DISSIPATIVE EFFECTS ON THE PROPAGATION OF ENTANGLED PHOTON STATE IN FIBER OPTICS

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หัวข้อวิทยานิพนธ์	**ผลของการเสื่อมสลายต่อการแผ่ของสถานะเอนแทงเกิลโฟตอนในใยแก้วนำแสง**
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บทคัดย่อ

ได้วิเคราะห์ผลที่เกิดจากสภาพเสื่อมสลายที่กระทำต่อการแผ่ของสถานะเอนแทงเกิลโฟตอนในโพรงวงแหวนสั่นพร้องเส้นใยแก้วนำแสงที่ไม่เชิงเส้น โดยสร้างแฮมิลโทเนียนทั้งในส่วนของระบบเอนแทงเกิลโฟตอนและส่วนของอ่างพลังงานจากแหล่งภายนอกที่เกิดจากอุปกรณ์ทางแสงและส่วนที่เกิดจากการควบกันของทั้งคู่ ณ สภาวะสมดุลเชิงความร้อน จากนั้นได้ทำการหาสมการการเคลื่อนที่ของเอนแทงเกิลโฟตอนซึ่งถูกผลิตโดยกระบวนการ โฟร์เวฟมิกซิงภายในระบบ ทั้งในกรณีของการสถานะซ้ำซ้อนและสถานะไม่ซ้ำซ้อน เมื่อสมการหลักอยู่ในตัวแทนอันตรกิริยาได้กำจัดตัวแปรต่างๆของอ่างพลังงานจากแหล่งภายนอกทิ้งโดยใช้การประมาณของมาร์คอฟ จากนั้นได้ประยุกต์ใช้ตัวดำเนินการเชิงบวกพี เพื่อทำการแปลงจากตัวดำเนินการเมทริกซ์หนาแน่นที่ลดรูปแล้วไปสู่ฟังก์ชันการแจกแจงความน่าจะเป็น พบว่าตัวดำเนินการต่างๆในสมการหลักถูกลดรูปให้เป็นตัวแปรเชิงซ้อนในรูปของสมการฟอกเกอร์—แพลงค์ ในปริภูมิเฟส—สเปซ ซึ่งให้สมการการเคลื่อนที่เชิงวิเคราะห์แบบสมการเชิงอนุพันธ์สโตแคสติกที่สมนัยกัน ในรูปแบบของสมการแลงเกอแวงและสมการอิโตะซึ่งสามารถวิเคราะห์ได้ง่าย

ระบบดังกล่าวถูกใช้ประโยชน์สำหรับเอนแทงเกิลโฟตอนได้ในหลายกรณีที่ผลของการเสื่อมสลายสามารถเหนี่ยวนำให้เกิดสัญญาณรบกวนเข้าไปในระบบ ได้เสนอกรณีศึกษาที่เอนแทงเกิลโฟตอนถูกผลิตขึ้นในอุปกรณ์สั่นพร้องในระดับไมครอนในสองรูปแบบ รูปแบบแรกได้เอนแทงเกิลโฟตอนจำนวนสองอนุภาคและรูปแบบที่สองได้เอนแทงเกิลโฟตอนจำนวนสี่อนุภาค เอนแทงเกิลโฟตอนที่ให้ผลลัพธ์ดีที่สุดสามารถนำมาจัดการได้ด้วยการวิเคราะห์ดังวิธีที่กล่าวมาสำหรับส่วนที่ประกอบด้วยระบบและสัญญาณรบกวนจากภายนอก

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ABSTRACT

Dissipative effects on the propagation of the entangled photons in the nonlinear optical fiber ring resonator is analyzed. The system of the entangled photons and reservoir from the optical devices, and their coupling interaction Hamiltonian in the thermal equilibrium environment is established and analyzed. To obtain the corresponding equation of motion of the entangled photons generated by a four wave mixing process within the system, we propose the Markov approximation to eliminate the reservoir operators, where the system master equation in the interaction picture for both degenerate and non-degenerate cases. For the positive *P* representation, the master equation is reduced to the corresponding complex number in the form of a Fokker-Planck equation in phase-space description. From this point, the corresponding analytic stochastic differential equations in the Langevin equation and Itô types are analyzed.

The above established system can be used to utilize the optimum entangled photons in some cases where the dissipation effects induce noise into the system. We have proposed the case when the entangled photons are generated within the micro ring device. The entangled photons can be generated in two forms, firstly, the two entangled photon states is generated, the other, the four entangled photon states can be easily generated. The final point is the optimum entangled photon can be managed with the proposed analysis, where the system and external noises are involved and discussed.

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CHAPTER 1

Introduction

1.1 Motivation

There are many testable results from the vital EPR paradox controversialism [1] of the limitations of quantum mechanics principle and its application have became the birth of a big new faster growing field, quantum information [2] such as quantum teleportation, quantum cloning, quantum errors correction, quantum cryptography, quantum gate and quantum key distribution, which renders the fastest states sending and highest security communications. In this field, the aspect of qubits and entanglement especially for entangled photon [3] have major vast benefits, and there are many methods to generate and use them for many purposes. In the onset, the entangled photon generations is achieved via the photon-molecules or photon-atoms interaction, as in the laser process generation, by pumping of very short pulse through a nonlinear crystal of $\chi^{(2)}$ for spontaneous parametric down conversion (SPDC) process type I and type II as in the refs. [4, 5] and later from a nonlinear $\chi^{(3)}$ materials for four-wave mixing (FWM) process, where the jumping from the excited states to the lower state appeared most likely in the Λ shape emission [6, 7]. Most of them has the different advantages for fidelity communication purposes, optical fiber is the most another successful one of this goal for the reason of its economy cost and vast facilitation in the closed system as in the refs. [8, 9, 10].

The quantum theoretical aspect of the photon state has been established most perfectly by Roy J. Glauber [11, 12], who won the nobel prize in physics year 2005 for the success of this discovery, by using the method of second quantization of electromagnetic field to get its three different representations: the number states, the coherent states and the squeezed states. To apply his important idea and study photon dynamic in fiber optics, these representations must be used together with the master equation conception in the form of density operator approach in order to get all concerning operators and how they evolve in time. Fortunately, some open quantum system problems in nonlinear optical fibers, where the reservoir energy can always penetrate to the entangled photon system and vice versa, can be solved by using only an appropriate approximation methods. The dissipative solutions can tell us many important things about the evolving of them.

The best excellent example of using the above mention of photon representations which are applied to fiber optics to describe the soliton propagation is appeared in the classic works of P. D. Drummond *et. al* [13, 14], where the Raman-modified nonlinear Schrödinger equation occur more naturally. Also, for the overlap work of quantum optics and quantum information in squeezed states representation can see in the refs. [15, 16, 18]

The optimum method of produce the entangled photon state via optical fiber has appeared in the FWM process under the energy and momentum conservations, simultaneously. This can be attained by designing in the form of a fiber loop such as a Sagnac loop, where the beam splitter behavior can modulate two photons to get the Bell's states after they fly pass from the fiber loop to the normal straight line fiber. This process can also occur in a fiber ring resonator. However, because of the fiber nonlinearity any classical intense short pumped-pulse can excite photon-phonon interaction to be happen quite naturally and then induce any harm interaction to entangled photon correlation, in the other word, destroy the quality of entanglement pair.

1.2 Statement and Significance of the Problems

There always must be the lost phenomenon occur in an entangled photon generation via FWM process in nonlinear fiber optics and in general they cannot be avoided. This is the characterization of decoherence where the system starts from the time-reversible dynamics and finally ends up with irreversible situation. With some appropriate quantum mechanical approximation methods, we always can analyze this characteristic to obtain the corresponding dynamics.

1.3 The Goal of Thesis

The main objectives of this thesis can be divided into three parts:

1.3.1 To analyze the entangled photon state from FWM process in $\chi^{(3)}$ nonlinear fiber optic of both the degenerate and nondegenerate input pumping from the corresponding master equation.

1.3.2 To understand more deeply the effects of dissipation from the fiber's environment act on the propagation of entangled photon state.

1.3.3 To study the walk-off effect of entangled photon from FWM process by the inducing of heat reservoir and how to recovery it again after interact with the reservoir.

1.4 Hypothesis to be Tested

The interaction between entangled photon states and its reservoir in fiber optics, via phonon-photon interaction induced in $\chi^{(3)}$ Kerr nonlinear fiber optics, will bring about the dissipative effects on the propagation entangled photon state and can destroy the entangle quality.

1.5 Scope and Limitation of the Study

To attain the goals of this thesis, a moderate intensity of the short pumped fields launching into fiber and a copolarization photon propagation will be modelled and analyze at thermal equilibrium. The restriction is that the occurring photon-phonon interaction will not reach up to the Raman scattering domain. The study is also circumvented in the Markov assumption in order to eliminate the reservoir operators and leave only the concerning photon operators.

1.6 Process of the Study

We have planed the ordering studying to focus on the two following topics:

- First, we study theoretically the quantized electromagnetic field in the form of the number states, the coherent states and the squeezed states together with the positive P representation. From this point, it can always be reduced from the full quantum treatment to the complex number of the Fokker-Planck equation and its corresponding stochastic differential equations such as in Langevin, Itô and Stratonovich equations.
- Final, we study FWM process in nonlinear fiber optics to construct the necessary FWM Hamiltonian for both degenerate and nondegenerate cases.

1.7 Methods of Calculation

To attain the goals, we divide the operation into two parts, the first part will concern only with the system of pure FWM calculation as the following steps:

1. The pure nondegenerate and degenerate FWM Hamiltonian is established.
2. The master equation that corresponds to the first step is analyzed by appropriate methods.
3. The equation of motion for the involving operators resulted from the foregoing steps will be analyzed to get a quality of entanglement describing.

The remaining part of studying process is concerned about the system of FWM, reservoir and their interaction. The study process is the same as the first informed part.

1.8 Thesis Organization

To cover the story of the dissipative effects on the propagation of entangled photon state in nonlinear fiber optics, the thesis structure is planed as follows:

- Chapter 2 presents the basic quantum theory of light. Start from the field quantization to specify its three representations: the number or Fock states, the coherent states and the squeezed states, which will be used in this thesis. Also, the positive P representation which transforms the quantum operators to the corresponding c-number proves to be the invaluable tool and will be used through this thesis.

- Chapter 3 the Hamiltonian of entangled photon generation by four-wave mixing process in $\chi^{(3)}$ nonlinear fiber optics is established for both the degenerate and nondegenerate cases of which its equation of motion will be constructed from a classical aspect and then be transformed to its quantum counterpart.

- Since photon can interact with its environment such as fused- silica molecules or any heat reservoir, chapter 4 gives the knowledge of the open quantum system where the system and reservoir are coupling with each other by exchanging the energy. The Markov approximation, the non-memory process, is used to repel the reservoir variables from the coupling with the system operators.

- The results and discussions of this study is analyzed in chapter 5, and finally, the conclusion and suggestion are provided in chapter 6.

CHAPTER 2

Quantum Theory of Light

The study of the quantum features of light requires the quantization of electromagnetic field. In this chapter we quantize the field and introduce three possible sets of basis state, namely, the Fock or number states, the coherent states and the squeezed states. The properties of these states are discussed.

2.1 Field Quantization

The major emphasis is concerned with the uniquely quantum-mechanical properties of the electromagnetic field, which are not present in a classical treatment. As such we shall begin immediately by quantizing the electromagnetic field. We will make use of an expansion of the vector potential for the electromagnetic field in term of cavity modes. The problem then reduces to the quantization of the harmonic oscillator corresponding to each individual cavity mode.

We shall also introduce states of the electromagnetic field appropriate to the description of optical fields. The first set of states we introduce are the number states corresponding to having a definite number of photons in the field. It turns out that it is extremely difficult to create experimentally a number state of the field, though fields containing a very small number of photons have been generated. A more typical optical field will involve a superposition of number states. One such field is the coherent state of the field which has the minimum uncertainty in amplitude and phase allowed by the uncertainty principle, and hence is the closest possible quantum mechanical state to a classical field. It also possesses a high degree of optical coherence hence the name coherent state. The coherent state plays a fundamental role in quantum optics and has a practical significance in that the highly stabilized laser operating well above threshold generates a coherent state.

A rather more exotic set of states of the electromagnetic field are the squeezed state. These are also minimum-uncertainty states but unlike the coherent states the quantum noise is not uniformly distributed in phase. Squeezed states may have less noise in one quadrature than the vacuum. As a consequence the noise in the other quadrature is increased. We introduce the basic properties of squeezed states in this chapter.

A convenient starting point for the quantization of the electromagnetic field is the classical field equations. The free electromagnetic field obeys the source free Maxwell equations.

