A COUPLE MATHEMATICAL MODELS FOR WATER QUALITY MEASUREMENT IN A UNIFORM RESERVOIR

ARPARAVEE INVONG

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นักศึกษา	นางสาวอาภารวี อินวงศ์
รหัสประจำตัว	52650707
ปริญญา	วิทยาศาสตรมหาบัณฑิต
สาขาวิชา	คณิตศาสตร์ประยุกต์
พ.ศ.	2557
อาจารย์ที่ปรึกษา	ผศ.ดร.นพรัตน์ โพธิ์ชัย

บทคัดย่อ

การเติบโตของโรงงานอุตสาหกรรมถือเป็นสาเหตุหลักของการเพิ่มขึ้นของมลพิษ การควบคุมมลพิษจึงมีความจำเป็นต่อการรักษาสิ่งแวดล้อม เช่น แม่น้ำ ทะเลสาบ อ่างเก็บน้ำ ลำธาร บริเวณปากแม่น้ำ มหาสมุทรและน้ำใต้ดิน ในงานวิทยานิพนธ์นี้กล่าวถึงตัวแบบเชิงคณิตศาสตร์ของการตรวจวัดคุณภาพน้ำในอ่างเก็บน้ำเอกรูป โดยพิจารณากรณีที่มีการไหลไม่คงตัวในบริเวณที่มีขอบเขตปกติ วิทยานิพนธ์นี้นำเสนอตัวแบบเชิงคณิตศาสตร์คู่ควบเพื่อการจำลองแบบมลพิษที่เกิดจากน้ำเสียในอ่างเก็บน้ำที่มี ความเร็วของกระแสน้ำที่แปรผัน โดยตัวแบบแรกคือตัวแบบอุทกพลวัตชนิดที่มีสัมประสิทธิ์เป็นตัวแปรที่มีผลลัพธ์เป็นความเร็วและระดับการยกตัวของน้ำในแต่ละตำแหน่งและช่วงเวลา และตัวแบบที่สองคือตัวแบบการกระจายสำหรับจำลองความเข้มข้นของมลพิษในแต่ละตำแหน่งและช่วงเวลา โดยวิทยานิพนธ์นี้จะนำเสนอวิธีผลต่างจำกัดที่เหมาะสม เพื่อคำนวณหาผลเฉลยเชิงตัวเลขของตัวแบบอุทกพลวัตและตัวแบบการกระจาย

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Student	Arparavee Invong
Student ID	52650707
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Thesis Advisor	Asst.Prof.Dr.Nopparat Pochai

ABSTRACT

The increases in an industrial occupation is the principal reason for the growth of pollution. The control of pollution is necessary to protect the environment in rivers, lakes, reservoirs, streams, estuaries, oceans, and groundwater. In this thesis, a mathematical model of water quality in a uniform reservoir is proposed. The case of unsteady flows with regular boundary is considered. The couple mathematical models are used to simulate pollution due to sewage effluent in the uniform reservoir with varied current velocities. The first is the hydrodynamic model with variable coefficients that provides the velocity field and elevation of the water flow. The second is the dispersion model that gives the pollutant concentration field. In the simulating processes, we will use the finite difference methods to determine both solutions of hydrodynamic model and the dispersion model.

Keywords: Hydrodynamic Model/Dispersion Model/Couple Mathematical Models/Water Quality Model/Uniform Reservoir

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CHAPTER 1

Introduction

1.1 Water pollution

The increases in an industrial occupation is the principal reason for the growth of pollution. Waterquality must be protected and maintained for several uses, the principal ones being domestic water supply, energy production, industry, agriculture, fish and wildlife. The highest priority use is domestic water supply, with priorities for other uses depending largely on local or reginal conditions and factors. Water pollution can affect humans in many ways, depending on the purpose for which the water resources are to be used. Since it affects human lives, it is health problem.

The term to pollute may be defined as to destroy the purity of or to make foul or dirty. Water pollution may therefore be defined as the alteration of the characteristics of a receiving water body in such a way as to make it unfit for one or more specific uses. To state it another way, pollution refers to the changes in the natural physical, chemical, and biological characteristics of a receiving water caused by the discharge of any material into that water that detracts from beneficial use.

Control of pollution is necessary of the protection of the water environment and the maintenance of acceptable quality in rivers, lakes, reservoirs, streams, estuaries, oceans, and groundwater. The standard, in turn, will depend on thee uses to be made of the receiving water: water supply, fishing-wildlife, industrial, and other uses.

The methods to detect the amount of pollutant both in the air and water mostly are conducted by a field measurement and a mathematical simulation.In essence the mathematical model offers a quantitative framework to integrate the diverse physical, chemical and biological information that constitute complex environment systems. Beyond solving a particular pollution problem, models provide a vehicle for an enhanced understanding of how the environment works as a unit. Consequently they can be of great value in both research and management contexts.

Civil engineers become involved in the development of water treatment plants, distribution networks and wastewater collection systems. They began to design urban water and wastewater systems.They also has a wastewater treatment plant. The waste water discharged to rivers, lakes and estuaries into large sewers. However, the high cost of building sewage treatment plant. Consequently some design goal had to be established that would protect that environment adequately but economically.

Today water-quality management has moved well beyond the urban point-source problem to encompass many other types of pollution. In addition to waste-water, we now deal with other point sources such as industrial wastes as well as nonpoint inputs such as agricultural runoff. However, as depicted water-quality model still provides the essential link to predict concentration as a function of loadings.

1.2 Objectives

To apply the hydrodynamic model with variable coefficients and the advection-diffusion equations with variable coefficients to the water pollution problem in a uniform reservoir. The convection and diffusion of the pollutant, the concentration of pollutant at any point in the domain can be approximated by the finite difference method. The unsteady state flows can be considered. The velocity of the current can be formulated and computed from the hydrodynamic model as the input data to the dispersion model.

