THE APPROXIMATING METHOD FOR A COMMON ELEMENT OF THE SOLUTION SETS OF VARIATIONAL INEQUALITY PROBLEMS AND NONLINEAR MAPPINGS

PREEYAPORN SURBKIRD

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หัวข้อวิทยานิพนธ์	วิธีการประมาณค่าสำหรับสมาชิกร่วมของเซตผลเฉลยของปัญหาอสมการการแปรผันและการส่งไม่เชิงเส้น
นักศึกษา	นางสาวปรียาภรณ์ สืบเกิด
รหัสประจำตัว	55650704
ปริญญา	วิทยาศาสตรมหาบัณฑิต
สาขาวิชา	คณิตศาสตร์ประยุกต์
พ.ศ.	2557
อาจารย์ที่ปรึกษาวิทยานิพนธ์	ดร.อาทิตย์ แข็งธัญการ

บทคัดย่อ

จุดประสงค์ของวิทยานิพนธ์นี้เพื่อศึกษาทฤษฎีบทการลู่เข้าอย่างเข้มในปริภูมิฮิลเบิร์ตและแนะนำกระบวนการทำซ้ำแบบใหม่ในการหาสมาชิกร่วมของเซตของจุดตรึงของการส่งกึ่งหดเทียมโดยแท้และเซตของผลเฉลยของปัญหาอสมการการแปรผัน ผู้วิจัยได้พิสูจน์ทฤษฎีบทการลู่เข้าอย่างเข้มที่เกี่ยวข้องกับเซตของจุดตรึงของการส่งกึ่งไม่ขยาย T ในเซตย่อยนูนปิดของปริภูมิฮิลเบิร์ตโดยไม่ใช้เงื่อนไข $T_\varpi = (1-\omega)I + \omega T$ เมื่อ $\omega \in [0,1]$ และ T เป็นการส่งกึ่งปิด นอกจากนี้ผู้วิจัยได้พิสูจน์ทฤษฎีบทการลู่เข้าอย่างเข้มสำหรับวงศ์จำกัดของการส่งไม่กระจายและได้ยกตัวอย่างของทฤษฎีบทหลักเพื่อใช้ในการหาผลเฉลยทางเชิงตัวเลขอีกด้วย

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Student	Preeyaporn Surbkird
Student ID	55650704
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Thesis Advisor	Dr. Aitd Kangtunyakran

ABSTRACT

The purpose of this thesis is to study the strong convergence theorems in a real Hilbert space and introduce a new iterative scheme for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mappings and the sets of solutions of variational inequality. We also prove the strong convergence theorem involving the set of fixed points of quasi - nonexpansive mapping T of a nonempty closed convex subset of Hibert spaces into itself without assumptions $T_{\omega} = (1-\omega)I + \omega T$ where $\omega \in [0,1]$ and T is a demiclosed mapping. Moreover, we prove strong convergence theorems for a finite family of nonspreading mappings. Furthermore, we utilize our main theorem for the numerical example.

Keywords : Strictly pseudo - contractive mapping, Quasi - nonexpansive mapping,
Fixed point problem, Variational inequality problem

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CHAPTER I

INTRODUCTION

1.1 Background

Various problems in science and engineering can be formulated in forms of equations and inequalities. We can apply the fixed point theory to resolve those problems.

Let C be a nonempty closed convex subset of a real Hilbert space H. For a mapping T of C into itself is called *fixed point* if and only if $Tx = x$. We denote by $F(T)$ the set of fixed point of T, i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings, pseudocontractive mappings, quasi - nonexpansive mapping and κ - quasi strictly pseudo - contractive mapping have been studied by many mathematicians. So the fixed point problems concerning those mappings are interesting for investigating. So we aim to investigate the existence problem, structure of fixed points set, and approximation method for finding fixed point of the mappings. Then the fixed point plays an important role to solve many problems in mathematical, science and applied science.

Variational inequality were introduced and investigated by Stampacchia [20] in 1964.

Let $B : C \to H$ be a nonlinear mapping. The set of solutions of the *variational inequality* which is denoted by $VI(C,B)$ is to find a point $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0, \ \forall v \in C.$$

The variational inequality has been extensively studied in the branch of mathematical and engineering sciences with applications in industry, finance, economic, social, mathematical programming, ecology, regional, mechanics, optimal control, optimization, and applied sciences; see for instance [6], [7], [27].

Let H be a Hilbert space and let C be a nonempty closed convex subset of a real Hilbert space H. A mapping T of C into itself is called *nonexpansive* if

$$\|Tx-Ty\|\le\|x-y\|,\quad \text{for all } x,y\in C.$$

The mapping $T:C\to C$ is said to be *quasi - nonexpansive* if

$$\|Tx-y\|\le\|x-y\|,\quad \text{for all } x\in C \text{ and } y\in F(T).$$

The mapping $T:C\to C$ is said to be κ *- quasi strictly pseudo - contractive* if there exists a constant $\kappa\in[0,1)$ such that

$$\|Tx-y\|^2\le\|x-y\|^2+\kappa\|x-Tx\|^2,\quad \text{for all } x\in C \text{ and } y\in F(T).$$

In recent years, the strong convergence theorems of quasi - nonexpansive mapping in a real Hilbert spaces has been studied by many authors as follows:

In 2010, Paul-Emile Mainge [17] proved the strong convergence of the viscosity approximation method as follows:

Theorem 1.1 Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be the sequence generated by $x_0\in C$ and

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T_\omega x_n,\quad \forall n\ge 0,$$

where $\{\alpha_n\}\subset(0,1)$ such that

$$\lim_{n\to\infty}\alpha_n=0 \text{ and } \sum_{n=1}^{\infty}\alpha_n=\infty.$$

Also $\omega\in(0,1)$, $T_\omega=(1-\omega)I+\omega T$ with T be a quasi - nonexpansive and demiclosed on C and let $f:C\to C$ is a contractive mapping with $\rho\in[0,1)$. Then $\{x_n\}$ converges strongly to the unique element $z_0\in F(T)$, where $z_0=P_{F(T)}f(z_0)$, which equivalently solves the following variational inequality problem:

$$z_0\in F(T) \text{ and } \langle (I-f)z_0, v-z_0\rangle\ge 0,\quad \forall v\in F(T).$$

In 2010, Sun, Li and Zhou [21] proved the strong convergence of the viscosity approximation method in a real Hilbert spaces as follows:

Theorem 1.2 Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be the sequence generated by $x_0\in C$ and

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T_\omega x_n,\quad \forall n\ge 0,$$

where $\{\alpha_n\} \subset (0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Also $\omega \in (0,1]$, $T_\omega = (1-\omega)I + \omega T$ with T be a quasi - nonexpansive and demiclosed on C and let $f : C \to C$ is a contractive mapping with $\rho \in [0,1)$. Then $\{x_n\}$ converges strongly to the unique element $z_0 \in F(T)$, where $z_0 = P_{F(T)} f(z_0)$, which equivalently solves the following variational inequality problem:

$$z_0 \in F(T) \text{ and } \langle (I-f)z_0, v - z_0 \rangle \geq 0, \quad \forall v \in F(T).$$

After study research above and the research in the same way, we consider that many authors proved the strong convergence theorems involving quasi - nonexpansive mapping by using the conditions:

i. $T_\omega = (1-\omega)I + \omega T$;

ii. T is demiclosed on C;

see for instance [16], [24], [25].

In recently, Kangtunyakarn [12] modified the sets of variational inequalities as follows:

$$VI(C, aA + (1-a)B) = \{x \in C : \langle y - x, (aA + (1-a)B)x \rangle \geq 0\},$$

for all $y \in C, a \in (0,1)$ where A and B are the mappings of C into H. He also proved the strong convergence theorem of a new iterative scheme for finding a common element of the set of fixed points of κ - strictly pseudo - contractive mappings and two sets of solutions of variational inequalities as follows:

Theorem 1.3 Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β - inverse strongly monotone, respectively. Let T be a κ - strictly pseudo - contractive mapping with $\mathbb{F} = F(T) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C (I - \lambda_n (I - T)) P_C (I - r_n (aA + (1-a)B)) x_n, \qquad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$, $\{a_n\} \in (0,1)$, $\lambda \in (0, 1-\kappa)$, $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$, and $\{r_n\} \subset [0, 2\gamma]$, $\gamma = \min\{\alpha, \beta\}$ satisfy:

i. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii. $0 < c \le \beta_n \le d < 1$ and $0 < e \le r_n \le f < 2\gamma$;

iii. $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\kappa < \theta \le \lambda_n \le \eta < 1$;

iv. $\lim_{n\to\infty} |r_{n+1} - r_n| = 0.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} u$.

Motivated by Theorem 1.3, we prove the strong convergence theorem for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mappings and the set of solutions of variational inequalities in a real Hilbert space. By using our main result, we also prove the strong convergence theorem involving the set of fixed points of quasi - nonexpansive mapping T without assumptions $T_\omega = (1-\omega)I + \omega T$ where $\omega \in [0,1]$ and T is a demiclosed mapping. Moreover, we prove the strong convergence theorems for a finite family of nonspreading mappings. Furthermore, we utilize our main theorem for the numerical example.

1.2 Purpose of the Study

1.2.1 To study the fixed point theorems in a real Hilbert space.

1.2.2 To construct and study a new iterative scheme for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mappings and the set of solutions of variational inequalities in a real Hilbert space.

1.2.3 To prove strong convergence theorems involving the set of fixed points of quasi - nonexpansive mappings and nonspreading mappings in a real Hilbert space.

1.2.4 To give numerical example with guarantee convergence for the main result.

1.3 Scope of the Study

In this thesis, we study and introduce a new iterative scheme in a real Hilbert space. We also prove the strong convergence theorem for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mappings and the set of solutions of variational inequalities in a real Hilbert space. Moreover, we prove the strong convergence theorem involving the set of fixed points of quasi - nonexpansive mappings and nonspreading mappings. Furthermore, we utilize our main theorem for the numerical example.

1.4 Process of the Study

1.4.1 Study research about fixed point theorems.

1.4.2 Construct a new iteration scheme for finding a common element of the set of solutions of related problems.

1.4.3 Prove the strong convergence theorem for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mappings and the set of solutions of variational inequalities in a real Hilbert space.

1.4.4 Applying the main theorem in 1.4.3 to the set of fixed points of quasi - nonexpansive mappings and nonspreading mappings.

1.4.5 Give numerical example to create programs for guarantee the main result.

1.4.6 Write the thesis.

1.5 Utilization of the Study

1.5.1 The iteration scheme would be useful for finding a common element of the set of fixed points problems, the set of solutions of variation inequalities in a real Hilbert space.

1.5.2 Be able to use some numerical methods and programming to confirm the theoretical results.

1.5.3 To obtain a new convergence theorems for fixed point of nonlinear mappings.

1.5.4 To obtain new knowledge and applied research tools.

This thesis is divided into V chapters. Chapter I is an introduction to the research problems. In chapter II deals with some preliminaries and give some useful results that will be used in the next chapters. In chapter III, we introduce a new iterative scheme for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mapping and the set of solutions of variational inequalities in a real Hilbert space. In chapter IV, by using the main result, we prove the strong convergence theorems involving the set of fixed points of quasi - nonexpansive mappings and nonspreading mappings and we give numerical example to create programs for guarantee the main result. Finally, in chapter V, we give the conclusion of this thesis.

CHAPTER II

PRELIMINARIES

In this chapter, we introduce preliminaries which is a basic knowledge about this thesis and give some useful results that will be used in the next chapters.

2.1 Real Numbers

Definition 2.1.1 [4] The usual ordering of the field $\mathbb{R}$ of real numbers, the absolute value function on $\mathbb{R}$ is defined by the formulas

$$|a| = \begin{cases} a & if \quad a \geq 0, \\ -a & if \quad a < 0. \end{cases}$$

Lemma 2.1.2 [4] Let $a, b \in \mathbb{R}$.

1) $|a| = 0 \Leftrightarrow a = 0$ and $|a| > 0 \Leftrightarrow a \neq 0$,
2) $|ab| = |a||b|$,
3) If $b \geq 0$, then $|a| \leq b \Leftrightarrow -b \leq a \leq b$,
4) $|a+b| \leq |a| + |b|$.

Remark 2.1.3 [23] Using (2) and (4), we have that for any $a, b \in \mathbb{R}$,

$$\big||a| - |b|\big| \leq |a - b| \leq |a| + |b|.$$

Theorem 2.1.4 [23] If x and y are real numbers and $x < y$, then there is a rational number r such that $x < r < y$.

Let A be a nonempty subset of $\mathbb{R}$. Then, A is bounded from above if there exists a real number a such that $x \leq a$ for all $x \in A$. Such a is called an *upper bound* of A. A is bounded from below if there exists a real number b such that $b \leq x$ for all $x \in A$. Such b is called an *lower bound* of A.

Further, an upper bound of A is called the *least upper bound* of A if it is less than or equal to every upper bound of A. The least upper bound of A is call its *supremum* and denote by $\sup A$. A lower bound of A is called the *greatest lower bound* of A if it is greater than or equal to every lower bound of A. The greatest lower bound of A is call its *infimum* and denote by $\inf A$.

Let A be a nonempty subset of $\mathbb{R}$. Assume that A is bounded from above and bounded from below. Then A is called *bounded.*

Theorem 2.1.5 [23] Let A be a nonempty subset of $\mathbb{R}$. Suppose that A is bounded from above. Then, A has the least upper bound. Simiary, if A is bounded from below, then A has the greatest lower bound.