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= 0, && (2.1.1a)\\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, && (2.1.1b)\\
\nabla \cdot \mathbf{D} &= 0, && (2.1.1c)\\
\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, && (2.1.1d)
\end{aligned}$$

where $\mathbf{B} = \mu_0 \mathbf{H}$, $\mathbf{D} = \epsilon_0 \mathbf{E}$, μ_0 and ϵ_0 being the magnetic permeability and electric permittivity of free space, and $\mu_0 \epsilon_0 = c^{-2}$. Maxwell's equations are gauge invariant when no sources are present. A convenient choice of gauge for problems in quantum optics is the Coulomb gauge i.e. both $\mathbf{B}$ and $\mathbf{E}$ may be determined from a vector potential $\mathbf{A}(\mathbf{r}, t)$ as follows

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A}, && (2.1.2)\\
\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t}, && (2.1.3)
\end{aligned}$$

with the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0. \qquad (2.1.4)$$

Substituting Eq. (2.1.2) into Eq. (2.1.1d) we find that $\mathbf{A}(\mathbf{r}, t)$ satisfies the wave equation

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A}(\mathbf{r}, t). \qquad (2.1.5)$$

We separate the vector potential into two complex terms

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}^{(+)}(\mathbf{r}, t) + \mathbf{A}^{(-)}(\mathbf{r}, t), \qquad (2.1.6)$$

where $\mathbf{A}^{(+)}(\mathbf{r}, t)$ contains all amplitudes which vary as $e^{-i\omega t}$ for $\omega > 0$ and $\mathbf{A}^{(-)}(\mathbf{r}, t)$ contains all amplitudes which vary as $e^{i\omega t}$ and $\mathbf{A}^{(-)} = (\mathbf{A}^{(+)})^*$.

It is more convenient to deal with a discrete set of variables rather than the whole continuum. We shall therefore describe the field restricted to a certain volume of space and expand the vector potential in terms of a discrete set of orthogonal mode function:

$$\mathbf{A}^{(+)}(\mathbf{r}, t) = \sum_k c_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}, \qquad (2.1.7)$$

where the Fourier coefficient c_k are constant for a free field. The set of vector mode functions $\mathbf{u}_k(\mathbf{r})$ which correspond to the frequency ω_k will satisfy the wave equation

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2} \right) \mathbf{u}_k(\mathbf{r}) = 0, \qquad (2.1.8)$$

provided the volume contains no refracting material. The mode functions are also required to satisfy the transversality condition,

$$\nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0. \qquad (2.1.9)$$

The mode functions form a complete orthogonal set

$$\int_{V_Q} \mathbf{u}_k^*(\mathbf{r})\mathbf{u}_{k'}(\mathbf{r})d\mathbf{r} = \delta_{kk'}. \tag{2.1.10}$$

The mode functions depend on the boundary conditions of the physical volume under consideration, e.g., periodic boundary conditions corresponding to travelling-wave modes or conditions appropriate to reflecting walls which lead to standing waves. The plane wave mode functions appropriate to a cubical volume of side L may be written as

$$\mathbf{u}_{k',\lambda}(\mathbf{r}) = L^{-3/2}\hat{\mathbf{e}}_\lambda e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.1.11}$$

where $\hat{\mathbf{e}}_\lambda = |H\rangle\,, |V\rangle$ is the unit horizontal and vertical polarization state vectors, respectively. The mode index k describes several discrete variables, the polarization index ($\lambda = H, V$) and the three cartesian components of the propagation vector $\mathbf{k}$. Each component of the wave vector $\mathbf{k}$ takes the values

$$k_x = \frac{2\pi n_x}{L}, \quad k_y = \frac{2\pi n_y}{L}, \quad k_z = \frac{2\pi n_z}{L}, \qquad n_x, n_y, n_z = 0, \pm 1, \pm 2... \tag{2.1.12}$$

The polarization vector $\hat{\mathbf{e}}_\lambda$ is required to be perpendicular to $\mathbf{k}$ by the transversality condition of Eq. (2.1.9).

The vector potential may now be written in the form

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r},t) = \sum_k \left(\frac{\hbar}{2\omega_k\epsilon_0}\right)^{1/2} [\hat{a}_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_k t} + \hat{a}_k^\dagger \mathbf{u}_k^*(\mathbf{r})e^{i\omega_k t}]. \tag{2.1.13}$$

The corresponding form for the electric field is

$$\hat{\mathbf{E}}(\mathbf{r},t) = i\sum_k \left(\frac{\hbar\omega_k}{2\epsilon_0}\right)^{1/2} [\hat{a}_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_k t} + \hat{a}_k^\dagger \mathbf{u}_k^*(\mathbf{r})e^{i\omega_k t}]. \tag{2.1.14}$$

The normalization factors have been chosen such that the amplitudes $\hat{a}_k$ and $\hat{a}_k^\dagger$ are dimensionless.

In classical electromagnetic theory these Fourier amplitudes are complex numbers. Quantization of the electromagnetic field is accomplished by choosing $\hat{a}_k$ and $\hat{a}_k^\dagger$ to be mutually adjoint operators. Since photons are bosons the appropriate commutation relations to choose for the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the boson commutation relations

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \quad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}. \tag{2.1.15}$$

The dynamical behavior of the electric-field amplitudes may be described by an ensemble of independent harmonic oscillators obeying the above commutation relations. The quantum state of each mode may now be discussed independently of one another. The state in each mode may be described by a state vector $|\Psi\rangle_k$ of the Hilbert space appropriate to that mode. The states of the entire field are then defined in the tensor product space of the Hilbert spaces for all of the modes.

The Hamiltonian for the electromagnetic field is given by

$$\hat{\mathcal{H}} = \frac{1}{2}\int(\epsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2)dr. \tag{2.1.16}$$

Substituting Eq. (2.1.14) for $\mathbf{E}$ and the equivalent expression for $\mathbf{H}$ and making use of the conditions of Eqs.(2.1.9) and (2.1.10), the Hamiltonian may be reduced to the form

$$\hat{\mathcal{H}} = \sum_k \hbar\omega_k(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}). \tag{2.1.17}$$

This represents the sum of the number of photons in each mode multiplied by the energy of a photon in that mode, plus $\frac{1}{2}\hbar\omega_k$ representing the energy of the vacuum fluctuations in each mode. We shall now consider three possible representations of the electromagnetic field.

2.2 Fock or Number States

The Hamiltonian of Eq. (2.1.16) has the eigenvalues $\hbar\omega_k(n_k + \frac{1}{2})$ where n_k is an integer ($n_k = 0, 1, 2, ..., \infty$). The eigenstates are written as $|n_k\rangle$ and are known as number or Fock states. They are eigenstates of the number operator $n_k = \hat{a}_k^\dagger \hat{a}_k$

$$\hat{a}_k^\dagger \hat{a}_k |n_k\rangle = n_k |n_k\rangle . \tag{2.2.1}$$

The ground state of the oscillator (or vacuum state of the field mode) is defined by

$$\hat{a}_k |0\rangle = 0. \tag{2.2.2}$$

From Eqs. (2.1.17) and (2.2.2) we see that the energy of the ground state is given by

$$\langle 0|\hat{\mathcal{H}}|0\rangle = \frac{1}{2}\sum_k \hbar\omega_k. \tag{2.2.3}$$

Since there is no upper bound to the frequencies in the sum over electromagnetic field modes, the energy of the ground state is infinite, a conceptual difficulty of quantized radiation field theory. However, since practical experiments measure a change in the total energy of the electromagnetic field the infinite zero-point energy does not lead to any divergence in practice. $\hat{a}_k$ and $\hat{a}_k^\dagger$ are raising and lowering operators for the harmonic oscillator ladder of eigenstates. In terms of photons they represent the annihilation and creation of a photon with the wave vector $\mathbf{k}$ and a polarization $\mathbf{e}_{k,\lambda}$. Hence the terminology, annihilation and creation operators. Application of the creation and annihilation operators to the number states yield

$$\hat{a}_k |n_k\rangle = n_k^{1/2} |n_k - 1\rangle , \quad \hat{a}_k^\dagger |n_k\rangle = (n_k + 1)^{1/2} |n_k + 1\rangle . \tag{2.2.4}$$

The state vectors for the higher excited states may be obtained from the vacuum by successive application of the creation operator

$$|n_k\rangle = \frac{(\hat{a}^\dagger)^{n_k}}{(n_k!)^{1/2}}|0\rangle\,, \quad n_k = 0, 1, 2, ..., \infty. \tag{2.2.5}$$

The number states are orthogonal

$$\langle n_k|m_k\rangle = \delta_{mn}, \tag{2.2.6}$$

and a completeness relation

$$\sum_{n_k=0}^{\infty} |n_k\rangle\langle n_k| = 1. \tag{2.2.7}$$

Since the norm of these eigenvectors is finite, they form a complete set of basis vectors for a Hilbert space.

While the number states form a useful representation for high-energy photons, e.g. γ rays where the number of photons is very small, they are not the most suitable representation for optical fields where the total number of photons is large. Experimental difficulties have prevented the generation of photon number states with more than a small number of photons. Most optical fields are either a superposition of number states (pure state) or mixture of number states (mixed state). Despite this the number states of the electromagnetic field have been used as a basis for several problems in quantum optics including some laser theories.

2.3 Coherent States

A more appropriate basis for many optical fields are the coherent states. These states have an indefinite number of photons which allows them to have a more precisely defined phase than a number state where the phase is completely random. The product of the uncertainty in amplitude and phase for a coherent state is the minimum allowed by the uncertainty principle. In this sense they are the closest quantum mechanical states to a classical description of the field. We shall outline the basic properties of the coherent states below. These states are most easily generated using the unitary displacement operator

$$D(\alpha) = \mathrm{e}^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}, \tag{2.3.1}$$

where α is an arbitary complex number.

Using the operator theorem

$$\mathrm{e}^{(\mathbf{A}+\mathbf{B})} = \mathrm{e}^{\mathbf{A}}\mathrm{e}^{\mathbf{B}}\mathrm{e}^{[\mathbf{A},\mathbf{B}]/2}, \tag{2.3.2}$$

which holds when

$$[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{B}]] = 0,$$

we can write $D(\alpha)$ as

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}}. \tag{2.3.3}$$

The displacement operator $D(\alpha)$ has the following properties

$$\begin{aligned} D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha), \qquad D^\dagger(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha, \\ D^\dagger(\alpha)\hat{a}^\dagger D(\alpha) = \hat{a}^\dagger + \alpha^*. \end{aligned} \tag{2.3.4}$$

The coherent state α is generated by operating with $D(\alpha)$ on the vacuum state

$$|\alpha\rangle = D(\alpha)\,|0\rangle\,. \tag{2.3.5}$$

The coherent states are eigenstates of the annihilation operator $\hat{a}$. This may be proved as follows:

$$D^\dagger(\alpha)\hat{a}\,|\alpha\rangle = D^\dagger(\alpha)\hat{a}D(\alpha)\,|0\rangle = (\hat{a} + \alpha)\,|0\rangle = \alpha\,|0\rangle\,. \tag{2.3.6}$$

Multiplying both sides by $D(\alpha)$ we arrive at the eigenvalue equation

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{2.3.7}$$

Since $\hat{a}$ is a non-Hermitian operator its eigenvalues α are complex.

Another useful property which follows using Eq. (2.3.2) is

$$D(\alpha + \beta) = D(\alpha)D(\beta)e^{-i\Im\mathrm{m}\{\alpha\beta^*\}}. \tag{2.3.8}$$

The coherent states contain an indefinite number of photons. This may be made apparent by considering an expansion of the coherent state in the number-states basis.

Taking the scalar product of both sides of Eq. (2.3.7) with $\langle n|$ we find the recursion relation

$$(n+1)^{1/2}\,\langle n+1|\alpha\rangle = \alpha\,\langle n|\alpha\rangle\,. \tag{2.3.9}$$

It follows that

$$\langle n|\alpha\rangle = \frac{\alpha^n}{(n!)^{1/2}}\,\langle 0|\alpha\rangle\,. \tag{2.3.10}$$

We may expand $|\alpha\rangle$ in terms of the number states $|n\rangle$ with expansion coefficients $\langle n|\alpha\rangle$ as follows

$$|\alpha\rangle = \sum |n\rangle\,\langle n|\alpha\rangle = \langle 0|\alpha\rangle \sum_n \frac{\alpha^n}{(n!)^{1/2}}\,|n\rangle\,. \tag{2.3.11}$$

The squared length of the vector $|\alpha\rangle$ is thus

$$|\,\langle\alpha|\alpha\rangle\,|^2 = |\,\langle 0|\alpha\rangle\,|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |\,\langle 0|\alpha\rangle\,|^2 e^{|\alpha|^2}. \tag{2.3.12}$$

It is easily seen that

$$\begin{aligned} \langle 0|\alpha\rangle &= \langle 0|D(\alpha)|0\rangle \\ &= e^{(-|\alpha|^2/2)}. \end{aligned} \tag{2.3.13}$$

Thus $|\langle\alpha|\alpha\rangle|^2 = 1$ and the coherent states are normalized.

The coherent state may then be expanded in terms of the number states as

$$|\alpha\rangle = e^{(-|\alpha|^2/2)} \sum_n \frac{\alpha^n}{(n!)^{1/2}} |n\rangle . \tag{2.3.14}$$

We note that the probability distribution of photons in a coherent state is a Poisson distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}, \tag{2.3.15}$$

where $|\alpha|^2$ is the mean number of photons ($\langle n\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$).

The scalar product of two coherent states is

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(\beta)D(\alpha)|0\rangle . \tag{2.3.16}$$

Using Eq. (2.3.3) this becomes

$$\langle\beta|\alpha\rangle = e^{\left[\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\beta^*\right]}. \tag{2.3.17}$$

The absolute magnitude of the scalar product is

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}. \tag{2.3.18}$$

Thus the coherent states are not orthogonal although two states $|\alpha\rangle$ and $|\beta\rangle$ become approximately orthogonal in the limit $|\alpha - \beta| \gg 1$. The coherent states form a two-dimensional continuum of states and are, in fact, overcomplete. The completeness relation

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = 1, \tag{2.3.19}$$

may be proved as follows.