1.3 Scope of the thesis

The scope of the thesis restricted to the application of the finite difference method to the water pollution problem in reservoir, measurement, in the cases of unsteady flows with rectangular boundaries.

1.4 Plan of the thesis

In this thesis, the computation of the unsteady water quality measurement involved the hydrodynamic model and the dispersion model with variable coefficients in the case of simply domain boundary and uniform bottom topography. Two mathematical models are used to simulate pollution due to sewage effluent in the uniform channel and the uniform reservoir with varied current velocity. The first is the hydrodynamic model that provides the velocity field and elevation of the water flow. The second is the dispersion model with variable coefficients that gives the pollutant concentration field. In the simulating processes, we used the finite difference methods to both of hydrodynamic model and the dispersion model. The accuracy of the models are compared with the analytical solution.

All of the computer programs which appears in this thesis are coded by following the modified numerical method. These are developed and constructed by using MATLAB programming.

1.5 Expected results

The expected results of this thesis are the applicable model of water pollution assessment water pollution treatment in the reservoir, and the applicable of the hydrodynamic model for water current direction.

CHAPTER 2

A couple mathematical models of water quality assessment in a uniform reservoir

2.1 Surface motions on shallow water

The tidal current is modeled in a two-dimensional uniform reservoir as Fig 2.1, that represent a regular shape domain boundary domain, by using the shallow water equations. In this section, we will give the basic equations which describe the continuity equation and the equation of motion. Both of the equation are basic to describe the shallow water equations.

2.2 The water pollution computation in the uniform reservoir

In this section, two mathematical models are used to simulate pollution due to sewage effluent in the uniform reservoir with varied current velocity. The first is the hydrodynamic model that provides the velocity field and elevation of the water flow. The second is the dispersion model with variable coefficients that gives the pollutant concentration fields.

Averaging the equation over the depth, and discarding the term Coriolis force, shearing stresses and surface wind, it follows that the two-dimensional shallow water and advection-diffusion equations are applicable.

Since the reservoir is near the equator, it is not too deep and its underneath is flat, then we can omit Coriolis parameters, shearing stresses. Moreover, our reservoir is in rural place which is so far from the sea, then there is very low land breeze. Consequently, we do not consider surface wind.

Fig 2.1: A uniform reservoir

2.2.1 The hydrodynamic model in the uniform reservoir

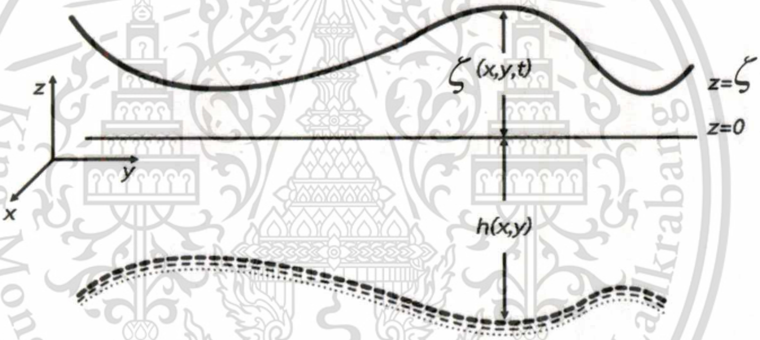


Fig 2.2: Shallow water current

The continuity and momentum equations are governed by the hydrodynamic behavior of the reservoir. Averaging the equations over the depth, discarding the term due to Coriolis parameters, shearing stresses and surface wind. We now introduce the well-known two-dimensional shallow water equations as Fig 2.2 [1]

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}[(h+\zeta)u]+\frac{\partial}{\partial y}[h+\zeta)v]=0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}=0, \tag{2.2}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial \zeta}{\partial y}=0, \tag{2.3}$$

where $h(x,y)$ be the depth measured from the mean water level to the bed of the reservoir, $\zeta(x,y,t)$ is the elevation from the mean water level to the temporary water surface or the tidal elevation, g is the acceleration due to gravity, and $u(x,y,t)$ and $v(x,y,t)$ are the velocity in x and y directions, respectively, for all $(x,y)\in[0,l]\times[0,l]$ and l is a length the regtangular.We assume that h is a constant and the water elevation so smaller than the depth, $\zeta \ll h$. Then $\zeta(x,y,t)+h(x,y)\approx h(x,y,t)$. Then equations

$$\frac{\partial\zeta}{\partial t}+h\frac{\partial u}{\partial x}+h\frac{\partial v}{\partial y}=0, \tag{2.4}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial\zeta}{\partial x}=0, \tag{2.5}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial\zeta}{\partial y}=0. \tag{2.6}$$

|

We will consider the equation in dimensionless problem by letting $U=u/\sqrt{gh}$, $V=v/\sqrt{gh}$, $X=x/l$, $Y=y/l$, $Z=\zeta/h$ and $T=t\sqrt{gh}/l$. Substituting them into equation (2.4-2.6)

From Eq. (2.4),we have $$\frac{\partial\zeta}{\partial t}+h\frac{\partial u}{\partial x}+h\frac{\partial v}{\partial y}=0.$$

Then $$\frac{\partial Zh}{\partial\frac{Tl}{\sqrt{gh}}}+\frac{\partial hU\sqrt{gh}}{\partial Xl}+\frac{\partial hV\sqrt{gh}}{\partial Yl}=0, \tag{2.7}$$

that is $$\frac{h\sqrt{gh}}{l}\frac{\partial Z}{\partial T}+\frac{h\sqrt{gh}}{l}\frac{\partial U}{\partial X}+\frac{h\sqrt{gh}}{l}\frac{\partial V}{\partial Y}=0, \tag{2.8}$$

it follows that $$\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0. \tag{2.9}$$