Definition 2.1.6 [23] Let $\{a_n\}$ be a sequence of real numbers. This sequence $\{a_n\}$ *converges* to 0 if and only if for any $\varepsilon > 0$ there exists a positive integer $n_0 \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for any $n > n_0$ and denote by $a_n \to 0$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = 0$.

Definition 2.1.7 [23] Let $\{a_n\}$ be a sequence of real numbers and let a be a real number. We say that $\{a_n\}$ converges to a if and only if for any $\varepsilon > 0$ there exists a positive integer $n_0 \geq 1$ such that $|a_n - a| < \varepsilon$ for any $n > n_0$ and denote by $a_n \to a$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = a$. The point a is call the *limit* of the sequence $\{a_n\}$. We also say that $\{a_n\}$ is *converges* if there exists a point a in $\mathbb{R}$ such that $|a_n - a| \to 0$. A sequence which does not converge to any real number is said to *diverge.*

Theorem 2.1.8 [9] The limit of a sequence, if it exists, is unique.

Definition 2.1.9 [2] Let $\{a_n\}$ be a sequence of real number. If $\{n_k\}$ is a strictly increasing sequence of natural numbers ($n_1 < n_2 < n_3 < \ldots$), then the sequence $\{a_{n_k} : k \geq 1\}$ is called a *subsequence* of $\{a_n\}$.

Theorem 2.1.10 [9] A sequence $\{a_n\}$ converges if and only if every subsequence $\{a_{n_k}\}$, converges to the same limit.

Remark 2.1.11 [23] If a sequence $\{a_n\}$ converges to a, then every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converges to a.

Lemma 2.1.12 [10] Suppose $\{a_n\}$ is a sequence for which every subsequence $\{a_{n_k}\}$ has a sub of subsequence $\{a_{n_{k_l}}\}$ which converges strongly to X. Then $x_n \to x$ strongly.

Definition 2.1.13 [2] A sequence $\{a_n\}$ is bounded if there exists a positive number M such that

$$|a_n| \leq M \text{ for all } n \geq 1.$$

A sequence that is not bounded is said to be unbounded.

Remark 2.1.14 [23] If $a_n \to a$, then $\{a_n\}$ is bounded.

Theorem 2.1.15 [9] Every convergent sequence is bounded.

Theorem 2.1.16 [9] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences such that for every $n \geq 1$, $a_n \leq b_n \leq c_n$. If $\{a_n\}$ and $\{c_n\}$ both converge to A, then $\{b_n\}$ also converges to A.

Definition 2.1.17 [9] A sequence $\{a_n\}$ is said to be *monotone increasing* if for all $n, m \geq 1$, $n \leq m$ implies $a_n \leq a_m$ and a sequence $\{a_n\}$ is said to be *monotone decreasing* if for all $n, m \geq 1$, $n \leq m$ implies $a_n \geq a_m$.

Definition 2.1.18 [9] A sequence $\{a_n\}$ is said to be *strictly monotone increasing* if for all $n, m \geq 1$, $n < m$ implies $a_n < a_m$ and a sequence $\{a_n\}$ is said to be *strictly monotone decreasing* if for all $n, m \geq 1$, $n < m$ implies $a_n > a_m$.

Theorem 2.1.19 [23] **(The convergence of monotone sequence).**

A monotone is increasing sequence bounded from above is convergent and a monotone decreasing sequence bounded from below is convergent.

Theorem 2.1.20 [9] Let $\{a_n\}_{n=1}^{\infty}$ be a monotone sequence. If $\{a_n\}$ is increasing and bounded above, then $\{a_n\}$ converges to $\sup\{a_n : n \geq 1\}$. If $\{a_n\}$ is decreasing and bounded below, then $\{a_n\}$ converges to $\inf\{a_n : n \geq 1\}$.

Theorem 2.1.21 [23] Let $\{a_n\}$ be a bounded sequence of real numbers and define

$$\alpha_n = \sup\{a_k : k \geq n\} = \sup_{k\geq n} a_k \qquad (n = 1, 2, \ldots).$$

Then, the sequence $\{\alpha_n\}$ is convergent. Similarly, define

$$\beta_n = \inf\{a_k : k \geq n\} = \inf_{k\geq n} a_k \qquad (n = 1, 2, \ldots).$$

Then, the sequence $\{\beta_n\}$ is convergent.

Definition 2.1.22 [4] Let $\{a_n\}$ be a bounded sequence of real numbers and define

$$A = \sup\{a_n : n \in \mathbb{N}\} = \sup_{n\geq 1} a_n \text{ and } B = \inf\{a_n : n \in \mathbb{N}\} = \inf_{n\geq 1} a_n.$$

For any sequence $\{a_n\}$ of real number, B is call the *limit superior* and denote by

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1}\left(\sup_{k\geq n} a_k\right),$$

and A is call the *limit inferior* and denote by

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1}\left(\inf_{k\geq n} a_k\right).$$

Theorem 2.1.23 [23] Let $\{a_n\}$ be a bounded sequence of real numbers. Then, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{k\to\infty} a_{n_k}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Theorem 2.1.24 [23] Let $\{a_n\}$ be a bounded sequence of real numbers. Then, the following holds:

$$\inf_{n\geq 1} a_n \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < \sup_{n\geq 1} a_n.$$

Corollary 2.1.25 [4] If $\{a_n\}$ is a bounded sequence in $\mathbb{R}$, then,

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \Leftrightarrow \{a_n\} \text{ is converges in } \mathbb{R}.$$

When this is the case,

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Theorem 2.1.26 [9] Let $A = \sup\{a_n : n \in \mathbb{N}\}$ and $B = \inf\{a_n : n \in \mathbb{N}\}$. If $\{a_n\}_{n=1}^{\infty}$ convergences to a, then $B \le a \le A$.

Definition 2.1.27 [5] The sequence $\{a_n\}$ is called a *Cauchy sequence* if for every positive number ε, there exists a positive integer $n_0 \ge 1$ such that

$$|a_m - a_n| < \varepsilon \text{ for all } m,n > n_0.$$

Lemma 2.1.28 [2] Every Cauchy sequence is bounded.

Theorem 2.1.29 [23] Let $\{a_n\}$ be a Cauchy sequence. Then $\{a_n\}$ converges.

Theorem 2.1.30 [23] If a sequence $\{a_n\}$ of real numbers converges, then $\{a_n\}$ is a Cauchy sequence.

Theorem 2.1.31 [28] A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

2.2 Metric Space

Definition 2.2.1 [11] A metric space is a pair (X, d), where X is a nonempty set and d is a *metric* on X (or distance function on X), that is, a real valued function defined on $\mathrm{X} \times \mathrm{X}$ such that for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$, we have:

1) $d(x,y) \ge 0$
2) $d(x,y) = 0$ if and only if $x = y$
3) $d(x,y) = d(y,x)$ (symmetry)
4) $d(x,z) \le d(x,y) + d(y,z)$ for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$ (triangle inequality).

A metric space consists of two objects, a nonempty set X and a metric d on X. The value of metric d at (x, y) is called *distance* between x and y, and the order pair (X, d) is called *metric space.*

Remark 2.2.2 [23] In a metric space X, we have for $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$,

$$|d(x,y) - d(x,z)| \leq d(y,z).$$

Definition 2.2.3 [23] Let X and Y be metric space and let f be a mapping of X into Y. Then f is said to be *continuous at* x_0 in X if

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

A mapping f of X into Y is said to be *continuous* if it is continuous at each x in X, that is,

$$x_n \to x \Rightarrow f(x_n) \to f(x).$$

Definition 2.2.4 [1] Let X be a metric space and let $\{x_n\}$ be a sequence of X. Then $\{x_n\}$ is called *Cauchy sequence* if for every $\varepsilon > 0$ there exists an integer $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$.

Theorem 2.2.5 [11] Every convergent sequence in a metric space is a Cauchy sequence.

Definition 2.2.6 [11] If a metric space has the property that all of its Cauchy sequence converges, then the metric spaces is called *complete metric spaces.*

Theorem 2.2.7 [23] Let X be a complete metric space and let A be a subspace of X. Then

A is complete $\Leftrightarrow$ A is closed.

2.3 Banach Space

Definition 2.3.1 [23] **(Vector space or Linear space)** Let E be a nonempty set, and assume that each pair of element x and y in E can be combined by a process call *addition* to yield an element z in E denote by $x+y$. Assume also that this operation of addition satisfies the following condition $(1-4)$:

1) $(x+y)+z=x+(y+z)$
2) $x+y=y+x$
3) There exists a unique element in E denote by 0 and called the *zero element*, such that $x+0=x$ for all $x\in E$
4) To each $x\in E$ there corresponds a unique element in E, denote by $-x$ and call the *negative* of x, such that $x+(-x)=0$

We also assume that each scalar and each element x in E can be combined by a process call *scalar multiplication* to yield an element y in E denote by $y=\alpha x$ satisfying $(5-8)$:

5) $\alpha(\beta x)=(\alpha\beta)x$
6) $1\cdot x=x$
7) $(\alpha+\beta)x=\alpha x+\beta x$
8) $\alpha(x+y)=\alpha x+\alpha y$

The algebraic system E defined by these operations and axims is called a *linear space.* A linear space is often called a *vector space.*

Definition 2.3.2 [23] A normed linear space is a linear space (or vector space) E in which to each vector x, there corresponds a real number, denoted by $\|x\|$ and called the *norm* of x, such that

1) $\|x\|\geq 0$ and $\|x\|=0$ if and only if $x=0$,
2) $\|\alpha x\|=|\alpha|\|x\|$,
3) $\|x+y\|\leq\|x\|+\|y\|$ (triangle inequality).

A normed linear space is also called a *normed space* or *normed vector space* or *normed linear space.*

Theorem 2.3.3 [23] Let E be a norm space. For any $x,\mathrm{y}\in E$, define metric $d(\mathrm{x},\mathrm{y})$ by

$$d(\mathrm{x},\mathrm{y})=\|x-y\|.$$

Then $(\mathrm{E,d})$ is a metric space.

Theorem 2.3.4 [23] The norm is a continuous function, and addition and scalar multiplication are jointly continuous :

1) $x_n \to x \Rightarrow \|x\| \to \|x\|,$
2) $x_n \to x$ and $y_n \to y \Rightarrow \mathrm{x}_n + y_n \to x + y,$
3) $\alpha_n \to \alpha$ and $x_n \to x \Rightarrow \alpha \mathrm{x}_n \to \alpha x.$

Definition 2.3.5. [23] **(Banach Space).** A complete normed space is called a *Banach space.*

Definition 2.3.6. [23] Let E and F be linear space with the same scalars and let T be a mapping of E into $F.$ Then T is called *linear* if for any $x, \mathrm{y} \in E$ and any scalar $\alpha \in \mathbb{R}$,

$$T(x+\mathrm{y}) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x).$$

Definition 2.3.7. [1] A sequence $\{x_n\}$ in a norm space E is said to be convergent to x if $\lim_{n\to\infty} \|x_n - x\| = 0,$ and we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x.$

2.4 Hilbert Space

Definition 2.4.1 [19] **(Inner Product Space).** Let E be a complex vector space. A mapping $\langle \cdot,\cdot \rangle : E \times E \to \mathbb{C}$ is called *inner product* on E if for every $x, y, z \in E$ and $\alpha \in \mathbb{C}$:

1) $\langle x,x \rangle \geq 0$ and $\langle x,x \rangle = 0 \Leftrightarrow x = 0$
2) $\langle \alpha x, y \rangle = \alpha \langle x,y \rangle$
3) $\langle x,y \rangle = \overline{\langle y,x \rangle}$
4) $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle.$

A vector space with an inner product is called an *inner product space.*

Remark 2.4.2 [23] An inner product space is called a *real inner product space* for the case when the scalars are the real numbers and $\langle x, \mathrm{y} \rangle$ is a real number. For the case $(3),$ mean

$$\langle x, \mathrm{y} \rangle = \langle \mathrm{y}, x \rangle$$

Remark 2.4.3 [23] Using $(2),(3)$ and (4), we obtain that for $x,y\in X$ and $\alpha,\beta\in C$,

$$\langle x,\alpha y+\beta z\rangle=\bar{\alpha}\langle x,y\rangle+\bar{\beta}\langle x,z\rangle.$$

Let us prove that the norm satisfies the conditions $(1),(2)$ and (4).

It is obvious that $\|x\|\geq 0$. We also have

$$\|\alpha x\|^2=\langle \alpha x,\alpha x\rangle=\alpha\bar{\alpha}\langle x,x\rangle=|\alpha|^2\|x\|^2,$$

and hence $\|\alpha x\|=|\alpha|\|x\|$.

Definition 2.4.4 [8] By the norm in an inner product space E. We mean the functional defined by $\|x\|=\sqrt{\langle x,x\rangle}$, and a metric on E defined by $d(x,y)=\|x-y\|=\sqrt{\langle x-y,x-y\rangle}$.

Theorem 2.4.5 [8] **(The Schwarz inequality).** For any two element x and y of an inner product space, we have

$$|\langle x,y\rangle|\leq\|x\|\|y\|.$$

Corollary 2.4.6 [8] **(Triangle Inequality).** For any two element x and y of an inner product space, we have

$$\|x+y\|\leq\|x\|+\|y\|.$$

Theorem 2.4.7 [8] **(Parallelogram Law).** For any two element x and y of inner product space H, the following holds:

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.$$

Remark 2.4.8 [8] We know that if the norm of a normed linear space satisfies the parallelogram law, its space becomes an inner product space.

