DETERMINANT OF ADJACENCY MATRIX OF SQUARE CYCLE GRAPH

NITIPHOOM ADSAWATITHISAKUL

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หัวข้อวิทยานิพนธ์	ดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟวงกำลังสอง
นักศึกษา	นายนิติภูมิ อัศวธิติสกุล
รหัสประจำตัว	53650703
ปริญญา	วิทยาศาสตรมหาบัณฑิต
สาขาวิชา	คณิตศาสตร์ประยุกต์
พ.ศ.	2556
อาจารย์ที่ปรึกษา	ดร.เดชา สมนะ

บทคัดย่อ

กราฟวงกำลังสอง C_n^2 คือ กราฟที่มีจำนวนจุดยอด n จุดและจุดยอดสองจุด u และ v มีเส้นเชื่อมกันก็ต่อเมื่อระยะทางระหว่าง u และ v มีค่าไม่เกิน 2 โดยในวิทยานิพนธ์นี้ได้ดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟวงกำลังสอง ดังนี้

$$\det(A(C_n^2)) = \begin{cases} 0 & ; n \equiv 0,2,4 \pmod 6 \\ 16 & ; n \equiv 3 \pmod 6 \\ 4 & ; n \equiv 1,5 \pmod 6 \end{cases}$$

เมื่อ n เป็นจำนวนเต็มบวก

นอกจากนี้ เราได้มีการหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟที่เกิดจากการดำเนินการการบวกระหว่างกราฟวงกำลังสองกับกราฟ G $\left(C_n^2 + G\right)$ และกราฟวงกำลังสอง $C_{n_1}^2$ ประชิดกับวิถี P_{n_2} ที่จุด v_1 $\left(C_{n_1}^2 \cup_{v_1} P_{n_2}\right)$ โดยที่ $v_1 \in V\left(P_{n_2}\right)$

คำสำคัญ : ดีเทอร์มิแนนต์ กราฟวงกำลังสอง เมทริกซ์ประชิด การดำเนินการการบวก

Thesis Title	Determinant of Adjacency Matrix of Square Cycle Graph
Student	Nitiphoom Adsawatithisakul
Student ID	53650703
Degree	Master of Science
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Thesis Advisor	Dr. Decha Samana

ABSTRACT

Square cycle graph denoted by C_n^2 is a graph which has n vertices and two vertices u and v are adjacent if and only if distance between u and v not greater than 2. In this thesis, we show that the determinant of adjacency matrix of square cycle graph C_n^2 are as follows

$$\det(A(C_n^2)) = \begin{cases} 0 & ; n \equiv 0,2,4 (\bmod 6) \\ 16 & ; n \equiv 3 (\bmod 6) \\ 4 & ; n \equiv 1,5 (\bmod 6) \end{cases}$$

where n is a positive integer.

Furthermore, we show that the determinant of adjacency matrix of graph $C_n^2 \dotplus G$ and the determinant of adjacency matrix of graph $C_{n_1}^2 \cup_{v_1} P_{n_2}$ which is a square cycle graph $C_{n_1}^2$ adjacent a path P_{n_2} at vertex v_1, where $v_1 \in V\left(P_{n_2}\right)$.

Keywords : Determinant, Square cycle graph, Adjacency matrix, Sum operation

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CHAPTER I

INTRODUCTION

Associated with every square matrix is a number called its determinant. The determinant of a matrix is a tool that is used in many branches of mathematics, science and engineering. Determinant of graph in graph theory and linear algebra which two subjects are mixed and we will change graph into matrix for example, adjacency matrix, incident matrix, laplacian matrix, distance matrix, detour matrix etc. Example

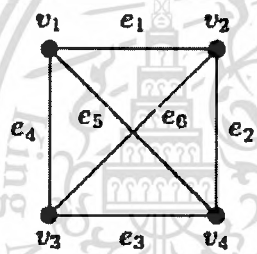


Figure 1.1 Complete graph K_4

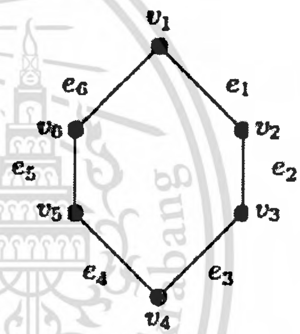


Figure 1.2 Cycle graph C_6

Form figure 1.1, we write adjacency matrix, incident matrix and laplacian matrix, respectively

$$A(K_4)=\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad B(K_4)=\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$Q(K_4)=\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

and figure 1.2,

$$A(C_6)=\begin{bmatrix}0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\end{bmatrix} \qquad B(C_6)=\begin{bmatrix}0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\end{bmatrix}$$

$$Q(C_6)=\begin{bmatrix}-2&1&0&0&0&1\\1&-2&1&0&0&0\\0&1&-2&1&0&0\\0&0&1&-2&1&0\\0&0&0&1&-2&1\\1&0&0&0&1&-2\end{bmatrix}$$

Furthermore,

$\det(A(K_4))=-3$,

$\det(B(K_4))$ is not exist because this matrix is not a square matrix,

$\det(Q(K_4))=0$,

$\det(A(C_6))=-4$,

$\det(B(C_6))=-4$,

and $\det(Q(C_6))=0$.

From this example, we see that determinant of graph is dependent on the numbers of vertices and edges in graph. If the numbers of vertices and edges in graph are large, determinant of graph will find difficult. It is interesting to study determinant of adjacency matrix of square cycle graph. Moreover, we study determinant of adjacency matrix of some graphs which obtained from operation between two graphs.

In 1974 and In 1980, N.Biggs and D.M. Cvetkovic et.al have published about determinant of adjacency matrix of some graph such as K_n, P_n, C_n and W_n. In 2002, M.Doob had found determinant of adjacency matrix of graph $G(r,t)$. In 2009, A.Abdollahi had found determinant of adjacency matrices of graphs. In 2011, B. Gyurov and J. Cloud had found determinant of adjacency matrix of pin wheel graph.

This thesis is divided into 5 chapters. Chapter I, introduction, tells about the aim and reason for doing this thesis. Chapters II, preliminaries and basic concepts, compiles the data and fundamental of this thesis. The process of finding determinant of adjacency matrix of square cycle graph is in chapter III. The results of determinant of adjacency matrix of some graphs are in chapter IV and the conclusions of this thesis are in chapter V.

CHAPTER II

PRELIMINARIES AND BASIC CONCEPTS

In this chapter, we introduce preliminaries which is a basic knowledge about this thesis. We divide into 2 sections which the first section is linear algebra and the second section is graph theory.

2.1 Linear algebra

Linear algebra is a branch of mathematics that play a central role in modern mathematics, and also is importance to engineers and physical, social, and behavioral scientists. We begin with introduction about basic concepts of linear algebra which is used in this thesis.

2.1.1 Linear equation

An equation in the variables x and y that can be written in the form $ax+by=c$, where a,b and c are real constants (a and b not both zero), is called a **linear equation**. The graph of such an equation is a straight line in the xy plane. Consider the system of two linear equations,

$$x+y=5$$

$$2x-y=4$$

A pair of values of x and y that satisfies both equations is called a **solution**. It can be seen by substitution that $(x,y)=(3,2)$ is a solution of this system.

2.1.2 Matrix

Definition 2.1 Matrix consists of rows and columns. Rows are labeled from the top of matrix, columns from the left. The location of an element in a matrix is described by giving the row and the column in which it lies. The element in row i, column j of matrix A is denoted by a_{ij}.

We refer to a_{ij} as the (i,j)-th element of the matrix A. We can visualize an arbitrary $m \times n$ matrix A as in figure 2.1.

If the number of rows, m, is equal to the number of columns, n, matrix A is said to be a **square matrix**. The elements of a square matrix A, where the subscripts are equal, namely $a_{11}, a_{22}, \ldots, a_{nn}$ form the main diagonal, see figure 2.2

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Figure 2.1 $m \times n$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Figure 2.2 square $n \times n$ matrix A

Example 1 Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -3 & 4 \\ 2 & 7 & 5 \end{bmatrix}$$

a_{12} is the element in row 1, column 2, thus $a_{12} = -2$. We see that $a_{23} = 4$ and $a_{31} = 2$. Matrix A is a square matrix. The main diagonal of matrix A consists of the elements $a_{11} = 1, a_{22} = -3$ and $a_{33} = 5$.

2.1.3 System of linear equations

We can write a general system of m linear equations in n variables

$$\begin{matrix} a_{11}x_1 + \ldots + a_{1n}x_x = b_1 \\ \vdots \quad \ldots \quad \vdots \quad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{matrix}$$

Using matrix notation as follows.

Represent each side of this equation as a column matrix,

$$\begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \quad \ddots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

The left matrix can be written as a product of the matrix of coefficients A and a column matrix of variables X. Let the column matrix of constants be B.

$$\overset{A}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \overset{X}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \overset{B}{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}$$

Thus, we can write the system of equations in matrix form

$$AX = B$$

Example 2

$$3x_1 + 2x_2 + 5x_3 = 7$$
$$x_1 - 8x_2 + 4x_3 = 9$$
$$2x_1 + 6x_2 - 7x_3 = -2$$

can be written

$$\begin{bmatrix} 3 & 2 & -5 \\ 1 & -8 & 4 \\ 2 & 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}.$$

We are now in a position to show all homogeneous systems of linear equations and nonhomogeneous system of linear equations.

Let $AX = B$ be a system of linear equations. Then

$AX = B$ and $B = 0$ be such a **homogeneous system of linear equations,**

$AX = B$ and $B \neq 0$ be such a **nonhomogeneous system of linear equations.**

2.1.4 Determinant

Definition 2.2 The **determinant** of a 2×2 matrix A is denoted by $|A|$ and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Observe that the determinant of a 2×2 matrix is given by the difference of the product of the two diagonals of the matrix. The notation $\det(A)$ or $|A|$ is also used for the determinant of A.

The determinant of a 3×3 matrix is defined in terms of determinants of 2×2 matrices. The determinant of a 4×4 matrix is defined in terms of determinants of 3×3 matrices, and so on. For these definitions, we need the following concepts of minor and cofactor.

Definition 2.3 Let A be a square matrix. The **minor** of the element a_{ij} is denoted by M_{ij} and is the determinant of the matrix that remains after deleting row i and column j of matrix A.

Definition 2.4 The **cofactor** of a_{ij} is denoted by C_{ij} and is given by $C_{ij}=(-1)^{i+j}M_{ij}$.

Note that the minor and cofactor differ in at most sign.

Definition 2.5 The **determinant of a square matrix** is the sum of the product of the elements of the first row and their cofactors.

If matrix A is 3×3, $|A|=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$

If matrix A is 4×4, $|A|=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}+a_{14}C_{14}$

$\vdots$

If matrix A is $n\times n$, $|A|=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}+\cdots+a_{1n}C_{1n}$

These equations are called cofactor expansions of $|A|$.

We have defined the determinant of a matrix in terms of its first row. It can be shown that the determinant can be found according to the following rules, using any row or column.

Definition 2.6 The determinant of a square matrix is the sum of the products of the elements of any row or column and their cofactors.

i-th row expansion : $|A|=a_{i1}C_{i1}+a_{i2}C_{i2}+\ldots+a_{in}C_{in}$

j-th column expansion : $|A|=a_{1j}C_{1j}+a_{2j}C_{2j}+\ldots+a_{nj}C_{nj}$

Example 3 Consider the matrix

$$A=\begin{bmatrix}3 & -1 & 2\\ 4 & 5 & 6\\ 7 & 1 & 2\end{bmatrix}$$

find M_{12}, M_{23}, M_{31} and $\det A$ by using cofactor.

Solution Minor M_{12} is the determinant of the matrix that remains after deleting row 1 and column 2 of matrix A, then $M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = -34$.

Minor M_{23} is the determinant of the matrix that remains after deleting row 2 and column 3 of matrix A, then $M_{23} = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 10$.

And minor M_{31} is the determinant of the matrix that remains after deleting row 3 and column 1 of matrix A, then $M_{31} = \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -16$.

$\det A$ is the sum of the product of the elements of the first row and their cofactors and uses i th row expansion : $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$, then

$$\begin{aligned}
\det A &= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13} \\
&= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\
&= 3\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & 5 \\ 7 & 1 \end{vmatrix} \\
&= 3(4) - (-1)(-34) + 2(-31) \\
&= -84.
\end{aligned}$$

2.1.5 Eigenvalues and Eigenvectors

Definition 2.7 Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there exists a nonzero vector x in $\mathbb{R}^n$ such that

$$Ax = \lambda x.$$

The vector x is called an **eigenvector** corresponding to λ.

Computation of eigenvalue and eigenvector

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector x. Thus

$$Ax = \lambda x$$

$$Ax - \lambda x = 0.$$

giving

$$(A-\lambda I_n)x=0.$$

This matrix equation represents a system of homogeneous linear equations $(A-\lambda I_n)x=0$ is a solution of this system. However, eigenvectors have been defined to be nonzero vectors. Further, nonzero solutions to this system of equations can only exist if the matrix of coefficients is singular, $|A-\lambda I_n|=0$. Hence, solving the equation $|A-\lambda I_n|=0$ for λ leads to all the eigenvalues of A.

On expanding the determinant $|A-\lambda I_n|$, we get a polynomial in λ. This polynomial is called the **characteristic polynomial** of A. The equation $|A-\lambda I_n|=0$ is called the **characteristic equation** of A.

Theorem 2.8 [6] Let $\lambda_1,\ldots,\lambda_n$ be eigenvalues of a square matrix A. Then

$$\det(A)=\lambda_1\lambda_2\ldots\lambda_n$$

2.2 Graph Theory

Graph theory is one of the major areas of combinatorics, it is developing into one of the major areas. In addition to its growing interest and increased importance as a mathematical subject, we introduce many of the basic concepts of graph which is used in this thesis.

Definition 2.9 A **graph** G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. We can write $G=(V,E)$.

Definition 2.10 A **multiple edges** are edges having the same pair of endpoints.

Definition 2.11 A **loop** is an edge whose endpoints are equal.

Definition 2.12 A **simple graph** is a graph having no loop and multiple edges.

Example 4

G H

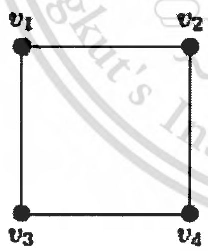
Figure. 2.3 graph G and graph H

From figure 2.3, graph G is a simple graph because it hasn't a loop and multiple edges and graph H isn't a simple graph because it has a loop and multiple edges.

Definition 2.13 The **degree of a vertex** v in a simple graph G is the number of edges of G incident with v, which is denoted by $\deg_G v$ or $\deg v$.

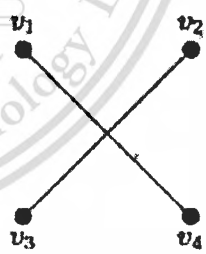
Definition 2.14 The **complement** $\bar{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.

Example 5



G

Figure 2.4 Graph G



$\bar{G}$

Figure 2.5 Complement graph $\bar{G}$

Definition 2.15 The **adjacency matrix** of graph G, written $A(G)$, is the $n \times n$ matrix in which entry a_{ij} where

$$a_{ij} = \begin{cases} 1; \text{ if } v_i v_j \in E(G) \\ 0; \text{ if } v_i v_j \notin E(G) \end{cases}$$

Remark An adjacency matrix is determined by a vertex ordering. Every adjacency matrix is symmetric ($a_{ij} = a_{ji}$ for all i, j).

From figure 2.4, the adjacency matrix of G is

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} .$$

Definition 2.16 A $v_1 - v_k$ **walk** is a list $v_1, e_1, v_2, e_2, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \le i \le k$,the edge e_i has endpoints v_{i-1} and v_i .