From Eq. (2.5),we have $$\frac{\partial u}{\partial t}+g\frac{\partial\zeta}{\partial x}=0,$$

we can obtain that $$\frac{\partial U\sqrt{gh}}{\partial\frac{Tl}{\sqrt{gh}}}+g\frac{\partial Zh}{\partial Xl}=0. \tag{2.10}$$

Then $$\frac{gh}{l}\frac{\partial U}{\partial T}+\frac{gh}{l}\frac{\partial Z}{\partial X}=0, \tag{2.11}$$

it follows that
$$\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}=0. \tag{2.12}$$

From Eq. (2.6), we have
$$\frac{\partial v}{\partial t}+g\frac{\partial \zeta}{\partial y}=0,$$

we can obtain that
$$\frac{\partial V\sqrt{gh}}{\partial \frac{Tl}{\sqrt{gh}}}+g\frac{\partial Zh}{\partial Yl}=0. \tag{2.13}$$

Then
$$\frac{gh}{l}\frac{\partial V}{\partial T}+\frac{gh}{l}\frac{\partial Z}{\partial Y}=0, \tag{2.14}$$

it follows that
$$\frac{\partial V}{\partial T}+\frac{\partial Z}{\partial Y}=0. \tag{2.15}$$

In order to solve the equations in $\Omega\times[0,T]$ where $\Omega=(0,1)\times(0,1)$. From (2.9), (2.12) and (2.15), we obtain

$$\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{2.16}$$

$$\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}=0, \tag{2.17}$$

$$\frac{\partial V}{\partial T}+\frac{\partial Z}{\partial Y}=0. \tag{2.18}$$

If there is the frictional force due to the drag of side of the reservoir and averaging the equation over the depth, then

$$\frac{1}{(1+XY)}\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{2.19}$$

$$\frac{\partial U}{\partial T}+(1+XY)\frac{\partial Z}{\partial X}=0, \tag{2.20}$$

$$\frac{\partial V}{\partial T}+(1+XY)\frac{\partial Z}{\partial Y}=0, \tag{2.21}$$

with the initial conditions $U = 0$, $V = 0$ and $Z = f(x, y)$. The boundary conditions $V = 0$, $\frac{\partial U}{\partial x} = 0$ at the planes $x = 0$ and $x = 1$, and $U = 0$, $\frac{\partial V}{\partial y} = 0$ at the planes $y = 0$ and $y = 1$, and $Z = 0$ at $\partial\Omega$

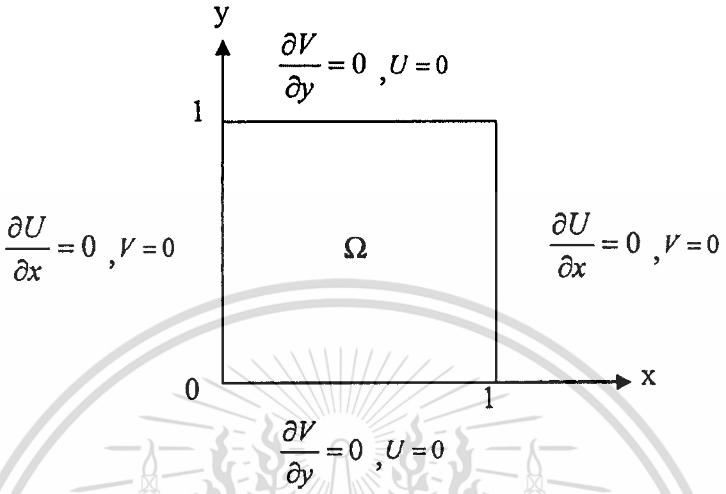


Fig 2.3: A domain of uniform reservoir

2.2.2 Dispersion model

2.2.2.1 Advection-Diffusion-Reaction equation

Numerous types of water motion transport matter within natural waters. Wind energy and gravity impart motion to the water that leads to mass transport. In the study within-system motion can be divided into two general categories: advection and diffusion.

The advection result is unidirectional and does not change the identity of the substance being transported. Advection moves matter from one position in space to another. Simple examples of transport primarily of this type are the flow of water through a lake's outlet and downstream transport due to flow in a river or estuary.

Diffusion refers to the movement of mass due to random water motion or mixing. Such transport causes the dye patch depicted to spread out and dilute over time with negligible net movement.

The dispersion of the concentration is described by the convection-diffusion equation[11] in an arbitrary domain $\Omega \subseteq \mathbb{R}^n, n = 1,2,3$

$$\frac{\partial \tau}{\partial t} + \gamma \cdot \nabla \tau = D \nabla^2 \tau, \tag{2.22}$$

where

∇ is defined by $\dfrac{\partial}{\partial x}i+\dfrac{\partial}{\partial y}j+\dfrac{\partial}{\partial z}k$,

∇^2 is the Laplacian operator $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$,

$\tau(x,t)$ is a concentration term,

$\gamma(x,t)$ is a velocity term,

D is a molecular diffusivity.

Both τ and γ are function of position x and time t. The domain boundary can be classified into two types, that are Ω_1 with specified concentration and Ω_2 with specified flux of concentration, and total boundary $\partial\Omega$ is $\partial\Omega=\Omega_1\cup\Omega_2$ and $\Omega_1\cup\Omega_2=\varnothing$. The boundary condition concerning C are given as follows: the concentration is specified in part of the boundary on Ω_1, and the diffusive flux in the exterior normal direction is specified on the rest of the boundary on Ω_2 are

$$C=C_B \text{ on } \Omega_1, \tag{2.23}$$

$$-D\frac{\partial C}{\partial n}=T_B \text{ on } \Omega_2. \tag{2.24}$$

If the boundary is a *non-absorbing* or *reflexive boundary*, then we can put $T_B=0$. If a discharge of the mass is instantaneous, the amount of the discharge can be considered in the initial condition $C_B=C_0$ is, where C_0 is a constant.