Theorem 2.4.9 [23] The inner product in an inner product space H is jointly continuous:

$$x_n\to x \text{ and } y_n\to y\Rightarrow\langle x_n,y_n\rangle\to\langle x,y\rangle.$$

Remark 2.4.10 [23] We of course obtain from Theorem 2.4.9 that if $x_n \to x,$ then for a fixed $y \in H,$

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ and } \langle y, x_n\rangle \to \langle y, x\rangle.$$

Definition 2.4.11 [8] **(Hillbert Space).** A complete inner product space is called a *Hillbert space.*

Definition 2.4.12 [8] **(Strong convergence).**Let $\{x_n\}$ be a sequence of vector in an inner product space E is called *strong convergence* to a vector x in E if $\|x_n - x\| \to 0$ as $n \to \infty$ and denoted by $x_n \to x.$

Definition 2.4.13 [8] **(Weak convergence).** Let $\{x_n\}$ be a sequence of vector in an inner product space E is called *weak convergence* to a vector x in E if $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$ for every $y \in E$ and denoted by $x_n \rightharpoonup x.$

Remark 2.4.14 [23] If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y,$ then $x = y.$ In fact, we have

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle\\ &= \langle x-x_n+x_n-y, x-y\rangle\\ &= \langle x-x_n, x-y\rangle + \langle x_n-y, x-y\rangle \to 0.\end{aligned}$$

Lemma 2.4.15 [23] Let H be an inner product space, let $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x.$ Then following inequality holds:

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

Lemma 2.4.16 [23] Let $\{x_n\}$ be a Cauchy sequence of an inner product space H such that

$$x_n \rightharpoonup x \text{ Then } x_n \to x.$$

Theorem 2.4.17 [8] A strong convergent sequence is weakly convergent (to the same limit), i.e.,

$$x_n \to x \text{ implies } x_n \rightharpoonup x.$$

Theorem 2.4.18 [8] If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|,$ then $x_n \to x.$

Theorem 2.4.19 [8] Weak convergent sequence in a Hilbert space are bounded, i.e., if $\{x_n\}$ is a weakly convergent sequence, then there exists a number M such that

$$\|x_n\| \le M \qquad \text{for all } n \in \mathbb{N}.$$

Theorem 2.4.20 [23] **(The nearest point theorem).** Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $x \in H$. Then there exists a unique element y_0 in C such that

$$d(x,C) = d(x,y_0).$$

where $d(x,C) = \inf_{y \in C} \|x-y\|$.

Definition 2.4.21 [23] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then for each point x in H, there corresponds a unique point x_0 in C such that

$$\|x-x_0\| = d(x,C).$$

We called such a mapping defined by $Px = x_0$, or $P_C x = x_0$, is the *metric projection* of H onto C.

Lemma 2.4.22 [23] Let C be a nonempty convex subset of a Hilbert space H. Then for $x \in H$ and $y \in C, \|x-y\| = d(x,C)$ if and only if

$$\langle x-y, y-z \rangle \ge 0 \qquad \text{for all } z \in C.$$

Theorem 2.4.23 [23] **(Properties of metric projection).** Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then the metric projection P of H onto C has the following properties (1) and (2):

1) P is a mapping of H onto C and $P^2 = P$,
2) $\|Px - Py\| \le \|x-y\|$ for all $x, y \in H$.

Theorem 2.4.24 [23] Let H be a Hilbert space and let $\{x_n\}$ be a bounded sequence of H. Then $\{x_n\}$ is weakly convergent if and only if each weakly convergent subsequence of $\{x_n\}$ has the same weak limit, that is, for $x \in H$.

$$x_n \rightharpoonup x \Leftrightarrow \left(x_{n_i} \rightharpoonup y \Rightarrow x = y\right).$$

Theorem 2.4.25 [23] Let H be an inner product space and let $x \in H$. Then

$$\|x\| = \sup_{\|y\|\le 1} \left|\langle x, y\rangle\right|.$$

Theorem 2.4.26 [23] Let H be a Hilbert space and let $\{x_n\}$ be a sequence of H. If $x_n \rightharpoonup x$, then $\{x_n\}$ is bounded.

Theorem 2.4.27 [23] **(Opial's theorem).** Let H be a Hilbert space and suppose $x_n \rightharpoonup x$. Then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \qquad \text{for all } y \in H \text{ with } x \neq y.$$

2.5 Nonlinear Mappings and Convex Analysis

Theorem 2.5.1 [23] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.5.2 [23] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping C into itself. Then $F(T)$ is closed and convex.

Definition 2.5.3 [8] **(Convex Sets).** A set U in a vector space is call *convex* if for any $x, y \in U$ and $\alpha \in (0,1)$ we have

$$\alpha x + (1-\alpha) y \in U.$$

Definition 2.5.4 [8] **(Convex Functions).** A function $f : E \to \mathbb{R}$, where E is a vector space, is call *convex* if

$$f\left(tx + (1-t)y\right) \le tf(x) + (1-t) f(y) \qquad \text{for all } x, y \in E \text{ and } t \in (0,1).$$

It is call *strictly convex* if

$$f(tx+(1-t)y)<tf(x)+(1-t)f(y) \quad \text{for all } x,y\in E \text{ and } t\in(0,1).$$

Definition 2.5.5 [23] **(Strongly monotone).**Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let A be an operator of C into H and there exists $\alpha>0$. Then, A is called α - *strongly monotone* if

$$\langle Ax-Ay,x-y\rangle \geq \alpha\|x-y\|^2, \quad \text{for all } x,y\in C.$$

Definition 2.5.6 [23] **(Inverse strongly monotone).**Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let A be an operator of C into H. Then, A is called an *inverse strongly monotone* operator if there exists $\alpha>0$ such that

$$\langle Ax-Ay,x-y\rangle \geq \alpha\|Ax-Ay\|^2, \quad \text{for all } x,y\in C.$$

Such an A is called α - *inverse strongly monotone.*

Remark 2.5.7 [23] If $T:C\to C$ be a nonexpansive mapping, Then $A=I-T$ is $\frac{1}{2}$ - inverse strongly monotone, that is, for all $x,y\in C$, we have,

$$\begin{aligned}\|Ax-Ay\|^2 &\leq \|x-y-(Tx-Ty)\|^2\\ &=\|x-y\|^2-2\langle x-y,Tx-Ty\rangle+\|Tx-Ty\|^2\\ &\leq\|x-y\|^2-2\langle x-y,Tx-Ty\rangle+\|x-y\|^2\\ &=2\langle x-y,x-y-(Tx-Ty)\rangle\\ &=2\langle x-y,Ax-Ay\rangle.\end{aligned}$$

So, A is $\frac{1}{2}-$ inverse strongly monotone.

Theorem 2.5.8. [23] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\alpha>0$ and let $A:C\to H$ be α - inverse strongly monotone. Then $VI(C,A)\neq\varnothing$.

Definition 2.5.9 [15] A mapping $T:C\to C$ is called *nonspreading* if

$$2\|Tx-Ty\|^2\leq\|Tx-y\|^2+\|Ty-x\|^2,\ \forall x,y\in C.$$

Lemma 2.5.10 [15] Let H be a Hilbert space, let C be a nonempty closed convex subset of H , and let T be a nonspreading mapping of C into itself. Then $F(T)$ is closed and convex.

Remark 2.5.11 A nonspreading mapping T with $F(T)\neq\varnothing$ is quasi - nonexpansive mapping T. But the converse is not true.

2.6 Some Useful Lemmas and Theorems

For every $x \in H$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \le \|x - y\|$, $\forall y \in C$. The mapping P_C is called the *metric projection* of H onto C.

Remark 2.6.1 [23] It is well-known that metric projection P_C has the following properties:

1) P_C is *firmly nonexpansive*, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

2) For each $x \in H$,

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \ge 0, \quad \forall y \in C.$$

Theorem 2.6.2 [23] Let H be a Hilbert space, let x and y be elements in H and let $\lambda \in \mathbb{R}$. Then

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

Lemma 2.6.3 [23] Let H be a real Hilbert space. Then, the following inequality holds:

1) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$,
2) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$, for all $x, y \in H$.

Lemma 2.6.4 [22] Let H be a real Hilbert space and let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a mappings of C into H.Then, for $u \in C$ and $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where is the metric projection of H onto C.

Lemma 2.6.5 [26] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\beta_n, \qquad \forall n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_n\}$ is a sequence such that

1) $\sum_{n=1}^{\infty}\alpha_n = \infty,$

2) $\limsup_{n\to\infty}\beta_n \le 0$ or $\sum_{n=1}^{\infty}|\alpha_n\beta_n| < \infty.$

Then, $\lim_{n\to\infty} s_n = 0.$

Lemma 2.6.6 [12] Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α, β - inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \cap VI(C, B) \neq \emptyset.$ Then,

$$VI\big(C, aA + (1-a)B\big) = VI(C, A) \cap VI(C, B), \qquad \forall a \in (0,1).$$

Furthermore, if $0 < \gamma < 2\eta,$ where $\eta = \min\{\alpha, \beta\},$ we have $I - \gamma\big(aA + (1-a)B\big)$ is nonexpansive mapping.

CHAPTER III

MAIN RESULT

In this chapter, we introduce a new iterative scheme for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mappings and the sets of solutions of variational inequality. We obtain the strong convergence theorems of the new iterative schemes in a real Hilbert space.

The following lemmas are useful to prove the main result.

Lemma 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H and let $\tilde{T}:C \to C$ be κ - quasi strictly pseudo - contractive mapping with $F(\tilde{T}) \neq \varnothing$. Define $\tilde{S}:C \to C$ by $\tilde{S}x = cx + (1-c)\tilde{T}x$, for all $x \in C$ and $c \in (\kappa, 1)$. Then,

1) $F(\tilde{S}) = F(\tilde{T})$;
2) $\tilde{S}$ is quasi - nonexpansive mapping;
3) $VI(C, I-\tilde{S}) = F(\tilde{S})$.

Proof. To show (1), from the definition of $\tilde{S}$, we have

$$\begin{aligned} x - \tilde{S}x &= x - cx - (1-c)\tilde{T}x \\ &= (1-c)x - (1-c)\tilde{T}x \\ &= (1-c)(x - \tilde{T}x), \qquad \forall x \in C \text{ and } c \in (\kappa, 1). \end{aligned} \tag{3.1}$$

From (3.1) and $c \in (\kappa, 1)$, we can conclude that $F(\tilde{S}) = F(\tilde{T})$.

To show (2), let $x \in C$ and $z \in F(\tilde{T})$, we have

$$\begin{aligned} \|\tilde{S}x - z\|^2 &= \|c(x-z) + (1-c)(\tilde{T}x - z)\|^2 \\ &= c\|x-z\|^2 + (1-c)\|\tilde{T}x - z\|^2 - c(1-c)\|\tilde{T}x - x\|^2 \\ &\leq c\|x-z\|^2 + (1-c)\left(\|x-z\|^2 + \kappa\|\tilde{T}x - x\|^2\right) - c(1-c)\|\tilde{T}x - x\|^2 \\ &= c\|x-z\|^2 + (1-c)\|x-z\|^2 + (1-c)\kappa\|\tilde{T}x - x\|^2 - c(1-c)\|\tilde{T}x - x\|^2 \end{aligned}$$

$$= c\|x-z\|^2 + (1-c)\|x-z\|^2 + (1-c)\kappa\|\tilde{T}x - x\|^2 - c(1-c)\|\tilde{T}x - x\|^2$$
$$= \|x-z\|^2 - (1-c)(c-\kappa)\|\tilde{T}x - x\|^2$$
$$\leq \|x-z\|^2.$$

Then $\tilde{S}$ is a quasi - nonexpansive mapping.

To show (3), it is easy to see that

$$F(\tilde{S}) \subseteq VI(C, \mathrm{I} - \tilde{S}). \tag{3.2}$$

Let $z \in VI(C, \mathrm{I} - \tilde{S})$ and $z^* \in F(\tilde{S})$, we have

$$\langle y - z, (I - \tilde{S})z \rangle \geq 0, \quad \forall y \in C. \tag{3.3}$$

Put $A = \mathrm{I} - \tilde{S}$. From $\tilde{S}$ is a quasi - nonexpansive mapping, we have

$$\|\tilde{S}z - z^*\|^2 = \|(I-A)z - z^*\|^2$$
$$= \|z - Az - z^*\|^2$$
$$= \|(z - z^*) - Az\|^2$$
$$= \|z - z^*\|^2 - 2\langle z - z^*, Az\rangle + \|Az\|^2$$
$$\leq \|z - z^*\|^2.$$

It implies that

$$\|Az\| \leq 2\langle z - z^*, Az\rangle \leq 0.$$

From (3.3), it implies that $z = \tilde{S}z$, that is, $z \in F(\tilde{S})$. Then,

$$VI(C, \mathrm{I} - \tilde{S}) \subseteq F(\tilde{S}). \tag{3.4}$$

From (3.2) and (3.4), we have $VI(C, I - \tilde{S}) = F(\tilde{S})$.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β - inverse strongly monotone mappings, respectively. Let $\tilde{T} : \mathrm{C} \to \mathrm{C}$ be a κ - quasi strictly pseudo - contractive mapping with $\mathbb{F} = F(\tilde{T}) \cap VI(C, \mathrm{A}) \cap VI(C, B) \neq \varnothing$. Define the mapping $\tilde{S} : C \to C$ by

$\tilde{S}x = cx + (1-c)\tilde{T}x$ for all $x \in C$ and $c \in (\kappa, 1)$. Let $f: C \to C$ be a d - contractive mapping with $d \in \left(0, \frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 \in C$ and

$$x_{n+1} = P_C\left(I - \lambda_n\left(I - \tilde{S}\right)\right)\left(\alpha_n f(x_n) + (1-\alpha_n) P_C\left(I - \rho_n\left(a_n A + (1-a_n)B\right)\right)x_n\right), \qquad (3.5)$$

for all $n \geq 1$, where $\{\alpha_n\}, \{a_n\} \subset [\kappa, 1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii. $\sum_{n=1}^{\infty} \lambda_n < \infty$, and $0 < \lambda < 1$;

iii. $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

iv. $0 < a \leq \rho_n \leq b < 2\eta$, where $\eta = \min\{\alpha, \beta\}$, $\forall n \geq 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} f(z_0)$.