Remark Often only the vertices of a walk are indicated because the edges present are then evident.

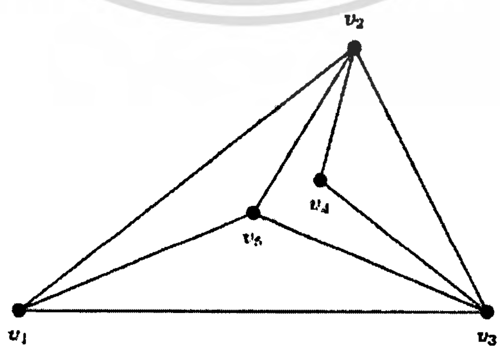
Definition 2.17 A $v_i - v_k$ **trail** is a $v_i - v_k$ walk in which no edge is repeated.

Definition 2.18 A $v_i - v_k$ **path** is a $v_i - v_k$ walk in which no vertex is repeated.

Remark A $v_i - v_k$ walk is a closed or open walk depending on whether $v_i = v_k$ or $v_i \neq v_k$.

Definition 2.19 If G has a $v_i - v_k$ path, then the **distance** from v_i to v_j , written $d_G(v_i, v_j)$ or simply $d(u,v)$, is the least length of a $v_i - v_k$ path.

Example 6



G

Figure 2.6 graph G

From figure 2.6, $v_1, v_2, v_3, v_2, v_5, v_3, v_4$ is a $v_1 - v_4$ walk that is not a $v_1 - v_4$ trail, v_1, v_3, v_4 is a $v_1 - v_4$ path and $v_1, v_2, v_5, v_1, v_3, v_4$ is a $v_1 - v_4$ trail that is not a $v_1 - v_4$ path.

Definition 2.20 A graph G is **bipartite** if $V(G)$ is the union of two disjoint (possibly empty) independent sets called partite sets of G.

Definition 2.21 A graph G is **k - partite** if $V(G)$ can be expressed as the union of k (possibly empty) independent sets.

Example 7

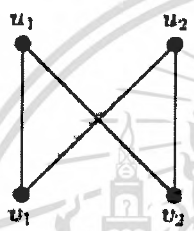


Figure 2.7 Bipartite graph

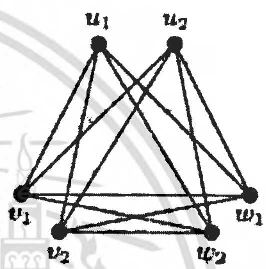


Figure 2.8 Tripartite graph

Definition 2.22 A graph G is **regular of degree** r if $\deg v = r$ for each vertex v of graph G. Such graphs are called r **- regular**.

Definition 2.23 A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if only if they are consecutive in the list, a path with n vertices is denoted by P_n.

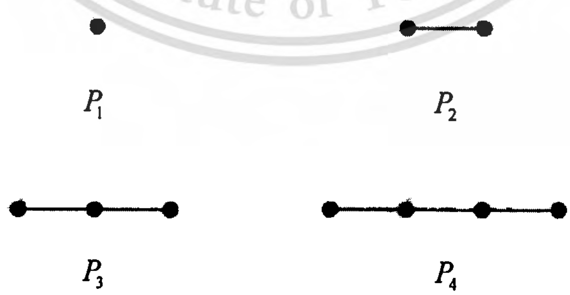


Figure 2.9 Paths P_1, P_2, P_3 and P_4

Definition 2.24 A **cycle** is a graph with equal numbers of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle, a cycle with n vertices is denoted by C_n.

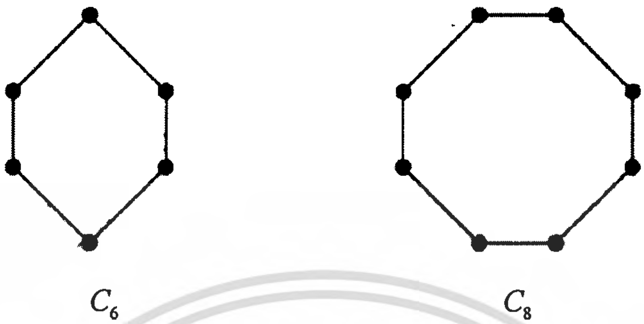


Figure 2.10 Cycle graphs C_6 and C_8

Definition 2.25 A **complete graph** is a simple graph whose vertices are pairwise adjacent; a complete graph with n vertices is denoted by K_n.

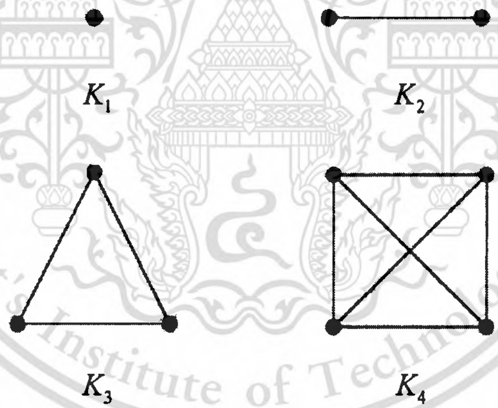


Figure 2.11 Complete graphs K_1, K_2, K_3 and K_4

Definition 2.26 A **complete bipartite graph** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the partite sets have m and n vertices, a complete bipartite graph is denoted by $K_{m,n}$.

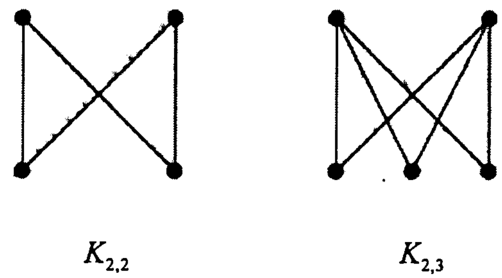


Figure 2.12 Complete bipartite graphs $K_{2,2}$ and $K_{2,3}$

Definition 2.27 A **wheel** is a graph that contains $n+1$ vertices, with n vertices forming a cycle of length n and the $(n+1)^{th}$ vertex adjacent to all n vertices of the C_n. We denote a wheel with $n+1$ vertices by W_n.

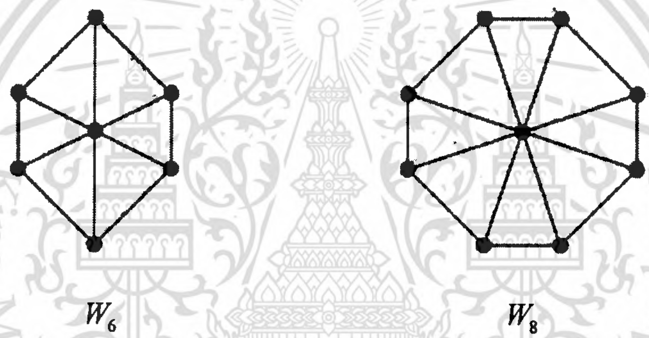
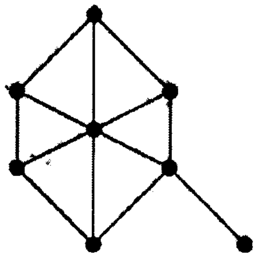
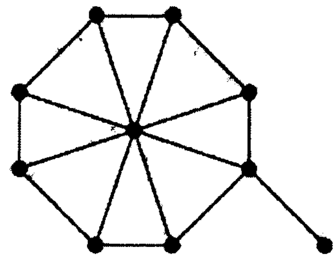


Figure 2.13 A wheel graph

Definition 2.28 [7] A **Pin – wheel graph** is a graph that contain $n+2$ vertices, with n vertices forming a cycle of length n, the $(n+1)^{th}$ vertex adjacent to all n vertices of the C_n and the $(n+2)^{th}$ vertex is adjacent only to single vertex of C_n. We denote the pin – wheel graph with $n+2$ vertices by W_n'.



W_6'



W_8'

Figure 2.14 Pin wheel graphs W_6' and W_8'

Definition 2.29 [7] A **generalized pin – wheel graph**, is denoted by W_n^m, is obtained from a Pin – wheel graph where the $(n+2)^{\text{th}}$ vertex is replaced by a path P_m of length $m-1$, or symbolically $W_n^m = W_n' \cup_{x_1} P_m$.

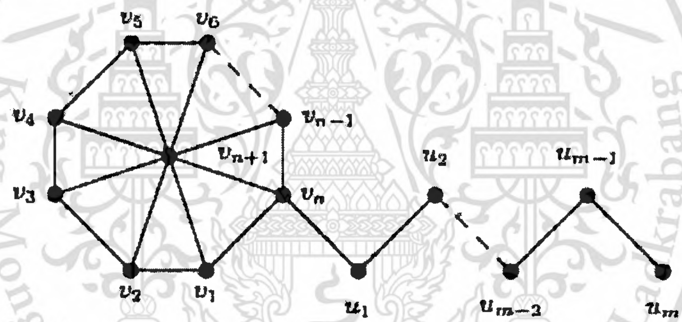
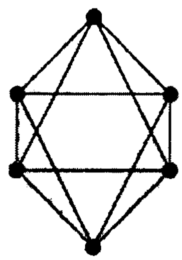


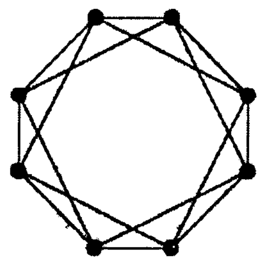
Figure 2.15 Generalized pin wheel graph

Definition 2.30 A **d - th power** of cycle is a simple graph and two vertices u and v are adjacent if and only if distance between u and v not greater than d, the d-th power of cycle C_n is denoted by C_n^d.

Definition 2.31 A **square cycle graph** is a simple graph and two vertices u and v are adjacent if and only if distance between u and v not greater than 2, the square cycle graph C_n is denoted C_n^2.



C_6^2

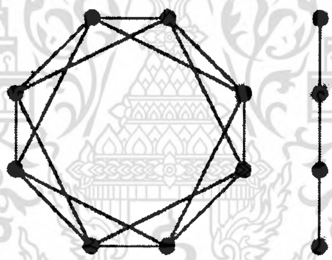


C_8^2

Figure 2.16 Square cycle graphs C_6^2 and C_8^2

Definition 2.32 [5] A **graph** $G(r,t)$ is a r- regular graph which has $n=(r-1)t+2$ vertices when t is a positive integer.

Definition 2.33 [4] The **direct sum** of graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, denoted by $G_1 \dot{+} G_2$, is the graph $G=(V,E)$ for which $V=V_1 \cup V_2$ and $E=E_1 \cup E_2$.



$C_8^2 \dot{+} P_4$

Figure 2.17 A direct sum graph

Definition 2.34 Let n_1 and n_2 be positive integers with $n_1 \geq 6$. A graph $C_{n_1}^2 \bigcup_{v_1} P_{n_2}$ is a graph obtained from $C_{n_1}^2 \dot{+} P_{n_2}$ by joining one vertex of a square cycle $C_{n_1}^2$ with one endpoint of a path P_{n_2}, say v_1.

Definition 2.35 [2] A **circulant matrix** is a matrix which all main diagonal are zero and entries in first row satisfy $a_{1j}=a_{1,(n-j+2)}$ for $j=2,\ldots,n$ and $a_{ij}=a_{i+1,j+1}$.

Definition 2.36 [2] A **circulant graph** is a graph G whose vertices can be ordered so that the adjacency matrix $A(G)$ is a circulant matrix.

From figure 2.16, we write adjacency matrices as

$$A(C_6^2)=\begin{bmatrix}0&1&1&0&1&1\\1&0&1&1&0&1\\1&1&0&1&1&0\\0&1&1&0&1&1\\1&0&1&1&0&1\\1&1&0&1&1&0\end{bmatrix}\qquad A(C_8^2)=\begin{bmatrix}0&1&1&0&0&0&1&1\\1&0&1&1&0&0&0&1\\1&1&0&1&1&0&0&0\\0&1&1&0&1&1&0&0\\0&0&1&1&0&1&1&0\\0&0&0&1&1&0&1&1\\1&0&0&0&1&1&0&1\\1&1&0&0&0&1&1&0\end{bmatrix}$$

Adjacency matrices of C_6^2 and C_8^2 are circulant matrices because main diagonal of matrices are equal to zero and entries in first row satisfy $a_{1j}=a_{1,n-j+2}$ for $j=2,3,\ldots,n$ and $a_{ij}=a_{i+1,j+1}$. Adjacency matrix of square cycle graph C_n^2 is circulant matrix because main diagonal of matrices are equal to zero and entries in first row satisfy $a_{1j}=a_{1,n-j+2}$ for $j=2,3,\ldots,n$ and $a_{ij}=a_{i+1,j+1}$, Thus these square cycle graphs are circulant graphs.

CHAPTER III

DETERMINANT OF ADJACENCY MATRIX OF SQUARE CYCLE GRAPH

The purpose of this chapter is to present literature review about determinant of adjacency matrix of some graphs and our results which are determinant of adjacency matrix of square cycle graphs.

3.1 Determinant of adjacency matrix of some graphs

Theorem 3.1 [4] The determinant of adjacency matrix of a complete graph K_n is

$$\det(A(K_n)) = (-1)^{n-1}(n-1).$$

Theorem 3.2 [4] The determinant of adjacency matrix of a path P_n is

$$\det(A(P_n)) = \begin{cases} (-1)^k & ; n = 2k \\ 0 & ; \text{otherwise.} \end{cases}$$

Theorem 3.3 [4] The determinant of adjacency matrix of a cycle C_n is

$$\det(A(C_n)) = \begin{cases} 0 & ; n \equiv 0 \pmod 4 \\ -4 & ; n \equiv 2 \pmod 4 \\ 2 & ; \text{otherwise.} \end{cases}$$

Theorem 3.4 [7] The determinant of adjacency matrix of a wheel graph W_n is

$$\det(A(W_n)) \begin{cases} 0 & ; n \equiv 0 \pmod 4 \\ -n & ; n \equiv 1 \pmod 4 \\ 2n & ; n \equiv 2 \pmod 4 \\ -n & ; n \equiv 3 \pmod 4). \end{cases}$$

Theorem 3.5 [5] The determinant of adjacency matrix of graph $G(r,t)$ is given by

$$\det(G(r,t)) = \begin{cases} (-1)^r r & ;\text{if } t \text{ is odd and } (n,rt)=1 \\ -r^2 & ;\text{if } t \text{ is even and } (n,rt)=1 \\ 0 & ;\text{otherwise.} \end{cases}$$

where $G(r,t), r \geq 2, t \geq 1$. Such a graph has $n=(r-1)t+2$ vertices and r is a regular of degree.

Furthermore, [1], [10], [13] showed determinants of adjacency matrix of some graphs.

3.2 Main results

In this section, we present the determinant of square cycle graph which use eigenvalue of square cycle graph to find its determinant. In chapter II, square cycle graph is a circulant graph. First, we present a proposition about eigenvalue of the adjacency matrix of a circulant graph.

Proposition 3.6 [2] Suppose that $[0, a_2, \ldots, a_n]$ is the first row of the adjacency matrix of a circulant graph G. Then the eigenvalues of graph G, denoted by $E(G;k)$, is defined by

$$E(G;k) = \sum_{j=1}^{n} a_j z^{j-1}$$

where $z = e^{\frac{2k\pi i}{n}}, k = 1,2,\ldots,n$.