2.2.2.2 The non-dimensional form of the two-dimensional unsteady advection-diffusion equation with constant coefficients

In this section, a mathematical model for described the dispersion of the concentration in two-dimensional domain eg. Lake, reservoir, estuaries or another nearly closed water area, will be present. In this case, the dispersion of the concentration is described by the steady convection-diffusion equation with constant coefficients in two-dimensional domain $\Omega \subset \mathbb{R}^2$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}-D_x\frac{\partial^2 C}{\partial x^2}-D_y\frac{\partial^2 C}{\partial y^2}+RC-Q=0, \tag{2.25}$$

where

$C(x,y)$: concentration at the point (x,y) in Ω (kg/m^3),

u,v : flow velocity in X,Y directions $(m/\sec)$,

D_x,D_y : diffusion coefficients in X,Y direction $(m^2/\sec)$s,

R : the substance decaying rate $(\sec^{-1})$,

Q : the increasing rate of substance concentration due to a source $(kg/m^3\sec)$.

The domain boundary can be of two types : S_1 with specified concentration and S_2 with specified flux of concentration, and total boundary Γ is $\Gamma = S_1 \cup S_2$. The boundary condition on S_1 and S_2 are

$$C = C_B \text{ on } S_1, \tag{2.26}$$

$$\frac{\partial C}{\partial n}=\frac{\partial C}{\partial x}\cos(x,n)+\frac{\partial C}{\partial y}\sin(y,n)=T_B \text{ on } S_2. \tag{2.27}$$

By omitting the term of reaction equation, we have

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right), \tag{2.28}$$

We will transform the equation (2.28) into dimensionless equation by letting,

$$C^*=\frac{C}{U} \quad \text{or} \quad C=C^*U, \tag{2.29}$$

$$u^*=\frac{u}{U} \quad \text{or} \quad u=u^*U, \tag{2.30}$$

$$v^*=\frac{v}{U} \quad \text{or} \quad v=v^*U, \tag{2.31}$$

$$x^*=\frac{x}{l} \quad \text{or} \quad x=x^*l, \tag{2.32}$$

$$y^*=\frac{y}{l} \quad \text{or} \quad y=y^*l, \tag{2.33}$$

$$t^*=\frac{tU}{l} \quad \text{or} \quad t=\frac{t^*l}{U}, \tag{2.34}$$

where U is absolute maximum of vector u and vector v, i.e, U = max{u,v}. Substituting Eq.(2.29)-(2.34) into Eq.(2.28), we have

$$\frac{\partial\left(C^*U\right)}{\partial\left(\frac{t^*l}{U}\right)}+\left(u^*U\right)\frac{\partial\left(C^*U\right)}{\partial\left(x^*l\right)}+\left(v^*U\right)\frac{\partial\left(C^*U\right)}{\partial\left(y^*l\right)}=D\left(\frac{\partial^2\left(C^*U\right)}{\partial\left(x^*l\right)^2}+\frac{\partial^2\left(C^*U\right)}{\partial\left(y^*l\right)^2}\right), \quad (2.35)$$

$$\frac{U}{\frac{l}{U}}\frac{\partial C^*}{\partial t^*}+\left(u^*U\right)\frac{U}{l}\frac{\partial C^*}{\partial x^*}+\left(v^*U\right)\frac{U}{l}\frac{\partial C^*}{\partial y^*}=D\left(\frac{U}{l}\frac{\partial^2 C^*}{\partial x^{*2}}+\frac{U}{l}\frac{\partial^2 C^*}{\partial y^{*2}}\right), \quad (2.36)$$

$$\frac{U^2}{l}\frac{\partial C^*}{\partial t^*}+u^*\frac{U^2}{l}\frac{\partial C^*}{\partial x^*}+v^*\frac{U^2}{l}\frac{\partial C^*}{\partial y^*}=D\frac{U}{l}\left(\frac{\partial^2 C^*}{\partial x^{*2}}+\frac{\partial^2 C^*}{\partial y^{*2}}\right), \quad (2.39)$$

The non-dimensional form of the two-dimensional unsteady advection-diffusion equation with constant coefficients becomes,

$$\frac{\partial C^*}{\partial t^*}+u^*\frac{\partial C^*}{\partial x^*}+v^*\frac{\partial C^*}{\partial y^*}=\frac{D}{U}\left(\frac{\partial^2 C^*}{\partial x^{*2}}+\frac{\partial^2 C^*}{\partial y^{*2}}\right). \quad (2.40)$$

CHAPTER 3

Numerical computations of the hydrodynamic model and the dispersion model

3.1 The numerical solution for the hydrodynamic model

The hydrodynamic model provides the velocity field and elevation of the water. The calculated result of the model are input to the dispersion model, which provides the pollutant concentration field.

