \qquad (3.17)$$

From the definition of x_n, (3.3) and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|P_C(I - \lambda_n(I - T))x_n - x^*\|^2 \\
&\quad + \gamma_n \left(a\|x_n - x^*\|^2 + (1-a)\|P_C(I - \lambda_2 B)x_n - x^*\|^2\right) \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n(a\|x_n - x^*\|^2 \\
&\quad + (1-a)(\|x_n - x^*\|^2 - \|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\quad + 2\lambda_2\langle x_n - P_C(I - \lambda_2 B)x_n, Bx_n - Bx^*\rangle - \lambda_2^2\|Bx_n - Bx^*\|^2)) \\
&\leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad - \gamma_n(1-a)\|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\quad + 2\lambda_2\gamma_n(1-a)\langle x_n - P_C(I - \lambda_2 B)x_n, Bx_n - Bx^*\rangle \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n(1-a)\|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\quad + 2\lambda_2\gamma_n(1-a)\|x_n - P_C(I - \lambda_2 B)x_n\|\|Bx_n - Bx^*\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\gamma_n(1-a)&\|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\lambda_2\gamma_n(1-a)\|x_n - P_C(I - \lambda_2 B)x_n\|\|Bx_n - Bx^*\| \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad + 2\lambda_2\gamma_n(1-a)\|x_n - P_C(I - \lambda_2 B)x_n\|\|Bx_n - Bx^*\|. \qquad (3.18)
\end{aligned}$$

From the condition (ii), (3.4) and (3.17), we derive

$$\lim_{n\to\infty} \|x_n - P_C(I - \lambda_2 B)x_n\| = 0.$$

Since

$$\begin{aligned}\|M_n - x_n\| &= \|ax_n + (1-a)P_C(I-\lambda_2 B)x_n - x_n\| \\ &= (1-a)\|P_C(I-\lambda_2 B)x_n - x_n\|\end{aligned}$$

and $\|P_C(I-\lambda_2 B)x_n - x_n\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|M_n - x_n\| = 0. \tag{3.19}$$

From (3.12) and (3.19), we have

$$\lim_{n\to\infty} \|W_n - x_n\| = 0. \tag{3.20}$$

Since

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \beta_n(P_C(I - \lambda_n(I-T))x_n - x_n) + \gamma_n(W_n - x_n),$$

it implies by the condition (ii), the condition (iii), (3.4) and (3.20) that

$$\lim_{n\to\infty} \|P_C(I - \lambda_n(I-T))x_n - x_n\| = 0. \tag{3.21}$$

Step 4. We show that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_{\mathcal{F}}u$. To show this inequality, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{j\to\infty}\langle u - z_0, x_{n_j} - z_0\rangle.$$

Without loss of generality, we may assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, where $\omega \in C$. First, we show that $\omega \in F(T)$. From Remark 2.9, we have $F(T) = VI(C, I-T) = F(P_C(I - \lambda_{n_j}(I-T)))$. Assume that $\omega \notin F(T)$, that $\omega \neq P_C(I - \lambda_{n_j}(I-T))\omega$. By $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, (3.21) and Opial's property, we have

$$\begin{aligned}\liminf_{j\to\infty}\left\|x_{n_j} - \omega\right\| &< \liminf_{j\to\infty}\left\|x_{n_j} - P_C(I-\lambda_{n_j}(I-T))\omega\right\| \\ &\le \liminf_{j\to\infty}(\left\|x_{n_j} - P_C(I-\lambda_{n_j}(I-T))x_{n_j}\right\| \\ &\quad + \left\|P_C(I-\lambda_{n_j}(I-T))x_{n_j} - P_C(I-\lambda_{n_j}(I-T))\omega\right\|) \\ &\le \liminf_{j\to\infty}(\left\|x_{n_j} - P_C(I-\lambda_{n_j}(I-T))x_{n_j}\right\| \\ &\quad + \left\|x_{n_j} - \omega\right\| + \lambda_{n_j}\left\|(I-T)x_{n_j} - (I-T)\omega\right\|) \\ &\le \liminf_{j\to\infty}\left\|x_{n_j} - \omega\right\|.\end{aligned}$$