Proof. Put $M_n = \alpha_n f(x_n) + (1-\alpha_n) P_C\left(I - \rho_n\left(a_n A + (1-a_n)B\right)\right)x_n$ and $D_n = a_n A + (1-a_n)B$ for all $n \geq 1$. From (3.5), we can rewrite the sequence $\{x_n\}$ by

$$x_{n+1} = P_C\left(I - \lambda_n\left(I - \tilde{S}\right)\right)M_n, \quad \text{for all } n \geq 1.$$

Let $z \in \mathbb{F}$. From Lemma 2.6.4 and 3.1, we have

$$F\left(\tilde{S}\right) = F\left(\tilde{T}\right) = F\left(P_C\left(I - \lambda_n\left(I - \tilde{S}\right)\right)\right). \qquad (3.6)$$

From Lemma 2.6.4 and 2.6.6, we have

$$VI(C,A) \cap VI(C,B) = F\left(P_C\left(I - \rho_n\left(a_n A + (1-a_n)B\right)\right)\right). \qquad (3.7)$$

Next, we divide the proof into five steps.

Step 1. We show that the sequence $\{x_n\}$ is bounded.

From the definition of $\{M_n\}$, we have

$$\begin{aligned}\left\|\tilde{S}M_n - \tilde{S}z\right\|^2 &= \left\|\left(I - \left(I - \tilde{S}\right)\right)M_n - \left(I - \left(I - \tilde{S}\right)\right)z\right\|^2 \\ &= \left\|(M_n - z) - \left(\left(I - \tilde{S}\right)M_n - \left(I - \tilde{S}\right)z\right)\right\|^2\end{aligned}$$

$$
\begin{aligned}
&= \|M_n - z\|^2 - 2\left\langle M_n - z, (I-\tilde{S})M_n - (I-\tilde{S})z \right\rangle + \left\|(I-\tilde{S})M_n - (I-\tilde{S})z\right\|^2 \\
&= \|M_n - z\|^2 - 2\left\langle M_n - z, (I-\tilde{S})M_n \right\rangle + \left\|(I-\tilde{S})M_n\right\|^2 \\
&\leq \|M_n - z\|^2.
\end{aligned}
$$

It follows that

$$
\left\|(I-\tilde{S})M_n\right\|^2 \leq 2\left\langle M_n - z, (I-\tilde{S})M_n \right\rangle. \tag{3.8}
$$

From the definition of $\{x_n\}$ and (3.8), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|P_C\left(I - \lambda_n\left(I-\tilde{S}\right)\right)M_n - z\right\|^2 \\
&= \left\|P_C\left(I - \lambda_n\left(I-\tilde{S}\right)\right)M_n - P_C\left(I - \lambda_n\left(I-\tilde{S}\right)\right)z\right\|^2 \\
&\leq \left\|\left(I - \lambda_n\left(I-\tilde{S}\right)\right)M_n - \left(I - \lambda_n\left(I-\tilde{S}\right)\right)z\right\|^2 \\
&= \left\|(M_n - z) - \lambda_n\left((I-\tilde{S})M_n - (I-\tilde{S})z\right)\right\|^2 \\
&= \|M_n - z\|^2 - 2\lambda_n\left\langle M_n - z, (I-\tilde{S})M_n \right\rangle + \lambda_n^2\left\|(I-\tilde{S})M_n\right\|^2 \\
&\leq \|M_n - z\|^2 - \lambda_n\left\|(I-\tilde{S})M_n\right\|^2 + \lambda_n^2\left\|(I-\tilde{S})M_n\right\|^2 \\
&\leq \|M_n - z\|^2.
\end{aligned} \tag{3.9}
$$

It implies that

$$
\begin{aligned}
\|x_{n+1} - z\| &\leq \|M_n - z\| \\
&= \left\|\alpha_n f(x_n) + (1-\alpha_n)P_C(I - \rho_n D_n)x_n - z\right\| \\
&= \left\|\alpha_n\left(f(x_n) - z\right) + (1-\alpha_n)\left(P_C(I - \rho_n D_n)x_n - z\right)\right\| \\
&\leq \alpha_n\left\|f(x_n) - z\right\| + (1-\alpha_n)\left\|P_C(I - \rho_n D_n)x_n - z\right\| \\
&\leq \alpha_n\left\|f(x_n) - f(z)\right\| + \alpha_n\left\|f(z) - z\right\| + (1-\alpha_n)\|x_n - z\| \\
&\leq \alpha_n d\|x_n - z\| + \alpha_n\left\|f(z) - z\right\| + (1-\alpha_n)\|x_n - z\| \\
&= \left(1 - \alpha_n(1-d)\right)\|x_n - z\| + \alpha_n\left\|f(z) - z\right\| \\
&\leq \max\left\{\|x_1 - z\|, \left\|\frac{f(z)-z}{1-d}\right\|\right\}.
\end{aligned}
$$

By induction, we can conclude that

$$
\|x_n - z\| \leq \max\left\{\|x_1 - z\|, \left\|\frac{f(z)-z}{1-d}\right\|\right\}, \ \forall \text{n} \geq 1.
$$

This implies that the sequence $\{x_n\}$ is bounded and so is $\{M_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From the definition of $\{x_n\}$, we have

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|P_C\left(I-\lambda_n\left(I-\tilde{S}\right)\right)M_n - P_C\left(I-\lambda_{n-1}\left(I-\tilde{S}\right)\right)M_{n-1}\right\| \\
&\le \left\|\left(I-\lambda_n\left(I-\tilde{S}\right)\right)M_n - \left(I-\lambda_n\left(I-\tilde{S}\right)\right)M_n\right\| \\
&= \left\|M_n - \lambda_n\left(I-\tilde{S}\right)M_n - M_{n-1} + \lambda_{n-1}\left(I-\tilde{S}\right)M_{n-1})\right\| \\
&\le \|M_n - M_{n-1}\| + \left\|\lambda_n\left(I-\tilde{S}\right)M_n - \lambda_{n-1}\left(I-\tilde{S}\right)M_{n-1}\right\| \\
&\le \|M_n - M_{n-1}\| + \left\|\lambda_n\left(I-\tilde{S}\right)M_n - \lambda_n\left(I-\tilde{S}\right)M_{n-1}\right\| + \\
&\quad \left\|\lambda_n\left(I-\tilde{S}\right)M_{n-1} - \lambda_{n-1}\left(I-\tilde{S}\right)M_{n-1}\right\| \\
&= \|M_n - M_{n-1}\| + \lambda_n\left\|\left(I-\tilde{S}\right)M_n - \left(I-\tilde{S}\right)M_{n-1}\right\| + \\
&\quad |\lambda_n - \lambda_{n-1}|\left\|\left(I-\tilde{S}\right)M_{n-1}\right\|.
\end{aligned}
\tag{3.10}
$$

From the definition of $\{M_n\}$, we have

$$
\begin{aligned}
\|M_n - M_{n-1}\| &= \|\alpha_n f(x_n) + (1-\alpha_n)P_C(I-\rho_n D_n)x_n - \alpha_{n-1}f(x_{n-1}) - \\
&\quad (1-\alpha_{n-1})P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\
&\le \|\alpha_n f(x_n) - \alpha_{n-1}f(x_{n-1})\| + \\
&\quad \|(1-\alpha_n)P_C(I-\rho_n D_n)x_n - (1-\alpha_{n-1})P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\
&\le \|\alpha_n f(x_n) - \alpha_n f(x_{n-1})\| + \|\alpha_n f(x_{n-1}) - \alpha_{n-1}f(x_{n-1})\| + \\
&\quad \|(1-\alpha_n)P_C(I-\rho_n D_n)x_n - (1-\alpha_n)P_C(I-\rho_n D_n)x_{n-1}\| + \\
&\quad \|(1-\alpha_n)P_C(I-\rho_n D_n)x_{n-1} - (1-\alpha_n)P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad \|(1-\alpha_n)P_C(I-\rho_{n-1}D_{n-1})x_{n-1} - (1-\alpha_{n-1})P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\
&\le \alpha_n d\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + (1-\alpha_n)\|x_n - x_{n-1}\| + \\
&\quad (1-\alpha_n)\|P_C(I-\rho_n D_n)x_{n-1} - P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad |\alpha_n - \alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\
&= (1-\alpha_n(1-d))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \\
&\quad (1-\alpha_n)\|P_C(I-\rho_n D_n)x_{n-1} - P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad |\alpha_n - \alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\|.
\end{aligned}
\tag{3.11}
$$

From the nonexpansive of P_C, we have

$$
\begin{aligned}
\|P_C(I-\rho_n D_n)x_{n-1} - P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| &\le \|(I-\rho_n D_n)x_{n-1} - (I-\rho_{n-1}D_{n-1})x_{n-1}\| \\
&= \|\rho_n D_n x_{n-1} - \rho_{n-1}D_{n-1}x_{n-1}\| \\
&\le \|\rho_n D_n x_{n-1} - \rho_n D_{n-1}x_{n-1}\| + \\
&\quad \|\rho_n D_{n-1}x_{n-1} - \rho_{n-1}D_{n-1}x_{n-1}\|
\end{aligned}
$$

$$\leq \rho_n \|D_n x_{n-1} - D_{n-1}x_{n-1}\| + |\rho_n - \rho_{n-1}|\|D_{n-1}x_{n-1}\|. \tag{3.12}$$

From the definition of $\{D_n\}$, we have

$$\|D_n x_{n-1} - D_{n-1}x_{n-1}\| = \|(a_n A + (1-a_n)B)x_{n-1} - (a_{n-1}A + (1-a_{n-1})B)x_{n-1}\| \\ \leq |a_n - a_{n-1}|\|Ax_{n-1}\| + |a_n - a_{n-1}|\|Bx_{n-1}\|. \tag{3.13}$$

From (3.10), (3.11), (3.12) and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|M_n - M_{n-1}\| + \lambda_n \|(I-\tilde{S})M_n - (I-\tilde{S})M_{n-1}\| + \\
&\quad |\lambda_n - \lambda_{n-1}|\|(I-\tilde{S})M_{n-1}\| \\
&\leq (1-\alpha_n(1-d))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \\
&\quad (1-\alpha_n)\|P_C(I-\rho_n D_n)x_{n-1} - P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad |\alpha_n - \alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad \lambda_n \|(I-\tilde{S})M_n - (I-\tilde{S})M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-\tilde{S})M_{n-1}\| \\
&\leq (1-\alpha_n(1-d))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \\
&\quad (1-\alpha_n)(\rho_n\|D_n x_{n-1} - D_{n-1}x_{n-1}\| + |\rho_n - \rho_{n-1}|\|D_{n-1}x_{n-1}\|) + \\
&\quad |\alpha_n - \alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad \lambda_n \|(I-\tilde{S})M_n - (I-\tilde{S})M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-\tilde{S})M_{n-1}\| \\
&= (1-\alpha_n(1-d))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \\
&\quad (1-\alpha_n)\rho_n\|D_n x_{n-1} - D_{n-1}x_{n-1}\| + (1-\alpha_n)|\rho_n - \rho_{n-1}|\|D_{n-1}x_{n-1}\| + \\
&\quad |\alpha_n - \alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad \lambda_n \|(I-\tilde{S})M_n - (I-\tilde{S})M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-\tilde{S})M_{n-1}\| \\
&\leq (1-\alpha_n(1-d))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \\
&\quad (1-\alpha_n)\rho_n|a_n - a_{n-1}|\|Ax_{n-1}\| + (1-\alpha_n)\rho_n|a_n - a_{n-1}|\|Bx_{n-1}\| + \\
&\quad (1-\alpha_n)|\rho_n - \rho_{n-1}|\|D_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad \lambda_n \|(I-\tilde{S})M_n - (I-\tilde{S})M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-\tilde{S})M_{n-1}\| \\
&\leq (1-\alpha_n(1-d))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \\
&\quad b|a_n - a_{n-1}|\|Ax_{n-1}\| + b|a_n - a_{n-1}|\|Bx_{n-1}\| + \\
&\quad |\rho_n - \rho_{n-1}|\|D_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| + \\
&\quad \lambda_n \|(I-\tilde{S})M_n - (I-\tilde{S})M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-\tilde{S})M_{n-1}\| \\
&\leq (1-\alpha_n(1-d))\|x_n - x_{n-1}\| + 2K|\alpha_n - \alpha_{n-1}| \\
&\quad 2K|a_n - a_{n-1}| + |\rho_n - \rho_{n-1}|K + \lambda_n K + |\lambda_n - \lambda_{n-1}|K,
\end{aligned}$$

where $K := \max_{n\in\mathbb{N}}\left\{\left\|f\left(x_n\right)\right\|, b\left\|Ax_n\right\|, b\left\|Bx_n\right\|, \left\|\dot{D}_n x_n\right\|, \left\|P_C\left(I-\rho_n D_n\right)x_n\right\|,\right.$

$$\left.\left\|\left(I-\tilde{S}\right)M_n-\left(I-\tilde{S}\right)M_{n-1}\right\|, \left\|\left(I-\tilde{S}\right)M_n\right\|\right\}.$$