The equations (2.19)-(2.21) can be written in the form

$$\frac{\partial Z}{\partial T}=0\frac{\partial W_1}{\partial x}-\frac{\partial W_2}{\partial x}+0\frac{\partial W_3}{\partial x}+0\frac{\partial W_1}{\partial y}+0\frac{\partial W_2}{\partial y}-\frac{\partial W_3}{\partial y}, \qquad (3.1)$$

$$\frac{\partial U}{\partial T}=-(1+xy)\frac{\partial W_1}{\partial x}+0\frac{\partial W_2}{\partial x}+0\frac{\partial W_3}{\partial x}+0\frac{\partial W_1}{\partial y}+0\frac{\partial W_2}{\partial y}+0\frac{\partial W_3}{\partial y}, \qquad (3.2)$$

$$\frac{\partial V}{\partial T}=0\frac{\partial W_1}{\partial x}+0\frac{\partial W_2}{\partial x}+0\frac{\partial W_3}{\partial x}-(1+xy)\frac{\partial W_1}{\partial y}+0\frac{\partial W_2}{\partial y}+0\frac{\partial W_3}{\partial y}. \qquad (3.3)$$

It can be obtain in the matrix form

$$\frac{\partial W}{\partial T}=A\frac{\partial W}{\partial X}+B\frac{\partial W}{\partial Y}, \qquad (3.4)$$

where

$$W=\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}, A=\begin{bmatrix} 0 & -1 & 0 \\ -(1+XY) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B=\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+XY) & 0 & 0 \end{bmatrix}, \qquad (3.5)$$

in which $W_1=Z$, $W_2=U$, $W_3=V$.

We now discretize by diving the interval $[0,1]$ into L and M subintervals such that $L\Delta X=1$ and $M\Delta Y=1$, and the interval $[0,T]$ into N subintervals such that $N\Delta T=T$. We can then approximate $W_1(X_l, \mathrm{Y}_m, \mathrm{T}_n)$ by $W_{1l,m}^{n}$, value of the difference approximation of $W_1(\mathrm{X},\mathrm{Y},\mathrm{T})$ at point $X=l\Delta X$, $Y=m\Delta Y$ and $T=n\Delta T$, where $0\le l\le L$, $0\le m\le M$ and

$0 \le n \le N$,and similarly defined for $W^{n}_{2l,m}$ and $W^{n}_{3l,m}$. The grid point $\left(X_l, \mathrm{Y}_m, \mathrm{T}_n\right)$ are defined by $X_l = l\Delta X$ for all $l = 0,1,2,\dots,L$, $Y_m = m\Delta Y$ for all $m = 0,1,2,\dots,M$ and $T_n = n\Delta T$ for all $n = 0,1,2,\dots,N$ in which L , M and N are positive integers. Using Lax-Wendroff method for variable coefficient[2], we get the following finite difference equation

$$W^{n+1}_{l,m} = \Big[I + \frac{1}{2} pA(\Delta_x + \nabla_x) + \frac{1}{2} pB(\Delta_y + \nabla_y) + \frac{1}{2} p^2 A^2 \Delta_x \nabla_x + \frac{1}{2} p^2 B^2 \Delta_y \nabla_y + \frac{1}{8}\left(AB + BA\right)(\Delta_x + \nabla_x)(\Delta_y + \nabla_y) \Big] U^{n}_{l,m}, \tag{3.6}$$

where

$$W^{n+1}_{l,m} = \begin{pmatrix} W^{n}_{1l,m} \\ W^{n}_{2l,m} \\ W^{n}_{3l,m} \end{pmatrix}, \Delta_x W^{n}_{l} = W^{n}_{l+1} - W^{n}_{l} \text{ and } \nabla_x W^{n}_{l} = W^{n}_{l} - W^{n}_{l-1}, \tag{3.7}$$

and $p = \Delta t / \Delta x$. Then

$$W^{n+1}_{l,m} = W^{n}_{l,m} + \frac{1}{2} pA_{l,m}(\Delta_x + \nabla_x) W^{n}_{l,m} + \frac{1}{2} pB_{l,m}(\Delta_y + \nabla_y) W^{n}_{l,m} + \frac{1}{2} p^2 A^2_{l,m} \Delta_x \nabla_x W^{n}_{l,m} + \frac{1}{2} p^2 B^2_{l,m} \Delta_y \nabla_y W^{n}_{l,m} + \frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})(\Delta_x + \nabla_x)(\Delta_y + \nabla_y) W^{n}_{l,m}, \tag{3.8}$$

Consider the second term, we have

$$\begin{aligned} \frac{1}{2} pA_{l,m}(\Delta_x + \nabla_x) W^{n}_{l,m} &= \frac{1}{2} pA_{l,m} \Delta_x W^{n}_{l,m} + \frac{1}{2} pA_{l,m} \nabla_x W^{n}_{l,m}, \\ &= \frac{1}{2} pA_{l,m} \left(W^{n}_{l+1,m} - W^{n}_{l,m} \right) + \frac{1}{2} pA_{l,m} \left(W^{n}_{l,m} - W^{n}_{l-1,m} \right), \\ &= \frac{1}{2} pA_{l,m} \left(W^{n}_{l+1,m} - W^{n}_{l-1,m} \right). \end{aligned} \tag{3.9}$$

Consider the third term, we have

$$\begin{aligned} \frac{1}{2} pB_{l,m}(\Delta_y + \nabla_y) W^{n}_{l,m} &= \frac{1}{2} pB_{l,m} \Delta_y W^{n}_{l,m} + \frac{1}{2} pB_{l,m} \nabla_y W^{n}_{l,m}, \\ &= \frac{1}{2} pB_{l,m} \left(W^{n}_{l,m+1} - W^{n}_{l,m} \right) + \frac{1}{2} pB_{l,m} \left(W^{n}_{l,m} - W^{n}_{l,m-1} \right), \\ &= \frac{1}{2} pB_{l,m} \left(W^{n}_{l,m+1} - W^{n}_{l,m-1} \right). \end{aligned} \tag{3.10}$$