Because square cycle graph is a circulant graph then eigenvalues of square cycle graph are

$$E(C_n^2;k) = \sum_{j=1}^{n} a_j z^{j-1} \tag{3.1}$$

where $z = e^{\frac{2k\pi i}{n}}, k = 1,2,\ldots,n$. Thus, consider first row of the adjacency matrix of a circulant graph, we see $a_j = 1$ for $j = 2,3,n-1$ and n. Then

$$E(C_n^2;k) = z + z^2 + z^{n-2} + z^{n-1} \tag{3.2}$$

From (3.2), we get

$$E(C_n^2;k) = e^{\frac{2k\pi i}{n}} + e^{\frac{4k\pi i}{n}} + e^{\frac{2k(n-2)\pi i}{n}} + e^{\frac{2k(n-1)\pi i}{n}}$$

$$= e^{\frac{2k\pi i}{n}} + e^{\frac{4k\pi i}{n}} + e^{\frac{2kn\pi i}{n}} \cdot e^{\frac{-4k\pi i}{n}} + e^{\frac{2kn\pi i}{n}} \cdot e^{\frac{-2k\pi i}{n}}.$$

By Euler's formula is $e^{\theta i} = \cos\theta + i\sin\theta$, we obtain

$$E(C_n^2;k) = \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) + \left(\cos\frac{4k\pi}{n} + i\sin\frac{4k\pi}{n}\right)$$

$$+\left(\cos\frac{-4k\pi}{n} + i\sin\frac{-4k\pi}{n}\right) + \left(\cos\frac{-2k\pi}{n} + i\sin\frac{-2k\pi}{n}\right).$$

And $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$. Then

$$= \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) + \left(\cos\frac{4k\pi}{n} + i\sin\frac{4k\pi}{n}\right)$$

$$+\left(\cos\frac{4k\pi}{n} - i\sin\frac{4k\pi}{n}\right) + \left(\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}\right)$$

$$E(C_n^2;k) = 2\cos\frac{2k\pi}{n} + 2\cos\frac{4k\pi}{n}.$$

We use $\cos\alpha + \cos\beta = 2\cos\left(\frac{1}{2}\right)(\alpha+\beta)\cos\left(\frac{1}{2}\right)(\alpha-\beta)$. Then

$$E(C_n^2;k) = 2\left[2\cos\left(\frac{1}{2}\right)\left(\frac{4k\pi+2k\pi}{n}\right)\cos\left(\frac{1}{2}\right)\left(\frac{4k\pi-2k\pi}{n}\right)\right]$$

$$E(C_n^2;k) = 4\left(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}\right) \tag{3.3}$$

Next, we present lemma that will use in the proof of determinant of adjacency matrix of square cycle graph.

Lemma 3.7 Let q be a positive integer. Then

$$\prod_{k=1}^{2q}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=2q+2}^{4q+1}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=4q+3}^{6q+2}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)=2^{-12q}.$$

Proof.

$$\prod_{k=1}^{2q}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=2q+2}^{4q+1}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=4q+3}^{6q+2}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)$$

$$=\prod_{k=1}^{2q}\left(\frac{2\sin\left(\frac{3k\pi}{6q+3}\right)\cos\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)\cos\left(\frac{k\pi}{6q+3}\right)}{2\sin\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)}\right)$$

$$\prod_{k=2q+2}^{4q+1}\left(\frac{2\sin\left(\frac{3k\pi}{6q+3}\right)\cos\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)\cos\left(\frac{k\pi}{6q+3}\right)}{2\sin\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)}\right)$$

$$\prod_{k=4q+3}^{6q+2}\left(\frac{2\sin\left(\frac{3k\pi}{6q+3}\right)\cos\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)\cos\left(\frac{k\pi}{6q+3}\right)}{2\sin\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)}\right)$$

$$=\prod_{k=1}^{2q}\left(\frac{\sin\left(\frac{6k\pi}{6q+3}\right)\sin\left(\frac{2k\pi}{6q+3}\right)}{2\sin\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)}\right)\prod_{k=2q+2}^{4q+1}\left(\frac{\sin\left(\frac{6k\pi}{6q+3}\right)\sin\left(\frac{2k\pi}{6q+3}\right)}{2\sin\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)}\right)$$

$$\prod_{k=4q+3}^{6q+2}\left(\frac{\sin\left(\frac{6k\pi}{6q+3}\right)\sin\left(\frac{2k\pi}{6q+3}\right)}{2\sin\left(\frac{3k\pi}{6q+3}\right)2\sin\left(\frac{k\pi}{6q+3}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\left[\sin\left(\frac{6\pi}{6q+3}\right)\sin\left(\frac{12\pi}{6q+3}\right)\ldots\sin\left(\frac{6(2q)\pi}{6q+3}\right)\right]\left[\sin\left(\frac{2\pi}{6q+3}\right)\sin\left(\frac{4\pi}{6q+3}\right)\ldots\sin\left(\frac{2(2q)\pi}{6q+3}\right)\right]}{\left[\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{6\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q)\pi}{6q+3}\right)\right]\left[\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{2\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q)\pi}{6q+3}\right)\right]}\right)$$

$$\left(\frac{\sin\left(\frac{6(2q+2)\pi}{6q+3}\right)\sin\left(\frac{6(2q+3)\pi}{6q+3}\right)\ldots\sin\left(\frac{6(4q+1)\pi}{6q+3}\right)}{\sin\left(\frac{3(2q+2)\pi}{6q+3}\right)\sin\left(\frac{3(2q+3)\pi}{6q+3}\right)\ldots\sin\left(\frac{3(4q+1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{2(2q+2)\pi}{6q+3}\right)\sin\left(\frac{2(2q+3)\pi}{6q+3}\right)\ldots\sin\left(\frac{2(4q+1)\pi}{6q+3}\right)}{\sin\left(\frac{(2q+2)\pi}{6q+3}\right)\sin\left(\frac{(2q+3)\pi}{6q+3}\right)\ldots\sin\left(\frac{(4q+1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{6(4q+3)\pi}{6q+3}\right)\sin\left(\frac{6(4q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{6(6q+2)\pi}{6q+3}\right)}{\sin\left(\frac{3(4q+3)\pi}{6q+3}\right)\sin\left(\frac{3(4q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{3(6q+2)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{2(4q+3)\pi}{6q+3}\right)\sin\left(\frac{2(4q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{2(6q+2)\pi}{6q+3}\right)}{\sin\left(\frac{(4q+3)\pi}{6q+3}\right)\sin\left(\frac{(4q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{(6q+2)\pi}{6q+3}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\sin\left(\frac{6(q+1)\pi}{6q+3}\right)\sin\left(\frac{6(q+2)\pi}{6q+3}\right)\ldots\sin\left(\frac{6(2q)\pi}{6q+3}\right)}{\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{9\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(12q+12)\pi}{6q+3}\right)\sin\left(\frac{(12q+18)\pi}{6q+3}\right)\ldots\sin\left(\frac{(24q+6)\pi}{6q+3}\right)}{\sin\left(\frac{(6q+6)\pi}{6q+3}\right)\sin\left(\frac{(6q+9)\pi}{6q+3}\right)\ldots\sin\left(\frac{(12q+3)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(24q+18)\pi}{6q+3}\right)\sin\left(\frac{(24q+24)\pi}{6q+3}\right)\ldots\sin\left(\frac{(36q+12)\pi}{6q+3}\right)}{\sin\left(\frac{(12q+9)\pi}{6q+3}\right)\sin\left(\frac{(12q+12)\pi}{6q+3}\right)\ldots\sin\left(\frac{(18q+6)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(2q+2)\pi}{6q+3}\right)\sin\left(\frac{(2q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{4q\pi}{6q+3}\right)}{\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{3\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(4q+4)\pi}{6q+3}\right)\sin\left(\frac{(4q+6)\pi}{6q+3}\right)\ldots\sin\left(\frac{(8q+2)\pi}{6q+3}\right)}{\sin\left(\frac{(2q+2)\pi}{6q+3}\right)\sin\left(\frac{(2q+3)\pi}{6q+3}\right)\ldots\sin\left(\frac{(4q+1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(8q+6)\pi}{6q+3}\right)\sin\left(\frac{(8q+8)\pi}{6q+3}\right)\ldots\sin\left(\frac{(12q+4)\pi}{6q+3}\right)}{\sin\left(\frac{(4q+3)\pi}{6q+3}\right)\sin\left(\frac{(4q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{(6q+2)\pi}{6q+3}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{1}{\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{9\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(6q+6)\pi}{6q+3}\right)\sin\left(\frac{(6q+12)\pi}{6q+3}\right)\ldots\sin\left(\frac{(12q)\pi}{6q+3}\right)}{\sin\left(\frac{(6q+6)\pi}{6q+3}\right)\sin\left(\frac{(6q+9)\pi}{6q+3}\right)\ldots\sin\left(\frac{(12q+3)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(12q+12)\pi}{6q+3}\right)\sin\left(\frac{(12q+18)\pi}{6q+3}\right)\ldots\sin\left(\frac{(24q+6)\pi}{6q+3}\right)}{\sin\left(\frac{(12q+9)\pi}{6q+3}\right)\sin\left(\frac{(12q+12)\pi}{6q+3}\right)\ldots\sin\left(\frac{(18q+6)\pi}{6q+3}\right)}\right)$$

$$\sin\left(\frac{((18q+9)+(6q+9))}{6q+3}\pi\right)\sin\left(\frac{((18q+9)+(6q+15))}{6q+3}\pi\right)\ldots\sin\left(\frac{((18q+9)+(18q+3))}{6q+3}\pi\right)$$

$$\left(\frac{1}{\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{3\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(2q+2)\pi}{6q+3}\right)\sin\left(\frac{(2q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{4q\pi}{6q+3}\right)}{\sin\left(\frac{(2q+2)\pi}{6q+3}\right)\sin\left(\frac{(2q+3)\pi}{6q+3}\right)\ldots\sin\left(\frac{(4q+1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(4q+4)\pi}{6q+3}\right)\sin\left(\frac{(4q+6)\pi}{6q+3}\right)\ldots\sin\left(\frac{(8q+2)\pi}{6q+3}\right)}{\sin\left(\frac{(4q+3)\pi}{6q+3}\right)\sin\left(\frac{(4q+4)\pi}{6q+3}\right)\ldots\sin\left(\frac{(6q+2)\pi}{6q+3}\right)}\right)$$

$$\left(\sin\left(\frac{((6q+3)+(2q+3))}{6q+3}\pi\right)\sin\left(\frac{((6q+3)+(2q+5))}{6q+3}\pi\right)\ldots\sin\left(\frac{((6q+3)+(6q+1))}{6q+3}\pi\right)\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{1}{\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{9\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{1}{\sin\left(\frac{(6q+9)\pi}{6q+3}\right)\sin\left(\frac{(6q+15)\pi}{6q+3}\right)\ldots\sin\left(\frac{(12q+3)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(18q+12)\pi}{6q+3}\right)\sin\left(\frac{(18q+18)\pi}{6q+3}\right)\ldots\sin\left(\frac{(24q+6)\pi}{6q+3}\right)}{\sin\left(\frac{(12q+9)\pi}{6q+3}\right)\sin\left(\frac{(12q+15)\pi}{6q+3}\right)\ldots\sin\left(\frac{(18q+3)\pi}{6q+3}\right)}\right)$$

$$\left(\sin\left(3\pi+\frac{(6q+9)\pi}{6q+3}\right)\sin\left(3\pi+\frac{(6q+15)\pi}{6q+3}\right)\ldots\sin\left(3\pi+\frac{(12q+3)\pi}{6q+3}\right)\right)$$

$$\left(\sin\left(3\pi+\frac{(12q+9)\pi}{6q+3}\right)\sin\left(3\pi+\frac{(12q+15)\pi}{6q+3}\right)\ldots\sin\left(3\pi+\frac{(18q+3)\pi}{6q+3}\right)\right)$$

$$\left(\frac{1}{\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{3\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{1}{\sin\left(\frac{(2q+3)\pi}{6q+3}\right)\sin\left(\frac{(2q+5)\pi}{6q+3}\right)\ldots\sin\left(\frac{(4q+1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(6q+4)\pi}{6q+3}\right)\sin\left(\frac{(6q+6)\pi}{6q+3}\right)\ldots\sin\left(\frac{(8q+2)\pi}{6q+3}\right)}{\sin\left(\frac{(4q+3)\pi}{6q+3}\right)\sin\left(\frac{(4q+5)\pi}{6q+3}\right)\ldots\sin\left(\frac{(6q+1)\pi}{6q+3}\right)}\right)$$

$$\left(\sin\left(\pi+\frac{(2q+3)\pi}{6q+3}\right)\sin\left(\pi+\frac{(2q+5)\pi}{6q+3}\right)\ldots\sin\left(\pi+\frac{(4q+1)\pi}{6q+3}\right)\right)$$

$$\left(\sin\left(\pi+\frac{(4q+3)\pi}{6q+3}\right)\sin\left(\pi+\frac{(4q+5)\pi}{6q+3}\right)\ldots\sin\left(\pi+\frac{(6q+1)\pi}{6q+3}\right)\right)$$

$$=\frac{1\cdot(-1)^{4q}}{2^{12q}}\left(\frac{\sin\left(\frac{((18q+9)+3)}{6q+3}\pi\right)\sin\left(\frac{((18q+9)+9)}{6q+3}\pi\right)\ldots\sin\left(\frac{((18q+9)+(6q-3))}{6q+3}\pi\right)}{\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{9\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(6q+9)\pi}{6q+3}\right)\sin\left(\frac{(6q+15)\pi}{6q+3}\right)\ldots\sin\left(\frac{(12q+3)\pi}{6q+3}\right)}{\sin\left(\frac{(6q+9)\pi}{6q+3}\right)\sin\left(\frac{(6q+15)\pi}{6q+3}\right)\ldots\sin\left(\frac{(12q+3)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(12q+9)\pi}{6q+3}\right)\sin\left(\frac{(12q+15)\pi}{6q+3}\right)\ldots\sin\left(\frac{(18q+3)\pi}{6q+3}\right)}{\sin\left(\frac{(12q+9)\pi}{6q+3}\right)\sin\left(\frac{(12q+15)\pi}{6q+3}\right)\ldots\sin\left(\frac{(18q+3)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{((6q+3)+1)}{6q+3}\pi\right)\sin\left(\frac{((6q+3)+3)}{6q+3}\pi\right)\ldots\sin\left(\frac{((6q+3)+(2q-1))}{6q+3}\pi\right)}{\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{3\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\left(\sin\left(\frac{(2q+3)\pi}{6q+3}\right)\sin\left(\frac{(2q+5)\pi}{6q+3}\right)\ldots\sin\left(\frac{(4q+1)\pi}{6q+3}\right)\right)}{\sin\left(\frac{(2q+3)\pi}{6q+3}\right)\sin\left(\frac{(2q+5)\pi}{6q+3}\right)\ldots\sin\left(\frac{(4q+1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\left(\sin\left(\frac{(4q+3)\pi}{6q+3}\right)\sin\left(\frac{(4q+5)\pi}{6q+3}\right)\ldots\sin\left(\frac{(6q+1)\pi}{6q+3}\right)\right)}{\sin\left(\frac{(4q+3)\pi}{6q+3}\right)\sin\left(\frac{(4q+5)\pi}{6q+3}\right)\ldots\sin\left(\frac{(6q+1)\pi}{6q+3}\right)}\right)$$

$$=\frac{1\cdot(-1)^{4q}}{2^{12q}}\left(\frac{\sin\left(3\pi+\frac{3\pi}{6q+3}\right)\sin\left(3\pi+\frac{9\pi}{6q+3}\right)\ldots\sin\left(3\pi+\frac{3(2q-1)\pi}{6q+3}\right)}{\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{9\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\pi+\frac{\pi}{6q+3}\right)\sin\left(\pi+\frac{3\pi}{6q+3}\right)\ldots\sin\left(\pi+\frac{(2q-1)\pi}{6q+3}\right)}{\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{3\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q-1)\pi}{6q+3}\right)}\right)$$

$$=\frac{1\cdot(-1)^{6q}}{2^{12q}}\left(\frac{\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{9\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q-1)\pi}{6q+3}\right)}{\sin\left(\frac{3\pi}{6q+3}\right)\sin\left(\frac{9\pi}{6q+3}\right)\ldots\sin\left(\frac{3(2q-1)\pi}{6q+3}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{3\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q-1)\pi}{6q+3}\right)}{\sin\left(\frac{\pi}{6q+3}\right)\sin\left(\frac{3\pi}{6q+3}\right)\ldots\sin\left(\frac{(2q-1)\pi}{6q+3}\right)}\right)$$

$$=2^{-12q}$$ □

Lemma 3.8 Let q be a positive integer. Then

$$\prod_{k=1}^{6q}\left(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}\right)=2^{-12q}.$$