We use the expansion Eq. (2.3.14) to give

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{|n\rangle\langle m|}{\pi\sqrt{n!m!}} \int e^{-|\alpha|^2} \alpha^{*m}\alpha^n d^2\alpha. \tag{2.3.20}$$

Changing to polar coordinates this becomes

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{|n\rangle\langle m|}{\pi\sqrt{n!m!}} \int_0^\infty r dr e^{-r^2} r^{n+m} \int_0^{2\pi} d\theta e^{i(n-m)\theta}. \tag{2.3.21}$$

Using

$$\int_0^{2\pi} d\theta e^{i(n-m)\theta} = 2\pi\delta_{nm}, \tag{2.3.22}$$

we have

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^\infty d\varepsilon e^{-\varepsilon}\varepsilon^n, \tag{2.3.23}$$

where we let $\varepsilon = r^2$. The integral equals $n!$. Hence we have

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1, \tag{2.3.24}$$

following from the completeness relation for the number states.

An alternative proof of the completeness of the coherent states may be given as follows. Using the relation

$$e^{\zeta \mathbf{B}} \mathbf{A} e^{-\zeta \mathbf{B}} = \mathbf{A} + \zeta[\mathbf{B}, \mathbf{A}] + \frac{\zeta^2}{2!}[\mathbf{B}, [\mathbf{B}, \mathbf{A}]] + \ldots \quad , \tag{2.3.25}$$

it is easy to see that all the operators $\mathbf{A}$ such that

$$D^\dagger(\alpha)\mathbf{A}D(\alpha) = \mathbf{A} \tag{2.3.26}$$

are proportional to the identity.

We consider

$$\mathbf{A} = \int d^2\alpha \, |\alpha\rangle\langle\alpha|$$

then

$$D^\dagger(\beta) \int d^2\alpha \, |\alpha\rangle\langle\alpha| \, D(\beta) = \int d^2\alpha \, |\alpha - \beta\rangle\langle\alpha - \beta| = \int d^2\alpha \, |\alpha\rangle\langle\alpha| \, . \tag{2.3.27}$$

Then using the above result we conclude that

$$\int d^2\alpha \, |\alpha\rangle\langle\alpha| \propto I. \tag{2.3.28}$$

The constant of proportionality is easily seen to be π.

The coherent states have a physical significance in that the field generated by a highly stabilized laser operating well above threshold is a coherent state. They form a useful basis for expanding the optical field in problems in laser physics and nonlinear optics.

2.4 Squeezed States

A general class of minimum-uncertainty states are known as *squeezed states*. In general, a squeezed state may have less noise in one quadrature than a coherent state. To satisfy the requirements of a minimum-uncertainty state the noise in the other quadrature is greater than that of a coherent state. The coherent states are a particular member of this more general class of minimum uncertainty states with equal noise in both quadratures. We will begin our discussion by defining a family of minimum-uncertainty states. Let us calculate the variances for the position and momentum operators for the harmonic oscillator

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger). \tag{2.4.1}$$

The variances are defined by

$$V(A) = (\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 . \tag{2.4.2}$$

In a coherent state we obtain

$$(\Delta \hat{q})^2_{\mathrm{coh}} = \frac{\hbar}{\omega}, \quad (\hat{p})^2_{\mathrm{coh}} = \frac{\hbar\omega}{2}. \tag{2.4.3}$$

Thus the product of the uncertainties is a minimum

$$(\Delta \hat{p} \Delta \hat{q})_{\mathrm{coh}} = \frac{\hbar}{2}. \tag{2.4.4}$$

Thus, there exists a sense in which the description of the state of an oscillator by a coherent state represents as close an approach to classical localization as possible. We shall consider the properties of a single-mode field. We may write the annihilation operator $\hat{a}$ as a linear combination of two Hermitian operators

$$\hat{a} = \frac{\hat{X}_1 + i\hat{X}_2}{2}. \tag{2.4.5}$$

X_1 and X_2, the real and imaginary parts of the complex amplitude, give dimensionless amplitudes for the modes' two quadrature phases. They obey the following commutation relation

$$[\hat{X}_1, \hat{X}_2] = 2i. \tag{2.4.6}$$

The corresponding uncertainty principle is

$$\Delta \hat{X}_1 \Delta \hat{X}_2 \geq 1. \tag{2.4.7}$$

This relation with the equals sign defines a family of minimum-uncertainty states. The coherent states are a particular minimum-uncertainty state with

$$\Delta \hat{X}_1 = \Delta \hat{X}_2 = 1. \tag{2.4.8}$$

The coherent state $|\alpha\rangle$ has the mean complex amplitude α and it is a minimum-uncertainty state for $\hat{X}_1$ and $\hat{X}_2$, with equal uncertainties in the two quadrature phases. A coherent state may be represented by an "error circle" in a complex amplitude plane whose axes are $\hat{X}_1$ and $\hat{X}_2$ (Fig. 2.1(a)). The center of the error circle lies at $\frac{1}{2}\langle \hat{X}_1 + \hat{X}_2 \rangle = \alpha$ and the radius $\Delta \hat{X}_1 = \Delta \hat{X}_2 = 1$ accounts for the uncertainties in $\hat{X}_1$ and $\hat{X}_2$.

There is obviously a whole family of minimum-uncertainty states defined by $\Delta \hat{X}_1 \Delta \hat{X}_2 = 1$. If we plot $\Delta \hat{X}_1$ against $\Delta \hat{X}_2$ the minimum-uncertainty states lie on a hyperbola (Fig. 2.2). Only points lying to the right of this hyperbola correspond to the physical states. The coherent state

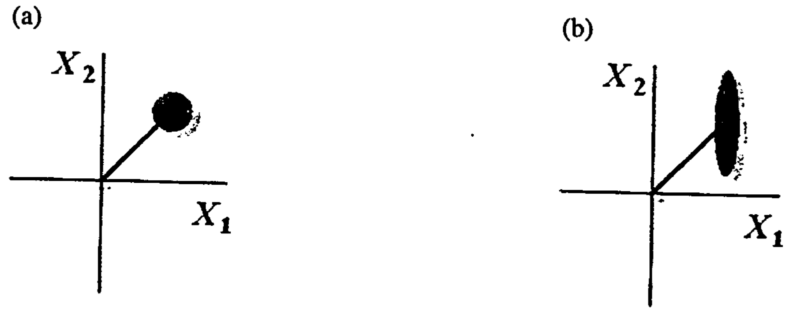

Figure 2.1 Phase-space plot showing the uncertainty in (a) a coherent state $|\alpha\rangle$, and (b) a squeezed state $|\alpha, r\rangle$.

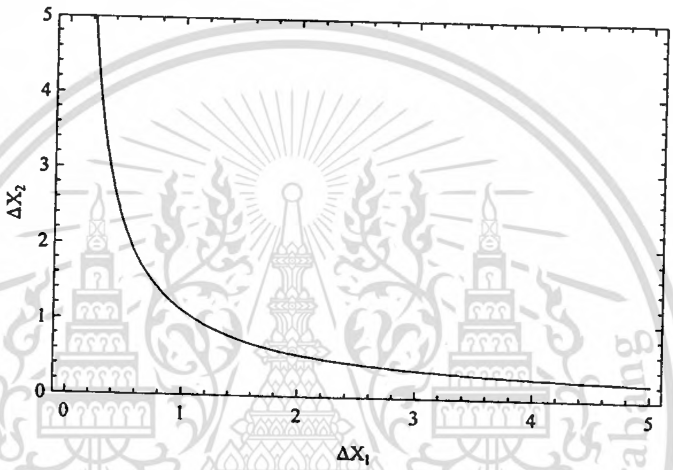

Figure 2.2 Plot of $\Delta\hat{X}_1$ versus $\Delta\hat{X}_2$ for the minimum-uncertainty states.

with $\Delta\hat{X}_1 = \Delta\hat{X}_2$ is a special case of a more general class of states which may have reduced uncertainty in the other ($\Delta\hat{X}_1 < 1 < \Delta\hat{X}_2$). These squeezed states may be generated by using the unitary squeeze operator

$$S(\varepsilon) = e^{\frac{1}{2}(\varepsilon^*\hat{a}^2 - \varepsilon\hat{a}^{\dagger 2})}, \tag{2.4.9}$$

where $\varepsilon = re^{2i\phi}$. Note the squeeze operator obeys the relations

$$S^{\dagger}(\varepsilon) = S^{-1}(\varepsilon) = S(-\varepsilon), \tag{2.4.10}$$

and has the following useful transformation properties

$$\begin{aligned} S^{\dagger}(\varepsilon)\hat{a}S(\varepsilon) &= \hat{a}\cosh r - \hat{a}^{\dagger}e^{-2i\phi}\sinh r, \\ S^{\dagger}(\varepsilon)\hat{a}^{\dagger}S(\varepsilon) &= \hat{a}^{\dagger}\cosh r - \hat{a}e^{-2i\phi}\sinh r, \\ S^{\dagger}(\varepsilon)(Y_1 + iY_2)S(\varepsilon) &= Y_1 e^{-r} + iY_2 e^{r}, \end{aligned} \tag{2.4.11}$$

where

$$Y_1 + iY_2 = (X_1 + iX_2)e^{-i\phi} \tag{2.4.12}$$

is the rotated complex amplitude. The squeeze operator attenuates one component of the rotated complex amplitude, and it amplifies the other component. The degree of attenuation and amplification is determined by $r = |\varepsilon|$, which will be called the *squeeze factor*. The squeeze state $|\alpha, \varepsilon\rangle$ is obtained by first squeezing the vacuum and then displacing it

$$|\alpha, \varepsilon\rangle = D(\alpha)S(\varepsilon)\,|0\rangle\,. \tag{2.4.13}$$

A squeezed state has the following expectation values and variances

$$\langle X_1 + iX_2\rangle = \langle Y_1 + iY_2\rangle\, e^{-i\phi} = 2\alpha, \tag{2.4.14}$$

$$\Delta Y_1 = e^{-r}, \quad \Delta Y_2 = e^{r}, \tag{2.4.15}$$

$$\langle \Delta N\rangle = |\alpha|^2 + \sinh^2 r, \tag{2.4.16}$$

$$(\Delta N)^2 = |\alpha\cosh r - \alpha e^{2i\phi}\sinh r|^2 + 2\cosh^2 r\sinh^2 r. \tag{2.4.17}$$

Thus the squeezed state has unequal uncertainties for Y_1 and Y_2 as seen in the error ellipse shown in Fig. 2.1. The principal axes of the ellipse lie along the Y_1 and Y_2 axes, and the principal radii are ΔY_1 and ΔY_2.

2.5 Quantum Theory of Coherence

This theory was formulated originally by R. J. Glauber, where he considers the process of photon detection, which plays a central role. The basic process involved in the detection is the absorption of a photon and the corresponding generation of a photoelectron, measured via an electric current. This simple model is an ideal detector, sensitive to what we define as the positive frequency component of the field (proportional to the annihilation operator of field)

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) = i\sum_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}\, \hat{a}_k \mathbf{u}_{k,\lambda}(\mathbf{r}) e^{i\omega_k t} \tag{2.5.1}$$

where

$$\mathbf{u}_{k,\lambda}(\mathbf{r}) = \frac{\hat{\mathbf{e}}_\lambda e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V_Q}}, \tag{2.5.2}$$

with $\hat{\mathbf{e}}_\lambda$ being the polarization vector. Then, the total electric field operator can be written as

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) + \hat{\mathbf{E}}^{(-)}(\mathbf{r}, t), \tag{2.5.3}$$

with

$$\hat{\mathbf{E}}^{(-)}(\mathbf{r}, t) = (\hat{\mathbf{E}}^{(+)})^\dagger(\mathbf{r}, t). \tag{2.5.4}$$

In this model, the detector atoms are in the ground state, so that only absorption take place. Since it is only the annihilation part $\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)$ and $\hat{\mathbf{E}}^{(-)}(\mathbf{r},t)$ in a way that the actual detection is more closely related to $\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)$ than the total field $\hat{\mathbf{E}}(\mathbf{r},t)$. An ideal photodetector would also have infinite bandwidth, responding to a field at time t, and a negligible spatial extension.

The transition probability from an initial state $|\psi_{\mathrm{i}}\rangle$ to a final state $|\psi_{\mathrm{f}}\rangle$ is proportional to

$$\mathcal{W}_{\mathrm{if}} = |\langle\psi_{\mathrm{f}}|\hat{\mathbf{E}}^{(+)}|\psi_{\mathrm{i}}\rangle|^2. \tag{2.5.5}$$

As we will see, this is the first-order approximation. In general, the final state of the field is not know, so we have to sum over all the possible final states:

$$\begin{aligned} I_{\mathrm{i}}(\mathbf{r},t) &= \sum_{\mathrm{f}} \left|\langle\psi_{\mathrm{f}}|\hat{\mathbf{E}}^{(+)}|\psi_{\mathrm{i}}\rangle\right|^2 \\ &= \sum_{\mathrm{f}} \left\langle\psi_{\mathrm{i}}\left|\hat{\mathbf{E}}^{(-)}\right|\psi_{\mathrm{f}}\right\rangle \left\langle\psi_{\mathrm{f}}\left|\hat{\mathbf{E}}^{(+)}\right|\psi_{\mathrm{i}}\right\rangle \\ &= \langle\psi_{\mathrm{i}}|\hat{\mathbf{E}}^{(-)}\hat{\mathbf{E}}^{(+)}|\psi_{\mathrm{f}}\rangle, \end{aligned} \tag{2.5.6}$$

giving us an average field intensity. In the last step, we made use of the completeness of the final states.