This is a contradiction, we have

$$\omega \in F(T). \tag{3.22}$$

Next, we show that $\omega \in VI(C,A) \cap VI(C,B)$. From Lemma 2.6, we have $VI(C,A) \cap VI(C,B) = F(G)$. From (3.20), we have $W_{n_j} \rightharpoonup \omega$ as $j \to \infty$.

$$\begin{aligned}\|W_n - G(W_n)\| &= \|P_C(I-\lambda_1 A)(ax_n + (1-a)P_C(I-\lambda_2 B)x_n) - G(W_n)\| \\ &= \|G(x_n) - G(W_n)\| \\ &\le \|x_n - W_n\|.\end{aligned}$$

From (3.20), we have

$$\lim_{n\to\infty} \|W_n - G(W_n)\| = 0.$$

From $W_{n_j} \rightharpoonup \omega$ as $j \to \infty$ and Lemma 2.4, we have

$$\omega \in F(G) = VI(C,A) \cap VI(C,B). \tag{3.23}$$

From (3.22) and (3.23), we have $\omega \in \mathcal{F}$. Since $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle &= \lim_{j\to\infty}\langle u - z_0, x_{n_j} - z_0\rangle \\ &= \langle u - z_0, \omega - z_0\rangle \le 0. \end{aligned} \tag{3.24}$$

Step 5. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$. From the definition of x_n and $z_0 = P_{\mathcal{F}}u$, we have

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\alpha_n(u - z_0) + \beta_n(P_C(I-\lambda_n(I-T))x_n - z_0) + \gamma_n(W_n - z_0)\|^2 \\ &\le \|\beta_n(P_C(I-\lambda_n(I-T))x_n - z_0) + \gamma_n(W_n - z_0)\|^2 \\ &\quad + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\le \beta_n\|P_C(I-\lambda_n(I-T))x_n - z_0\|^2 + \gamma_n\|W_n - z_0\|^2 \\ &\quad + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\le \beta_n\|x_n - z_0\|^2 + \gamma_n\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &= (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.\end{aligned}$$

From the condition *(ii)*, (3.24) and Lemma 2.5, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$. This completes the proof. □

From our main result, Lemma 1.1 and Lemma 2.6, we have the following corollary:

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I-\lambda_1 A)(ax + (1-a)P_C(I-\lambda_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = F(G) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I-\lambda_n(I-T))x_n + \gamma_n Gx_n, \ \forall n \ge 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions holds:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$ *and* (z_0, y_0) *is a solution of (1.3), where* $y_0 = P_C(I - \lambda_2 B) z_0$.

4 Application

In this section, we prove strong convergence theorems involving the set of fixed points of nonspreading mapping.

A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall x, y \in C.$$

The such mapping is defined by Kohsaka and Takahashi [19].

The following lemma is needed to prove in application.

Lemma 4.1 ([19]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

In 2009, Kangtunyakarn and Suantai [20] introduced the S-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as following. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\ \ \vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an *S-mapping* generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

For every $i = 1, 2, ..., N$. Put $\alpha_3^i = 0$ in Definition 4.1, then the S-mapping is reduced to the K-mapping defined by Kangtunyakarn and Suantai [21] as following. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself, and let $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, ..., N$. We define a mapping $K : C \to C$ as follows:

$$
\begin{aligned}
U_0 &= I,\\
U_1 &= \lambda_1 T_1 + (1-\lambda_1)I,\\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1,\\
U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3)U_2,\\
&\ \ \vdots\\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2},\\
K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}
$$

Such a mapping K is called the *K-mapping* generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$.