Proof.

$$\prod_{k=1}^{6q}\left(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}\right)=\prod_{k=1}^{6q}\left(\frac{2\sin\left(\frac{3k\pi}{6q+1}\right)\cos\left(\frac{3k\pi}{6q+1}\right)2\sin\left(\frac{k\pi}{6q+1}\right)\cos\left(\frac{k\pi}{6q+1}\right)}{2\sin\left(\frac{3k\pi}{6q+1}\right)2\sin\left(\frac{k\pi}{6q+1}\right)}\right)$$

$$=\prod_{k=1}^{6q}\left(\frac{\sin\left(\frac{6k\pi}{6q+1}\right)\sin\left(\frac{2k\pi}{6q+1}\right)}{2\sin\left(\frac{3k\pi}{6q+1}\right)2\sin\left(\frac{k\pi}{6q+1}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\sin\left(\frac{6\pi}{6q+1}\right)\sin\left(\frac{12\pi}{6q+1}\right)\ldots\sin\left(\frac{6(6q)\pi}{6q+1}\right)}{\sin\left(\frac{3\pi}{6q+1}\right)\sin\left(\frac{6\pi}{6q+1}\right)\ldots\sin\left(\frac{3(6q)\pi}{6q+1}\right)}\right)\left(\frac{\sin\left(\frac{2\pi}{6q+1}\right)\sin\left(\frac{4\pi}{6q+1}\right)\ldots\sin\left(\frac{2(6q)\pi}{6q+1}\right)}{\sin\left(\frac{\pi}{6q+1}\right)\sin\left(\frac{2\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q)\pi}{6q+1}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\sin\left(\frac{6(3q+1)\pi}{6q+1}\right)\sin\left(\frac{6(3q+2)\pi}{6q+1}\right)\ldots\sin\left(\frac{6(6q)\pi}{6q+1}\right)}{\sin\left(\frac{3\pi}{6q+1}\right)\sin\left(\frac{9\pi}{6q+1}\right)\ldots\sin\left(\frac{3(6q-1)\pi}{6q+1}\right)}\right)\left(\frac{\sin\left(\frac{2(3q+1)\pi}{6q+1}\right)\sin\left(\frac{2(3q+2)\pi}{6q+1}\right)\ldots\sin\left(\frac{2(6q)\pi}{6q+1}\right)}{\sin\left(\frac{\pi}{6q+1}\right)\sin\left(\frac{3\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q-1)\pi}{6q+1}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\sin\left(\frac{(18q+6)\pi}{6q+1}\right)\sin\left(\frac{6(18q+12)\pi}{6q+1}\right)\ldots\sin\left(\frac{(36q)\pi}{6q+1}\right)}{\sin\left(\frac{3\pi}{6q+1}\right)\sin\left(\frac{9\pi}{6q+1}\right)\ldots\sin\left(\frac{3(6q-1)\pi}{6q+1}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(6q+2)\pi}{6q+1}\right)\sin\left(\frac{(6q+4)\pi}{6q+1}\right)\ldots\sin\left(\frac{(12q)\pi}{6q+1}\right)}{\sin\left(\frac{\pi}{6q+1}\right)\sin\left(\frac{3\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q-1)\pi}{6q+1}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\sin\left(\frac{((18q+3)+(3))\pi}{6q+1}\right)\sin\left(\frac{((18q+3)+(9))\pi}{6q+1}\right)\ldots\sin\left(\frac{((18q+3)+(18q-3))\pi}{6q+1}\right)}{\sin\left(\frac{3\pi}{6q+1}\right)\sin\left(\frac{9\pi}{6q+1}\right)\ldots\sin\left(\frac{3(6q-1)\pi}{6q+1}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(6q+1)+(1)\pi}{6q+1}\right)\sin\left(\frac{(6q+1)+(3)\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q+1)+(6q-1)\pi}{6q+1}\right)}{\sin\left(\frac{\pi}{6q+1}\right)\sin\left(\frac{3\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q-1)\pi}{6q+1}\right)}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\sin\left(3\pi+\frac{3\pi}{6q+1}\right)\sin\left(3\pi+\frac{9\pi}{6q+1}\right)\ldots\sin\left(3\pi+\frac{3(6q-1)\pi}{6q+1}\right)}{\sin\left(\frac{3\pi}{6q+1}\right)\sin\left(\frac{9\pi}{6q+1}\right)\ldots\sin\left(\frac{3(6q-1)\pi}{6q+1}\right)}\right)$$

$$\left(\frac{\sin\left(\pi+\frac{\pi}{6q+1}\right)\sin\left(\pi+\frac{3\pi}{6q+1}\right)\ldots\sin\left(\pi+\frac{(6q-1)\pi}{6q+1}\right)}{\sin\left(\frac{\pi}{6q+1}\right)\sin\left(\frac{3\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q-1)\pi}{6q+1}\right)}\right)$$

$$=\frac{1\cdot(-1)^{6q}}{2^{12q}}\left(\frac{\sin\left(\frac{3\pi}{6q+1}\right)\sin\left(\frac{9\pi}{6q+1}\right)\ldots\sin\left(\frac{3(6q-1)\pi}{6q+1}\right)}{\sin\left(\frac{3\pi}{6q+1}\right)\sin\left(\frac{9\pi}{6q+1}\right)\ldots\sin\left(\frac{3(6q-1)\pi}{6q+1}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{\pi}{6q+1}\right)\sin\left(\frac{3\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q-1)\pi}{6q+1}\right)}{\sin\left(\frac{\pi}{6q+1}\right)\sin\left(\frac{3\pi}{6q+1}\right)\ldots\sin\left(\frac{(6q-1)\pi}{6q+1}\right)}\right)$$

$$=2^{-12q}. \qquad \square$$

Lemma 3.9 Let q be a positive integer. Then

$$\prod_{k=1}^{6q+4}\left(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}\right)=2^{-2(6q+4)}.$$

Proof. $\prod_{k=1}^{6q+4}\left(\cdot\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}\right)$

$$=\prod_{k=1}^{6q+4}\left(\frac{2\sin\left(\frac{3k\pi}{6q+5}\right)\cos\left(\frac{3k\pi}{6q+5}\right)2\sin\left(\frac{k\pi}{6q+5}\right)\cos\left(\frac{k\pi}{6q+5}\right)}{2\sin\left(\frac{3k\pi}{6q+5}\right)2\sin\left(\frac{k\pi}{6q+5}\right)}\right)$$

$$=\prod_{k=1}^{6q+4}\left(\frac{\sin\left(\frac{6k\pi}{6q+5}\right)\sin\left(\frac{2k\pi}{6q+5}\right)}{2\sin\left(\frac{3k\pi}{6q+5}\right)2\sin\left(\frac{k\pi}{6q+5}\right)}\right)$$

$$=\frac{1}{2^{2(6q+4)}}\left(\frac{\sin\left(\frac{6\pi}{6q+5}\right)\sin\left(\frac{12\pi}{6q+5}\right)\ldots\sin\left(\frac{6(6q+4)\pi}{6q+5}\right)}{\sin\left(\frac{3\pi}{6q+5}\right)\sin\left(\frac{6\pi}{6q+5}\right)\ldots\sin\left(\frac{3(6q+4)\pi}{6q+5}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{2\pi}{6q+5}\right)\sin\left(\frac{4\pi}{6q+5}\right)\ldots\sin\left(\frac{2(6q+4)\pi}{6q+5}\right)}{\sin\left(\frac{\pi}{6q+5}\right)\sin\left(\frac{2\pi}{6q+5}\right)\ldots\sin\left(\frac{(6q+4)\pi}{6q+5}\right)}\right)$$

$$=\frac{1}{2^{2(6q+4)}}\left(\frac{\sin\left(\frac{6(3q+3)\pi}{6q+5}\right)\sin\left(\frac{6(3q+4)\pi}{6q+5}\right)\ldots\sin\left(\frac{6(6q+4)\pi}{6q+5}\right)}{\sin\left(\frac{3\pi}{6q+5}\right)\sin\left(\frac{9\pi}{6q+5}\right)\ldots\sin\left(\frac{3(6q+3)\pi}{6q+5}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{2(3q+3)\pi}{6q+5}\right)\sin\left(\frac{2(3q+4)\pi}{6q+5}\right)\cdots\sin\left(\frac{2(6q+4)\pi}{6q+5}\right)}{\sin\left(\frac{\pi}{6q+5}\right)\sin\left(\frac{3\pi}{6q+5}\right)\cdots\sin\left(\frac{(6q+3)\pi}{6q+5}\right)}\right)$$

$$=\frac{1}{2^{2(6q+4)}}\left(\frac{\sin\left(\frac{(18q+18)\pi}{6q+5}\right)\sin\left(\frac{(18q+24)\pi}{6q+5}\right)\ldots\sin\left(\frac{(36q+24)\pi}{6q+5}\right)}{\sin\left(\frac{3\pi}{6q+5}\right)\sin\left(\frac{9\pi}{6q+5}\right)\ldots\sin\left(\frac{3(6q+3)\pi}{6q+5}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{(6q+5+1)\pi}{6q+5}\right)\sin\left(\frac{(6q+5+3)\pi}{6q+5}\right)\ldots\sin\left(\frac{(6q+5+6q+3)\pi}{6q+5}\right)}{\sin\left(\frac{\pi}{6q+5}\right)\sin\left(\frac{3\pi}{6q+5}\right)\ldots\sin\left(\frac{(6q+3)\pi}{6q+5}\right)}\right)$$

$$=\frac{1}{2^{2(6q+4)}}\left(\frac{\sin\left(3\pi+\frac{3\pi}{6q+5}\right)\sin\left(3\pi+\frac{9\pi}{6q+5}\right)\ldots\sin\left(3\pi+\frac{3(6q+3)\pi}{6q+5}\right)}{\sin\left(\frac{3\pi}{6q+5}\right)\sin\left(\frac{9\pi}{6q+5}\right)\ldots\sin\left(\frac{3(6q+3)\pi}{6q+5}\right)}\right)$$

$$\left(\frac{\sin\left(\pi+\frac{\pi}{6q+5}\right)\sin\left(\pi+\frac{3\pi}{6q+5}\right)\ldots\sin\left(\pi+\frac{(6q+3)\pi}{6q+5}\right)}{\sin\frac{\pi}{6q+5}\sin\frac{3\pi}{6q+5}\ldots\sin\frac{(6q+3)\pi}{6q+5}}\right)$$

$$=\frac{1\cdot(-1)^{6q+4}}{2^{2(6q+4)}}\left(\frac{\sin\left(\frac{3\pi}{6q+5}\right)\sin\left(\frac{9\pi}{6q+5}\right)\ldots\sin\left(\frac{3(6q+3)\pi}{6q+5}\right)}{\sin\left(\frac{3\pi}{6q+5}\right)\sin\left(\frac{9\pi}{6q+5}\right)\ldots\sin\left(\frac{3(6q+3)\pi}{6q+5}\right)}\right)$$

$$\left(\frac{\sin\left(\frac{\pi}{6q+5}\right)\sin\left(\frac{3\pi}{6q+5}\right)\ldots\sin\left(\frac{(6q+3)\pi}{6q+5}\right)}{\sin\frac{\pi}{6q+5}\sin\frac{3\pi}{6q+5}\ldots\sin\frac{(6q+3)\pi}{6q+5}}\right)$$

$$=2^{-2(6q+4)}. \qquad \square$$

Theorem 3.10 Let C_n^2 be a square cycle graph with n vertices and n be a positive integer. Then

$$\det(A(C_n^2)) = \begin{cases} 0 & ; n \equiv 0,2,4 \pmod 6 \\ 16 & ; n \equiv 3 \pmod 6 \\ 4 & ; n \equiv 1,5 \pmod 6 \end{cases}$$

Proof. Let $E(C_n^2;k)$ be a k^{th} eigenvalue of adjacency matrix of square cycle graph C_n^2.

From (3.3), we have

$$\det(A(C_n^2)) = \prod_{k=1}^{n} E(C_n^2;k)$$

$$= \prod_{k=1}^{n} 4\left(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}\right). \tag{3.5}$$

Consider the following 3 cases.

Case I, $n \equiv 0,2,4 \pmod 6$.

Since n is even and $1 \le k \le n$, consider (3.3) when $k = \frac{n}{2}$. Then

$$E\left(C_n^2;\frac{n}{2}\right) = 4\left(\cos\frac{3\left(\frac{n}{2}\right)\pi}{n}\cos\frac{\left(\frac{n}{2}\right)\pi}{n}\right)$$

$$= 0.$$

From (3.5), we obtain

$$\det(A(C_n^2)) = \prod_{k=1}^{n} E(C_n^2;k)$$

$$= 0.$$

Therefore, $\det(A(C_n^2)) = 0$ when $n \equiv 0,2,4 \pmod 6$.

Case II, $n \equiv 3 \pmod 6$ Then $n = 6q+3, \exists q \in \mathbb{Z}^+$.

From (3.5), we obtain

$$\det(A(C_n^2)) = \prod_{k=1}^{n} E(C_n^2;k)$$

$$= \prod_{k=1}^{6q+3} 4\left(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}\right)$$

$$= \left(4\cos\frac{3(2q+1)\pi}{6q+3}\cos\frac{(2q+1)\pi}{6q+3}\right)\left(4\cos\frac{3(4q+2)\pi}{6q+3}\cos\frac{(4q+2)\pi}{6q+3}\right)$$
$$\left(4\cos\frac{3(6q+3)\pi}{6q+3}\cos\frac{(6q+3)\pi}{6q+3}\right)\cdot 2^{12q}\prod_{k=1}^{2q}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)$$
$$\prod_{k=2q+2}^{4q+1}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=4q+3}^{6q+2}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)$$

$$= (-2)\cdot(-2)\cdot(4)\cdot 2^{12q}\prod_{k=1}^{2q}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)$$
$$\prod_{k=2q+2}^{4q+1}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=4q+3}^{6q+2}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right).$$

By using Lemma 3.7, we have

$$= (-2)\cdot(-2)\cdot(4)\cdot 2^{12q}\cdot 2^{-12q}$$

$$= 16\ .$$

Therefore $\det(A(C_n^2)) = 16$ when $n \equiv 3 \pmod 6$.

Case III, $n \equiv 1 \pmod 6$ or $n \equiv 5 \pmod 6$. We consider 2 subcases.

Subcase I: $n \equiv 1 \pmod 6$. By (3.5), we obtain

$$\det(A(C_n^2)) = \prod_{k=1}^{n} E(C_n^2;k)$$

$$= \prod_{k=1}^{6q+1} 4\left(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}\right)$$

$$= \left(4\cos\frac{3(6q+1)\pi}{6q+1}\cos\frac{(6q+1)\pi}{6q+1}\right)\cdot 2^{12q}\prod_{k=1}^{6q}\left(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}\right).$$

By using Lemma 3.8, we have

$$\det(A(C_n^2)) = 4.$$

Subcase II: $n \equiv 5(\mathrm{mod}\,6)$ By (3.5), we obtain

$$\det(A(C_n^2)) = \prod_{k=1}^{n} E(C_n^2;k)$$

$$= \prod_{k=1}^{6q+5} 4\left(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}\right)$$

$$= \left(4\cos\frac{3(6q+5)\pi}{6q+5}\cos\frac{(6q+5)\pi}{6q+5}\right)\cdot 2^{2(6q+4)}\prod_{k=1}^{6q+4}\left(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}\right)$$

$$= (4)\cdot 2^{2(6q+4)}\prod_{k=1}^{6q+4}\left(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}\right).$$

By using Lemma 3.9, we have

$$\det(A(C_n^2)) = (4)\cdot 2^{2(6q+4)}\cdot 2^{-2(6q+4)}$$

$$= 4.$$

From subcase I and II, we obtain $\det(A(C_n^2)) = 4$ for $n \equiv 1,5(\mathrm{mod}\,6)$.