If the initial state is a mixed state, then we have to use the density matrix ρ, and we write

$$\langle I_{\mathrm{i}}(\mathbf{r},t)\rangle = \mathrm{Tr}\left\{\rho\hat{\mathbf{E}}^{(-)}(\mathbf{r},t)\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)\right\}. \tag{2.5.7}$$

We now define the first-oder coherence function:

$$G^{(1)}(x,x') \equiv \mathrm{Tr}\left\{\rho\hat{\mathbf{E}}^{(-)}(x)\hat{\mathbf{E}}^{(+)}(x')\right\}, \tag{2.5.8}$$

where x and x' are $x = (\mathbf{r},t)$ and $x' = (\mathbf{r}',t')$.

The first-order coherence function appears typically in interference experiments. In order to describe more sophisticated experiments, like the coincidence experiments of Handbury, Brown and Twiss, it is useful to define an n^{th} order coherence function

$$G^{(n)}(x_1,x_2,...,x_n;x_{n+1},...,x_{2n}) \equiv \mathrm{Tr}\left\{\rho\hat{\mathbf{E}}^{(-)}(x_1),...,\hat{\mathbf{E}}^{(-)}(x_n)\hat{\mathbf{E}}^{(+)}(x_{n+1}),...,\hat{\mathbf{E}}^{(+)}(x_{2n})\right\}, \tag{2.5.9}$$

where again $x_j = (\mathbf{r}_j,t_j)$.

2.6 General Properties of the Correlation Function

The n^{th} order coherence function was defined as the expectation value of Eq. (2.5.9). As a first property, we notice if there is an upper bound on the number of photons present in the field, then $G^{(n)}(x_1,...,x_n$

$; x_{n+1}, ..., x_{2n})$ must vanish identically for n larger than the upper bound M. To be more specific, if the field density operator is written as

$$\rho = \sum_{nm} C_{n,m} |n\rangle \langle m| , \tag{2.6.1}$$

and if $C_{n,m} = 0$ for n or m larger than M, then

$$\hat{\mathrm{E}}^{(+)}(x_1)...\hat{\mathrm{E}}^{(+)}(x_p)\rho = 0, \tag{2.6.2}$$

for $p > M$, simply because the number of times the annihilation operator is applied to the density matrix is larger than the number of photons available in the field, thus $G^{(p)} = 0$.

Another property can be derived from the identity $\mathrm{Tr}[A^\dagger] = \mathrm{Tr}[A]^*$, which is valid for any linear operator A. Applying this identity to $G^{(n)}(x_1, ..., x_n; x_{n+1}, ..., x_{2n})$ we get

$$\begin{aligned}\left[G^{(n)}(x_1, ..., x_n; x_{n+1}, ..., x_{2n})\right]^* &= \mathrm{Tr}\left\{\hat{\mathrm{E}}^{(-)}(x_{2n}), ..., \hat{\mathrm{E}}^{(-)}(x_{n+1})\hat{\mathrm{E}}^{(+)}(x_n), ..., \hat{\mathrm{E}}^{(+)}(x_1)\rho\right\} \\ &= \mathrm{Tr}\left\{\rho\hat{\mathrm{E}}^{(-)}(x_{2n}), ..., \hat{\mathrm{E}}^{(-)}(x_{n+1})\hat{\mathrm{E}}^{(+)}(x_n), ..., \hat{\mathrm{E}}^{(+)}(x_1)\right\} \\ &= G^{(n)}(x_{2n}, ..., x_{n+1}; x_n, ..., (x_1)),\end{aligned} \tag{2.6.3}$$

where we made use of the Hermitian character of ρ and the invariance of the trace under cyclic permutation. As a consequence of commutation properties of $\hat{\mathrm{E}}^{(-)}$ and $\hat{\mathrm{E}}^{(+)}$, we can freely permute the arguments $(x_1, x_2, ..., x_n)$ and $(x_{n+1}, x_{n+2}, ..., x_{2n})$ without changing $G^{(n)}$, but we cannot interchange any of the first n arguments with any of the remaining n, since the corresponding operators do not commute.

Another set of properties can be derived from the positive definite character of the operator $A^\dagger A$, such that $\mathrm{Tr}[A^\dagger A] \geq 0$, for any linear operator A. To show the above inequality, we write

$$\begin{aligned}\mathrm{Tr}[\rho A^\dagger A] &= \sum_k p_k \langle k|A^\dagger A|k\rangle \\ &= \sum_{k,m} p_k \langle k|A^\dagger |m\rangle \langle m| A|k\rangle \\ &= \sum_{k,m} p_k | \langle m|A|k\rangle |^2 \geq 0.\end{aligned} \tag{2.6.4}$$

Since p_k and $| \langle m|A|k\rangle |^2 \geq 0$, there are several interesting cases:

i). Let $A = \hat{\mathrm{E}}^{(+)}(x_1)$, then applying the inequality $\mathrm{Tr}[A^\dagger A] \geq 0$, we get

$$G^{(1)}(x_1, x_1) \geq 0. \tag{2.6.5}$$

ii). Let $A = \hat{\mathrm{E}}^{(+)}(x_1), ..., \hat{\mathrm{E}}^{(+)}(x_n)$, we get directly

$$G^{(n}(x_1, ..., x_n; x_n, ..., x_1) \geq 0. \tag{2.6.6}$$

iii). Let $A = \sum_{j=1}^{n} \lambda_j \hat{\mathbf{E}}^{(+)}(x_j)$, where λ_j is a set of arbitrary complex numbers. In this case, we get

$$\sum_{i,j} \lambda_i^* \lambda_j G^{(1}(x_i, x_j) \geq 0, \tag{2.6.7}$$

thus the set of correlation function $G^{(1)}(x_i, x_j)$ forms a matrix coefficient for the quadratic form of the set λs. Such a matrix has a positive determinant; thus: for $n = 1$ we get Eq. (2.6.5); for $n = 2$ we get

$$G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2) \geq |G^{(1)}(x_1, x_2)|^2, \tag{2.6.8}$$

which is a simple generalization of Schwartz's identity.

2.7 Second-Order Correlations. Photon Bunching and Antibunching

The second-order normalized correlation function is defined as

$$g^{(2)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2; \mathbf{r}_2 t_2, \mathbf{r}_1 t_1) = \frac{\langle \hat{\mathbf{E}}^{(-)}(\mathbf{r}_1, t_1) \hat{\mathbf{E}}^{(-)}(\mathbf{r}_2, t_2) \hat{\mathbf{E}}^{(+)}(\mathbf{r}_2, t_2) \hat{\mathbf{E}}^{(+)}(\mathbf{r}_1, t_1) \rangle}{\langle \hat{\mathbf{E}}^{(-)}(\mathbf{r}_1, t_1) \hat{\mathbf{E}}^{(+)}(\mathbf{r}_1, t_1) \rangle \, \langle \hat{\mathbf{E}}^{(-)}(\mathbf{r}_2, t_2) \hat{\mathbf{E}}^{(+)}(\mathbf{r}_2, t_2) \rangle}. \tag{2.7.1}$$

In this section we will consider only parallel light beams (z-direction), so that the space time coordinates (z_1, t_1), (z_2, t_2) enter in $g^{(2)}$ only as a phase difference:

$$\tau = t_2 - t_1 + \frac{z_1 - z_2}{c}. \tag{2.7.2}$$

We start the subject with a brief review of some classical aspects.

2.7.1 Classical Second-Order Coherence

We consider a beam of light described by a classical intensity $I_1(t)$, which is time-dependent and averaged over each cycle. In general, the intensity will show random fluctuations, if one is dealing, for example, with a source of chaotic light. We will assume that the light sources under study are stationary and ergodic, in such a way that ensemble averages are equal to time averages. Classically, the second-order correlation function may be defined as

$$g_{11}^{(2)}(t) = \frac{\langle I_1(t) I_1(0) \rangle}{\bar{I}_1^2}, \tag{2.7.3}$$

where the average is over a long series of pairs of intensity measurements separated by a fixed time t and $\bar{I}_1^2 = \langle I_1(t) \rangle$ is a time-independent average, due to the assumption of stationary sources.

In a different type of experiment, we may measure intensities at different positions $\mathbf{r}_1, \mathbf{r}_2$. Then the relevant second-order correlation function is

$$g_{12}^{(2)}(t) = \frac{\langle I_1(t) I_2(0) \rangle}{\bar{I}_1 \bar{I}_2}. \tag{2.7.4}$$

The classical correlation function satisfy a series of inequalities:

(a) Since the intensity is positive

$$g_{12}^{(2)}(t) \geq 0. \quad (2.7.5)$$

(b) Since $\langle I_1^2(t)\rangle \geq \bar{I}_1^2$, then

$$g_{11}^{(2)}(0) \geq 1. \quad (2.7.6)$$

(c) In the more general case, and according to Cauchy's inequality,

$$\langle I_1^2\rangle \langle I_2^2\rangle \geq \langle I_1 I_2\rangle^2 . \quad (2.7.7)$$

Fixing the time t between the two measurements on the two beams, we have

$$g_{11}^{(2)}(0) g_{12}^{(2)}(0) \geq \left[g_{12}^{(2)}(t)\right]^2 , \quad (2.7.8)$$

and for a single beam $g_{11}^{(2)}(0) = g_{22}^{(2)}(0)$ and

$$g_{11}^{(2)}(0) \geq g_{12}^{(2)}(t). \quad (2.7.9)$$

In many cases, the fluctuations of the cycle-averaged intensities are too rapid for direct observation and the measurement reflects some average of the fluctuations over some typical response time of the detector. However, we do have fast detectors nowadays, so let us assume that its response time is much faster than the coherent time of the light. If, furthermore, the ergodic hypothesis is satisfied, then the time average may be replaced by statistical averages, denoted by angle brackets. We take a model of chaotic light emitted by a collision-broadened light source. In this model, the elastic collisions break up the wave radiated by single atoms, in discrete sections, where each section has a constant phase that abruptly ends with a collision. Suppose that we have light of intensity I_0 from n radiating atoms, the phase of the field emitted from the i^{th} atom being a random variable ϕ_i. Then, one can write

$$\begin{aligned} E(t) &= E_1(t) + E_1(t) + ... + E_n(t) \\ &= E_0\{e^{i\phi_1(t)} + e^{i\phi_2(t)} + ... + e^{i\phi_n(t)}\}, \end{aligned} \quad (2.7.10)$$

where each atom is associated with the same amplitude and field frequency but with phases which are completely independent.

The instantaneous average value of the square of the intensity is

$$\overline{I_1^2(0)} = I_0^2 \overline{|e^{i\phi_1} + e^{i\phi_2} + ... + e^{i\phi_n}|^4}. \quad (2.7.11)$$

The only non-zero contributions come from the terms in which each factor is multiplied by its complex conjugate. These are

$$\begin{aligned} \overline{I_1^2(0)} &= I_0^2 \left[\sum_{i=1}^{n} |e^{i\phi_i}|^4 + \sum_{i\neq j} |2e^{i(\phi_i-\phi_j)}|^2\right] \\ &= I_0^2[2n^2 - n]. \end{aligned} \quad (2.7.12)$$

If we further average, considering a Poissonian distribution of incoming atoms, with a mean $\bar{n}$, and $\overline{n^2} = \bar{n}^2 + \bar{n}$, then

$$\overline{\langle I_1^2(0)\rangle}_{\text{Poiss}} = I_0^2(2\bar{n}^2 + \bar{n}). \tag{2.7.13}$$

Also, since $\overline{\langle I_1^2(0)\rangle}_{\text{Poiss}} = \bar{n}I_0$, we have

$$g_{11}^{(2)}(0) = 2 + \frac{1}{\bar{n}}. \tag{2.7.14}$$

The standard theory of chaotic light considers a very large number of atoms radiating, that is, the limit $\bar{n} \to \infty$, $g_{11}^{(2)}(0)$ =2. More generally, one can consider a large number of radiating atoms and the summation over the phases is treated as a random walk. As a result of such a theory, one gets the probability distribution intensity I_1:

$$P(I) = \frac{1}{\bar{I}_1} e^{-\left(\frac{I_1}{\bar{I}_1}\right)}, \tag{2.7.15}$$

with giving $g_{11}^{(2)}(0) = 2$, in agreement with our previous result.
Normally, when we deal with a single beam of light, we skip the lower indices, and for chaotic light, we will just write $g^{(2)}(0) = 2$.

2.7.2 Quantum Theory of Second-Order Coherence

The quantum mechanical normalized second-order correlation function $g^{(2)}$ is positive, so that the inequality

$$0 \leq g^{(2)}(\tau) \leq \infty, \tag{2.7.16}$$

is identical to the classical range. However, the classical inequalities given in Eq. (2.7.6), (2.7.9) are, in general, no longer true. Even for zero time delay, in the quantum mechanic, the only true inequality is

$$0 \leq g^{(2)}(0) \leq \infty, \tag{2.7.17}$$

and since classically $g^{(2)}(0)|_{\text{class}} \geq 1$, there is an interest rang:

$$0 \leq g^{(2)}(0\rangle < 1, \tag{2.7.18}$$

which is purely quantum mechanical range.