Consider the forth term, we have

$$\frac{1}{2}p^2A_{l,m}^2\Delta_x\nabla_x W_{l,m}^n = \frac{1}{4}p^2\left(A_{l,m}\Delta_x A_{l,m}\nabla_x + A_{l,m}\nabla_x A_{l,m}\Delta_x\right)W_{l,m}^n,$$

$$= \frac{1}{4}p^2A_{l,m}\left[\Delta_x A_{l,m}\left(W_{l,m}^n - W_{l-1,m}^n\right) + \nabla_x A_{l,m}\left(W_{l+1,m}^n - W_{l,m}^n\right)\right],$$

$$= \frac{1}{4}p^2A_{l,m}\left[\Delta_x A_{l,m}W_{l,m}^n - \Delta_x A_{l,m}W_{l-1,m}^n + \nabla_x A_{l,m}W_{l+1,m}^n - \nabla_x A_{l,m}W_{l,m}^n\right],$$

$$= \frac{1}{4}p^2A_{l,m}[\left(A_{l+1,m}W_{l+1,m}^n - A_{l,m}W_{l,m}^n\right) - \left(A_{l+1,m}W_{l,m}^n A_{l,m}W_{l-1,m}^n\right)$$

$$+ (A_{l,m}W_{l+1,m}^n - A_{l-1,m}W_{l,m}^n) - (A_{l,m}W_{l,m}^n - A_{l-1,m}W_{l-1,m}^n)]. \tag{3.11}$$

Consider the fifth term, we have

$$\frac{1}{2}p^2B_{l,m}^2\Delta_y\nabla_y W_{l,m}^n = \frac{1}{4}p^2(B_{l,m}\Delta_y B_{l,m}\nabla_y + B_{l,m}\nabla_y B_{l,m}\Delta_y)W_{l,m}^n,$$

$$= \frac{1}{4}p^2B_{l,m}[\Delta_y B_{l,m}(W_{l,m}^n - W_{l,m-1}^n) + \nabla_y B_{l,m}(W_{l,m+1}^n - W_{l,m}^n)],$$

$$= \frac{1}{4}p^2B_{l,,m}[\Delta_y B_{l,m}W_{l,m}^n - \Delta_y B_{l,m}W_{l,m-1}^n + \nabla_y B_{l,m}W_{l,m+1}^n - \nabla_y B_{l,m}W_{l,m}^n],$$

$$= \frac{1}{4}p^2B_{l,m}[(B_{l,m+1}W_{l,m+1}^n - B_{l,m}W_{l,m}^n) - (B_{l,m+1}W_{l,m}^n - B_{l,m}W_{l,m-1}^n). \tag{3.12}$$

Consider the sixth term, we have

$$\frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(\Delta_x + \nabla_x)(\Delta_y + \nabla_y)W_{l,m}^n$$

$$= \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[\Delta_x\Delta_y + \Delta_x\nabla_y + \nabla_x\Delta_y + \nabla_x\nabla_y]W_{l,m}^n,$$

$$= \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[\Delta_x\Delta_y W_{l,m}^n + \Delta_x\nabla_y W_{l,m}^n + \nabla_x\Delta_y W_{l,m}^n + \nabla_x\nabla_y W_{l,m}^n],$$

$$= \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[\Delta_x(W_{l,m+1}^n - W_{l,m}^n) + \Delta_x(W_{l,m}^n - W_{l,m-1}^n)$$

$$+\nabla_x(W_{l,m+1}^n - W_{l,m}^n) + \nabla_x(W_{l,m+1}^n - W_{l,m-1}^n)],$$

$$= \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[\Delta_x W_{l,m+1}^n - \Delta_x W_{l,m}^n + \Delta_x W_{l,m}^n - \Delta_x W_{l,m-1}^n$$

$$+\nabla_x W_{l,m+1}^n - \nabla_x W_{l,m}^n + \nabla_x W_{l,m}^n - \nabla_x W_{l,m-1}^n],$$

$$= \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[(W_{l+1,m+1}^n - W_{l,m+1}^n) - (W_{l+1,m-1}^n - W_{l,m-1}^n)$$

$$+(W_{l,m+1}^n - W_{l-1,m+1}^n) - (W_{l,m-1}^n - W_{l-1,m-1}^n)],$$

$$= \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[W_{l+1,m+1}^n - W_{l+1,m-1}^n - W_{l-1,m+1}^n + W_{l-1,m-1}^n]. \tag{3.13}$$

From equation (3.8)-(3.13) and I is the unit matrix of order 3 and $p = \Delta t / \Delta x$. It obtained the general form of finite difference equation,

$$
\begin{aligned}
W_{l,m}^{n+1} = {} & W_{l,m}^{n} + \frac{1}{2} p A_{l,m}\left(W_{l+1,m}^{n} - W_{l-1,m}^{n}\right) + \frac{1}{2} p B_{l,m}\left(W_{l,m+1}^{n} - W_{l,m-1}^{n}\right) \\
& + \frac{1}{4} p^{2} A_{l,m}\Big[\left(A_{l+1,m}W_{l+1,m}^{n} - A_{l,m}W_{l,m}^{n}\right) - \left(A_{l+1,m}W_{l,m}^{n} - A_{l,m}W_{l-1,m}^{n}\right) \\
& + (A_{l,m}W_{l+1,m}^{n} - A_{l-1,m}W_{l,m}^{n}) - (A_{l,m}W_{l,m}^{n} - A_{l-1,m}W_{l-1,m}^{n})\Big] \\
& + \frac{1}{4} p^{2} B_{l,m}\Big[\left(B_{l,m+1}W_{l,m+1}^{n} - B_{l,m}W_{l,m}^{n}\right) - \left(B_{l,m+1}W_{l,m}^{n} - B_{l,m}W_{l,m-1}^{n}\right) \\
& + \left(B_{l,m}W_{l,m+1}^{n} - B_{l,m-1}W_{l,m}^{n}\right) - \left(B_{l,m}W_{l,m}^{n} - B_{l,m-1}W_{l,m-1}^{n}\right)\Big] \\
& + \frac{1}{8} p^{2}(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})[W_{l+1,m+1}^{n} - W_{l+1,m-1}^{n} - W_{l-1,m+1}^{n} + W_{l-1,m-1}^{n}]. \qquad (3.14)
\end{aligned}
$$