From case I, II and III,

Therefore,

$$\det(A(C_n^2)) = \begin{cases} 0 & ; n \equiv 0,2,4(\mathrm{mod}\,6) \\ 16 & ; n \equiv 3(\mathrm{mod}\,6) \\ 4 & ; n \equiv 1,5(\mathrm{mod}\,6) \end{cases}$$

where n is a positive integer. □

CHAPTER IV

DETERMINANT OF ADJACENCY MATRIX OF SOME GRAPH

The aim of this chapter is to present literature review about some graphs and determinant of adjacency matrix on sum operation of C_n^2 and G $\left(C_n^2 \dotplus G\right)$, determinant of adjacency matrix $C_{n_1}^2 \cup_{v_1} P_{n_2}$.

4.1 Determinant of adjacency matrix of some graphs

Theorem 4.1 [7] Let W_n' be a pin – wheel graph and $n \geq 3$. Then

$$\det(A(W_n')) = \begin{cases} 0 & ; n \equiv 0 (\bmod 4) \\ \dfrac{n-1}{2} & ; n \equiv 1 (\bmod 4) \\ 1 & ; n \equiv 2 (\bmod 4) \\ -\dfrac{(n+1)}{2} & ; n \equiv 3 (\bmod 4). \end{cases}$$

Theorem 4.2 [7] Let W_n^m be a generalized Pin – wheel graph. Then

$$\det(A(W_n^m)) = \begin{cases} 0 & ;\text{if } n \equiv 0 (\bmod 4) \text{ and } m \equiv 0,2 (\bmod 4) \\ (-1)^{\frac{m+2}{2}} n & ;\text{if } n \equiv 1 (\bmod 4) \text{ and } m \equiv 0,2 (\bmod 4) \\ (-1)^{\frac{m}{2}} 2n & ;\text{if } n \equiv 2 (\bmod 4) \text{ and } m \equiv 0,2 (\bmod 4) \\ (-1)^{\frac{m+2}{2}} n & ;\text{if } n \equiv 3 (\bmod 4) \text{ and } m \equiv 0,2 (\bmod 4) \\ 0 & ;\text{if } n \equiv 0 (\bmod 4) \text{ and } m \equiv 1,3 (\bmod 4) \\ (-1)^{\frac{m-1}{2}} (\frac{n-1}{2}) & ;\text{if } n \equiv 1 (\bmod 4) \text{ and } m \equiv 1,3 (\bmod 4) \\ (-1)^{\frac{m-1}{2}} & ;\text{if } n \equiv 2 (\bmod 4) \text{ and } m \equiv 1,3 (\bmod 4) \\ (-1)^{\frac{m+1}{2}} (\frac{n+1}{2}) & ;\text{if } n \equiv 3 (\bmod 4) \text{ and } m \equiv 1,3 (\bmod 4). \end{cases}$$

where m, n are positive integers.

4.2 Main results

In this section, we study some graphs which obtains from the direct sum of two graphs and $C_{n_1}^2 \cup_{v_1} P_{n_2}$, for example

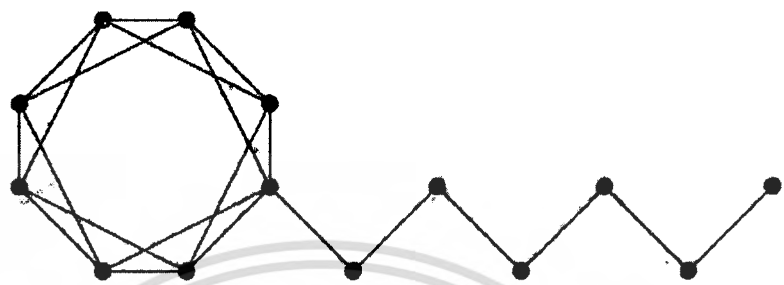


Figure 4.1 $C_8^2 \cup_{v_1} P_6$

Figure 4.2 graph $C_{n_1}^2$ adjacent graph P_{n_2} with vertex v_1 or graph $C_{n_1}^2 \cup_{v_1} P_{n_2}$

We use the following theorem to find determinant of adjacency matrix of some graphs which obtained from an operation between two graphs.

Theorem 4.3 [8] Let A, B, C and D are matrices order $n\times n, n\times m, m\times n$ and $m\times m$, respectively. Then

$$\det\begin{bmatrix} A & B \\ \underline{0} & D \end{bmatrix} = \det\begin{bmatrix} A & \underline{0} \\ C & D \end{bmatrix} = \det(A)\det(D)$$

where $\underline{0}$ is a zero matrix.

Theorem 4.4 [13] Let G be a graph of order $n > 2$ and let x_1, x_2 be distinct vertices in G such that $N(x_1) \subseteq N(x_2)$. Let G' be the graph obtained from G by removing all the edges $x_2 y$, where $y \in N(x_1)$. Then $\det(A(G)) = \det(A(G'))$.

Theorem 4.5 Let G_1 and G_2 be simple graphs.Tthen

$$\det(A(G_1 \dot{+} G_2)) = \det(A(G_1))\det(A(G_2)).$$

Proof. Let G_1 be a simple graph have with vertex set $\{u_1, u_2, \ldots, u_k\}$ and let G_2 be a simple graph with vertex set $\{u_{k+1}, u_{k+2}, \ldots, u_n\}$. Then graph $G_1 \dot{+} G_2$ is shown in figure 4.1.

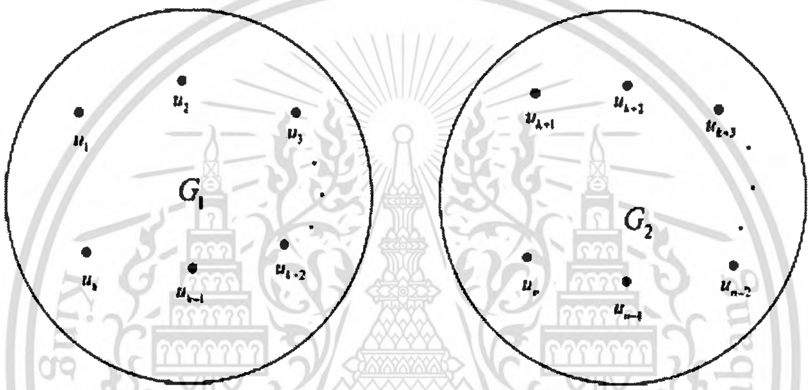


Figure 4.3 Graph $G_1 \dot{+} G_2$

Since each vertex in G_1 is not adjacent to any vertex in G_2, so we obtain an adjacency matrix as follows:

$$A(G_1 \dot{+} G_2) = \begin{bmatrix} & & & & 0 & 0 & 0 & \cdots & 0 \\ & & & & 0 & & & & 0 \\ & A(G_1) & & & 0 & & \ddots & & 0 \\ & & & & \vdots & & & & \vdots \\ & & & & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & & & & \\ 0 & & & & 0 & & & & \\ 0 & & \ddots & & 0 & & A(G_2) & & \\ \vdots & & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & & & & \end{bmatrix}.$$

We rewrite

$$A(G_1 \dot{+} G_2) = \begin{bmatrix} A(G_1) & \underline{0} \\ \underline{0} & A(G_2) \end{bmatrix}$$

where $\underline{0}$ is a zero matrix.

From Theorem 4.3, we obtain

$$\det(A(G_1 \dot{+} G_2)) = \det(A(G_1))\det(A(G_2)). \qquad \square$$

Theorem 4.6 Let $G_1, G_2, \ldots, G_n$ be simple graphs, then

$$\det(A(G_1 \dot{+} G_2 \dot{+} \cdots \dot{+} G_n)) = \det(A(G_1))\det(A(G_2))\cdots\det(A(G_n))$$

Proof . Let $G_1, G_2, \ldots, G_n$ be the simple graphs have order $n_1, n_2, \ldots, n_n$, respectively. Then $G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_n$ have $n_1 \dot{+} n_2 \dot{+} \ldots \dot{+} n_n$ vertices and $V(G_i) \cap V(G_j) = \varnothing$, where $i \neq j$, is shown in figure 4.4.

Figure 4.4 $G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_n$

That is

$$A(G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_n) = \begin{bmatrix} A(G_1) & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & A(G_2) & \underline{0} & \cdots & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & & \cdots & \underline{0} & \underline{0} & \underline{0} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \underline{0} & \underline{0} & \underline{0} & \cdots & & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & A(G_{n-1}) & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} & A(G_n) \end{bmatrix}$$

where $\underline{0}$ is a matrix zero.

From Theorem 4.5, then

$$\det(A(G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_n)) = \det(A(G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_{n-1}))\det(A(G_n))$$
$$= \det(A(G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_{n-2}))\det(A(G_{n-1}))\det(A(G_n))$$
$$\vdots \qquad \vdots$$
$$= \det(A(G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_{n-k}))\det(A(G_{n-k+1}))\ldots\det(A(G_{n-1}))\det(A(G_n))$$
$$\vdots \qquad \vdots$$
$$= \det(A(G_1))\det(A(G_2))\ldots\det(A(G_n))$$
$$\det(A(G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_n)) = \prod_{i=1}^{n}\det(A(G_i))$$ □

From Theorem 3.2, Theorem 3.10 and Theorem 4.5, we have

Corollary 4.7 Let $C_{n_1}^2$ be a square cycle graph and P_{n_2} be a path. Then

$$\det(A(C_{n_1}^2 \dot{+} P_{n_2})) = \begin{cases} 0 & ; n_1 \text{ is even or } n_2 \text{ is odd} \\ 16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 \\ -16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 - 2 \\ 4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 \\ -4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 - 2 \end{cases}$$

where q_1, q_2 are positive integers.

From Theorem 3.10 and Theorem 4.5, we get

Corollary 4.8 Let $C_{n_1}^2$ and $C_{n_2}^2$ be square cycle graphs which have the numbers of vertices n_1 and n_2 vertices, respectively. Then

$$\det(A(C_{n_1}^2 \dot{+} C_{n_2}^2)) = \begin{cases} 0 & ; n_1 = 2q_1 + 4 \text{ or } n_2 = 2q_2 + 4 \\ 256 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 6q_2 + 3 \\ 16 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 6q_2 + 1 \text{ or } n_2 = 6q_2 + 5 \\ 64 & ; \text{otherwise} \end{cases}$$

where q_1, q_2 are positive integers.

Next, we show that determinant of adjacency matrix of $C_{n_1}^2 \cup_{v_1} P_{n_2}$ as follows

Corollary 4.10 Let $C_{n_1}^2$ be a square cycle graph with n_1 vertices and P_{n_2} be a path with n_2 vertices where n_1, n_2 are positive integers. Then

$$\det(A(C_{n_1}^2 \cup_{v_1} P_{n_2})) = \begin{cases} 0 & ; n_1 \text{ is even} \\ 16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 \\ -16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 - 2 \\ 4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 \\ -4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 - 2 \end{cases}$$

where q_1, q_2 are positive integers.

Proof. Let C_n^2 be graph with vertices $u_1, u_2, \ldots, u_{n_1}$ and P_{n_2} be path with vertices $v_1, v_2, \ldots, v_{n_2}$. Graph P_{n_2} adjacent graph $C_{n_1}^2$ at v_1 where $v_1 \in V(P_{n_2})$ as in figure 4.5.

Figure 4.5 A graph $C_{n_1}^2 \cup_{v_1} P_{n_2}$

Consider a vertex v_{n_2} and a vertex v_{n_2-2}, of graph $C_{n_1}^2 \cup_{v_1} P_{n_2}$ such that $N(v_{n_2}) \subseteq N(v_{n_2-2})$. By Theorem 4.4, we have

$$\det(A(C_{n_1}^2 \cup_{v_1} P_{n_2})) = \det(A(C_{n_1}^2 \cup_{v_1} P_{n_2} - v_{n_2-2}v_{n_2-1})).$$

That is,

$$\det(A(C_{n_1}^2 \cup_{v_1} P_{n_2})) = \det(A(C_{n_1}^2 \cup_{v_1} P_{n_2-2}) \dot{+} P_2)$$

as in figure 4.6.

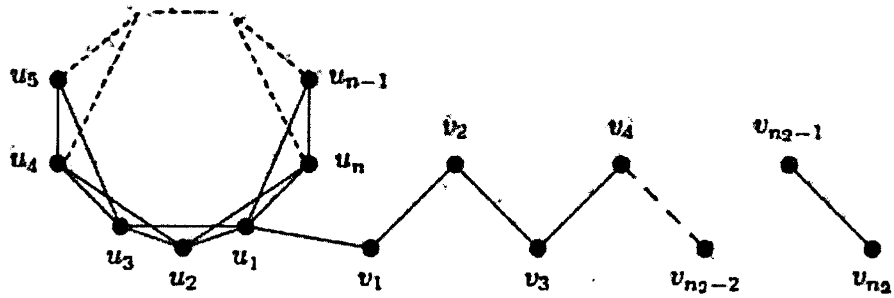


Figure 4.6 A graph $C_{n_1}^2 \bigcup_{v_1} P_{n_2} - \{v_{n_2-2}v_{n_2-1}\}$

Next, consider a vertex v_{n_2-2} and a vertex v_{n_2-4}, such that $N(v_{n_2-2}) \subseteq N(v_{n_2-4})$. By Theorem 4.4, we have

$$\det(A(C_{n_1}^2 \bigcup_{v_1} P_{n_2-2})) = \det(A(C_{n_1}^2 \bigcup_{v_1} P_{n_2-2} - v_{n_2-3}v_{n_2-4})).$$

That is,

$$\det(A(C_{n_1}^2 \bigcup_{v_1} P_{n_2-2})) = \det(A(C_{n_1}^2 \bigcup_{v_1} P_{n_2-4} \dot{+} P_2)).$$

By continuing this process $\frac{n_2}{2}-1$ steps and considering a vertex v_4 and a vertex v_2, we have

$$\det(A(C_{n_1}^2 \bigcup_{v_1} P_4)) = \det(A(C_{n_1}^2 \bigcup_{v_1} P_4 - v_3v_4)).$$

That is,

$$\det(A(C_{n_1}^2 \bigcup_{v_1} P_4)) = \det(A(C_{n_1}^2 \bigcup_{v_1} P_2 \dot{+} P_2)).$$

Finally, consider u_1 and v_2, such that $N(v_2) \subseteq N(u_1)$. By Theorem 4.4, we have

$$\det(A(C_{n_1}^2 \bigcup_{v_1} P_2)) = \det(A(C_{n_1}^2 \bigcup_{v_1} P_2 - v_1u_1)).$$

That is,

$$\det(A(C_{n_1}^2 \bigcup_{v_1} P_2)) = \det(A(C_{n_1}^2 \dot{+} P_2))$$

as in figure 4.7.