For a single-mode field, the normalized correlation functions become simpler, and one can write

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle^2}, \tag{2.7.19}$$

which can also be written in terms of the photon-number operator

$$\begin{aligned} g^{(2)}(\tau) &= \frac{\langle n(n+1)\rangle}{\langle n^2\rangle} \\ &= 1+\frac{\langle\Delta(n)^2\rangle-\langle n\rangle}{\langle n\rangle^2} \end{aligned} \tag{2.7.20}$$

We observe that for a single-mode field, there is no time dependence (the τ dependent phase factor cancels) in $g^{(2)}(\tau)$.

A few simple examples of $g^{(2)}(\tau)$ are:

(a) For an $|n\rangle$ state

$$g^{(2)}(\tau)=\frac{(n-1)}{n}, \quad n\geq 2 \tag{2.7.21}$$

and $g^{(2)}(\tau)=0$ for $n=0,1$.

(b) For a coherent state $|\alpha\rangle$, $\langle\Delta(n)^2\rangle=\langle n\rangle$ and $g^{(2)}(\tau)=1$. It is convenient to define the moment of generating function $Q(s)$ as

$$Q(s)=\sum_{n=0}^{\infty}(1-s)^n P(n), \tag{2.7.22}$$

where $P(n)$ is the probability of having n photons in the field. One immediately sees that

$$\begin{aligned} \langle n\rangle &= -\frac{d}{ds}Q(s)\Big|_{s=0}, \\ \langle\Delta(n)^2\rangle &= \langle n^2\rangle-\langle n\rangle^2 \\ &= \left(\frac{d}{ds}\right)^2 Q(s)\Big|_{s=0}-\langle n\rangle\left(\langle n\rangle-1\right), \end{aligned} \tag{2.7.23}$$

and also

$$g^{(2)}=\frac{1}{\langle n\rangle^2}\left(\frac{d}{ds}\right)^2 Q(s)\Big|_{s=0}. \tag{2.7.24}$$

In general, light with $g^{(2)}=1$ is second-order coherent or Poissonian (as in the case of the coherent state), $g^{(2)}>1$ super-Poissonian and $g^{(2)}<1$ sub-Poissonian.

(c) Squeezed state. We have seen that

$$(\Delta n)^2=|\alpha|^2\left[e^{-2r}\cos^2\left(\phi-\frac{\theta}{2}\right)+e^{2r}\sin^2\left(\phi-\frac{\theta}{2}\right)\right]+2\sinh^2(r)\cosh^2(r), \tag{2.7.25}$$

with

$$\alpha=|\alpha|e^{i\phi}. \tag{2.7.26}$$

with the above expression and Eq. (2.7.20), one can evaluate $g^{(2)}(0)$.

If $g^{(2)}(\tau) < g^{(2)}(0)$, there is a tendency for photons to arrive in pairs, a situation referred to as *photon bunching*. The reverse situation $g^{(2)}(\tau) > g^{(2)}(0)$ is called *photon antibunching*, and this occurs typically when an atom emits a photon and right after that there is an anticorrelation for a second photon to be emitted, considering that the atom requires a finite time to go back to its excited state to be ready to emit a second photon. For very long times, there is no longer any correlation and $g^{(2)}(\tau)|_{\tau\to\infty} \longrightarrow 1$. A field with $g^{(2)}(0) < 1$, then, will always be antibunched over some time scale, which is the quantum mechanical case with no classical analogue.

Photon antibunching and sub-Poissonian statistics sometimes get mixed up in the literature, giving the wrong impression that they correspond to the same thing. Although they are related, they are not the same.

Mandel derived a formula for stationary fields:

$$V(n) - \langle n\rangle = \frac{\langle n\rangle^2}{T^2}\int_{-T}^{+T} d\tau(T - |\tau|)\left[g^{(2)}(\tau) - 1\right], \tag{2.7.27}$$

with $V(n) = \langle n^2\rangle - \langle n\rangle^2$. When a field has $g^{(2)}(\tau) < 1$ for $\forall\tau$, then $V(n) - \langle n\rangle < 0$ and exhibits sub-Poissonnian statistics. However, we may have the case $g^{(2)}(\tau) > g^{(2)}(0)$ (antibunching) which exhibits super-Poissonian statistics $\left(g^{(2)}(\tau)\text{ and } g^{(2)}(0) > 1\right)$, for some time interval τ.

2.8 Generalized *P* Representations

Another representation which like the *R* representation uses an expansion non-diagonal coherent state projection operators was suggested by *Drummond* and *Gardiner*. The representation is defined as follows

$$\rho = \int_D \Lambda(\alpha,\beta)P(\alpha,\beta)d\mu(\alpha,\beta) \tag{2.8.1}$$

where

$$\Lambda(\alpha,\beta) = \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}$$

and $d\mu(\alpha,\beta)$ is the integration measure which may be chosen to define different classes of possible representations and D is the domain of integration. The projection operator $\Lambda(\alpha,\beta)$ is analytic in α and β. It is clear that the normalization condition on ρ leads to the following normalization condition on $P(\alpha,\beta)$

$$\int_D P(\alpha,\beta)d\mu(\alpha,\beta) = 1. \tag{2.8.2}$$

Thus, the $P(\alpha,\beta)$ is normalizable and we shall see in the next chapter that it gives rise to Fokker-Planck equations. The definition given in (2.8.1) leads to different representations depending on the integration measure.

Useful choices of integration measure are

1. **Glauber-Sudarshan *P* Representation:**

$$d\mu(\alpha,\beta) = \delta^2(\alpha^* - \beta)d^2\alpha d^2\beta. \tag{2.8.3}$$

This measure corresponds to the diagonal Glauber-Sudarshan *P* representation defined as

$$\rho = \int P(\alpha)\,|\alpha\rangle\,\langle\alpha|\,d^2\alpha, \tag{2.8.4}$$

where $d^2\alpha = d\,[\Re\mathrm{e}(\alpha)]\,d\,[\Im\mathrm{m}(\alpha)]$.

2. **Complex *P* Representation:**

$$d\mu(\alpha,\beta) = d\alpha d\beta. \tag{2.8.5}$$

Here (α,β) are treated as complex variables which are to be integrated on individual contours C, C'. The conditions for the existence of this representation are discussed in the appendix. This particular representation may take on complex values so in no sense can it have any physical interpretation as a probability distribution. However, as we shall see it is an extremely useful representation giving exact results for certain problems and physical observable such as all the single time correlation functions.

3. **Positive *P* Representation:**

$$d\mu(\alpha,\beta) = d^2\alpha d^2\beta. \tag{2.8.6}$$

This enables stochastic differential equations, and a correspondence between the quantum Markov process and ordinary diffusion processes to be derived.

In all representations, it is, of course, true that observable moments are given by

$$\langle(\hat{a}^\dagger)^m\hat{a}^n\rangle = \int_D d\mu(\alpha,\beta)\beta^m\alpha^n P(\alpha,\beta). \tag{2.8.7}$$

Master equation may be converted to a c-number representation using the complex *P* representation by an analogous set of operator rules used for the diagonal *P* representation. The nondiagonal coherent state projection operator is defined as

$$\Lambda(\boldsymbol{\alpha}) = \frac{|\alpha\rangle\,\langle\beta^*|}{\langle\beta^*|\alpha\rangle} \tag{2.8.8}$$

where $\boldsymbol{\alpha} \equiv (\alpha,\beta)$. Use a property of creation and annihilation operators that operate on the coherent state, $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and $\langle\alpha|\,\hat{a}^\dagger = \langle\alpha|\,\alpha^*$. Then, the following identities hold

$$\begin{aligned} \hat{a}\Lambda(\boldsymbol{\alpha}) &= \alpha\Lambda(\boldsymbol{\alpha}), & \hat{a}^\dagger\Lambda(\boldsymbol{\alpha}) &= \left(\beta + \frac{\partial}{\partial\alpha}\right)\Lambda(\boldsymbol{\alpha}), \\ \Lambda(\boldsymbol{\alpha})\hat{a}^\dagger &= \Lambda(\boldsymbol{\alpha})\beta, & \Lambda(\boldsymbol{\alpha})\hat{a} &= \left(\frac{\partial}{\partial\beta} + \alpha\right)\Lambda(\boldsymbol{\alpha}). \end{aligned} \tag{2.8.9}$$

By substituting the above identities into (2.8.1) defining the generalized *P* representation, and using partial integration (providing the boundary terms vanish) these identities can be used to generate operations on the *P* function depending on the representation.

2.8.1 Complex *P* Representation

$$\hat{a}\rho \leftrightarrow \alpha P(\alpha), \qquad \hat{a}^{\dagger}\rho \leftrightarrow \left(\beta - \frac{\partial}{\partial\alpha}\right) P(\alpha),$$
$$\rho\hat{a}^{\dagger} \leftrightarrow \beta P(\alpha), \qquad \rho\hat{a} \leftrightarrow \left(\alpha - \frac{\partial}{\partial\beta}\right) P(\alpha). \tag{2.8.10}$$

This procedure yields a very similar equation to that for the Glauber-Sudarshan *P* function. We assume that, by appropriate reordering of the differential operators, we can reduce an operator master equation to the form [where $(\alpha,\beta) = \boldsymbol{\alpha} = (\alpha^{(1)}, \alpha^{(2)})$]

$$\frac{\partial\rho}{\partial t} = \int_C \int_{C'} \Lambda(\boldsymbol{\alpha}) \frac{\partial P(\boldsymbol{\alpha})}{\partial t} d\alpha d\beta \tag{2.8.11}$$
$$= \int_C \int_{C'} d\alpha^{(1)} d\alpha^{(2)} P(\boldsymbol{\alpha}) \left\{ A^{\mu}(\boldsymbol{\alpha}) \frac{\partial}{\partial\alpha^{\mu}} + \frac{1}{2} D^{\mu\nu}(\boldsymbol{\alpha}) \frac{\partial}{\partial\alpha^{\mu}} \frac{\partial}{\partial\alpha^{\nu}} \right\} \Lambda(\boldsymbol{\alpha}). \tag{2.8.12}$$

We now integrate by parts and if we can neglect boundary terms, which may be made possible by an appropriate choice of contours, C and C', at least one solution is obtained by equating the coefficients of $\Lambda(\boldsymbol{\alpha})$

$$\frac{\partial P(\boldsymbol{\alpha})}{\partial t} = \left\{ -\frac{\partial}{\partial\alpha^{\mu}} A^{\mu}(\boldsymbol{\alpha}) + \frac{1}{2} \frac{\partial}{\partial\alpha^{\mu}} \frac{\partial}{\partial\alpha^{\nu}} D^{\mu\nu}(\boldsymbol{\alpha}) \right\} P(\boldsymbol{\alpha}) \tag{2.8.13}$$

This equation is sufficient to imply (2.8.12) but is not a unique equation because the $\Lambda(\boldsymbol{\alpha})$ are not linearly independent. The Fokker-Planck equation has the same form as that derived using the diagonal *P* representation with α^* replaced by β.

It should be noted that for the complex *P* representation, $A^{\mu}(\boldsymbol{\alpha})$ and $D^{\mu\nu}(\boldsymbol{\alpha})$ are always analytic in $\boldsymbol{\alpha}$, if $P(\boldsymbol{\alpha})$ is initially analytic (2.8.13) preserves this analyticity as time develops.

2.8.2 Positive *P* Representation

This integration measure is chosen as

$$d\mu(\alpha,\beta) = d^2\alpha d^2\beta. \tag{2.8.14}$$

This representation allows α, β to vary independently over the whole complex plane. It was proved that $P(\alpha,\beta)$ always exists for a physical density operator and can always be chosen positive. For this reason we call it the positive *P* representation. $P(\alpha,\beta)$ has all the mathematical properties of a genuine probability. It may also have an interpretation as a probability distribution. It proves a most useful representation, in particulary, for problems where the Fokker-Planck equation in other representations may have a non-positive equation definite diffusion matrix. It may be shown that provided any Fokker-Planck equation exists for the time development in the Glauber-Sudarshan representation, a corresponding Fokker-Planck equation exists with a positive semi-definite diffusion coefficient for the positive P representation.

The operator identities for the positive P representation are the same as (2.8.10) for the complex P representation. In addition, using the analyticity of $\Lambda(\alpha, \beta)$ and noting that if

$$\alpha = \alpha_x + i\alpha_y, \qquad \text{and} \qquad \beta = \beta_x + i\beta_y,$$

then

$$\frac{\partial}{\partial \alpha}\Lambda(\alpha) = \frac{\partial}{\partial \alpha_x}\Lambda(\alpha) = -i\frac{\partial}{\partial \alpha_y}\Lambda(\alpha)$$

and

$$\frac{\partial}{\partial \beta}\Lambda(\alpha) = \frac{\partial}{\partial \beta_x}\Lambda(\alpha) = -i\frac{\partial}{\partial \beta_y}\Lambda(\alpha). \tag{2.8.15}$$

Thus in addition to (2.8.10) we also have

$$\begin{aligned} \hat{a}^\dagger\rho &\longleftrightarrow \left(\beta - \frac{\partial}{\partial \alpha_x}\right)P(\alpha) \longleftrightarrow \left(\beta + i\frac{\partial}{\partial \alpha_y}\right)P(\alpha), \\ \rho\hat{a} &\longleftrightarrow \left(\alpha - \frac{\partial}{\partial \beta_x}\right)P(\alpha) \longleftrightarrow \left(\alpha + i\frac{\partial}{\partial \beta_y}\right)P(\alpha). \end{aligned} \tag{2.8.16}$$

The positive P representation may be used to give a Fokker-Planck equation with a positive definition diffusion matrix. We shall demonstrate this in the following.