Lemma 4.2 ([22]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.*

Lemma 4.3 ([23]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$ and K is quasi-nonexpansive mapping.*

By using these results, we obtain the following theorems

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in$*

$(0,1], \alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ *for all* $j = 1,2,...,N$. *Let* S *be the mapping generated by* $T_1, T_2, ..., T_N$ *and* $\alpha_1, \alpha_2, ..., \alpha_N$. *Let* $A, B : C \to H$ *be* α, β*-inverse strongly monotone mappings, respectively. Define the mapping* $G : C \to C$ *by* $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ *for all* $x \in C$. *Assume* $\mathcal{F} = VI(C,A) \cap VI(C,B) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Suppose that* $x_1, u \in C$ *and let* $\{x_n\}$ *be sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - S))x_n + \gamma_n Gx_n, \ \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in* $[0,1]$. *Suppose the following conditions hold:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}} u$.

Proof. By using Theorem 3.1 and Lemma 4.2, we obtain the conclusion. □

Theorem 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonspreading mappings of* C *into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $\lambda_1, \lambda_2, ..., \lambda_N$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1,2,...,N-1$ *and* $0 < \lambda_N \leq 1$. *Let* K *be the K-mapping generated by* $T_1, T_2, ..., T_N$ *and* $\lambda_1, \lambda_2, ..., \lambda_N$. *Let* $A, B : C \to H$ *be* α, β*-inverse strongly monotone mappings, respectively. Define the mapping* $G : C \to C$ *by* $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ *for all* $x \in C$. *Assume* $\mathcal{F} = VI(C,A) \cap VI(C,B) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Suppose that* $x_1, u \in C$ *and let* $\{x_n\}$ *be sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - K))x_n + \gamma_n Gx_n, \ \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in* $[0,1]$. *Suppose the following conditions hold:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}} u$.

Proof. By using Theorem 3.1 and Lemma 4.3, we obtain the conclusion. □

The following result is direct proved from Theorem 4.4. Therefore, we omit the prove.

Corollary 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space. Let T be a nonspreading mappings of C into itself with $F(T) \neq \emptyset$. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n G x_n, \ \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

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APPENDIX B.

THE NUMERICAL RESULTS

Table 1. The values of the sequence $\{x_n\}$ in Example 4.2.1.

n	x_n		n	x_n	
	$u = x_1 = -1$	$u = x_1 = 5$		$u = x_1 = -1$	$u = x_1 = 5$
1	-1	5	**26**	1.8056	2.1944
2	-0.4562	4.4562	**27**	1.8135	2.1865
3	0.0152	3.9848	**28**	1.8208	2.1792
4	0.3844	3.6156	**29**	1.8276	2.1724
5	0.6701	3.3299	**30**	1.8338	2.1662
6	0.8911	3.1089	**31**	1.8396	2.1604
7	1.0629	2.9371	**32**	1.8451	2.1549
8	1.1975	2.8025	**33**	1.8501	2.1499
9	1.304	2.696	**34**	1.8549	2.1451
10	1.3891	2.6109	**35**	1.8593	2.1407
11	1.4577	2.5423	**36**	1.8635	2.1365
12	1.5138	2.4862	**37**	1.8674	2.1326
13	1.5602	2.4398	**38**	1.8711	2.1289
14	1.5988	2.4012	**39**	1.8747	2.1253
15	1.6315	2.3685	**40**	1.878	2.122
16	1.6593	2.3407	**41**	1.8811	2.1189
17	1.6833	2.3167	**42**	1.8841	2.1159
18	1.7041	2.2959	**43**	1.887	2.113
19	1.7223	2.2777	**44**	1.8897	2.1103
20	1.7384	2.2616	**45**	1.8923	2.1077
21	1.7527	2.2473	**46**	1.8947	2.1053
22	1.7655	2.2345	**47**	1.8971	2.1029
23	1.777	2.223	**48**	1.8993	2.1007
24	1.7874	2.2126	**49**	1.9015	2.0985
25	1.7969	2.2031	**50**	1.9035	2.0965