3.2 The numerical method for dispersion model

The distributed pollutant process satisfies the mass transfer equation, which includes transportation and diffusion. We now introduced the unsteady advection-diffusion equation with variable coefficient in a uniform reservoir,

$$\frac{\partial C}{\partial t} + \left(\frac{u}{1+xy}\right)\frac{\partial C}{\partial x} + \left(\frac{v}{1+xy}\right)\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \qquad (3.15)$$

where $C(x,y,t)$ is the concentration averaged in depth at the point (x,y) and at time t, D is the diffusion coefficient and $u(x,y,t)$ and $v(x,y,t)$ are the velocity component in x and y direction respectively. The initial condition is $C(x,y,0) = g(x,y)$ at $t = 0$ and the non-absorbing boundary condition is assume to be $\frac{\partial C}{\partial n} = 0$ on $\partial\Omega$, where $g(x,y)$ is given.

We now discretize by diving the interval $[0,1]$ into L and M subintervals such that $L\Delta x = 1$ and $M\Delta y = 1$, and the interval $[0,T]$ into N subintervals such that. $N\Delta t = T$

We can then approximate $c(x_l, y_m, t_n)$ by $C_{l,m}^{n}$, value of the difference approximation of $C(x,y,t)$ at point $x = l\Delta x$, $y = m\Delta y$ and $t = n\Delta t$, where $0 \le l \le L$, $0 \le m \le M$ and $0 \le n \le N$ where $u_{l,m}^{n} = U_{l,m}^{n}\sqrt{gh}$ and $v_{l,m}^{n} = V_{l,m}^{n}\sqrt{gh}$. Using Forward difference in time and Backward difference in space (FTBS), we get the following finite difference equation where $\lambda = \Delta t / \Delta x^2$. We can obtain the explicit form as

$$\frac{C_{l,m}^{n+1} - C_{l,m}^{n}}{\Delta t} + \left(\frac{u_{l,m}^{n}}{1+xy}\right)\left(\frac{C_{l,m}^{n} - C_{l-1,m}^{n}}{\Delta x}\right) + \left(\frac{v_{l,m}^{n}}{1+xy}\right)\left(\frac{C_{l,m}^{n} - C_{l,m-1}^{n}}{\Delta y}\right)$$

$$= \frac{D}{U}\left(\frac{C_{l+1,m}^{n} - 2C_{l,m}^{n} + C_{l-1,m}^{n}}{(\Delta x)^2} + \frac{C_{l,m+1}^{n} - 2C_{l,m}^{n} + C_{l,m-1}^{n}}{(\Delta y)^2}\right), \tag{3.16}$$

$$C_{l,m}^{n+1} = \left[\frac{D}{U}\left(\frac{C_{l+1,m}^{n} - 2C_{l,m}^{n} + C_{l-1,m}^{n}}{(\Delta x)^2} + \frac{C_{l,m+1}^{n} - 2C_{l,m}^{n} + C_{l,m-1}^{n}}{(\Delta y)^2}\right)\right.$$

$$\left. - \left(\frac{u_{l,m}^{n}}{1+x_{l,m}y_{l,m}}\right)\left(\frac{C_{l,m}^{n} - C_{l-1,m}^{n}}{\Delta x}\right) - \left(\frac{v_{l,m}^{n}}{1+x_{l,m}y_{l,m}}\right)\left(\frac{C_{l,m}^{n} - C_{l,m-1}^{n}}{\Delta y}\right)\right]\Delta t + C_{l,m}^{n}, \tag{3.17}$$

$$= \frac{\Delta t D C_{l+1,m}^{n}}{U(\Delta x)^2} - \frac{2\Delta t D C_{l,m}^{n}}{U(\Delta x)^2} + \frac{\Delta t D C_{l-1,m}^{n}}{U(\Delta x)^2} + \frac{\Delta t D C_{l,m+1}^{n}}{U(\Delta y)^2} - \frac{2\Delta t D C_{l,m}^{n}}{U(\Delta y)^2} + \frac{\Delta t D C_{l,m-1}^{n}}{U(\Delta y)^2} - \frac{\Delta t u_{l,m}^{n} C_{l,m}^{n}}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}$$

$$+ \frac{\Delta t u_{l,m}^{n} C_{l-1,m}^{n}}{\Delta x\left(1+x_{l,m}y_{l,m}\right)} - \frac{\Delta t v_{l,m}^{n} C_{l,m}^{n}}{\Delta y\left(1+x_{l,m}y_{l,m}\right)} + \frac{\Delta t v_{l,m}^{n} C_{l,m-1}^{n}}{\Delta y\left(1+x_{l,m}y_{l,m}\right)} + C_{l,m}^{n}, \tag{3.18}$$

$$= \lambda\frac{D}{U}C_{l+1,m}^{n} - 2\lambda\frac{D}{U}C_{l,m}^{n} + \lambda\frac{D}{U}C_{l-1,m}^{n} + \lambda\frac{D}{U}C_{l,m+1}^{n} - 2\lambda\frac{D}{U}C_{l,m}^{n} + \lambda\frac{D}{U}C_{l,m-1}^{n}$$

$$- \frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n}C_{l,m}^{n} + \frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n}C_{l-1,m}^{n} - \frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n}C_{l,m}^{n}$$

$$+ \frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n}C_{l,m-1}^{n} + C_{l,m}^{n}, \tag{3.19}$$

$$= \left(\lambda\frac{D}{U} + \frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n}\right)C_{l-1,m}^{n} + \left(4\lambda\frac{D}{U} - \frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n} - \frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n} + 1\right)C_{l,m}^{n}$$

$$+ \lambda\frac{D}{U}C_{l+1,m}^{n} + \left(\lambda\frac{D}{U} + \frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n}\right)C_{l,m-1}^{n} + \lambda\frac{D}{U}C_{l,m+1}^{n}. \tag{3.20}$$

CHAPTER4

Numerical experiment of a couple mathematical models

In chapter 3, it is mentioned that the quality of water by industrial waste water is released from the plant. This chapter has presented a measurement water quality a mathematical model for measurement water quality.