We assume that the same equation

$$\frac{\partial}{\partial t}P(\mathrm{x}) = \left\{-\frac{\partial}{\partial x_j}A_j(\mathrm{x}) + \frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}D_{ij}(\mathrm{x})\right\}P(\mathrm{x})$$

is being considered but with a positive P representation. The symmetric diffusion matrix can always be factorized in the form

$$\mathbf{D}(\alpha) = \mathbf{B}(\alpha)\mathbf{B}^{\mathrm{T}}(\alpha).$$

We now write

$$\mathbf{A}(\alpha) = \mathbf{A}_x(\alpha) + i\mathbf{A}_y(\alpha), \tag{2.8.17}$$

$$\mathbf{B}(\alpha) = \mathbf{B}_x(\alpha) + i\mathbf{B}_y(\alpha), \tag{2.8.18}$$

where $\mathbf{A}_x, \mathbf{A}_y, \mathbf{B}_x, \mathbf{B}_y$ are real. We then find that the master equation yields

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \iint d^2\alpha d^2\beta \Lambda(\alpha)\frac{\partial P(\alpha)}{\partial t} \\ &= \iint P(\alpha)\Big[A_x^\mu(\alpha)\partial_\mu^x + A_y^\mu(\alpha)\partial_\mu^y + \frac{1}{2}\Big(B_x^{\mu\sigma}(\alpha)B_x^{\nu\sigma}(\alpha)\partial_\mu^x\partial_\nu^x + B_y^{\mu\sigma}(\alpha)B_y^{\nu\sigma}(\alpha)\partial_\mu^y\partial_\nu^y \\ &\quad + 2B_x^{\mu\sigma}(\alpha)B_y^{\nu\sigma}(\alpha)\partial_\mu^x\partial_\nu^y\Big)\Big]\Lambda(\alpha)d^2\alpha d^2\beta. \end{aligned} \tag{2.8.19}$$

Here we have, for notational simplicity, written $\partial/\partial\alpha_x^\mu = \partial_\mu^x$ etc., and have used the analyticity of $\Lambda(\alpha)$ to make either of the replacements

$$\partial/\partial\alpha_x^\mu \longleftrightarrow \partial_\mu^x \longleftrightarrow -i\partial_\mu^y \tag{2.8.20}$$

in such a way as to yields (2.8.19). Now, provided partial integration is permissible, we deduce the Fokker-Planck equation:

$$\begin{aligned}\frac{\partial P(\alpha)}{\partial t} &= \Big[-\partial^x_\mu A^\mu_x(\alpha) - \partial^y_\mu A^\mu_y(\alpha) + \frac{1}{2}\Big(\partial^x_\mu\partial^x_\nu B^{\mu\sigma}_x(\alpha)B^{\nu\sigma}_x(\alpha)\\ &\quad + 2\partial^x_\mu\partial^y_\nu B^{\mu\sigma}_x(\alpha)B^{\nu\sigma}_y(\alpha) + \partial^y_\mu\partial^y_\nu B^{\mu\sigma}_y(\alpha)B^{\nu\sigma}_y(\alpha)\Big)\Big]P(\alpha). \qquad (2.8.21)\end{aligned}$$

Again, this is not a unique time-development equation but (2.8.19) is a consequence of (2.8.21).

However, the Fokker-Planck equation (2.8.21) now possesses a positive semidefinite diffusion matrix in a four-dimensional space whose vectors are

$$\left(\alpha^{(1)}_x, \alpha^{(2)}_x, \alpha^{(1)}_y, \alpha^{(2)}_y\right) \equiv (\alpha_x, \beta_x, \alpha_y, \beta_y)\,. \qquad (2.8.22)$$

We find the drift vector is

$$\mathscr{A}(\alpha) \equiv \left(A^{(1)}_x(\alpha), A^{(2)}_x(\alpha), A^{(1)}_y(\alpha), A^{(2)}_y(\alpha)\right) \qquad (2.8.23)$$

and the diffusion matrix

$$\mathscr{D}(\alpha) = \begin{pmatrix} \mathbf{B}_x\mathbf{B}^{\mathrm{T}}_x & \mathbf{B}_x\mathbf{B}^{\mathrm{T}}_y \\ \mathbf{B}_y\mathbf{B}^{\mathrm{T}}_x & \mathbf{B}_y\mathbf{B}^{\mathrm{T}}_y \end{pmatrix}(\alpha) \equiv \mathscr{B}(\alpha)\mathscr{B}^{\mathrm{T}}(\alpha) \qquad (2.8.24)$$

where

$$\mathscr{B}(\alpha) = \begin{pmatrix} \mathbf{B}_x & 0 \\ \mathbf{B}_y & 0 \end{pmatrix}(\alpha) \qquad (2.8.25)$$

and $\mathscr{D}$ is thus explicitly positive semi-definite (and not positive definite). The corresponding Itô stochastic differential equations can be written as (please also see Eqs. (2.8.17) and (2.8.18))

$$\frac{d}{dt}\begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \begin{pmatrix} \mathbf{A}_x(\alpha) \\ \mathbf{A}_y(\alpha) \end{pmatrix} + \begin{pmatrix} \mathbf{B}_x(\alpha)\xi(t) \\ \mathbf{B}_y(\alpha)\xi(t) \end{pmatrix}, \qquad (2.8.26)$$

or recombining real and imaginary parts

$$\frac{d\alpha}{dt} = \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\xi(t). \qquad (2.8.27)$$

Apart from the substitution $\alpha^* \to \beta$, (2.8.27) is just the stochastic differential equation which would be obtained by using the Glauber-Sudarshan P representation, and naively converting the Fokker-Planck equation with a non-positive definite diffusion matrix into an Itô stochastic differential equation.

In our derivation, the two formal variables (α, α^*) have been replaced by variables in the complex plane (α, β) that are allowed to fluctuate independently. The positive P representation as defined here thus appears as a mathematical justification of this procedure.

CHAPTER 3

Four-Wave Mixing in Nonlinear Fiber Optics

3.1 Origin of Four-Wave Mixing

The origin of FWM lies in the nonlinear response of bound electrons of a material to an electromagnetic field. The polarization induced in the medium contains terms whose magnitude is governed by the nonlinear susceptibilities. The resulting nonlinear effects can be classified as second or third order parametric processes, depending on whether the second susceptibility $\chi^{(2)}$, or the third order susceptibility $\chi^{(3)}$, is responsible for them. Because $\chi^{(2)}$ vanishes for an isotropic medium (in dipole approximation), the second order processes such as second harmonic generation should not occur in silica fibers. In practice, they do occur because of quadrupole and magnetic-dipole effects but with a relatively low conversion efficiency.

The third order parametric processes involve nonlinear interaction among four optical waves and include the phenomena such as FWM and third harmonic generation. Indeed, FWM in optical fibers was studied soon after low-loss fibers first became available. The main features of FWM can be understood from the third order polarization term as

$$\mathbf{P}_{\mathrm{NL}} = \epsilon_0 \chi^{(3)} \vdots \mathbf{EEE}, \tag{3.1.1}$$

where $\mathbf{E}$ is the electric field and $\mathbf{P}_{\mathrm{NL}}$ is the induced nonlinear polarization.

In general, FWM is polarization dependent and one must develop a full vector theory for it. However, considerable physical insight is gained by first considering the scalar case in which all four fields are linearly polarized along a principal axis of birefringent fiber such that they maintain their state of polarization. Consider four CW waves oscillating at frequencies ω_1, ω_2, ω_3, and ω_4 and linearly polarized along the same axis x. The total electric field can be written as

$$\mathbf{E} = \frac{1}{2}\hat{x}\sum_{j=1}^{4} E_j e^{i(\beta_j z - \omega_j t)} + \text{c.c.}, \tag{3.1.2}$$

where the propagation constant $\beta_j = \tilde{n}_j \omega_j / c$, $\tilde{n}_j$ being the mode index. If we substitute Eq. (3.1.2) in Eq. (3.1.1) and express $\mathbf{P}_{\mathrm{NL}}$ in the same form as $\mathbf{E}$ using

$$\mathbf{P}_{\mathrm{NL}} = \frac{1}{2}\hat{x}\sum_{j=1}^{4} P_j e^{i(\beta_j z - \omega_j t)} + \text{c.c.}, \tag{3.1.3}$$

we find that P_j ($j = 1$ to 4) consists of a large number of terms involving the products of three electric fields. For example, P_4 can be expressed as

$$\begin{aligned} P_4 &= \frac{3\epsilon_0}{4}\chi^{(3)}_{xxxx}\Big[|E_4|^2E_4 + 2\left(|E_1|^2 + |E_2|^2 + |E_3|^2\right)E_4 \\ &\quad + 2E_1E_2E_3\mathrm{e}^{i(\theta_+)} + 2E_1E_2E_3^*\mathrm{e}^{i(\theta_-)} + \ldots\Big], \end{aligned} \tag{3.1.4}$$

where θ_+ and θ_- are defined as

$$\theta_+ = (\beta_1 + \beta_2 + \beta_3 - \beta_4)z - (\omega_1 + \omega_2 + \omega_3 - \omega_4)t, \tag{3.1.5}$$

$$\theta_- = (\beta_1 + \beta_2 - \beta_3 - \beta_4)z - (\omega_1 + \omega_2 - \omega_3 - \omega_4)t. \tag{3.1.6}$$

The first four terms containing E_4 in Eq. (3.1.4) are responsible for the self phase modulation (SPM) and cross phase modulation (XPM) effects, but the remaining terms result from the frequency combinations (sum or difference) of all four waves. How many of these are effective during a FWM process depend on the phase mismatch between E_4 and P_4 governed by θ_+, θ_-, or a similar quantity.

Significant FWM occurs only if the phase mismatch nearly vanishes. This requires matching of the frequencies as well as of the wave vectors. The latter requirement is often referred to as phase matching. In quantum mechanical terms, FWM occurs when photons from one or more waves are annihilated and new photons are created at different frequencies such that the net energy and momentum are conserved during the parametric interaction. The main difference between a FWM process and a stimulated scattering process is that the phase matching condition is automatically satisfied in the case of Raman or Brillouin scattering as a result of the active participation of the nonlinear medium. In contrast, the phase matching condition requires a specific choice of input wavelengths and fiber parameters before FWM can occur with high efficiency.

There are two types of FWM terms in Eq. (3.1.4). The term containing θ_+ corresponds to the case in which three photons transfer their energy to a single photon at the frequency $\omega_4 = \omega_1 + \omega_2 + \omega_3$. This term is responsible for the phenomena such as third harmonic generation ($\omega_1 = \omega_2 = \omega_3$). In general, it is difficult to satisfy the phase matching condition for such processes to occur in fiber optics with high efficiencies. The term containing θ_- in Eq. (3.1.4) corresponds to the case in which two photons at frequencies ω_1 and ω_2 are annihilated, while two photons at frequencies ω_3 and ω_4 are created simultaneously such that

$$\omega_3 + \omega_4 = \omega_1 + \omega_2. \tag{3.1.7}$$

The phase matching requirement for this process is $\Delta k = 0$, where

$$\begin{aligned} \Delta k &= \beta_3 + \beta_4 - \beta_1 - \beta_2 \\ &= (\tilde{n}_3\omega_3 + \tilde{n}_4\omega_4 - \tilde{n}_1\omega_1 - \tilde{n}_2\omega_2)/c, \end{aligned} \tag{3.1.8}$$

and $\tilde{n}_j$ is the effective mode index at the frequency ω_j.

In the general case in which $\omega_1 \neq \omega_2$, one must launch two pump beams for FWM to occur. The special case, in which $\omega_1 = \omega_2$, is interesting because FWM can be initiated with a single pump beam. This degenerate case is often useful for optical fibers. Physically, it manifests in a way similar to stimulated Raman scattering (SRS). A strong pump wave at ω_1 creates two sidebands located symmetrically at frequencies ω_3 and ω_4 with a frequency shift

$$\Omega_s = \omega_1 - \omega_3 = \omega_4 - \omega_1, \tag{3.1.9}$$

where we assumed for definiteness $\omega_3 < \omega_4$. The low-frequency sideband at ω_3 and the high-frequency sideband at ω_4 are referred to as the Stokes and anti-Stokes bands in direct analogy with SRS. The degenerate FWM was originally called three-wave mixing as only three distinct frequencies are involved in the nonlinear processes. However, the term three-wave mixing should be reserved for the processes mediated by $\chi^{(2)}$. The name four-photon mixing is also used for FWM. The Stokes and anti-Stokes bands are often called the signal and idler waves or photons, borrowing the terminology from the field of microwaves.

3.2 Theory of Four-Wave Mixing

The degenerate FWM transfers energy from a strong pump wave to two waves, upshifted and down-shifted in frequency from the pump frequency ω_1 by an amount Ω_s given in Eq. (3.1.9). If only the pump wave is incident at the fiber, and the phase matching condition is satisfied, the Stokes and anti-Stokes waves at the frequencies ω_3 and ω_4 can be generated from noise, similar to the stimulated scattering processes. On the other hand, if a weak signal at ω_3 is also launched into the fiber together with the pump, the signal is amplified while a new idler wave at ω_4 is generated simultaneously. The gain responsible for such amplification is called the parameter gain. In this section, we consider the FWM mixing process in detail and derive an expression for the parametric gain. The nondegenerate case ($\omega_1 \neq \omega_2$) is considered for generality.