From the conditions (i), (ii), (iii) and Lemma 2.6.5, we can conclude that

$$\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=0. \tag{3.14}$$

From the definition of x_n and (3.14), we have

$$\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n\left(I-\tilde{S}\right)\right)M_n-x_n\right\|=0. \tag{3.15}$$

Step 3. We show that $\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n\left(I-\tilde{S}\right)\right)M_n-M_n\right\|=0.$

From the definition of M_n and (3.9), we have

$$\begin{aligned}\left\|x_{n+1}-z\right\|^2 &\le \left\|M_n-z\right\|^2\\ &=\left\|\alpha_n\left(f\left(x_n\right)-z\right)+\left(1-\alpha_n\right)\left(P_C\left(I-\rho_n D_n\right)x_n-z\right)\right\|^2\\ &\le \alpha_n\left\|f\left(x_n\right)-z\right\|^2+\left(1-\alpha_n\right)\left\|P_C\left(I-\rho_n D_n\right)x_n-z\right\|^2.\end{aligned} \tag{3.16}$$

From the definition of D_n and the condition (iv), we have

$$\begin{aligned}\left\langle D_n x_n-D_n z, x_n-z\right\rangle &=\left\langle\left(a_n A+(1-a_n)B\right)x_n-\left(a_n A+(1-a_n)B\right)z, x_n-z\right\rangle\\ &=a_n\left\langle Ax_n-Az, x_n-z\right\rangle+\left(1-a_n\right)\left\langle Bx_n-Bz, x_n-z\right\rangle\\ &\ge a_n\alpha\left\|Ax_n-Az\right\|^2+\left(1-a_n\right)\beta\left\|Bx_n-Bz\right\|^2\\ &\ge \eta\left\|a_n\left(Ax_n-Az\right)+\left(1-a_n\right)\left(Bx_n-Bz\right)\right\|^2\\ &=\eta\left\|D_n x_n-D_n z\right\|^2.\end{aligned} \tag{3.17}$$

From (3.17), we have

$$\begin{aligned}\left\|P_C\left(I-\rho_n D_n\right)x_n-z\right\|^2 &=\left\|P_C\left(I-\rho_n D_n\right)x_n-P_C\left(I-\rho_n D_n\right)z\right\|^2\\ &\le\left\|\left(I-\rho_n D_n\right)x_n-\left(I-\rho_n D_n\right)z\right\|^2\\ &=\left\|\left(x_n-z\right)-\left(\rho_n D_n x_n-\rho_n D_n z\right)\right\|^2\\ &=\left\|x_n-z\right\|^2-2\rho_n\left\langle x_n-z, D_n x_n-D_n z\right\rangle+\rho_n^2\left\|D_n x_n-D_n z\right\|^2\\ &\le\left\|x_n-z\right\|^2-2\rho_n\eta\left\|D_n x_n-D_n z\right\|^2+\rho_n^2\left\|D_n x_n-D_n z\right\|^2\\ &=\left\|x_n-z\right\|^2-\rho_n\left(2\eta-\rho_n\right)\left\|D_n x_n-D_n z\right\|^2.\end{aligned} \tag{3.18}$$

Substitute (3.18) into (3.16), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\|x_n-z\|^2-(1-\alpha_n)\rho_n(2\eta-\rho_n)\|D_nx_n-D_nz\|^2\\ &\le \alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-(1-\alpha_n)\rho_n(2\eta-\rho_n)\|D_nx_n-D_nz\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}(1-\alpha_n)\rho_n(2\eta-\rho_n)\|D_nx_n-D_nz\|^2 \le \alpha_n\|f(x_n)-z\|^2 +\\ \|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|). \qquad (3.19)\end{aligned}$$

From the conditions (i), (iv), (3.14) and (3.19), we have

$$\lim_{n\to\infty}\|D_nx_n-D_nz\|=0. \qquad (3.20)$$

Since P_C, is firmly nonexpansive, we have

$$\begin{aligned}\|P_C(I-\rho_nD_n)x_n-z\|^2 &= \|P_C(I-\rho_nD_n)x_n-P_C(I-\rho_nD_n)z\|^2\\ &\le \langle (I-\rho_nD_n)x_n-(I-\rho_nD_n)z, P_C(I-\rho_nD_n)x_n-z\rangle\\ &= \frac{1}{2}\Big(\|(I-\rho_nD_n)x_n-(I-\rho_nD_n)z\|^2+\|P_C(I-\rho_nD_n)x_n-z\|^2-\\ &\quad \|(I-\rho_nD_n)x_n-(I-\rho_nD_n)z-P_C(I-\rho_nD_n)x_n+z\|^2\Big)\\ &\le \frac{1}{2}\Big(\|x_n-z\|^2+\|P_C(I-\rho_nD_n)x_n-z\|^2-\\ &\quad \|(x_n-P_C(I-\rho_nD_nx_n))-\rho_n(D_nx_n-D_nz)\|^2\Big)\\ &= \frac{1}{2}\Big(\|x_n-z\|^2+\|P_C(I-\rho_nD_n)x_n-z\|^2-\|x_n-P_C(I-\rho_nD_n)x_n\|^2+\\ &\quad 2\rho_n\langle x_n-P_C(I-\rho_nD_n)x_n, D_nx_n-D_nz\rangle-\rho_n^2\|D_nx_n-D_nz\|^2\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\|P_C(I-\rho_nD_n)x_n-z\|^2 \le \|x_n-z\|^2-\|x_n-P_C(I-\rho_nD_n)x_n\|^2+\\ 2\rho_n\langle x_n-P_C(I-\rho_nD_n)x_n, D_nx_n-D_nz\rangle. \qquad (3.21)\end{aligned}$$

Substitute (3.21) into (3.16), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\|x_n-z\|^2-(1-\alpha_n)\|x_n-P_C(I-\rho_nD_n)x_n\|^2+\\ &\quad (1-\alpha_n)2\rho_n\langle x_n-P_C(I-\rho_nD_n)x_n, D_nx_n-D_nz\rangle\\ &\le \alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-(1-\alpha_n)\|x_n-P_C(I-\rho_nD_n)x_n\|^2+\\ &\quad 2\rho_n\|x_n-P_C(I-\rho_nD_n)x_n\|\|D_nx_n-D_nz\|,\end{aligned}$$

which implies that

$$(1-\alpha_n)\|x_n - P_C(I-\rho_n D_n)x_n\|^2 \le \alpha_n \|f(x_n)-z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|) + 2\rho_n \|x_n - P_C(I-\rho_n D_n)x_n\|\|D_n x_n - D_n z\|. \quad (3.22)$$

From the condition (i), (3.14), (3.20) and (3.22), we have

$$\lim_{n\to\infty}\|x_n - P_C(I-\rho_n D_n)x_n\| = 0. \quad (3.23)$$

From the definition of M_n, we have

$$\begin{aligned}\|M_n - x_n\| &= \|\alpha_n(f(x_n)-x_n) + (1-\alpha_n)(P_C(I-\rho_n D_n)x_n - x_n)\| \\ &\le \alpha_n\|f(x_n)-x_n\| + (1-\alpha_n)\|P_C(I-\rho_n D_n)x_n - x_n\|. \quad (3.24)\end{aligned}$$

From the condition (i), (3.23) and (3.24), we have

$$\lim_{n\to\infty}\|M_n - x_n\| = 0. \quad (3.25)$$

From (3.15) and (3.25), we can conclude that

$$\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n\left(I-\tilde{S}\right)\right)M_n - M_n\right\| = 0. \quad (3.26)$$

Step 4. We show that $\limsup_{n\to\infty}\langle f(z_0)-z_0, M_n - z_0\rangle \le 0$, where $z_0 = P_{\mathbb{F}} f(z_0)$.

To show this, take a subsequence $\{M_{n_j}\}$ of $\{M_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0, M_n - z_0\rangle = \lim_{j\to\infty}\langle f(z_0)-z_0, M_{n_j} - z_0\rangle.$$

Since $\{M_n\}$ is bounded, without loss of generality, we may assume that $M_{n_j} \rightharpoonup \omega$ as $j\to\infty$ where $\omega\in C$. First, we show that $\omega\in F(\tilde{T})$. From Lemma 2.6.4 and 3.1, we have $F(\tilde{T}) = F(\tilde{S}) = VI(C, I-\tilde{S}) = F\left(P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)\right)$. Assume that $\omega\notin F(\tilde{T})$, then we have $\omega \ne P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)\omega$. From the condition (ii), (3.26) and Opial's property, we have

$$\begin{aligned}\liminf_{j\to\infty}\|M_{n_j}-\omega\| &< \liminf_{j\to\infty}\left\|M_{n_j} - P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)\omega\right\| \\ &\le \liminf_{j\to\infty}\Big(\left\|M_{n_j} - P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)M_{n_j}\right\| + \\ &\quad \left\|P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)M_{n_j} - P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)\omega\right\|\Big)\end{aligned}$$

$$\leq \liminf_{j\to\infty}\left(\left\|M_{n_j} - P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)M_{n_j}\right\| + \left\|\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)M_{n_j} - \left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)\omega\right\|\right)$$

$$\leq \liminf_{j\to\infty}\left(\left\|M_{n_j} - P_C\left(I-\lambda_{n_j}\left(I-\tilde{S}\right)\right)M_{n_j}\right\| + \left\|M_{n_j}-\omega\right\| + \lambda_{n_j}\left\|(I-\tilde{S})M_{n_j} - (I-\tilde{S})\omega\right\|\right)$$

$$= \liminf_{j\to\infty}\left\|M_{n_j}-\omega\right\|.$$

This is a contradiction. Then, we have

$$\omega \in F\left(\tilde{T}\right). \tag{3.27}$$

Next, we show that $\omega \in VI(C,A)\cap VI(C,B)$. From Lemma 2.6.4 and 2.6.6, we have $VI(C,A)\cap VI(C,B) = F\left(P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)\right)$. Assume that $\omega \notin VI(C,A)\cap VI(C,B)$, then we have $\omega \neq P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)\omega$. From $M_{n_j} \rightharpoonup \omega$ as $j\to\infty$ and (3.25), we have $x_{n_j} \rightharpoonup \omega$ as $j\to\infty$. From Lemma 2.6.6, (3.23) and Opial's property, we have

$$\liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\| < \liminf_{j\to\infty}\left\|x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)\omega\right\|$$

$$\leq \liminf_{j\to\infty}\left(\left\|x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)x_{n_j}\right\| + \left\|P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)\omega\right\|\right)$$

$$\leq \liminf_{j\to\infty}\left(\left\|x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)x_{n_j}\right\| + \left\|x_{n_j}-\omega\right\|\right)$$

$$= \liminf_{j\to\infty}\left\|x_{n_j}-\omega\right\|.$$

This is a contradiction. Then, we have

$$\omega \in VI(C,A)\cap VI(C,B). \tag{3.28}$$

From (3.27) and (3.28), we can conclude that $\omega \in \mathbb{F}$.

Since $\omega \in \mathbb{F}$ and $M_{n_j} \rightharpoonup \omega$ as $j\to\infty$, we have

$$\limsup_{n\to\infty}\left\langle f(z_0)-z_0, M_n - z_0\right\rangle = \lim_{j\to\infty}\left\langle f(z_0)-z_0, M_{n_j}-z_0\right\rangle = \left\langle f(z_0)-z_0, \omega - z_0\right\rangle \leq 0. \tag{3.29}$$

Step 5. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} f(z_0)$.

From the definition of x_n and (3.9), we have

$$
\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le \|M_n - z_0\|^2 \\
&= \|\alpha_n (f(x_n) - z_0) + (1-\alpha_n)(P_C (I - \rho_n D_n) x_n - z_0)\|^2 \\
&\le (1-\alpha_n)^2 \|P_C (I - \rho_n D_n) x_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - z_0, M_n - z_0 \rangle \\
&\le (1-\alpha_n)^2 \|P_C (I - \rho_n D_n) x_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - f(z_0), M_n - z_0 \rangle + \\
&\quad 2\alpha_n \langle f(z_0) - z_0, M_n - z_0 \rangle \\
&\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n d \|x_n - z_0\| \|M_n - z_0\| + 2\alpha_n \langle f(z_0) - z_0, M_n - z_0 \rangle \\
&= (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n d \|x_n - z_0\| \|\alpha_n (f(x_n) - z_0) + \\
&\quad (1-\alpha_n)(P_C (I - \rho_n D_n) x_n - z_0)\| + 2\alpha_n \langle f(z_0) - z_0, M_n - z_0 \rangle \\
&\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n d \|x_n - z_0\| (\alpha_n \|f(x_n) - z_0\| + \\
&\quad (1-\alpha_n) \|P_C (I - \rho_n D_n) x_n - z_0\|) + 2\alpha_n \langle f(z_0) - z_0, M_n - z_0 \rangle \\
&\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n^2 d \|x_n - z_0\| \|f(x_n) - z_0\| + \\
&\quad 2\alpha_n d (1-\alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle f(z_0) - z_0, M_n - z_0 \rangle \\
&\le (1-\alpha_n) \|x_n - z_0\|^2 + \alpha_n (\alpha_n \|x_n - z_0\| \|f(x_n) - z_0\| + \langle f(z_0) - z_0, M_n - z_0 \rangle).
\end{aligned}
$$

From the condition (i), (3.29) and applying Lemma 2.6.5, we can conclude that $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} f(z_0)$. This completes the proof.

CHAPTER IV

APPLICATIONS AND NUMERICAL RESULT

In this chapter, by using the main result, we prove the strong convergence theorems involving the set of fixed points of quasi - nonexpansive mappings and nonspreading mappings. Moreover, we give a numerical example with guarantee convergence for the main result.