n	x_n		n	x_n	
	$u = x_1 = -1$	$u = x_1 = 5$		$u = x_1 = -1$	$u = x_1 = 5$
51	1.9055	2.0945	76	1.9375	2.0625
52	1.9074	2.0926	77	1.9384	2.0616
53	1.9092	2.0908	78	1.9392	2.0608
54	1.911	2.089	79	1.94	2.06
55	1.9127	2.0873	80	1.9407	2.0593
56	1.9143	2.0857	81	1.9415	2.0585
57	1.9159	2.0841	82	1.9422	2.0578
58	1.9174	2.0826	83	1.9429	2.0571
59	1.9188	2.0812	84	1.9436	2.0564
60	1.9202	2.0798	85	1.9443	2.0557
61	1.9216	2.0784	86	1.945	2.055
62	1.9229	2.0771	87	1.9456	2.0544
63	1.9242	2.0758	88	1.9463	2.0537
64	1.9254	2.0746	89	1.9469	2.0531
65	1.9266	2.0734	90	1.9475	2.0525
66	1.9277	2.0723	91	1.9481	2.0519
67	1.9289	2.0711	92	1.9487	2.0513
68	1.9299	2.0701	93	1.9492	2.0508
69	1.931	2.069	94	1.9498	2.0502
70	1.932	2.068	95	1.9503	2.0497
71	1.933	2.067	96	1.9508	2.0492
72	1.9339	2.0661	97	1.9514	2.0486
73	1.9349	2.0651	98	1.9519	2.0481
74	1.9358	2.0642	99	1.9524	2.0476
75	1.9367	2.0633	100	1.9529	2.0471

Table 2. The values of sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ in Example 4.2.2.

n	$u = x_1 = -1$			n	$u = x_1 = 5$		
	x_n	u_n	v_n		x_n	u_n	v_n
1	-1	0	0.5	1	5	4	3.5
2	-0.475	0.425	0.35	2	4.475	3.575	3.65
3	0.017	0.7606	0.5127	3	3.983	3.2394	3.4873
4	0.3988	1.0088	0.719	4	3.6012	2.9912	3.281
5	0.6896	1.1936	0.908	5	3.3104	2.8064	3.092
6	0.9116	1.3329	1.0671	6	3.0884	2.6671	2.9329
7	1.0823	1.4392	1.197	7	2.9177	2.5608	2.803
8	1.2149	1.5213	1.3022	8	2.7851	2.4787	2.6978
9	1.3192	1.5856	1.3872	9	2.6808	2.4144	2.6128
10	1.4021	1.6366	1.4564	10	2.5979	2.3634	2.5436
11	1.4688	1.6775	1.5131	11	2.5312	2.3225	2.4869
12	1.5233	1.7108	1.5599	12	2.4767	2.2892	2.4401
13	1.5682	1.7383	1.599	13	2.4318	2.2617	2.401
14	1.6057	1.7612	1.632	14	2.3943	2.2388	2.368
15	1.6374	1.7805	1.66	15	2.3626	2.2195	2.34
16	1.6644	1.797	1.6841	16	2.3356	2.203	2.3159
17	1.6877	1.8111	1.705	17	2.3123	2.1889	2.295
18	1.7079	1.8235	1.7233	18	2.2921	2.1765	2.2767
19	1.7257	1.8343	1.7394	19	2.2743	2.1657	2.2606
20	1.7414	1.8438	1.7537	20	2.2586	2.1562	2.2463
21	1.7554	1.8523	1.7665	21	2.2446	2.1477	2.2335
22	1.7679	1.8599	1.778	22	2.2321	2.1401	2.222
23	1.7792	1.8667	1.7884	23	2.2208	2.1333	2.2116
24	1.7894	1.8729	1.7978	24	2.2106	2.1271	2.2022
25	1.7987	1.8786	1.8064	25	2.2013	2.1214	2.1936