3.2.1 Coupled Amplitude Equations

The starting point is, usual, the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \left\{ \frac{\partial^2 \mathbf{P}_\mathrm{L}}{\partial t^2} + \frac{\partial^2 \mathbf{P}_\mathrm{NL}}{\partial t^2} \right\}, \tag{3.2.1}$$

where $\mathbf{E}(\mathbf{r}, t)$ is the total electric field and

$$\mathbf{P}_\mathrm{L} = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(t - t') \cdot \mathbf{E}(\mathbf{r}, t') dt' \tag{3.2.2}$$

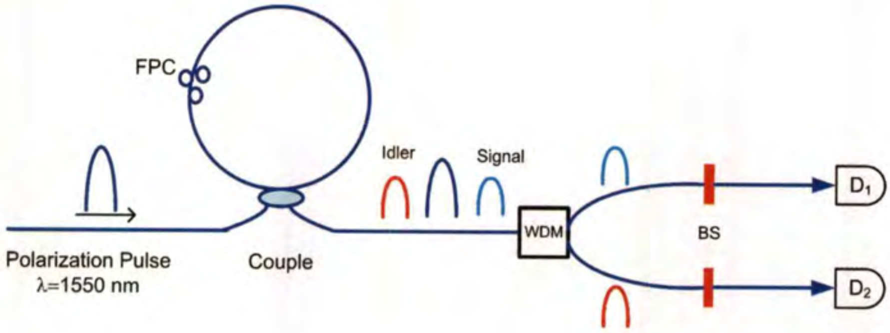

Figure 3.1 The ultra short pumped pulses at 1550 nm are launched into fiber and then circulating in a fiber ring resonator, and then an entangled photon pair i.e., signal and idler photons, is generated via FWM process. After a ring resonator the pumped pulses are blocked by a wavelength division multiplexing (WDM) device leaving only the signal and idler photons pass to the detector D_1 and D_2.

is the linear part with the nonlinear part $\mathbf{P}_{\mathrm{NL}}$ given in Eq. (3.1.1). We substitute Eqs. (3.1.2) and (3.1.3) in the wave equation, together with a similar expression for the the linear part of the polarization, and neglect the time dependence of the field components E_j ($j = 1$ to 4) assuming quasi-CW conditions. Their spatial dependence is, however, included using $E_j(\mathrm{r}) = F_j(x, y)A_j(z)$, where $F_j(x, y)$ is the spatial distribution of the fiber mode in which the j^{th} field propagates inside the fiber. Integrating over the spatial mode profiles, the evolution of the amplitude $A_j(z)$ inside an optical fiber is governed by the following set of four coupled equations:

$$\frac{\partial A_{p_1}}{\partial z} = i\gamma|A_{p_1}|^2 A_{p_1}, \tag{3.2.3a}$$

$$\frac{\partial A_{p_2}}{\partial z} = i\gamma|A_{p_2}|^2 A_{p_2}, \tag{3.2.3b}$$

$$\frac{\partial A_s}{\partial z} = i\gamma\Big[2A_s\left(|A_{p_1}|^2 + |A_{p_2}|^2\right) + A_{p_1}A_{p_2}A_i^* \mathrm{e}^{-i\Delta kz}\Big], \tag{3.2.3c}$$

$$\frac{\partial A_i}{\partial z} = i\gamma\Big[2A_i\left(|A_{p_1}|^2 + |A_{p_2}|^2\right) + A_{p_1}A_{p_2}A_s^* \mathrm{e}^{-i\Delta kz}\Big], \tag{3.2.3d}$$

where A_j $(j = p_1, p_2, s, i)$ denote electric field amplitudes of mode j for the pumps, signal and idler, respectively and the magnitude of wave vector difference that account for momentum conservation is $\Delta k = k_s + k_i - k_{p_1} - k_{p_2}$ and the nonlinear parameter of interaction is $\gamma = 2\pi n_2/\lambda A_{\mathrm{eff}}$ where the nonlinear index $n_2 = (3/4n^2\epsilon_0 c)\Re(\chi^{(3)}_{xxxx})$ and ϵ_0 the vacuum permittivity, A_{eff} is the effective mode area of the optical fiber, $\lambda \approx \lambda_{p_1,p_2,s,i}$ is the wavelength involved in the FWM interaction.

3.3 Nondegenerate Four-Wave Mixing Hamiltonian

Having got the classical equation of motion of the classical fields as appearing in Eq. (3.2.3), then its counterpart quantum equation of motion can be found from the second quantized rule for all four classical fields. It is easy to rewrite the Heisenberg equation of motion of FWM in the form

$$\begin{aligned}
\frac{\partial \hat{E}_{p_1}^{(+)}}{\partial z} &= i\eta \hat{E}_{p_1}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{p_1}^{(+)}, && (3.3.1\text{a})\\
\frac{\partial \hat{E}_{p_2}^{(+)}}{\partial z} &= i\eta \hat{E}_{p_2}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{p_2}^{(+)}, && (3.3.1\text{b})\\
\frac{\partial \hat{E}_{s}^{(+)}}{\partial z} &= i\eta\Big[2\hat{E}_{p_1}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{s}^{(+)} + 2\hat{E}_{p_2}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{s}^{(+)} \\
&\quad + \hat{E}_{i}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{p_2}^{(+)} + \hat{E}_{i}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{p_1}^{(+)}\Big], && (3.3.1\text{c})\\
\frac{\partial \hat{E}_{i}^{(+)}}{\partial z} &= i\eta\Big[2\hat{E}_{p_1}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{i}^{(+)} + 2\hat{E}_{p_2}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{i}^{(+)} \\
&\quad + \hat{E}_{s}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{p_2}^{(+)} + \hat{E}_{s}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{p_1}^{(+)}\Big], && (3.3.1\text{d})
\end{aligned}$$

where $\hat{E}_j^{(+)} = \sqrt{(\hbar\omega_j/2\epsilon_0 V_Q)}\,\hat{a}_j$ $(j = p_1, p_2, s, i)$ are the positive frequency electric-field operators, corresponding to photon annihilation operators, and V_Q is the quantization volume. Here we omit the Hermitian-conjugate equations corresponding to Eqs. (3.3.1) for simplicity. We have assumed that the photon fields phase match, i.e., $\Delta k = 0$. In Eqs. (3.3.1), $\eta = -\chi^{(3)} A_{\text{eff}} n L\omega/2V_Q c$ is a constant similar to γ in the classical Eqs. (3.2.3); the exact form of this constant differs from its classical cousin to compensate for the unit discrepancy between the two sets of equations (note that the operator $\hat{E}_j^{(+)}$ is of unit V/m, and the amplitude A_j is of unit $\sqrt{\text{W}}$. The correct form of the interaction Hamiltonian that we are seeking should lead to Eqs. (3.3.1) via the Heisenberg equation of motion for the field operators, namely, $i\hbar(\partial\hat{E}/\partial t) = [\hat{E}, \hat{\mathcal{H}}_{\text{I}}]$, where $\hat{E}$ stands for any electric-field operator. Utilizing the mathematical facts $\partial/\partial t \equiv (c/n)(\partial/\partial z)$ and $[\hat{E}_j^{(+)}(z), \hat{E}_k^{(-)}(z')] = (\hbar\omega/2\epsilon_0 V_Q)\delta(z-z')\delta_{jk}$, we arrive at the following form for our interaction Hamiltonian:

$$\begin{aligned}
\hat{\mathcal{H}}_{\text{I}} &= \beta\epsilon_0\chi^{(3)}\int_V dV\Big[\hat{E}_{p_1}^{(-)}\hat{E}_{p_1}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{p_1}^{(+)} + \hat{E}_{p_2}^{(-)}\hat{E}_{p_2}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{p_2}^{(+)} \\
&\quad + \hat{E}_{s}^{(-)}\hat{E}_{i}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{p_2}^{(+)} + \hat{E}_{s}^{(-)}\hat{E}_{i}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{p_1}^{(+)} \\
&\quad + 4\hat{E}_{p_1}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{i}^{(-)}\hat{E}_{i}^{(+)} + 4\hat{E}_{p_2}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{i}^{(-)}\hat{E}_{i}^{(+)} \\
&\quad + 4\hat{E}_{p_1}^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{s}^{(-)}\hat{E}_{s}^{(+)} + 4\hat{E}_{p_2}^{(-)}\hat{E}_{p_2}^{(+)}\hat{E}_{s}^{(-)}\hat{E}_{s}^{(+)}\Big], \qquad (3.3.2)
\end{aligned}$$

where β is an overall unknown constant related to the specific experimental details, which will be determined later when we compare our theory with the experiment; $\chi^{(3)}$ is the nonlinear electric susceptibility whose tensorial nature is ignored since all the optical fields are assumed to be linearly copolarized. The integral is taken over the entire volume of interaction, namely, the effective volume of the optical fiber. We label the first two terms in the integrand of Eq. (3.3.2) as the self-phase modulation (SPM) of the pump field, the next two terms as the four-photon scattering (FPS) among the optical fields, and the last four terms as the cross-phase modulation (XPM) between pump and signal (idler)fields. The undepleted-pump approximation holds because the loss in the fiber is negligible and only a few photons are scattered ($\sim$ 1 out of 10^8) through the nonlinear interaction.

The two-photon state at the output of the fiber is calculated by means of first-order perturbation theory, i.e.,

$$|\Psi\rangle = |0\rangle + \frac{1}{i\hbar}\int_{-\infty}^{\infty} \hat{\mathcal{H}}_{\mathrm{I}}(t)dt\,|0\rangle\,. \tag{3.3.3}$$

Retaining of higher-order terms in the perturbation series involves generation of multi-photon states, which will be ignored in our calculation owing to their smallness. We can see that only the FPS terms in the interaction Hamiltonian contribute to the formation of the signal/idler two-photon state. This is because all terms vanish when acting on the vacuum state $|0\rangle$ with the exception of $\hat{E}_s^{(-)}\hat{E}_i^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{p_2}^{(+)}$ + H.c., which we denote as

$$\hat{\mathcal{H}}_{\mathrm{FPS}} \equiv \alpha\epsilon_0\chi^{(3)}\int_V dV(\hat{E}_s^{(-)}\hat{E}_i^{(-)}\hat{E}_{p_1}^{(+)}\hat{E}_{p_2}^{(+)} + \text{H.c.}), \tag{3.3.4}$$

where $\alpha = 2\beta$, and H.c. stands for Hermitian conjugate.

The state vector is then given by

$$|\Psi\rangle = |0\rangle + \frac{1}{i\hbar}\int_{-\infty}^{\infty} \hat{\mathcal{H}}_{\mathrm{FPS}}(t)dt\,|0\rangle\,, \tag{3.3.5}$$

which is a superposition of the vacuum and the two-photon state. The following field operators

$$\hat{E}_s^{(-)} = \sum_{\omega_s}\sqrt{\frac{\hbar\omega_s}{2\epsilon_0 V_Q}}\frac{\hat{a}_{k_s}^\dagger}{n(\omega_s)}\,\mathrm{e}^{-i[k_s(\omega_s)z-\omega_s t]}, \tag{3.3.6}$$

$$\hat{E}_i^{(-)} = \sum_{\omega_i}\sqrt{\frac{\hbar\omega_i}{2\epsilon_0 V_Q}}\frac{\hat{a}_{k_i}^\dagger}{n(\omega_i)}\,\mathrm{e}^{-i[k_i(\omega_i)z-\omega_i t]}, \tag{3.3.7}$$

$$\hat{E}_p^{(+)} = \mathrm{e}^{-i\Omega_p t}\mathrm{e}^{-i\gamma_p t z}E_{0p}\int d\nu_p \mathrm{e}^{-\nu_p^2/2\sigma_p^2}\mathrm{e}^{i(k_p z-\nu_p t)} \tag{3.3.8}$$

are applied to Eq. (3.3.5) where $\hat{a}_{k_s}^\dagger$ is the creation operator for the signal mode with frequency ω_s, $k_s(\omega_s) = n(\omega_s)\omega_s/c$ is its wave-vector magnitude, $P_p = 2\sqrt{\pi}A_{\mathrm{eff}}\epsilon_0 c n\sigma_p^2 E_{0p}^2$ is the peak power of the pump pulse, which is treated as a constant under the undepleted pump approximation, and σ_p is the optical bandwidth of the pump. The Eq. (3.3.6)- (3.3.8) and Eqs. (3.3.4), when put into Eq. (3.3.5), after some algebra, lead to the following form of the state vector:

$$|\Psi\rangle = |0\rangle + \frac{1}{i\hbar}\sum_{k_s,k_i} F(k_s,k_i)\hat{a}_{k_s}^\dagger\hat{a}_{k_i}^\dagger\,|0\rangle\,, \tag{3.3.9}$$

$$F(k_s, k_i) = g\int_{-L}^{0} dz \frac{1}{\sqrt{1 - ik''(\Omega_p)\sigma_p^2 z}} \exp\bigg\{ -\frac{ik''(\Omega_p)z}{4}(\nu_s - \nu_i + \Delta)^2 - 2i\gamma P_p z - \frac{(\nu_s + \nu_i)^2}{4\sigma_p^2}\bigg\}, \tag{3.3.10}$$

$$g = \frac{\alpha\pi^2\chi^{(3)}P_p}{i\epsilon_0 V_Q n^3 \lambda_p \sigma_p}. \tag{3.3.11}$$

The function $F(k_s, k_i)$ is called the two-photon spectral function. Here $k''(\Omega_p) = (d^2k/d\omega^2)|_{\omega=\Omega_p}$ is the second order dispersion at the pump central frequency also known as the group-velocity dispersion, or GVD for short. λ_0 is the zero-dispersion wavelength of the dispersion-shifted fiber (DSF). $\Delta \equiv \Omega_s - \Omega_i$ is the central-frequency difference between signal and idler fields. The ν_s and ν_i are related to ω_s and ω_i through the following relation: $\nu_s = \omega_s - \Omega_s$, $\nu_i = \omega_i - \Omega_i$.