4.1 Strong Convergence of κ - Quasi Strictly Pseudo - Contractive Mappings

Corollary 4.1 Let C be a nonempty closed convex subset of a real Hilbert space H and let $A:C\to H$ be α - inverse strongly monotone mapping. Let $\tilde{T}:C\to C$ be a κ - quasi strictly pseudo - contractive mapping with $\mathbb{F}=F\left(\tilde{T}\right)\cap VI\left(C,A\right)\neq\varnothing$. Define the mapping $\tilde{S}:C\to C$ by $\tilde{S}x=cx+(1-c)\tilde{T}x$ for all $x\in C$ and $c\in(\kappa,1)$. Let $f:C\to C$ be a d - contractive mapping with $d\in\left(0,\frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1\in C$ and

$$x_{n+1}=P_C\left(I-\lambda_n\left(I-\tilde{S}\right)\right)\left(\alpha_n f\left(x_n\right)+\left(1-\alpha_n\right)P_C\left(I-\rho_n A\right)x_n\right),$$

for all $n\geq 1$, where $\{\alpha_n\}\subset[\kappa,1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;

ii. $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda<1$;

iii. $\sum_{n=1}^{\infty}\left|\lambda_{n+1}-\lambda_n\right|<\infty,\ \sum_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_n\right|<\infty,\ \sum_{n=1}^{\infty}\left|\rho_{n+1}-\rho_n\right|<\infty$;

iv. $0<a\leq\rho_n\leq b<2\alpha,\ \forall n\geq 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathbb{F}}f\left(z_0\right)$.

Proof. Put $A\equiv B$ in Theorem 3.2., we obtain the desired result.

4.2 Strong Convergence of Quasi - Nonexpansive Mappings

Theorem 4.2 Let C be a nonempty closed convex subset of a real Hilbert space H and let $A,B:C\to H$ be α and β - inverse strongly monotone mappings, respectively. Let $T:\mathrm{C}\to\mathrm{C}$ be a quasi - nonexpansive mapping with $\mathbb{F}=F(T)\cap VI(C,\mathrm{A})\cap VI(C,B)\neq\varnothing$. Let $f:C\to C$ be a d - contractive mapping with $d\in\left(0,\frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1\in C$ and

$$x_{n+1}=P_C\left(I-\lambda_n\left(I-T\right)\right)\left(\alpha_n f\left(x_n\right)+\left(1-\alpha_n\right)P_C\left(I-\rho_n\left(a_n A+\left(1-a_n\right)B\right)\right)x_n\right),$$

for all $n\geq 1$, where $\{\alpha_n\},\{a_n\}\subset[\kappa,1]$. Assume the following conditions hold:

i. $\lim\limits_{n\to\infty}\alpha_n=0$ and $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$;

ii. $\sum\limits_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n<1$;

iii. $\sum\limits_{n=1}^{\infty}\left|\lambda_{n+1}-\lambda_n\right|<\infty,\ \sum\limits_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_n\right|<\infty,$
$\sum\limits_{n=1}^{\infty}\left|\rho_{n+1}-\rho_n\right|<\infty,\ \sum\limits_{n=1}^{\infty}\left|a_{n+1}-a_n\right|<\infty;$

iv. $0<a\leq\rho_n\leq b<2\eta$, where $\eta=\min\{\alpha,\beta\}$, $\forall n\geq 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathbb{F}}f\left(z_0\right)$.

Proof. By using Theorem 3.2., we can conclude the desired conclusion.

4.3 Strong Convergence of Finite Family of Nonspreading Mappings

In 2009, Kangtunyakarn and Suantai [14] introduced the S - mapping generated by $T_1,T_2,\dots,T_N$ and $\alpha_1,\alpha_2,\dots,\alpha_N$ as follows:

Definition 4.3 Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself. For each $j=1,2,\dots,N,$ and let $\alpha_j=\left(\alpha_1^j,\alpha_2^j,\alpha_3^j\right)\in I\times I\times I,$ where $I\in[0,1]$ and $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$. Define the mapping $S:C\to C$ as follows:

$$
\begin{aligned}
&U_0 = I,\\
&U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
&U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
&U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\quad\vdots\\
&U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
&S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}
$$

This mapping is called an S - *mapping* generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.4 [13] Let C be a nonempty closed convex subset of real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1], \alpha_3^N \in [0,1), \alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is quasi - nonexpansive mapping.

By using Theorem 4.2. and Lemma 4.4, we obtain the following theorem:

Theorem 4.5 Let C be a nonempty closed convex subset of real Hilbert space H and let $A, B : C \to H$ be α and β - inverse strongly monotone mappings, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\mathbb{F} = \bigcap_{i=1}^N F(T_i) \cap VI(C, \mathrm{A}) \cap VI(C, B) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1], \alpha_3^N \in [0,1) \alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $f : C \to C$ be a d - contractive mapping with $d \in \left(0, \frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 \in C$ and

$$x_{n+1} = P_C\left(I - \lambda_n (I - S)\right)\left(\alpha_n f(x_n) + (1 - \alpha_n) P_C\left(I - \rho_n\left(a_n A + (1 - a_n) B\right)\right) x_n\right),$$

for all $n \geq 1$, where $\{\alpha_n\}, \{a_n\} \subset [\kappa, 1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii. $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n<1$;

iii. $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty,\ \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,$

$\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty,\ \sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty;$

iv. $0<a\le\rho_n\le b<2\eta$, where $\eta=\min\{\alpha,\beta\}$, $\forall n\ge 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathbb{F}}f(z_0)$.

4.4 Strong Convergence of Nonspreading Mappings

The following result is a direct consequence from Theorem 4.5.

Corollary 4.6 Let C be a nonempty closed convex subset of real Hilbert space H and let $A,B:C\to H$ be α and β - inverse strongly monotone mappings, respectively. Let $\hat{T}$ be a nonspreading mappings of C into C with $\mathbb{F}=F(\hat{T})\cap VI(C,A)\cap VI(C,B)\neq\varnothing$. Let $f:C\to C$ be a d - contractive mapping with $d\in\left(0,\frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1\in C$ and

$$x_{n+1}=P_C\left(I-\lambda_n\left(I-\hat{T}\right)\right)\left(\alpha_n f(x_n)+(1-\alpha_n)P_C\left(I-\rho_n\left(a_nA+(1-a_n)B\right)\right)x_n\right),$$

for all $n\ge 1$, where $\{\alpha_n\},\{a_n\}\subset[\kappa,1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;

ii. $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n<1$;

iii. $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty,\ \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,$

$\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty,\ \sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty;$

iv. $0<a\le\rho_n\le b<2\eta$, where $\eta=\min\{\alpha,\beta\}$, $\forall n\ge 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathbb{F}}f(z_0)$.

Proof. Put $N=1$ and $T_1=\hat{T}$ in Theorem 4.5. Then we can conclude the desired conclusion.

4.5 Numerical Result

Example 4.7 Let $\mathbb{R}$ be the set of real numbers and let the mappings $A,B:\mathbb{R}\to\mathbb{R}$ be defined by $Ax=\frac{2(x-1)}{3}$ and $Bx=\frac{2(x-1)}{5}$, for all $x\in\mathbb{R}$, respectively. Let the mapping $\tilde{T}:\mathbb{R}\to\mathbb{R}$ defined by $\tilde{T}x=\frac{2x+1}{3}$ for all $x\in\mathbb{R}$. Define the mapping $\tilde{S}:\mathbb{R}\to\mathbb{R}$ by $\tilde{S}x=cx+(1-c)\tilde{T}x$ for all $x\in\mathbb{R}$ and $c\in(0,1)$. Let $f:\mathbb{R}\to\mathbb{R}$ be a d - contractive mapping with $d\in\left(0,\frac{1}{2}\right)$ and $f(x)=\frac{x}{3}$. Suppose that $\{x_n\}$ is the sequence generated by $x_1\in\mathbb{R}$ and

$$x_{n+1}=\left(I-\lambda_n\left(I-\tilde{S}\right)\right)\left(\alpha_n f(x_n)+(1-\alpha_n)\left(I-\rho_n\left(a_nA+(1-a_n)B\right)\right)x_n\right), \quad (4.1)$$

for all $n\geq 1$, where $\lambda_n=\frac{1}{n^2}, \alpha_n=\frac{1}{n}, a_n=\frac{1}{2n}, \rho_n=\frac{n+2}{2n+1}$.

Then the sequence $\{x_n\}$ converges strongly to 1.

Solution. It is easy to see that the sequences $\{\lambda_n\},\{\alpha_n\},\{a_n\}$ and $\{\rho_n\}$ satisfy the conditions $(i),(ii),(iii)$ and (iv) in the theorem 3.2. and A, B are 1 - inverse strongly monotone mappings with $VI(C,A)\cap VI(C,B)=\{1\}$ and $\tilde{T}$ is a κ - quasi strictly pseudo - contractive mapping with $F(\tilde{T})=1$. Then, we have $VI(C,A)\cap VI(C,B)\cap F(\tilde{T})=\{1\}$. From the theorem 3.2., we can conclude that the sequence $\{x_n\}$ converge strongly to 1.

For numerical results, choose $x_1=-2$ and $x_1=2$ in (4.1), respectively. We can rewrite the sequence $\{x_n\}$ by

$$x_{n+1}=\left(I-\frac{1}{n^2}\left(I-\tilde{S}\right)\right)\left(\frac{1}{n}\left(\frac{x_n}{3}\right)+\left(1-\frac{1}{n}\right)\left(I-\frac{n+2}{2n+1}\left(\frac{1}{2n}A+\left(1-\frac{1}{2n}\right)B\right)\right)x_n\right), \forall n\geq 1. (4.2)$$

The following graphs show the values of the sequence $\{x_n\}$ defined by (4.2), where $x_1 = -2$ and $x_1 = 2$, respectively.

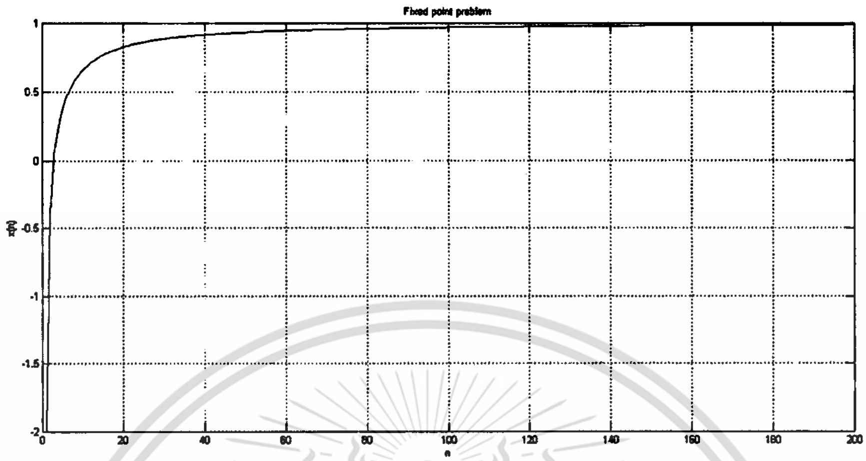

Figure 1: The convergence of the sequence $\{x_n\}$ with initial value $x_1 = -2$ and $n = 200$.

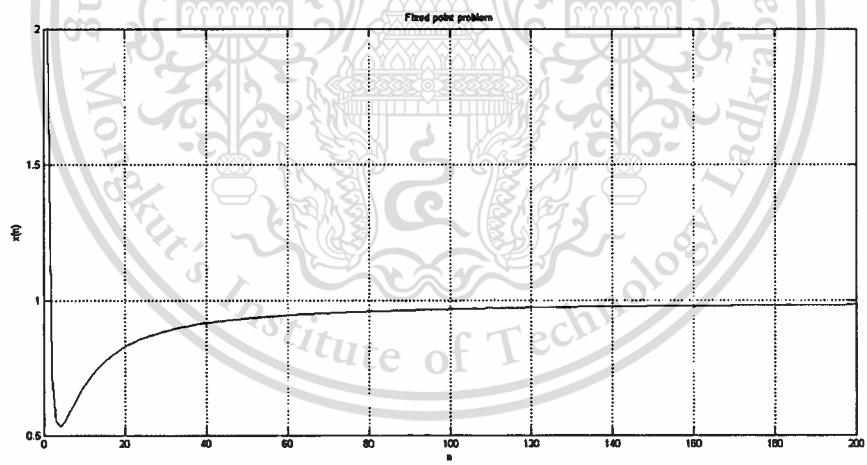

Figure 2: The convergence of the sequence $\{x_n\}$ with initial value $x_1 = 2$ and $n = 200$.

The following table show the value of the sequence $\{x_n\}$ with $n=200$, $x_1=-2$ and $x_1=2$.

Table 1: The values of $\{x_n\}$ with $n=200$, $x_1=-2$ and $x_1=2$.

n	$x_1=-2$	$x_1=2$
	x_n	x_n
1	-2.0000	2.0000
2	-0.3889	0.7222
3	0.0417	0.5528
4	0.2494	0.5334
5	0.3770	0.5503
⋮	⋮	⋮
50	0.9322	0.9322
⋮	⋮	⋮
96	0.9650	0.9650
97	0.9653	0.9653
98	0.9657	0.9657
99	0.9660	0.9660
100	0.9664	0.9664
⋮	⋮	⋮
150	0.9777	0.9777
⋮	⋮	⋮
196	0.9829	0.9829
197	0.9830	0.9830
198	0.9831	0.9831
199	0.9832	0.9832
200	0.9833	0.9833

CHAPTER V

CONCLUSION

In this research, we study the strong convergence theorems in real Hilbert spaces and introduce a new iterative scheme for finding a common element of the set of fixed points of κ - quasi strictly pseudo - contractive mappings and the sets of solutions of variational inequality. We also prove the strong convergence theorem involving the set of fixed point of quasi-nonexpansive mapping T of a nonempty closed convex subset of Hibert spaces into itself without assumptions $T_\omega = (1-\omega)I + \omega T$ where $\omega \in [0,1]$ and T is a demiclosed mapping. Moreover, we prove strong convergence theorems for a finite family of nonspreading mappings. Also, this chapter summarizes the results overall study in chapter III and chapter IV.