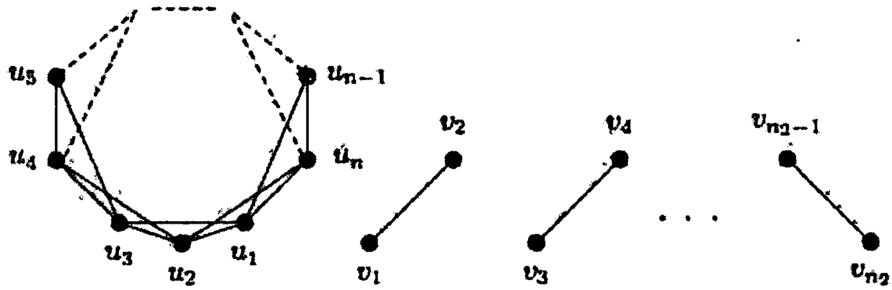


Figure 4.7 A graph $C_{n_1}^2 \dot{+} \underbrace{P_2 \dot{+} P_2 \dot{+} \cdots \dot{+} P_2}_{n/2}$

By using Theorem 4.6, we get $\det(A(C_{n_1}^2 \bigcup_{v_1} P_{n_2})) = \det(A(C_{n_1}^2 \dot{+} \underbrace{P_2 \dot{+} P_2 \dot{+} \ldots \dot{+} P_2}_{n/2}))$.

Thus,

$$\det(A(C_{n_1}^2 \bigcup_{v_1} P_{n_2})) = \begin{cases} 0 & ; n_1 \text{ is even} \\ 16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 \\ -16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 - 2 \\ 4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 \\ -4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 - 2 \end{cases}$$

where q_1, q_2 are positive integers. □

CHAPTER V

CONCLUSIONS

The terminus of this chapter is the conclusion from chapter III and chapter IV which is result of determinant of adjacency matrix of square cycle graph and some graphs which is operation with direct sum of two graphs. We aggregate all results in this chapter.

5.1 Determinant of adjacency matrix of square cycle graph

(1) Let q be a positive integer. Then

$$\prod_{k=1}^{2q}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=2q+2}^{4q+1}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)\prod_{k=4q+3}^{6q+2}\left(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}\right)=2^{-12q}.$$

(2) Let q be a positive integer. Then

$$\prod_{k=1}^{6q}\left(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}\right)=2^{-12q}.$$

(3) Let q be a positive integer. Then

$$\prod_{k=1}^{6q+4}\left(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}\right)=2^{2(6q+4)}.$$

(4) Let C_n^2 be a square cycle graph with n vertices and q be a positive integer. Then

$$\det(A(C_n^2))=\begin{cases}0 & ;n\equiv 0,2,4 \pmod 6\\ 16 & ;n\equiv 3 \pmod 6\\ 4 & ;n\equiv 1,5 \pmod 6).\end{cases}$$

5.2 Determinant of adjacency matrix of some graph

(5) Let G_1 and G_2 be simple graphs. Then

$$\det(A(G_1 \dot{+} G_2)) = \det(A(G_1))\det(A(G_2)).$$

(6) Let $G_1, G_2, \ldots, G_n$ be simple graphs. Then

$$\det(A(G_1 \dot{+} G_2 \dot{+} \cdots \dot{+} G_n)) = \det(A(G_1))\det(A(G_2))\cdots\det(A(G_n)).$$

(7) Let $C_{n_1}^2$ be a square cycle graph and P_{n_2} be a path. Then

$$\det(A(C_{n_1}^2 \dot{+} P_{n_2})) = \begin{cases} 0 & ; n_1 \text{ is even or } n_2 \text{ is odd} \\ 16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 \\ -16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 - 2 \\ 4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 \\ -4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 - 2 \end{cases}$$

where q_1, q_2 are positive integers.

(8) Let $C_{n_1}^2$ and $C_{n_2}^2$ be square cycle graphs which have n_1 and n_2 vertices, respectively. Then

$$\det(A(C_{n_1}^2 \dot{+} C_{n_2}^2)) = \begin{cases} 0 & ; n_1 = 2q_1 + 4 \text{ or } n_2 = 2q_2 + 4 \\ 256 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 6q_2 + 3 \\ 16 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 6q_2 + 1 \text{ or } n_2 = 6q_2 + 5 \\ 64 & ; \text{otherwise} \end{cases}$$

where q_1, q_2 are positive integers.

(9) Let $C_{n_1}^2$ be a square cycle graph with n_1 vertices and P_{n_2} be a path with n_2 vertices where n_1, n_2 are positive integers. Then

$$\det(A(C_{n_1}^2 \cup_{v_1} P_{n_2})) = \begin{cases} 0 & ; n_1 \text{ is even} \\ 16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 \\ -16 & ; n_1 = 6q_1 + 3 \text{ and } n_2 = 4q_2 - 2 \\ 4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 \\ -4 & ; n_1 = 6q_1 + 1 \text{ or } n_1 = 6q_1 + 5 \text{ and } n_2 = 4q_2 - 2 \end{cases}$$

where q_1, q_2 are positive integers.

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APPENDIX

THE RESEARCH PAPER

Determinant of Adjacency Matrix of Square Cycle Graph

Nitiphoom Adsawatithisakul and Decha Samana

***Abstract*—Square Cycle, C_n^2 is a graph that has n vertices and two vertices u and v are adjacent if and only if distance between u and v not greater than 2. In this paper, we show that the determinant of adjacency matrix of square cycle C_n^2 are as follows**

$$det(A(C_n^2)) = \begin{cases} 0 & ; n \equiv 0,2,4 \bmod 6 \\ 16 & ; n \equiv 3 \bmod 6 \\ 4 & ; n \equiv 1,5 \bmod 6. \end{cases}$$

***Index Terms*—Determinant, Square cycle graph, Adjacency matrix.**

I. Introduction

Let G be a simple graph with n vertices. We denote $det(A(G))$ is the determinant of adjacency matrix of G and $E(G;k)$ is k^{th} eigenvalues of the adjacency matrix which $det(A(G))$ and $E(G;k)$ are independent of the choice of vertices in adjacency matrix and are an invariant of G.

In [2] and [4] , they determined the determinant of adjacency matrix of some graphs, such as K_n, C_n, P_n and W_n. B. Gyurov and J. Cloud [7] has determined determinant of Pin-wheel graph. Moreover, there are studies of graph which satisfy some properties of determinant for example, M. Doob [5] construct circulant graph with $det(A(G)) = -deg(G)$, S. Hu [9] and A. Abdollahi [1] have found that the determinant of graphs with exactly one cycle and exactly two cycles, respectively.

Cycle power, C_n^d is a graph that has n vertices and distance each pair of vertex is less or equal d . For example,

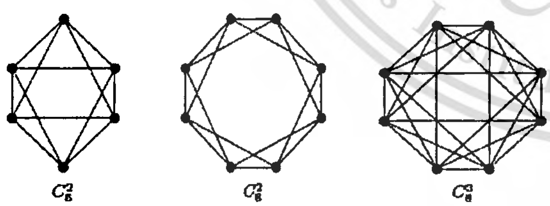

Figure 1. *d*-th power of cycle graph

If $d = 2$, $n \geq 6$, it is called square cycle graph.

Furthermore, there are studies of cycle power such as, C.N.Campos and C.P.de Mello [3] , M.Krivelevich and A.Nachmias [10] studied about the colouring in cycle power, Y.Hoa, C.Woo and P.Chen [8] investigate the sandpile group in cycle power , D.Li and M.Liu [11] consider cycle power and their complements which satisfy Hadwiger's conjecture.

From figure 1 graph C_6^2 and graph C_8^2, we write to adjacency matrix

$$A(C_6^2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$A(C_8^2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

We see adjacency matrix of C_6^2 and C_8^2 is a circulant matrix because a main diagonal of matrix is equal to zero and entries in first row satisfy $a_{1j} = a_{1,(n-j+2)}$ for $j = 2, ..., n$ and $a_{ij} = a_{i+1,j+1}$, then a square cycle graph is a circulant graph. It is interesting to study determinant of adjacency matrix of square cycle graph.

Proposition 1. *[2] Suppose that $[0, a_2, ..., a_n]$ is the first row of the adjacency matrix of a circulant graph G. Then the eigenvalues of graph G is denoted $E(G;k)$,*

$$E(G;k) = \sum_{j=1}^{n} a_j z^{j-1}$$

where $z = e^{\frac{2k\pi i}{n}}, k = 1, 2, ..., n$

square cycle graph is a circulant graph then eigenvalues of square cycle graph is

$$E(C_n^2;k) = \sum_{j=1}^{n} a_j z^{j-1} \tag{1}$$

where $z = e^{\frac{2k\pi i}{n}}, k = 1, 2, 3, ...$ Thus

$$E(C_n^2;k) = z + z^2 + z^{n-2} + z^{n-1}. \tag{2}$$

Theorem 2. *[6] Let $\lambda_1, ..., \lambda_n$ be a eigenvalues of a square matrix A. Then*

$$det(A) = \lambda_1 \lambda_2 ... \lambda_n$$

Next, we present lemma that will be used in the proof of determinant of adjacency matrix of square cycle graph.

N. Adsawatithisakul is with the Department of Mathematics, King Mongkut's Institute of Technology Ladkrabang, Bangkok, Thailand 10520 Email: na.adsawatithisakul@gmail.com.

D. Samana is with the Department of Mathematics, King Mongkut's Institute of Technology Ladkrabang, Bangkok, Thailand 10520 Email: dechasamana@hotmail.com.

II. Main Results

Lemma 3. *Let q be a positive number. Then*

$$\prod_{k=1}^{2q}(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})\prod_{k=2q+2}^{4q+1}(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})\prod_{k=4q+3}^{6q+2}(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3})=2^{-12q}. \quad (3)$$

Proof: The left hand side of (3) is

$$\prod_{k=1}^{2q}\frac{\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{2\sin\frac{3k\pi}{6q+3}2\sin\frac{k\pi}{6q+3}}\prod_{k=2q+2}^{4q+1}\frac{\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{2\sin\frac{3k\pi}{6q+3}2\sin\frac{k\pi}{6q+3}}\prod_{k=4q+3}^{6q+2}\frac{\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{2\sin\frac{3k\pi}{6q+3}2\sin\frac{k\pi}{6q+3}}$$

$$=\frac{1}{2^{12q}}\left(\frac{\prod_{k=q+1}^{2q}\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{\prod_{k=1}^{q}\sin\frac{(6k-3)\pi}{6q+3}\sin\frac{(2k-1)\pi}{6q+3}}\right)\left(\frac{\prod_{k=2q+2}^{4q+1}\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{\prod_{k=2q+2}^{4q+1}\sin\frac{3k\pi}{6q+3}\sin\frac{k\pi}{6q+3}}\right)\left(\frac{\prod_{k=4q+3}^{6q+2}\sin\frac{6k\pi}{6q+3}\sin\frac{2k\pi}{6q+3}}{\prod_{k=4q+3}^{6q+2}\sin\frac{3k\pi}{6q+3}\sin\frac{k\pi}{6q+3}}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\prod_{k=1}^{q}\sin\frac{(6k-3)\pi}{6q+3}\sin\frac{(2k-1)\pi}{6q+3}}{\prod_{k=1}^{q}\sin\frac{(6k-3)\pi}{6q+3}\sin\frac{(2k-1)\pi}{6q+3}}\right)\left(\frac{\prod_{k=2q+2}^{3q+1}\sin\frac{(6k-(6q+3))\pi}{6q+3}\sin\frac{(2k-(2q+1))\pi}{6q+3}}{\prod_{k=2q+2}^{3q+1}\sin\frac{(6k-(6q+3))\pi}{6q+3}\sin\frac{(2k-(2q+1))\pi}{6q+3}}\right)\left(\frac{\prod_{k=4q+3}^{5q+2}\sin\frac{(6k-(12q+9))\pi}{6q+3}\sin\frac{(2k-(4q+3))\pi}{6q+3}}{\prod_{k=4q+3}^{5q+2}\sin\frac{(6k-(12q+9))\pi}{6q+3}\sin\frac{(2k-(4q+3))\pi}{6q+3}}\right)$$

$$=2^{-12q}.$$

■

Lemma 4. *Let q be a positive integer. Then*

$$\prod_{k=1}^{6q}(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1})=2^{-12q}.$$

Proof: It can be proved by

$$\prod_{k=1}^{6q}(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1})=\prod_{k=1}^{6q}\frac{\sin\frac{6k\pi}{6q+1}\sin\frac{2k\pi}{6q+1}}{2\sin\frac{3k\pi}{6q+1}2\sin\frac{3k\pi}{6q+1}}$$

$$=\frac{1}{2^{12q}}\left(\frac{\prod_{k=3q+1}^{6q}\sin\frac{6k\pi}{6q+1}\sin\frac{2k\pi}{6q+1}}{\prod_{k=1}^{3q}\sin\frac{(6k-3)\pi}{6q+1}\sin\frac{(2k-1)\pi}{6q+1}}\right)$$

$$=\frac{1}{2^{12q}}\left(\frac{\prod_{k=1}^{3q}\sin\frac{(6k-3)\pi}{6q+1}\sin\frac{(2k-1)\pi}{6q+1}}{\prod_{k=1}^{3q}\sin\frac{(6k-3)\pi}{6q+1}\sin\frac{(2k-1)\pi}{6q+1}}\right)$$

$$=2^{-12q}.$$

■

Lemma 5. *Let q be a positive integer. Then*

$$\prod_{k=1}^{6q+4}(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5})=2^{-2(6q+4)}.$$

Proof: It can be proved by

$$\prod_{k=1}^{6q+4}(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5})=\prod_{k=1}^{6q+4}\frac{\sin\frac{6k\pi}{6q+5}\sin\frac{2k\pi}{6q+5}}{2\sin\frac{3k\pi}{6q+5}2\sin\frac{3k\pi}{6q+5}}$$

$$=\frac{1}{2^{2(6q+4)}}\left(\frac{\prod_{k=3q+3}^{6q+4}\sin\frac{6k\pi}{6q+5}\sin\frac{2k\pi}{6q+5}}{\prod_{k=1}^{3q+2}\sin\frac{(6k-3)\pi}{6q+5}\sin\frac{(2k-1)\pi}{6q+5}}\right)$$

$$=\frac{1}{2^{2(6q+4)}}\left(\frac{\prod_{k=1}^{3q+2}\sin\frac{(6k-3)\pi}{6q+5}\sin\frac{(2k-1)\pi}{6q+5}}{\prod_{k=1}^{3q+2}\sin\frac{(6k-3)\pi}{6q+5}\sin\frac{(2k-1)\pi}{6q+5}}\right)$$

$$=2^{-2(6q+4)}.$$

■

Theorem 6. *Let C_n^2 be a square cycle graph with n vertices and n be a positive integer where $n \geq 6$. Then*

$$det(A(C_n^2))=\begin{cases}0 & ;\ n\equiv 0,2,4 \bmod 6\\ 16 & ;\ n\equiv 3 \bmod 6\\ 4 & ;\ n\equiv 1,5 \bmod 6\end{cases}$$

Proof: Let $E(C_n^2;k)$ be a k^{th} eigenvalue of adjacency matrix of square cycle graph C_n^2. From (2) We get

$$E(C_n^2;k)=e^{\frac{2k\pi i}{n}}+e^{\frac{4k\pi i}{n}}+e^{\frac{2k(n-2)\pi i}{n}}+e^{\frac{2k(n-1)\pi i}{n}}$$
$$=e^{\frac{2k\pi i}{n}}+e^{\frac{4k\pi i}{n}}+e^{\frac{2kn\pi i}{n}}\cdot e^{\frac{-4k\pi i}{n}}+e^{\frac{2kn\pi i}{n}}\cdot e^{\frac{-2k\pi i}{n}}$$

By Euler's formula, we obtain

$$E(C_n^2;k)=(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n})+(\cos\frac{4k\pi}{n}+i\sin\frac{4k\pi}{n})+(\cos\frac{-4k\pi}{n}+i\sin\frac{-4k\pi}{n})+(\cos\frac{-2k\pi}{n}+i\sin\frac{-2k\pi}{n})$$
$$=2\cos\frac{2k\pi}{n}+2\cos\frac{4k\pi}{n}$$

We can rewrite

$$E(C_n^2;k)=4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}) \quad (4)$$

From (4) We have

$$det(A(C_n^2))=\prod_{k=1}^{n}E(C_n^2;k)=\prod_{k=1}^{n}4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}) \quad (5)$$

Consider n as follows

Case I, $n\equiv 0,2,4 \bmod 6$

Since n is even and $1\leq k\leq n$, consider (4) when $k=\frac{n}{2}$. Then

$$E(C_n^2;\frac{n}{2})=4(\cos\frac{3\frac{n}{2}\pi}{n}\cos\frac{\frac{n}{2}\pi}{n})=0.$$

From (5), we obtain

$$det(A(C_n^2))=\prod_{k=1}^{n}E(C_n^2;k)=0.$$

Therefore, $det(A(C_n^2))=0$ when $n\equiv 0,2,4 \bmod 6$.