n	$u = x_1 = -1$			n	$u = x_1 = 5$		
	x_n	u_n	v_n		x_n	u_n	v_n
26	1.8072	1.8837	1.8143	**26**	2.1928	2.1163	2.1857
27	1.815	1.8885	1.8216	**27**	2.185	2.1115	2.1784
28	1.8222	1.8928	1.8283	**28**	2.1778	2.1072	2.1717
29	1.8288	1.8968	1.8345	**29**	2.1712	2.1032	2.1655
30	1.835	1.9006	1.8403	**30**	2.165	2.0994	2.1597
31	1.8407	1.904	1.8457	**31**	2.1593	2.096	2.1543
32	1.8461	1.9073	1.8507	**32**	2.1539	2.0927	2.1493
33	1.8511	1.9103	1.8554	**33**	2.1489	2.0897	2.1446
34	1.8557	1.9131	1.8599	**34**	2.1443	2.0869	2.1401
35	1.8601	1.9158	1.864	**35**	2.1399	2.0842	2.136
36	1.8643	1.9183	1.8679	**36**	2.1357	2.0817	2.1321
37	1.8682	1.9206	1.8716	**37**	2.1318	2.0794	2.1284
38	1.8718	1.9228	1.8751	**38**	2.1282	2.0772	2.1249
39	1.8753	1.9249	1.8784	**39**	2.1247	2.0751	2.1216
40	1.8786	1.9269	1.8816	**40**	2.1214	2.0731	2.1184
41	1.8817	1.9288	1.8845	**41**	2.1183	2.0712	2.1155
42	1.8847	1.9306	1.8874	**42**	2.1153	2.0694	2.1126
43	1.8875	1.9323	1.8901	**43**	2.1125	2.0677	2.1099
44	1.8902	1.9339	1.8926	**44**	2.1098	2.0661	2.1074
45	1.8927	1.9355	1.8951	**45**	2.1073	2.0645	2.1049
46	1.8952	1.9369	1.8974	**46**	2.1048	2.0631	2.1026
47	1.8975	1.9383	1.8997	**47**	2.1025	2.0617	2.1003
48	1.8997	1.9397	1.9018	**48**	2.1003	2.0603	2.0982
49	1.9019	1.941	1.9038	**49**	2.0981	2.059	2.0962
50	1.9039	1.9422	1.9058	**50**	2.0961	2.0578	2.0942
51	1.9059	1.9434	1.9077	**51**	2.0941	2.0566	2.0923

n	$u = x_1 = -1$			n	$u = x_1 = 5$		
	x_n	u_n	v_n		x_n	u_n	v_n
52	1.9078	1.9445	1.9095	**52**	2.0922	2.0555	2.0905
53	1.9096	1.9456	1.9113	**53**	2.0904	2.0544	2.0887
54	1.9113	1.9467	1.9129	**54**	2.0887	2.0533	2.0871
55	1.913	1.9477	1.9146	**55**	2.087	2.0523	2.0854
56	1.9146	1.9486	1.9161	**56**	2.0854	2.0514	2.0839
57	1.9162	1.9496	1.9176	**57**	2.0838	2.0504	2.0824
58	1.9177	1.9505	1.9191	**58**	2.0823	2.0495	2.0809
59	1.9191	1.9514	1.9205	**59**	2.0809	2.0486	2.0795
60	1.9205	1.9522	1.9218	**60**	2.0795	2.0478	2.0782
61	1.9218	1.953	1.9231	**61**	2.0782	2.047	2.0769
62	1.9232	1.9538	1.9244	**62**	2.0768	2.0462	2.0756
63	1.9244	1.9546	1.9256	**63**	2.0756	2.0454	2.0744
64	1.9256	1.9553	1.9268	**64**	2.0744	2.0447	2.0732
65	1.9268	1.956	1.9279	**65**	2.0732	2.044	2.0721
66	1.928	1.9567	1.929	**66**	2.072	2.0433	2.071
67	1.9291	1.9574	1.9301	**67**	2.0709	2.0426	2.0699
68	1.9301	1.958	1.9311	**68**	2.0699	2.042	2.0689
69	1.9312	1.9586	1.9322	**69**	2.0688	2.0414	2.0678
70	1.9322	1.9592	1.9331	**70**	2.0678	2.0408	2.0669
71	1.9332	1.9598	1.9341	**71**	2.0668	2.0402	2.0659
72	1.9341	1.9604	1.935	**72**	2.0659	2.0396	2.065
73	1.9351	1.961	1.9359	**73**	2.0649	2.039	2.0641
74	1.936	1.9615	1.9368	**74**	2.064	2.0385	2.0632
75	1.9368	1.962	1.9377	**75**	2.0632	2.038	2.0623
76	1.9377	1.9625	1.9385	**76**	2.0623	2.0375	2.0615
77	1.9385	1.963	1.9393	**77**	2.0615	2.037	2.0607