(1) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β - inverse strongly monotone mappings, respectively. Let $\tilde{T} : C \to C$ be a κ - quasi strictly pseudo - contractive mapping with $\mathbb{F} = F(\tilde{T}) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Define the mapping $\tilde{S} : C \to C$ by $\tilde{S}x = cx + (1-c)\tilde{T}x$ for all $x \in C$ and $c \in (\kappa, 1)$. Let $f : C \to C$ be a d - contractive mapping with $d \in \left(0, \frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 \in C$ and

$$x_{n+1} = P_C\left(I - \lambda_n\left(I - \tilde{S}\right)\right)\left(\alpha_n f(x_n) + (1-\alpha_n) P_C\left(I - \rho_n\left(a_n A + (1-a_n)B\right)\right)x_n\right),$$

for all $n \geq 1$, where $\{\alpha_n\}, \{a_n\} \subset [\kappa, 1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii. $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$;

iii. $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

iv. $0 < a \le \rho_n \le b < 2\eta$, where $\eta = \min\{\alpha, \beta\}$, $\forall n \ge 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} f(z_0)$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A: C \to H$ be α - inverse strongly monotone mappings. Let $\tilde{T}: C \to C$ be a κ - quasi strictly pseudo - contractive mapping with $\mathbb{F} = F(\tilde{T}) \cap VI(C, A) \neq \varnothing$. Define the mapping $\tilde{S}: C \to C$ by $\tilde{S}x = cx + (1-c)\tilde{T}x$ for all $x \in C$ and $c \in (\kappa, 1)$. Let $f: C \to C$ be a d - contractive mapping with $d \in \left(0, \frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 \in C$ and

$$x_{n+1} = P_C\left(I - \lambda_n\left(I - \tilde{S}\right)\right)\left(\alpha_n f(x_n) + (1-\alpha_n) P_C(I - \rho_n A) x_n\right),$$

for all $n \ge 1$, where $\{\alpha_n\} \subset [\kappa, 1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii. $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$;

iii. $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$;

iv. $0 < a \le \rho_n \le b < 2\alpha$, $\forall n \ge 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} f(z_0)$.

(3) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B: C \to H$ be α and β - inverse strongly monotone mappings, respectively. Let $T: C \to C$ be a quasi - nonexpansive mapping with $\mathbb{F} = F(T) \cap VI(C, A) \cap VI(C, B) \neq \varnothing$. Let $f: C \to C$ be a d - contractive mapping with $d \in \left(0, \frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 \in C$ and

$$x_{n+1} = P_C\left(I - \lambda_n (I - T)\right)\left(\alpha_n f(x_n) + (1-\alpha_n) P_C\left(I - \rho_n\left(a_n A + (1-a_n) B\right)\right) x_n\right),$$

for all $n \ge 1$, where $\{\alpha_n\}, \{a_n\} \subset [\kappa, 1]$. Assume the following conditions hold:

v. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

vi. $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n<1$;

vii. $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$,

$\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty$, $\sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty$;

viii. $0<a\le\rho_n\le b<2\eta$, where $\eta=\min\{\alpha,\beta\}$, $\forall n\ge 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathbb{F}}f(z_0)$.

(4) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A,B:C\to H$ be α and β - inverse strongly monotone mappings, respectively. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into C with $\mathbb{F}=\bigcap_{i=1}^{N}F(T_i)\cap VI(C,A)\cap VI(C,B)\neq\varnothing$ and let $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I$, $j=1,2,\ldots,N$, where $I=[0,1]$, $\alpha_1^j+\alpha_2^j+\alpha_3^j=1, \alpha_1^j,\alpha_3^j\in(0,1)$ for all $j=1,2,\ldots,N-1$ and $\alpha_1^N\in(0,1],\alpha_3^N\in[0,1)\alpha_2^j\in[0,1)$ for all $j=1,2,\ldots,N$. Let S be the mapping generated by $T_1,T_2,\ldots,T_N$ and $\alpha_1,\alpha_2,\ldots,\alpha_N$. Let $f:C\to C$ be a d - contractive mapping with $d\in\left(0,\frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1\in C$ and

$$x_{n+1}=P_C\left(I-\lambda_n(I-S)\right)\left(\alpha_n f(x_n)+(1-\alpha_n)P_C\left(I-\rho_n\left(a_nA+(1-a_n)B\right)\right)x_n\right),$$

for all $n\ge 1$, where $\{\alpha_n\},\{a_n\}\subset[\kappa,1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;

ii. $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $0<\lambda_n<1$;

iii. $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$,

$\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty$, $\sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty$;

iv. $0<a\le\rho_n\le b<2\eta$, where $\eta=\min\{\alpha,\beta\}$, $\forall n\ge 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathbb{F}}f(z_0)$.

(5) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β - inverse strongly monotone mappings, respectively. Let $\hat{T}$ be a nonspreading mappings of C into C with $\mathbb{F} = F(\hat{T}) \cap VI(C, \mathrm{A}) \cap VI(C, B) \neq \varnothing$. Let $f : C \to C$ be a d - contractive mapping with $d \in \left(0, \frac{1}{2}\right)$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 \in C$ and

$$x_{n+1} = P_C\left(I - \lambda_n\left(I - \hat{T}\right)\right)\left(\alpha_n f(x_n) + (1-\alpha_n) P_C\left(I - \rho_n\left(a_n A + (1-a_n)B\right)\right)x_n\right),$$

for all $n \geq 1$, where $\{\alpha_n\}, \{a_n\} \subset [\kappa, 1]$. Assume the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii. $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$;

iii. $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

iv. $0 < a \leq \rho_n \leq b < 2\eta$, where $\eta = \min\{\alpha, \beta\}$, $\forall n \geq 1$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} f(z_0)$.

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APPENDICES

APPENDIX A.

THE RESEARCH PAPER

กระบวนการทำซ้ำสำหรับปัญหาอสมการการแปรผันและผลลัพธ์ทางเชิงตัวเลข

Iteration Method for Variational Inequalities Problems and Numerical Results

ปรียาภรณ์ สืบเกิด[1]*, และ อาทิตย์ แข็งธัญการ[1]

Preeyaporn Surbkird[1]* and Atid Kangtunyakarn[1]

[1]*ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง*

บทคัดย่อ

ในงานวิจัยนี้ เราศึกษาทฤษฎีบทการลู่เข้าแบบเข้มของวิธีการประมาณค่าความหน่วงในปริภูมิฮิลเบิร์ต และยังได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มของกระบวนการทำซ้ำสำหรับการหาสมาชิกร่วมของเซตของปัญหาอสมการการแปรผันสำหรับการส่งแบบผกผันทางเดียว นอกจากนี้โดยการใช้ทฤษฎีบทหลักทำให้เราได้ผลลัพธ์ทางเชิงตัวเลขดังกล่าว

คำสำคัญ : อสมการการแปรผัน / การส่งแบบผกผันทางเดียว / จุดตรึง

ABSTRACT

In this paper, we study the strong convergence theorem with the viscosity approximation method in a real Hilbert spaces. We also prove the strong convergence theorem of a general iterative scheme for finding a common element of the sets of solutions of variational inequalities problems for an inverse-strongly monotone mapping. Moreover, by using our main theorem, we obtain the numerical results.

Key Words : Inverse - strongly monotone / Variational Inequality / Fixed Point

*Corresponding author. E-mail : pypsk@hotmail.com

1. INTRODUCTION

Throughout this paper, we denote weak and strong convergence by notations " $\rightharpoonup$ " and " $\to$ ", respectively. Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of a real Hilbert space H. A self-mapping $T:C \to C$ is called nonexpansive if

$$\|Tx-Ty\| \le \|x-y\|, \qquad \forall x,y \in C. \tag{1.1}$$

Let $f:C \to C$ is a β - contraction on C if there exists a constant $0<\beta<1$ such that

$$\| f(x)-f(y)\| \le \beta \| x-y\|, \qquad \forall x,y \in C. \tag{1.2}$$

A mapping $A:C \to H$ is called α - inverse strongly monotone if there exists a positive real number $\alpha>0$ such that

$$\langle Ax-Ay, x-y\rangle \ge \alpha \|Ax-Ay\|^2, \qquad \forall x,y \in C. \tag{1.3}$$

Let $B:C \to H$. The variational inequality is to find a point $u \in C$ such that

$$\langle Bu, v-u\rangle \ge 0, \qquad \forall v \in C. \tag{1.4}$$

The set of solutions of the variational inequality is denoted by $VI(C,B)$. We use $F(T)=\{x\in C: Tx=x\}$ to denote the set of fixed point of T.

Kangtunyakam (2012) modified the sets of variational inequalities problems as follows:

Lemma 1.1 Let C be a nonempty closed convex subset of a real Hilbert space H and let $A,B:C \to H$ be α,β - inverse strongly monotone mappings, respectively, with $\alpha,\beta>0$ and $VI(C,A)\cap VI(C,B) \ne \emptyset$. Then,

$$VI(C, aA+(1-a)B) = VI(C,A)\cap VI(C,B), \qquad \forall a \in (0,1).$$

Furthermore, if $0<\gamma<2\eta$, where $\eta=\min\{\alpha,\beta\}$, we have $I-\gamma(aA+(1-a)B)$ is nonexpansive mapping.

Motivated by Kangtunyakarn (2012), we study the strong convergence theorem with the viscosity approximation method in a real Hilbert spaces. We also prove the strong convergence theorem of a general iterative scheme for finding a common element of the sets of solutions of variational inequalities problems for an inverse-strongly monotone mapping. Moreover, by using our main theorem, we obtain the numerical results.

2. PRELIMINARIES

Let C be a nonempty closed convex subset of a real Hilbert space H. Then, for every $x \in H$, there exists a unique nearest point in C, denote by $P_C(x)$, such that

$$\|x - P_C x\| \le \|x - y\|, \qquad \forall y \in C. \tag{2.1}$$

The mapping P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping. Moreover, for every $x \in H$ and $z \in C$,

$$z = P_C(x) \iff \langle x - z, z - y \rangle \ge 0, \qquad \forall y \in C. \tag{2.2}$$

Recall that H satisfies Opial's condition (Opial, 1967) i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{2.3}$$

holds for every $y \in H$ with $y \ne x$.

Lemma 2.1 Let H be a real Hilbert space. Then, the following inequality holds:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle, \qquad \forall x, y \in H.$$

Lemma 2.2 (Takahashi, 2000) Let A be a mapping of C into H and $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \iff u \in VI(C, A).$$

Lemma 2.3 (Xu, 2003) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n \beta_n, \qquad \forall n \ge 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are a sequence in $(0,1)$ satisfy the conditions :

$$(i) \ \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad (ii) \ \limsup_{n\to\infty} \beta_n \le 0 \ \ or \ \ \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then, $\lim_{n\to\infty} s_n = 0$.

3. MAIN RESULT

Theorem 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α, β - inverse strongly monotone mappings, respectively. Assume $F = VI(C, A) \cap VI(C, B) \ne \emptyset$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_C\left(I - \rho_n\left(a_n A + (1 - a_n)B\right)\right)x_n, \qquad \forall n \ge 1, \tag{3.1}$$

where $\{\alpha_n\},\{a_n\}\subset(0,1)$. Let $f:C\to C$ is a contractive mapping with $d\in\left(0,\frac{1}{2}\right)$. Assume the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$, (ii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty, \sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty$, (iii) $0<a\le\rho_n\le b<2\eta$, *where* $\eta=\min\{\alpha,\beta\}$.

Then, the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathrm{F}}f(z_0)$.