Case II, $n\equiv 3 \bmod 6$ Then $n=6q+3, \exists q\in\mathbb{Z}^+$.

From (5), we obtain

$$
\begin{aligned}
det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\
&= \prod_{k=1}^{6q+3} 4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}) \\
&= 4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n})4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n}) \\
&\quad 4(\cos\frac{3k\pi}{n}\cos\frac{k\pi}{n})2^{12q} \\
&\quad \prod_{k=1}^{2q} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) \\
&\quad \prod_{k=2q+2}^{4q+1} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) \\
&\quad \prod_{k=4q+3}^{6q+2} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) \\
&= (-2)(-2)(4)2^{12q}\prod_{k=1}^{2q} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) \\
&\quad \prod_{k=2q+2}^{4q+1} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}) \\
&\quad \prod_{k=4q+3}^{6q+2} 4(\cos\frac{3k\pi}{6q+3}\cos\frac{k\pi}{6q+3}).
\end{aligned}
$$

Using Lemma 3, we have

$$
\begin{aligned}
det(A(C_n^2)) &= (-2)(-2)(4)(2^{12q})(2^{-12q}) \\
&= 16.
\end{aligned}
$$

Therefore $det(A(C_n^2)) = 16$ when $n \equiv 3 \bmod 6$.

Case III, $n \equiv 1 \bmod 6$ and $n \equiv 5 \bmod 6$. We consider 2 subcases.

Subcase 3.1, $n \equiv 1 \bmod 6$, by (5), we obtain

$$
\begin{aligned}
det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\
&= \prod_{k=1}^{6q+1} 2^2(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}) \\
&= 4(\cos\frac{3(6q+1)\pi}{6q+1}\cos\frac{(6q+1)\pi}{6q+1})2^{12q} \\
&\quad (\prod_{k=1}^{6q} 4(\cos\frac{3k\pi}{6q+1}\cos\frac{k\pi}{6q+1}).
\end{aligned}
$$

Using Lemma 4, we have

$$
\begin{aligned}
det(A(C_n^2)) &= 4(2^{12q})(2^{-12q}) \\
&= 4.
\end{aligned}
$$

Subcase 3.2, $n \equiv 5 \bmod 6$, by (5), we obtain

$$
\begin{aligned}
det(A(C_n^2)) &= \prod_{k=1}^{n} E(C_n^2;k) \\
&= \prod_{k=1}^{6q+5} 2^2(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5}) \\
&= 4(\cos\frac{3(6q+5)\pi}{6q+5}\cos\frac{(6q+5)\pi}{6q+5})2^{2(6q+4)} \\
&\quad (\prod_{k=1}^{6q+4} 4(\cos\frac{3k\pi}{6q+5}\cos\frac{k\pi}{6q+5})).
\end{aligned}
$$

Using Lemma 5, we have

$$
\begin{aligned}
det(A(C_n^2)) &= 4(2^{2(6q+4)})(2^{-2(6q+4)}) \\
&= 4.
\end{aligned}
$$

From subcase 3.1 and 3.2, we obtain
$det(A(C_n^2)) = 4$ for $n \equiv 1, 5 \bmod 6$.
From case I, II and III,

$$
det(A(C_n^2)) = \begin{cases} 0 & ; n \equiv 0,2,4 \bmod 6 \\ 16 & ; n \equiv 3 \bmod 6 \\ 4 & ; n \equiv 1,5 \bmod 6 \end{cases}
$$

where $n \geq 6$. ■

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ดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟบางกราฟ

Determinant of Adjacency Matrix of Some Graphs

นิติภูมิ อัศวธิติสกุล[1] และเดชา สมนะ[2]

Nitiphoom Adsawatithisakul[1] and Decha Samana[2]

[1]นักศึกษาปริญญาโท สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง กรุงเทพฯ 10520

[2]อาจารย์ สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง กรุงเทพฯ 10520

Abstract

Square cycle power, C_n^2 is a graph having n vertices and the distance between each vertex pair is less than or equal to 2 and $G \bigcup_{v_1} P_{n_2}$ is a graph G adjacent v_1 where $v_1 \in V(P_{n_2})$. In this paper, we show that the determinant of the adjacency matrix $C_{n_1}^2 \bigcup_{v_1} P_{n_2}$.

Keywords: *Determinant, Square cycle graph, Adjacency matrix, Sum operation*

บทคัดย่อ

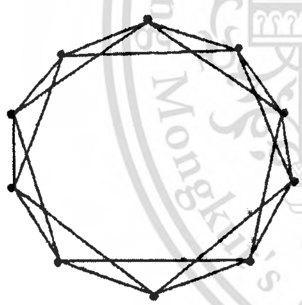
กราฟวงกำลังสอง C_n^2 คือ กราฟที่มีจำนวนจุดยอด n จุดและจุดยอดสองจุดที่แตกต่างกันในกราฟมีเส้นเชื่อมกันก็ต่อเมื่อระยะทางระหว่างจุดยอดสองจุดที่แตกต่างกันมีค่าไม่เกิน 2 และ $G \bigcup_{v_1} P_{n_2}$ คือ กราฟที่เกิดจากกราฟ G ประชิดกับจุด v_1 โดยที่ $v_1 \in V(P_{n_2})$ โดยงานวิจัยนี้ เราได้ดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $C_{n_1}^2 \bigcup_{v_1} P_{n_2}$

คำสำคัญ: *ดีเทอร์มิแนนต์, กราฟวงกำลังสอง, เมทริกซ์ประชิด, การดำเนินการการบวก*

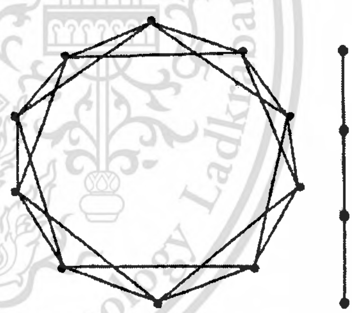
บทนำ

ให้กราฟ G เป็นกราฟเชิงเดียวและเป็นกราฟที่ไม่มีทิศทาง เราจะกำหนดให้ $V(G)$ เป็นเซตของจุดยอดในกราฟ G และ $E(G)$ เป็นเซตของเส้นเชื่อมในกราฟ G สำหรับจุดยอด $u,v \in V(G)$ เรากำหนดให้ $d(u,v)$ จะเป็นระยะทางระหว่างจุดยอด u และจุดยอด v และให้ $A(G)$ แทนเมทริกซ์ประชิดของกราฟ G และกำหนดให้ $\det(A(G))$ เป็นการหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ G

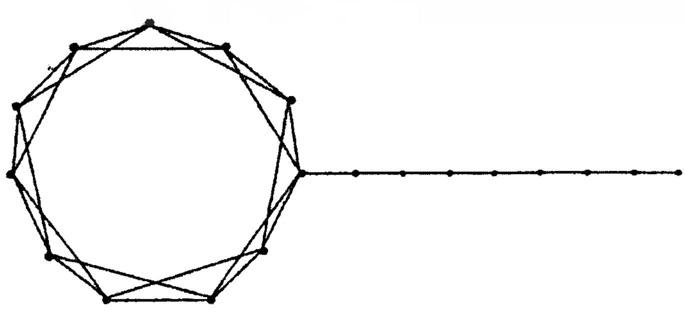
กราฟวงกำลังสอง C_n^2 คือ กราฟที่มีจำนวนจุดยอด n จุดและจุดยอดสองจุด u และ v มีเส้นเชื่อมกันก็ต่อเมื่อระยะทางระหว่าง u และ v มีค่าไม่เกิน 2 เราจะนำกราฟ C_n^2 และกราฟ G มาดำเนินการการบวก ซึ่งนิยามของการดำเนินการการบวก คือ กราฟทั้งสองจะต้องมีจุดยอดแตกต่างกันทั้งหมด คือ $V(G_1) \cap V(G_2) = \varnothing$ และผลของการดำเนินการการบวก คือ $V(G_1) \cup V(G_2)$ และ $E(G_1) \cup E(G_2)$ เขียนแทนด้วย $G_1 + G_2$ และเราจะนำกราฟวงกำลังสอง $C_{n_1}^2$ ประชิดกับกราฟวิถี P_{n_2} ที่จุด v_1 โดยที่ $v_1 \in V\left(P_{n_2}\right)$ เขียนแทนด้วย $C_{n_1}^2 \cup_{v_1} P_{n_2}$ ตัวอย่างเช่น



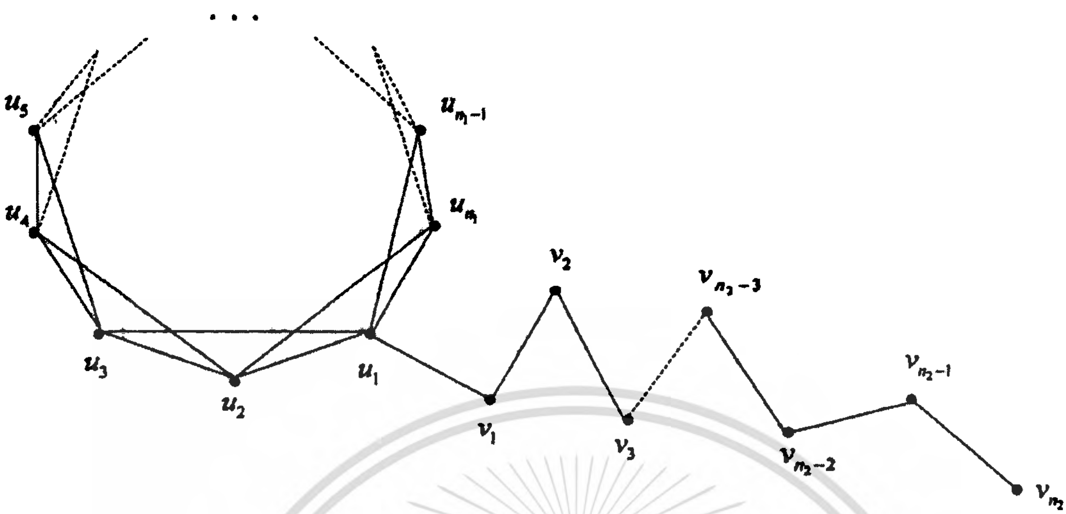
รูปที่ 1 กราฟ C_{10}^2



รูปที่ 2 กราฟ $C_{10}^2 + P_4$



รูปที่ 3 กราฟ $C_{11}^2 \cup_{v_1} P_8$



รูปที่ 4 กราฟวงกำลังสอง ประชิดกับกราฟวิถีที่จุด v_1 หรือกราฟ $C_{n_1}^2 \cup_{v_1} P_{n_2}$

การหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟต่างๆ ได้มีการศึกษาอย่างกว้างขวาง เช่น N.Biggs [2] และ D.Cvetkovic [3] ได้มีการเผยแพร่การหาดีเทอร์มิแนนต์ของกราฟต่าง ๆ เช่น K_n, C_n, P_n ดังนี้

$$\det(A(K_n)) = (-1)^{n-1}(n-1)$$

$$\det(A(C_n)) = \begin{cases} 0 & ; n \equiv 0 \pmod 4 \\ -4 & ; n \equiv 2 \pmod 4 \\ 2 & ; \text{อื่น ๆ} \end{cases}$$

$$\det(A(P_n)) = \begin{cases} (-1)^k ; n = 2k \\ 0 \quad ; \text{อื่น ๆ} \end{cases}$$

นอกจากนี้ J.Cloud [4] ได้หาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ pin-wheel $W_n' \cup_{v_1} P_m$ และ N.Adsawatithisakul and D.Samana [1] ได้หาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ C_n^2 ได้ดังนี้

$$\det(A(C_n^2)) = \begin{cases} 0 & ; n = 2q+4 \\ 16 & ; n = 6q+3 \\ 4 & ; \text{อื่น ๆ} \end{cases} \qquad \text{โดยที่ } q \text{ เป็นจำนวนเต็มบวก}$$

ในงานวิจัยนี้ คณะผู้วิจัยได้หาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $G_1 \dotplus G_2$ โดยที่ G_1 และ G_2 เป็นกราฟเชิงเดียวใดๆ เพื่อใช้ในการหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $C_{n_1}^2 \cup_{v_1} P_{n_2}$

วัตถุประสงค์

1. เพื่อหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $G_1 + G_2$
2. เพื่อหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $C_{n_1}^2 \cup_{\eta} P_{n_2}$

วิธีการวิจัย

ทฤษฎีบทที่จำเป็นในการหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $G_1 + G_2$ และกราฟ $C_{n_1}^2 \cup_{\eta} P_{n_2}$ มีดังนี้

ทฤษฎีบท 1 [5] ให้ A, B, C และ D เป็นเมทริกซ์ขนาด $n \times n$, $n \times m$, $m \times n$ และ $m \times m$ ตามลำดับ แล้ว

$$\det\begin{bmatrix} A & B \\ \underline{0} & D \end{bmatrix} = \det\begin{bmatrix} A & \underline{0} \\ C & D \end{bmatrix} = \det(A)\det(D) \text{ โดยที่ } \underline{0} \text{ เป็นเมทริกซ์ศูนย์}$$

ทฤษฎีบท 2 [6] ให้กราฟ G เป็นกราฟที่มีจำนวนจุดยอดมากกว่า 2 จุด และจุดยอด v_1, v_2 เป็นจุดยอดที่แตกต่างกันในกราฟ G โดยที่ $N(v_1) \subseteq N(v_2)$ และให้ G' เป็นกราฟที่ได้จากกราฟ G โดยการนำเส้นออกทุกเส้นของ v_2y โดยที่ $y \in N(v_1)$ ดังนั้น $\det(A(G)) = \det(A(G'))$

เช่น กำหนดให้กราฟ G และกราฟ G' ดังรูปที่ 4

รูปที่ 5 กราฟ G และกราฟ G'

เมื่อพิจารณา G จะเห็นได้ว่า $N(v_1) = \{v_3\}$ และ $N(v_2) = \{v_3, v_4\}$ ซึ่ง $N(v_1) \subseteq N(v_2)$ จะนำเส้น v_2y ออกทุกเส้น โดยที่ $y \in N(v_1)$ นั่นคือ ลบเส้น v_2v_3 จะได้ กราฟ G' และ $\det(A(G)) = \det(A(G')) = 1$

ในการหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $C_n^2 \cup_\eta P_m$ จะเริ่มต้นโดยการหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟ $G_1 \dotplus G_2$ แล้วจะใช้ทฤษฎีบท 1 และ 2 ช่วยในการหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟดังกล่าว

ผลการวิจัย

ทฤษฎีบท 3 ให้ G_1 และ G_2 เป็นกราฟเชิงเดียว แล้ว $\det(A(G_1 \dotplus G_2)) = \det(A(G_1))\det(A(G_2))$