n	$u = x_1 = -1$			n	$u = x_1 = 5$		
	x_n	u_n	v_n		x_n	u_n	v_n
78	1.9393	1.9635	1.9401	**78**	2.0607	2.0365	2.0599
79	1.9401	1.964	1.9409	**79**	2.0599	2.036	2.0591
80	1.9409	1.9645	1.9416	**80**	2.0591	2.0355	2.0584
81	1.9416	1.9649	1.9423	**81**	2.0584	2.0351	2.0577
82	1.9424	1.9654	1.9431	**82**	2.0576	2.0346	2.0569
83	1.9431	1.9658	1.9437	**83**	2.0569	2.0342	2.0563
84	1.9438	1.9662	1.9444	**84**	2.0562	2.0338	2.0556
85	1.9444	1.9666	1.9451	**85**	2.0556	2.0334	2.0549
86	1.9451	1.967	1.9457	**86**	2.0549	2.033	2.0543
87	1.9457	1.9674	1.9464	**87**	2.0543	2.0326	2.0536
88	1.9464	1.9678	1.947	**88**	2.0536	2.0322	2.053
89	1.947	1.9682	1.9476	**89**	2.053	2.0318	2.0524
90	1.9476	1.9685	1.9482	**90**	2.0524	2.0315	2.0518
91	1.9482	1.9689	1.9488	**91**	2.0518	2.0311	2.0512
92	1.9488	1.9692	1.9493	**92**	2.0512	2.0308	2.0507
93	1.9493	1.9696	1.9499	**93**	2.0507	2.0304	2.0501
94	1.9499	1.9699	1.9504	**94**	2.0501	2.0301	2.0496
95	1.9504	1.9702	1.9509	**95**	2.0496	2.0298	2.0491
96	1.9509	1.9705	1.9514	**96**	2.0491	2.0295	2.0486
97	1.9515	1.9708	1.952	**97**	2.0485	2.0292	2.048
98	1.952	1.9711	1.9524	**98**	2.048	2.0289	2.0476
99	1.9525	1.9714	1.9529	**99**	2.0475	2.0286	2.0471
100	1.9529	1.9717	1.9534	**100**	2.0471	2.0283	2.0466

APPENDIX C.