Proof. Put $D_n=a_nA+(1-a_n)B$ for all $n\in\mathbf{N}$. From (3.1), we can rewrite the sequence $\{x_n\}$ by

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)P_C(I-\rho_nD_n)x_n. \tag{3.2}$$

Let $z\in\mathrm{F}$. From Lemma 1.1 and 2.2, we have

$$VI(C,A)\cap VI(C,B)=\mathrm{F}\left(P_C\left(I-\rho_n\left(a_nA+(1-a_n)B\right)\right)\right). \tag{3.3}$$

Next, we divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

From the definition of $\{x_n\}$, we have

$$\begin{aligned}\|x_{n+1}-z\|&\le\alpha_n\|f(x_n)-z\|+(1-\alpha_n)\|P_C(I-\rho_nD_n)x_n-z\|\\&\le\alpha_nd\|x_n-z\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|x_n-z\|\\&=(1-\alpha_n(1-d))\|x_n-z\|+\alpha_n(1-d)\frac{\|f(z)-z\|}{(1-d)}.\end{aligned} \tag{3.4}$$

By induction, we can conclude that

$$\|x_n-z\|\le\max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{(1-d)}\right\},\qquad\forall n\ge1. \tag{3.5}$$

This implies that the sequence $\{x_n\}$ is bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|\alpha_n f(x_n)-\alpha_{n-1}f(x_{n-1})\|+\|(1-\alpha_n)P_C(I-\rho_n D_n)x_n-(1-\alpha_{n-1})P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\ &\le \alpha_n d\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+(1-\alpha_n)\|x_n-x_{n-1}\| \\ &\quad +(1-\alpha_n)\|P_C(I-\rho_n D_n)x_{n-1}-P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\ &=(1-\alpha_n(1-d))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\ &\quad +(1-\alpha_n)\|P_C(I-\rho_n D_n)x_{n-1}-P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\|. \end{aligned} \tag{3.6}$$

From the nonexpansive of P_C, we have

$$\begin{aligned}\|P_C(I-\rho_n D_n)x_{n-1}-(P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| &\le \|(I-\rho_n D_n)x_{n-1}-(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\ &\le \rho_n\|D_n x_{n-1}-D_{n-1}x_{n-1}\|+|\rho_n-\rho_{n-1}|\|D_{n-1}x_{n-1}\|. \end{aligned} \tag{3.7}$$

From the definition of D_n, we have

$$\begin{aligned}\|D_n x_{n-1}-D_{n-1}x_{n-1}\| &= \|(a_n A+(1-a_n)B)x_{n-1}-(a_{n-1}A+(1-a_{n-1})B)x_{n-1}\| \\ &\le |a_n-a_{n-1}|\|Ax_{n-1}\|+|a_n-a_{n-1}|\|Bx_{n-1}\|. \end{aligned} \tag{3.8}$$

From (3.6), (3.7) and (3.8), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\alpha_n(1-d))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+(1-\alpha_n)\rho_n\|D_n x_{n-1}-D_{n-1}x_{n-1}\| \\ &\quad +(1-\alpha_n)|\rho_n-\rho_{n-1}|\|D_{n-1}x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\ &\le (1-\alpha_n(1-d))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+(1-\alpha_n)\rho_n|a_n-a_{n-1}|\|Ax_{n-1}\|+(1-\alpha_n)\rho_n|a_n-a_{n-1}|\|Bx_{n-1}\| \\ &\quad +(1-\alpha_n)|\rho_n-\rho_{n-1}|\|D_{n-1}x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\ &\le (1-\alpha_n(1-d))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+b|a_n-a_{n-1}|\|Ax_{n-1}\|+b|a_n-a_{n-1}|\|Bx_{n-1}\| \\ &\quad +|\rho_n-\rho_{n-1}|\|D_{n-1}x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\| \\ &\le (1-\alpha_n(1-d))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|2K+|a_n-a_{n-1}|2K+|\rho_n-\rho_{n-1}|K, \end{aligned} \tag{3.9}$$

where $K := \max_{n\in N}\{\|f(x_{n-1})\|, \|Ax_{n-1}\|, \|Bx_{n-1}\|, \|D_{n-1}x_{n-1}\|, \|P_C(I-\rho_{n-1}D_{n-1})x_{n-1}\|\}$. (3.10)

From the conditions (i), (ii), (iii) and Lemma 2.3, we can conclude that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}$$

Step 3. We show that $\lim_{n\to\infty}\|P_C(I-\rho_n D_n)x_n - x_n\| = 0$.

From the definition of x_n, we have

$$(1-\alpha_n)\|P_C(I-\rho_n D_n)x_n - x_n\| \le \alpha_n \|f(x_n)-x_n\| + \|x_{n+1}-x_n\|. \tag{3.12}$$

From the condition (i) and (3.11), we have

$$\lim_{n\to\infty}\|P_C(I-\rho_n D_n)x_n - x_n\| = 0. \tag{3.13}$$

Step 4. We show that $\limsup_{n\to\infty}\langle f(z_0)-z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_F f(z_0)$.

To show this, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0, x_n - z_0\rangle = \lim_{j\to\infty}\langle f(z_0)-z_0, x_{n_j} - z_0\rangle. \tag{3.14}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_j} \rightharpoonup \omega$ as $j\to\infty$ where $\omega\in C$. To show that $\omega\in VI(C,A)\cap VI(C,B)$. Assume that $\omega\notin VI(C,A)\cap VI(C,B)$. From (3.3), we have $\omega \ne P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)\omega$. From $x_{n_j} \rightharpoonup \omega$ as $j\to\infty$, (3.13) and Opial's property, we have

$$\begin{aligned}
\liminf_{j\to\infty}\|x_{n_j}-\omega\| &< \liminf_{j\to\infty}\left\|x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)\omega\right\| \\
&\le \liminf_{j\to\infty}\left(\left\|x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)x_{n_j}\right\| + \left\|P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)\omega\right\|\right) \\
&\le \liminf_{j\to\infty}\left(\left\|x_{n_j} - P_C\left(I-\rho_{n_j}\left(a_{n_j}A+\left(1-a_{n_j}\right)B\right)\right)x_{n_j}\right\| + \left\|x_{n_j}-\omega\right\|\right) \\
&\le \liminf_{j\to\infty}\|x_{n_j}-\omega\|.
\end{aligned} \tag{3.15}$$

This is a contradiction. Then, we can conclude that

$$\omega\in VI(C,A)\cap VI(C,B) = F. \tag{3.16}$$

Since $\omega \in F$, and $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\limsup_{n\to\infty}\langle f(z_0)-z_0, x_n-z_0\rangle = \lim_{j\to\infty}\langle f(z_0)-z_0, x_{n_j}-z_0\rangle = \langle f(z_0)-z_0, \omega-z_0\rangle \le 0. \tag{3.17}$$

Step 5. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0 = P_F f(z_0)$.

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &= \|\alpha_n(f(x_n)-z_0)+(1-\alpha_n)(P_C(I-\rho_n D_n)x_n-z_0)\|^2 \\
&\le (1-\alpha_n)^2\|P_C(I-\rho_n D_n)x_n-z_0\|^2+2\alpha_n\langle f(x_n)-z_0, x_{n+1}-z_0\rangle \\
&\le (1-\alpha_n)^2\|x_n-z_0\|^2+2\alpha_n\langle f(x_n)-z_0, x_{n+1}-z_0\rangle \\
&\le (1-\alpha_n)^2\|x_n-z_0\|^2+2\alpha_n\langle f(x_n)-f(z_0), x_{n+1}-z_0\rangle+2\alpha_n\langle f(z_0)-z_0, x_{n+1}-z_0\rangle \\
&\le (1-\alpha_n)^2\|x_n-z_0\|^2+2\alpha_n d\|x_n-z_0\|\|x_{n+1}-z_0\|+2\alpha_n\langle f(z_0)-z_0, x_{n+1}-z_0\rangle \\
&= (1-\alpha_n)^2\|x_n-z_0\|^2+\alpha_n d\left(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2\right)+2\alpha_n\langle f(z_0)-z_0, x_{n+1}-z_0\rangle \\
&= \left((1-\alpha_n)^2+\alpha_n d\right)\|x_{n+1}-z_0\|^2+2\alpha_n\langle f(z_0)-z_0, x_{n+1}-z_0\rangle,
\end{aligned} \tag{3.18}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &\le \frac{1-2\alpha_n+\alpha_n d}{1-\alpha_n d}\|x_n-z_0\|^2+\frac{\alpha_n^2}{1-\alpha_n d}\|x_n-z_0\|^2+\frac{2\alpha_n}{1-\alpha_n d}\langle f(z_0)-z_0, x_{n+1}-z_0\rangle \\
&\le \left(1-\frac{2(1-d)\alpha_n}{1-\alpha_n d}\right)\|x_n-z_0\|^2+\frac{2(1-d)\alpha_n}{1-\alpha_n d}\left(\frac{\alpha_n}{2(1-d)}M+\frac{1}{1-d}\langle f(z_0)-z_0, x_{n+1}-z_0\rangle\right),
\end{aligned} \tag{3.19}$$

where

$$M = \sup\left\{\|x_n-z_0\|^2 : n \in \mathbb{N}\right\}. \tag{3.20}$$

From the condition (i),(3.17) and Lemma 2.3, we can conclude that $\{x_n\}$ converges strongly to $z_0 = P_F f(z_0)$. This completes the proof.

4. NUMERICAL RESULTS

In this section, an example is given for supporting Theorem 3.1.

Example 4.1 Let $\mathbb{R}$ be the set of real numbers and let $A,B:\mathbb{R}\to\mathbb{R}$ define by $Ax=\frac{2(x-1)}{3}$ and $Bx=\frac{2(x-1)}{5}$, $\forall x\in\mathbb{R}$, respectively. Suppose that $\{x_n\}$ is the sequence generated by $x_1\in\mathbb{R}$ and

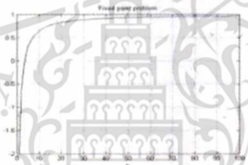
$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)P_{\mathbb{R}}\left(I-\rho_n\left(a_nA+(1-a_n)B\right)\right)x_n,\qquad \forall n\geq 1, \tag{4.1}$$

where $\alpha_n\in(0,1)$, $a_n\in(0,1)$, $\rho_n\in(0,2\eta)$ and $\alpha_n=\frac{1}{n}$, $a_n=\frac{1}{2n}$, $\rho_n=\frac{n+2}{2n+1}$, $\forall n\geq 1$. Let $f:C\to C$ is a contractive mapping with $d\in\left(0,\frac{1}{2}\right)$ and $f(x)=\frac{x}{3}$. Then the sequence $\{x_n\}$ converges strongly to 1.

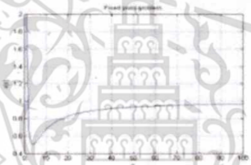
Solution. It is easy to see that the sequences $\{\alpha_n\}$, $\{a_n\}$ *and* $\{\rho_n\}$ satisfies the conditions (i),(ii) and (iii) in the theorem 3.1, A,B are 1-inverse strongly monotone mappings and $VI(C,A)\cap VI(C,B)=\{1\}$. From the theorem 3.1, we can conclude that the sequence $\{x_n\}$ converge strongly to 1. For numerical results, choose $x_1=-2$ and $x_1=2$ in the iterative (4.1), respectively, we can rewrite the sequence $\{x_n\}$ by

$$x_{n+1}=\frac{1}{n}\left(\frac{x_n}{3}\right)+\left(1-\frac{1}{n}\right)\left(x_n-\frac{n+2}{2n+1}\left(\frac{1}{2n}\left(\frac{2(x_n-1)}{3}\right)+\left(1-\frac{1}{2n}\right)\frac{2(x_n-1)}{5}\right)\right),\quad \forall n\geq 1. \tag{4.2}$$

The following graphs show the value of the sequence $\{x_n\}$ defined by (4.2), where $x_1=-2$ and $x_1=2$, respectively.



$n=100$ *and* $x_1=-2$



$n=100$ *and* $x_1=2$

Figure 1: The converges comparison with different initial value x_1.

The following table show the value of the sequence $\{x_n\}$ with $n=100$, $x_1=-2$ and $x_1=2$.

Table 1: The values of $\{x_n\}$ with $n=100$, $x_1=-2$ and $x_1=2$.

n	1	2	3	4	5	...	50	...	96	97	98	99	100
$x_1=-2$ x_n	-2.0000	-0.6667	-0.1333	0.1362	0.3006	...	0.9322	...	0.9650	0.9653	0.9657	0.9660	0.9664
$x_1=2$ x_n	2.0000	0.6667	0.5067	0.4985	0.5241	...	0.9322	...	0.9650	0.9653	0.9657	0.9660	0.9664

5. ACKNOWLEDGEMENTS

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APPENDIX B.

MATLAB PROGRAMMING

An example of Matlab program to find numerical solutions of Theorem 3.2 where all parameters satisfy all conditions in Theorem 3.2.

Function InverseStronglymonotone

```
x=[], p=[];
fprintf('Program computes the fixed point of alpha inverse strongly monotone \n\n');
    fprintf('Input Value:\n');
    fprintf('*******************************************\n');
x(1) = input('x(1) : ');
c = input('c : ');
N = input('round of calculation :  ');
    fprintf('******************************************\n\n');
for i=1:(N-1)
  PP =x(i)-(rho(i)*((a(i)*A(x(i)))+((1-a(i))*B(x(i)))));
  R = (Alpha(i)*f(x(i)))+((1-Alpha(i))*PP);
  x(i+1) = R-(lamda(i)*(R-S(R,c)));
end
p = 1:1:N;
fid = fopen('D:\myoutput.txt','w');
fprintf(fid,'n \t\t x(n) \r\n');
for i=1:N
  fprintf(fid,'%d \t\t %.4f \r\n',p(i),x(i));
end
fclose(fid);
plot(p,x);
grid on;
    title ('Fixed point problem');
    ylabel ('x(n)');
    xlabel ('n');
end
```

```
function m=rho(n)
        m=(n+2)/((2*n)+1);
end
function m=a(n)
        m=1/(2*n);
end
function m=A(x)
        m=(2*(x-1))/3;
end
function m=B(x)
        m=(2*(x-1))/5;
end
function m=Alpha(n)
        m=1/n;
end
function m=f(x)
        m=x/3;
end
function m=lamda(n)
        m=1/(n^2);
end
function m=S(x,c)
        m=(c*x)+((1-c)*T(x));
end
function m=T(x)
        m=((2*x)+1)/3;
end
```

BIOGRAPHY

NAME Miss. Preeyaporn Surbkird

DATE OF BIRTH March 15, 1990

PLACE OF BIRTH Phetchaburi, Thailand

ATTENDED INSTITUTION King Mongkut's Institute of Technology Ladkrabang, 2008-2011

Bachelor of Science

Applied Mathematics

King Mongkut's Institute of Technology Ladkrabang, 2012-2013

Master's Degree

Applied Mathematics

HOME ADDRESS 58 M.9, Sriwaree Road, Yangtan, Krokpha, Nakhon Sawan, 17601, Thailand

E-MAIL pypsk@hotmail.com