พิสูจน์ กำหนดให้กราฟ G_1 ประกอบด้วยจุดยอด $u_1, u_2, \ldots, u_k$ และให้กราฟ G_2 ประกอบด้วยจุดยอด $u_{k+1}, u_{k+2}, \ldots, u_n$ และเมื่อนำกราฟทั้งสองมาดำเนินการการบวก จะเป็นดังรูปข้างล่าง

รูปที่ 6 กราฟ $G_1 \dotplus G_2$

จะเห็นได้ว่าจุดยอดของกราฟ G_1 และกราฟ G_2 ไม่ประชิดกันเลยและเมื่อเขียนให้อยู่ในรูปของเมทริกซ์ประชิดจะได้

$$A(G_1 \dotplus G_2) = \left[\begin{array}{c|c} \overbrace{\qquad A(G_1) \qquad}^{k\ \text{สมาชิก}} & \overbrace{\begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & & & & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}}^{n-k\ \text{สมาชิก}} \\ \hline \begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & & & & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{matrix} & A(G_2) \end{array}\right]$$

สามารถเขียนใหม่ได้

$$A(G_1 \dot{+} G_2) = \begin{bmatrix} A(G_1) & \underline{0} \\ \underline{0} & A(G_2) \end{bmatrix}$$ โดยที่ $\underline{0}$ เป็นเมทริกซ์ศูนย์

จากทฤษฎีบท 1 ทำให้ได้ว่า

$$\det(A(G_1 \dot{+} G_2)) = \det(A(G_1))\det(A(G_2))$$

□

กำหนดให้ G_1 แทนด้วยกราฟ $C_{n_1}^2$ และกราฟ G_2 แทนด้วยกราฟ P_{n_2} จะได้

$$\det(A(C_{n_1}^2 \dot{+} P_{n_2})) = \begin{cases} 0 & ; n_1 \text{ เป็นเลขคู่ หรือ } n_2 \text{ เป็นเลขคี่} \\ 16 & ; n_1 = 6q_1 + 3 \text{ และ } n_2 = 4q_2 \\ -16 & ; n_1 = 6q_1 + 3 \text{ และ } n_2 = 4q_2 - 2 \\ 4 & ; n_1 = 6q_1 + 1 \text{ หรือ } n_1 = 6q_1 + 5 \text{ และ } n_2 = 4q_2 \\ -4 & ; n_1 = 6q_1 + 1 \text{ หรือ } n_1 = 6q_1 + 5 \text{ และ } n_2 = 4q_2 - 2 \end{cases} \quad (1)$$

โดยที่ q_1 และ q_2 เป็นจำนวนเต็มบวก

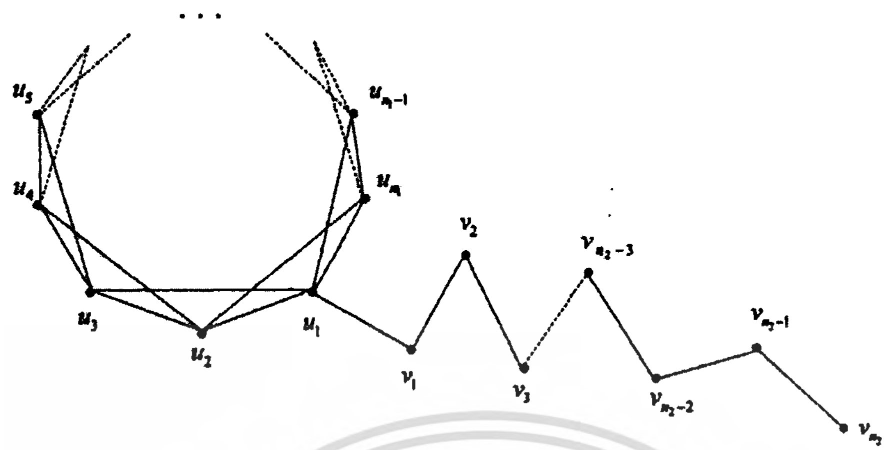
ลำดับถัดไป เราจะหาดีเทอร์มิแนนต์ของเมทริกซ์ประชิดของกราฟวงกำลังสอง $C_{n_1}^2$ ประชิดกับกราฟวิถี P_{n_2} ที่จุด v_1 โดยที่ $v_1 \in V(P_{n_2})$ นั่นก็คือ $\det(A(C_{n_1}^2 \cup_{v_1} P_{n_2}))$

บทแทรก 4 ให้ $C_{n_1}^2$ เป็นกราฟวงกำลังสองที่มีจุดยอด n_1 จุด และ P_{n_2} เป็นกราฟวิถีที่มีจุดยอด n_2 จุด โดยที่ n_2 เป็นจำนวนเต็มบวกคู่ ถ้า $C_{n_1}^2 \cup_{v_1} P_{n_2}$ คือกราฟวงกำลังสอง $C_{n_1}^2$ ประชิดกับกราฟวิถี P_{n_2} ที่จุด v_1 ซึ่ง $v_1 \in V(P_{n_2})$ แล้ว

$$\det(A(C_{n_1}^2 \cup_{v_1} P_{n_2})) = \begin{cases} 0 & ; n_1 \text{ เป็นจำนวนเต็มบวกคู่} \\ 16 & ; n_1 = 6q_1 + 3 \text{ และ } n_2 = 4q_2 \\ -16 & ; n_1 = 6q_1 + 3 \text{ และ } n_2 = 4q_2 - 2 \\ 4 & ; n_1 = 6q_1 + 1 \text{ หรือ } n_1 = 6q_1 + 5 \text{ และ } n_2 = 4q_2 \\ -4 & ; n_1 = 6q_1 + 1 \text{ หรือ } n_1 = 6q_1 + 5 \text{ และ } n_2 = 4q_2 - 2 \end{cases} \quad (2)$$

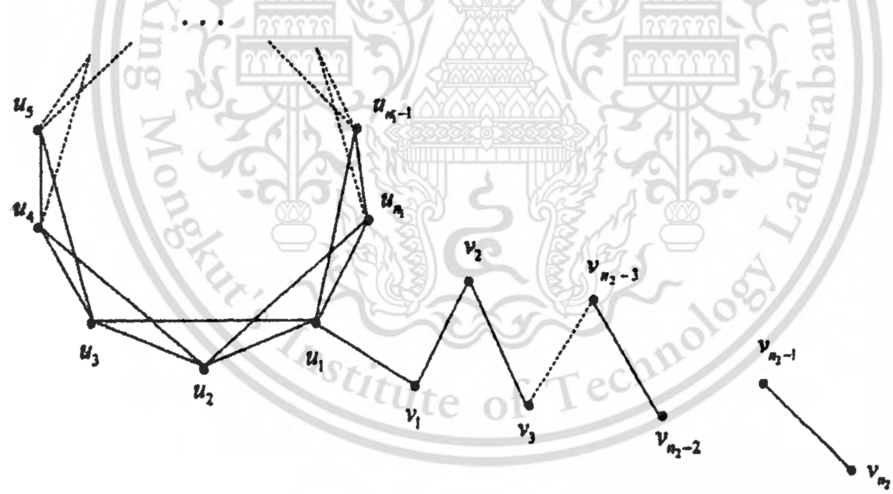
โดยที่ q_1 และ q_2 เป็นจำนวนเต็มบวก

พิสูจน์ กำหนดให้กราฟ C_n^2 ประกอบด้วยจุดยอด $u_1, u_2, \ldots, u_{n_1}$ และกราฟ P_{n_2} ประกอบด้วยจุดยอด $v_1, v_2, \ldots, v_{n_2}$ และเมื่อนำกราฟทั้งสองมาประชิดกันที่จุด v_1 โดยที่ $v_1 \in V(P_{n_2})$ เป็นดังรูปข้างล่าง



รูปที่ 7 กราฟวงกำลังสอง ประชิดกับกราฟวิถีที่จุด v_1 หรือกราฟ $C_{n_1}^2 \cup_{v_1} P_{n_2}$

พิจารณาจุดยอด v_{n_2} และจุดยอด v_{n_2-2} ของกราฟ $C_{n_1}^2 \cup_{v_1} P_{n_2}$ ซึ่งจะเห็นว่า $N(v_{n_2}) \subseteq N(v_{n_2-2})$ และเมื่อนำเส้น $v_{n_2}v_{n_2-2}$ ออก ทำให้ได้ดังรูปด้านล่าง



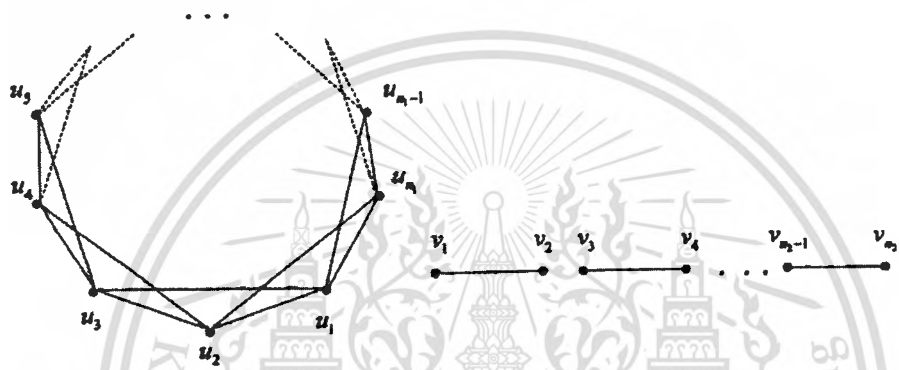
รูปที่ 8 รูปที่เกิดจากการตัดเส้นออกด้วยทฤษฎีบท 2

นั่นคือ กราฟ $C_{n_1}^2 \cup_{v_1} P_{n_2-2} \dot{+} P_2$ ต่อไปจะพิจารณาจุด v_{n_2-2} และ v_{n_2-4} ซึ่ง $N(v_{n_2-2}) \subseteq N(v_{n_2-4})$ และนำเส้น $v_{n_2-2}v_{n_2-4}$ ออก จะได้กราฟ $C_{n_1}^2 \cup_{v_1} P_{n_2-4} \dot{+} P_2 \dot{+} P_2$ ทำต่อไปเรื่อย ๆ จะทำให้ได้กราฟ $C_{n_1}^2 \dot{+} \underbrace{P_2 \dot{+} P_2 \dot{+} \ldots \dot{+} P_2}_{n/2}$ ดังรูปที่ 9 เมื่อใช้ทฤษฎีบท 2 และสมการ (1) จะได้ว่า

$$\det(A(C_{n_1}^2 \cup_{v_1} P_{n_2})) = \det(A(C_{n_1}^2 \dot{+} \underbrace{P_2 \dot{+} P_2 \dot{+} \ldots \dot{+} P_2}_{n/2}))$$

นั่นคือ

$$\det(A(C_{n_1}^2 \cup_\eta P_{n_2})) = \begin{cases} 0 & ;n_1 \text{ เป็นจำนวนเต็มบวกคู่} \\ 16 & ;n_1 = 6q_1+3 \text{ และ } n_2 = 4q_2 \\ -16 & ;n_1 = 6q_1+3 \text{ และ } n_2 = 4q_2-2 \\ 4 & ;n_1 = 6q_1+1 \text{ หรือ } n_1 = 6q_1+5 \text{ และ } n_2 = 4q_2 \\ -4 & ;n_1 = 6q_1+1 \text{ หรือ } n_1 = 6q_1+5 \text{ และ } n_2 = 4q_2-2 \end{cases}$$



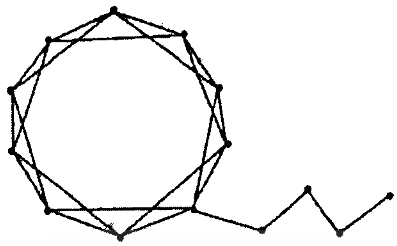
รูปที่ 9 กราฟ $C_{n_1}^2 \dotplus \underbrace{P_2 \dotplus P_2 \dotplus \ldots \dotplus P_2}_{n/2}$

สรุป

การหาดีเทอร์มินันต์ของเมทริกซ์ประชิดของกราฟ $C_{n_1}^2 \cup_\eta P_{n_2}$ หาได้ค่อนข้างยุ่งยากและซับซ้อน เนื่องจากเมทริกซ์ประชิดของกราฟ $C_{n_1}^2 \cup_\eta P_{n_2}$ มีขนาดใหญ่ เมื่อ n_1 และ n_2 เป็นจำนวนเต็มที่มีค่ามาก ดังเมทริกซ์ข้างล่าง

$$A(C_{n_1}^2 \cup_\eta P_{n_2}) = \left[\begin{array}{c|ccccc} & 1 & 0 & 0 & \cdots & 0 \\ & 0 & & & & 0 \\ A(C_{n_1}^2) & 0 & & \ddots & & 0 \\ & \vdots & & & & \vdots \\ & 0 & 0 & 0 & \cdots & 0 \\ \hline \begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{matrix} & & & A(P_{n_2}) & & \end{array}\right]$$

เมื่อใช้บทแทรก 4 จะสะดวกและรวดเร็ว เช่น กราฟ $C_{10}^2 \cup_{v_1} P_4$



รูปที่ 10 กราฟ $C_{10}^2 \cup_{v_1} P_4$

$$A(C_{10}^2 \cup_{v_1} P_4)=\begin{bmatrix}
0&1&1&0&0&0&0&0&1&1&1&0&0&0\\
1&0&1&1&0&0&0&0&0&1&0&0&0&0\\
1&1&0&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&1&0&0&0&0&0&0&0\\
0&0&0&1&1&0&1&1&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&1&0&0&0&0\\
1&0&0&0&0&0&1&1&0&1&0&0&0&0\\
1&1&0&0&0&0&0&1&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{bmatrix}$$

เนื่องจาก $n_1 = 10$ เป็นจำนวนเต็มบวกคู่ จะได้ว่า $\det(A(C_{10}^2 \cup_{v_1} P_4)) = 0$

รูปที่ 11 กราฟ $C_{11}^2 \cup_{v_1} P_2$

$$A(C_{11}^2 \cup_{v_1} P_2)=\begin{bmatrix}
0&1&1&0&0&0&0&0&0&1&1&1&0\\
1&0&1&1&0&0&0&0&0&0&1&0&0\\
1&1&0&1&1&0&0&0&0&0&0&0&0\\
0&1&1&0&1&1&0&0&0&0&0&0&0\\
0&0&1&1&0&1&1&0&0&0&0&0&0\\
0&0&0&1&1&0&1&1&0&0&0&0&0\\
0&0&0&0&1&1&0&1&1&0&0&0&0\\
0&0&0&0&0&1&1&0&1&1&0&0&0\\
0&0&0&0&0&0&1&1&0&1&1&0&0\\
1&0&0&0&0&0&0&1&1&0&1&0&0\\
1&1&0&0&0&0&0&0&1&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0
\end{bmatrix}$$

และเมื่อพิจารณากราฟ $C_{11}^2 \cup_{v_1} P_2$ จะได้ว่า $\det(A(C_{11}^2 \cup_{v_1} P_2)) = -4$ เนื่องจาก $n_1 = 11$ และ $n_2 = 2$

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BIOGRAPHY

NAME	Mr. Nitiphoom Adsawatithisakul
DATE OF BIRTH	October 17, 1987
PLACE OF BIRTH	Phetchabun, Thailand
ATTENDED INSTITUTION	Bachelor of Science (Applied Mathematics) King Mongkut's Institute of Technology Ladkrabang, in 2009, Bangkok, Thailand.
HOME ADDRESS	26/8, Soi Pinit, Thambom Lomsak, Amphur Lomsak, Phetchabun, 67110, Thailand
E-MAIL	na.adsawatithisakul@gmail.com

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