MATLAB PROGRAMMING

An example of Matlab program for finding numerical solutions of Theorem 3.1 with all parameters satisfy all conditions of Theorem 3.1.

```
function Theorem1
    x=[], p=[];
    fprintf('Program computes Theorem 3.1 \n \n');
    fprintf('Input Value:\n');
    fprintf('************************************************\n');
    u  = input('u : ');
    rho1  = input('rho1 : ');
    rho2  = input('rho2 : ');
    a = input('a : ');
    x(1) = input('x(1) : ');
    N = input('round of calculation :  ');
    fprintf('************************************************\n\n');
    for i=1:(N-1)
        be = (a*x(i))+((1-a)*(x(i)-(rho2*B(x(i)))));
        x(i+1) = (Alpha_n(i)*u)+(Beta_n(i)*(x(i)-(Lamma_n(i)*(x(i)-
                T(x(i))))))+(Gamma_n(i)*(be-(rho1*A(be))));
    end
    p = 1:1:N;
    fid = fopen('D:\myoutput1_2.txt','w');
    fprintf(fid,'n \t\t x(n) \r \n');
    for i=1:N
        fprintf(fid,'% d \t\t % .4f \r \n',p(i),x(i));
    end
    fclose(fid);
    plot(p,x);
```

```
        grid on;
        title ('Fixed point problem');
        ylabel ('x (n)');
        xlabel ('n');
end
function k1=A(t)
        k1=(t-2)/3;
end
function k2=B(t)
        k2=(t-2)/5;
end
function k3=Alpha_n(n)
        k3=1/(2*n);
end
function k4=Beta_n(n)
        k4=((3*n)-1)/(16*n);
end
function k5=Gamma_n(n)
        k5=((13*n)-7)/(16*n);
end
function k6=Lamma_n(n)
        k6=1/(2*(n^2));
end
function k7=T(n)
        k7=(n+2)/2;
end
```

Example of Matlab program to find numerical solutions of Theorem 3.3 with all parameters satisfy all conditions of Theorem 3.3.

```
function Theorem2
        x=[], p=[], u=[],v=[];
        fprintf('Program computes the fixed point of system of variational inequalities \n \n');
        fprintf('Input Value:\n');
        fprintf('**************************************************\n');
        ustart  = input('u : ');
        lamda1  = input('lamda1 : ');
        lamda2  = input('lamda2 : ');
        a = input('a : ');
        x(1) = input('x(1) : ');
        N = input('round of calculation :  ');
        fprintf('**************************************************\n\n');
        for i=1:(N-1)
                u(i) =((4*r_n(i))+x(i))/(1+(2*r_n(i)));
                v(i) =((6*s_n(i))+x(i))/(1+(3*s_n(i)));
                be = (a*v(i))+((1-a)*(v(i)-(lamda2*B(v(i)))));
                x(i+1) = (Alpha_n(i)*ustart)+(Beta_n(i)*x(i))+(Gamma_n(i)*(u(i)-
                        (Lamma_n(i)*(u(i)-T(u(i))))))+(Delta_n(i)*(be-(lamda1*A(be))));
        end
        p = 1:1:N;
        q = 1:1:(N-1);
        fid = fopen('D:\myoutputThm2_2.txt','w');
        fprintf(fid,'n \t\t x (n) \t\t\t u (n) \t\t\t v(n) \r \n');
        for i=1:(N-1)
                fprintf(fid,'% d \t\t % .4f \t\t % .4f \t\t % .4f \r \n',p(i),x(i),u(i),v(i));
```

```
        end
        fprintf(fid,'% d \t\t % .4f \r \n',p(i+1),x(i));
        fclose(fid);
        plot(p,x,'r',q,u,'b',q,v,'k')
        grid on;
        hleg1 = legend('x (n)','u (n)','v (n)');
        title ('Fixed point problem');
        ylabel ('x(n),u(n),v (n)');
        xlabel ('n');
end

function k1=A(t)
        k1=(t-2)/3;
end

function k2=B(t)
        k2=(t-2)/5;
end

function k3=Alpha_n(n)
        k3=1/(2*n);
end

function k4=Beta_n(n)
        k4=((3*n)-1)/(16*n);
end

function k5=Gamma_n(n)
        k5=((10*n)-3)/(16*n);
end
```

```
function k6=Delta_n(n)

        k6=((3*n)-4)/(16*n);

end

function k7=Lamma_n(n)

        k7=1/(2*(n^2));

end

function k8=r_n(n)

        k8=n/((3*n)+1);

end

function k9=s_n(n)

        k9=1/(3*n);

end

function k10=T(n)

        k10=(n+2)/2;

end
```

Biography

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