CHAPTER 4

Dissipation for Entangled Photon from FWM Process Generation

4.1 The Master Equation

Having laid the foundation about quantum aspect of photon and dissipation in the previous chapters, now it is time to apply them to this thesis. In this study the total Hamiltonian is established as

$$\hat{\mathcal{H}}_{\text{tot}} = \hat{\mathcal{H}}_{\text{fwm}} + \hat{\mathcal{H}}_{\text{pump}} + \hat{V}. \tag{4.1.1}$$

The first term $\hat{\mathcal{H}}_{\text{fwm}}$ is a four-wave mixing part as appeared in Eq. (3.3.4) can be rewritten in the

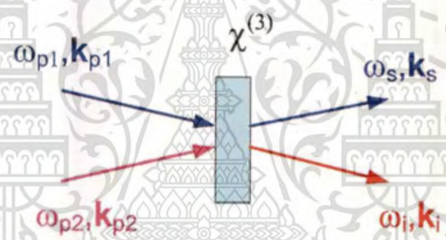

Figure 4.1 At particular point in the fiber ring resonator, the two ultra short-pumped pulses of photon mode p_1 and p_2 are annihilated and then transfer their momentum and energy to create an entangled photon of modes s and i.

form of creation and annihilation operators as

$$\hat{\mathcal{H}}_{\text{fwm}} = i\hbar\chi_0 \left(\hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_{p_1} \hat{a}_{p_2} - \hat{a}_s \hat{a}_i \hat{a}_{p_1}^\dagger \hat{a}_{p_2}^\dagger \right), \tag{4.1.2}$$

with the momentum conservation (phase matching) condition $k_s + k_i = k_{p_1} + k_{p_2}$ and the energy conservation $\omega_s + \omega_i = \omega_{p_1} + \omega_{p_2}$, simultaneously where the nonlinear coupling

$$\chi_0 = \frac{3\epsilon_0 \chi^{(3)} (\omega' k_0)^2}{8\epsilon^2 \Delta V},$$

and this part will be taken in to account for two different cases, nondegenerate and degenerate as shown in Fig. 4.2.

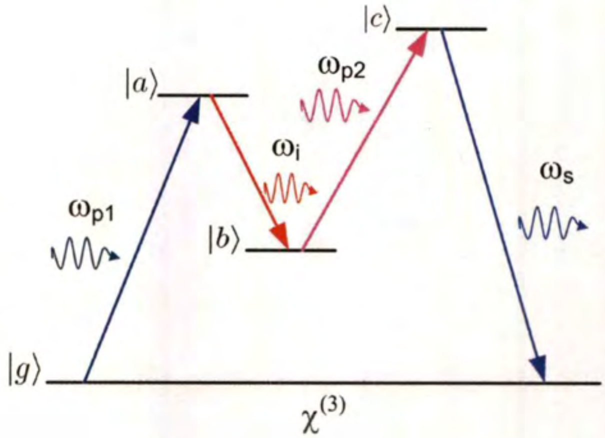

Figure 4.2 The $\chi^{(3)}$ nonlinear fiber optics ring resonator is excited by a pair of pumped photons ω_{p_1} and ω_{p_2} up to the virtual states and then cascade to the lower level states by radiating an entangled photon of frequencies ω_s and ω_i via FWM process subject to the momentum and energy conservations.

The second term is the nondegenerate ultra short pumped-pulse part of modes p_1 and p_2 and can be written as

$$\hat{\mathcal{H}}_{\text{pump}} = i\hbar \sum_{k=p_1,p_2} \mathcal{E}_k(\hat{a}_k^\dagger - \hat{a}_k), \tag{4.1.3}$$

where $\mathcal{E}_k$ is proportional to the amplitude of the driving laser field.

Finally the irreversible damping of the modes is represented by the Hamiltonian

$$\hat{V} = \hbar \sum_{k=i,s,p_1,p_2} (\hat{a}_k\hat{\Gamma}^\dagger + \hat{a}_k^\dagger\hat{\Gamma}), \tag{4.1.4}$$

where $\hat{\Gamma}$ and $\hat{\Gamma}^\dagger$ are heat-bath annihilation and creation operators, respectively and in general

$$\hat{\Gamma}(t) \equiv \sum_j g_j\hat{b}_j \mathrm{e}^{i(\omega-\omega_j)t}, \tag{4.1.5}$$

$\hat{b}_j$ is the reservoir operator of mode j and the g_j are taken as real. We assume that at $t = 0$ there is no correlation between the system and the reservoir bath, or $\widetilde{\rho}_{SR}(0) = \widetilde{\rho}_S(0) \otimes \widetilde{\rho}_R(0)$, where

$$\widetilde{\rho}_R(0) = \frac{\prod_j \mathrm{e}^{-\hbar\omega_j\hat{b}_j^\dagger\hat{b}_j/k_BT}}{\mathrm{Tr}_R\left[\prod_j \mathrm{e}^{-\hbar\omega_j\hat{b}_j^\dagger\hat{b}_j/k_BT}\right]} \tag{4.1.6}$$

We also assume that at time t the system does not depend on its past time to ensure that energy which goes into the reservoir will not return to the system, $\widetilde{\rho}_{SR}(t) = \widetilde{\rho}_S(t) \otimes \widetilde{\rho}_R(0)$, which is the Markovian assumption. After a straightforward calculation in the interaction picture one finds the master equation is in the form [19, 20]

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} &= \frac{1}{i\hbar}[\hat{\mathcal{H}}_{\mathrm{fwm}} + \hat{\mathcal{H}}_{\mathrm{pump}}, \hat{\rho}] \\ &\quad + \sum_{k=i,s,p_1,p_2} \gamma_k(2\hat{a}_k\hat{\rho}\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_k\hat{\rho} - \hat{\rho}\hat{a}_k^\dagger\hat{a}_k) \\ &\quad + \sum_{k=i,s,p_1,p_2} 2n_k^{\mathrm{th}}\gamma_k(\hat{a}_k\hat{\rho}\hat{a}_k^\dagger - \hat{\rho}\hat{a}_k\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_k\hat{\rho} + \hat{a}_k^\dagger\hat{\rho}\hat{a}_k), \end{aligned} \tag{4.1.7}$$

where the γ_k are the mode damping rates, and the mean number of thermal photons of frequency ω_k in the heat bath is $n_k^{\mathrm{th}}(\omega_k) = [\mathrm{e}^{\hbar\omega_k/k_B T} - 1]^{-1}$.

CHAPTER 5

Results and Discussion

5.1 The system of Equation of Motion of Photons

By applying $+P$ representation to Eq. (4.1.7), we get the corresponding Fokker-Planck equation

$$
\begin{aligned}
\frac{\partial P(\alpha,t)}{\partial t} = & \left\{ \frac{\partial}{\partial \alpha_{p_1}}(\chi\alpha_s\alpha_i\alpha_{p_2}^+ - \mathcal{E}_{p_1} + \gamma_{p_1}\alpha_{p_1}) + \frac{\partial}{\partial \alpha_{p_1}^+}(\chi\alpha_s^+\alpha_i^+\alpha_{p_2} - \mathcal{E}_{p_1} + \gamma_{p_1}\alpha_{p_1}^+) \right. \\
& + \frac{\partial}{\partial \alpha_{p_2}}(\chi\alpha_s\alpha_i\alpha_{p_1}^+ - \mathcal{E}_{p_2} + \gamma_{p_2}\alpha_{p_2}) + \frac{\partial}{\partial \alpha_{p_2}^+}(\chi\alpha_s^+\alpha_i^+\alpha_{p_1} - \mathcal{E}_{p_2} + \gamma_{p_2}\alpha_{p_2}^+) \\
& - \frac{\partial}{\partial \alpha_i}(\chi\alpha_s^+\alpha_{p_2}\alpha_{p_1} + \gamma_i\alpha_i) - \frac{\partial}{\partial \alpha_i^+}(\chi\alpha_s\alpha_{p_2}^+\alpha_{p_1}^+ + \gamma_i\alpha_i^+) \\
& - \frac{\partial}{\partial \alpha_s}(\chi\alpha_i^+\alpha_{p_2}\alpha_{p_1} + \gamma_s\alpha_s) - \frac{\partial}{\partial \alpha_s^+}(\chi\alpha_i\alpha_{p_2}^+\alpha_{p_1}^+ + \gamma_s\alpha_s^+) \\
& - \frac{\partial^2}{\partial \alpha_{p_2}\partial \alpha_{p_1}}\chi\alpha_s\alpha_i - \frac{\partial^2}{\partial \alpha_s\partial \alpha_i}\chi\alpha_{p_2}\alpha_{p_1} \\
& + \frac{\partial^2}{\partial \alpha_s^+\partial \alpha_i^+}\chi\alpha_{p_2}^+\alpha_{p_1}^+ - \frac{\partial^2}{\partial \alpha_{p_2}^+\partial \alpha_{p_1}^+}\chi\alpha_s^+\alpha_i^+ \\
& + \frac{\partial^2}{\partial \alpha_{p_1}\partial \alpha_{p_1}^+}2n_{p_1}^{th}\gamma_{p_1} + \frac{\partial^2}{\partial \alpha_{p_2}\partial \alpha_{p_2}^+}2n_{p_2}^{th}\gamma_{p_2} \\
& \left. + \frac{\partial^2}{\partial \alpha_i\partial \alpha_i^+}2n_i^{th}\gamma_i + \frac{\partial^2}{\partial \alpha_s\partial \alpha_s^+}2n_s^{th}\gamma_s \right\} P(\alpha,t). \qquad (5.1.1)
\end{aligned}
$$

From this point, we apply a Kramers-Moyal expansion to the Fokker-Planck equation, and finally get the multivariate Langevin equation of the corresponding stochastic process as

$$
\begin{aligned}
\frac{d}{dt}\alpha_1 &= -\chi\alpha_s\alpha_i\alpha_2^+ + \mathcal{E}_1 - \gamma_1\alpha_1 + i\sqrt{\chi}\alpha_s\eta_A + \sqrt{2n_1 + \chi|\alpha_s|^2}\eta_1, \\
\frac{d}{dt}\alpha_2 &= -\chi\alpha_s\alpha_i\alpha_1^+ + \mathcal{E}_2 - \gamma_2\alpha_2 + i\sqrt{\chi}\alpha_i\eta_A^+ + \sqrt{2n_2 + \chi|\alpha_i|^2}\eta_2, \\
\frac{d}{dt}\alpha_i &= \chi\alpha_1\alpha_2\alpha_s^+ - \gamma_i\alpha_i + i\sqrt{\chi}\alpha_1\eta_B + \sqrt{2n_i + \chi|\alpha_i|^2}\eta_i, \\
\frac{d}{dt}\alpha_s &= \chi\alpha_1\alpha_2\alpha_i^+ - \gamma_s\alpha_s + i\sqrt{\chi}\alpha_2\eta_B^+ + \sqrt{2n_s + \chi|\alpha_2|^2}\eta_s, \qquad (5.1.2)
\end{aligned}
$$

where the correlations of the complex noise variables are in the form of

$$
\begin{aligned}
\langle\eta_k\rangle &= \langle\eta_k^\dagger\rangle = 0, \\
\langle\eta_j(t)\eta_k(t')\rangle &= \langle\eta_j^+(t)\eta_k^+(t')\rangle = \delta_{jk}\delta(t-t'). \qquad (5.1.3)
\end{aligned}
$$

Consequently, the remaining four equations of Eq. (5.1.2) are easily found, just their complex conjugate, which can be used together to analyze further the validity area of entanglement criteria [21, 22].

CHAPTER 6

Conclusion and Suggestions

We have established FWM Hamiltonian both the degenerate and nondegenerate cases. By combining the effect of photon-phonon interaction to the total Hamiltonian which can be used incorporate with the system density matrix to get the master equation. At this point, we eliminate the phonon operators by using Markov approximation and then transform from the reduced density matrix by using $+P$ representation to the Fokker-Planck equation, in the language of phase-space description, which can next extract to get the stochastic differential equation of Langevin type that can solve numerically parallel to the Itô method. So, we finally can analyze the dissipative effects from the reservoir on the propagation of entangled photon state in fiber optics by calculation for the life-times and its correlation function, which depend heavily on the damping rate and environment temperature, of entangled photon pair which subject to the Reid-EPR criteria.

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List of Publications

1. Preecha P. Yupapin and Chatchawal Sripakdee, "Four entangled photons generation using weak light and an all fiber optic scheme for multi-entangled photons generation.", *International Journal for Light and Electron Optics*, (2008) (article inpress).

2. P. P. Yupapin and C. Sripakdee, "Thermal Dissipation Effects of the Entangled Photon Generated by a Nonlinear Ring Resonator.", *International Journal for Light and Electron Optics*, (2008) (accepted).

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