This chapter discusses the flow of water by considered a mathematical model two model is the hydrodynamic model and the dispersion model, which is to find numerical solutions to finite difference method with the two models. The hydrodynamic model provides the velocity field and elevation of the water. The calculated result of the model are input to the dispersion model, which provides the pollutant concentration field. The hydrodynamic model defined boundary conditions and the initial conditions of the model to be consistent with the actual conditions. The results showed that the velocity vector of the flow by the horizontal and vertical depth of the reservoir results into tables and graphs.

The dispersion model calculation to the speed of the flow in the horizontal and vertical depth of the model hydrodynamic represented in the model by considering the boundary conditions and the initial conditions of the model to be consistent with the conditions. the calculated results are shown to the concentration of water quality results into tables and graphs, the concentration of water quality.

4.1 Numerical treatment of a hydrodynamic model with drag force

The Lax-Wendroff method with variables coefficient is used to calculate equation (3.14), it obtained the general form into 9 cases as below

Case1: If $l = 1$ and $m = 1$, then

$$W_{1,1}^{n+1} = W_{1,1}^{n} + \frac{1}{2} pA_{1,1}(\mathrm{W}_{2,1}^{n} - W_{0,1}^{n}) + \frac{1}{2} pB_{1,1}(W_{1,2}^{n} - W_{1,0}^{n})$$
$$+\frac{1}{4} p^2 A_{1,1}\left[(A_{2,1}W_{2,1}^{n} - A_{1,1}W_{1,1}^{n}) - (A_{2,1}W_{1,1}^{n} - A_{1,1}W_{0,1}^{n}) + (A_{1,1}W_{2,1}^{n} - A_{0,1}W_{1,1}^{n}) - (A_{1,1}W_{1,1}^{n} - A_{0,1}W_{0,1}^{n})\right]$$
$$+\frac{1}{4} p^2 B_{1,1}\left[(B_{1,2}W_{1,2}^{n} - B_{1,1}W_{1,1}^{n}) - (B_{1,2}W_{1,1}^{n} - B_{1,1}W_{1,0}^{n}) + (B_{1,1}W_{1,2}^{n} - B_{1,0}W_{1,1}^{n}) - (B_{1,1}W_{1,1}^{n} - B_{1,0}W_{1,0}^{n})\right]$$
$$+\frac{1}{8} p^2 (A_{1,1}B_{1,1} + B_{1,1}A_{1,1})(W_{2,2}^{n} - W_{0,2}^{n} - W_{2,0}^{n} + W_{0,0}^{n}) . \qquad (4.1)$$

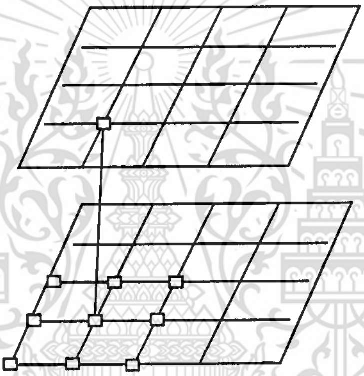


Fig.4.1: The stencil of case 1

From the boundary condition , we have $V = 0$ and $\frac{\partial U}{\partial x} = 0$. Then

$$V_{0,1} = 0 , \qquad (4.2)$$

and

$$\left(\frac{\partial U}{\partial x}\right)_{0,1} = 0 . \qquad (4.3)$$

From equation (4.3) are approximated by using the forward difference method, we obtain

$$\frac{U_{1,1} - U_{0,1}}{\Delta X} = 0 . \qquad (4.4)$$

Then

$$U_{0,1} = U_{1,1} . \qquad (4.5)$$

From equations (4.2) and (4.5), we have

$$W_{0,1} = \begin{bmatrix} Z_{0,1} \\ U_{1,1} \\ 0 \end{bmatrix}. \tag{4.6}$$

From the boundary condition , we have $U = 0$ and $\frac{\partial V}{\partial y} = 0$. Then

$$U_{1,0} = 0, \tag{4.6}$$

and

$$\left(\frac{\partial V}{\partial y}\right)_{1,0} = 0. \tag{4.7}$$

From equation (4.7) are approximated by using the forward difference method, we obtain

$$\frac{V_{1,1} - V_{1,0}}{\Delta Y} = 0. \tag{4.8}$$

Then

$$V_{1,0} = V_{1,1}. \tag{4.9}$$

From equations (4.6) and (4.9), we have

$$W_{1,0} = \begin{bmatrix} Z_{1,0} \\ 0 \\ V_{1,1} \end{bmatrix}. \tag{4.10}$$

Case2:If $l = 2, \ldots, L-2$ and $m = 1$, then

$$\begin{aligned}
W_{l,1}^{n+1} &= W_{l,1}^{n} + \frac{1}{2} pA_{l,1}(W_{l+1,1}^{n} - W_{l,1}^{n}) + \frac{1}{2} pB_{l,1}(W_{l,2}^{n} - W_{l,0}^{n}) \\
&+ \frac{1}{4} p^2 A_{l,1}\Big[(A_{l+1,1}W_{l+1,1}^{n} - A_{l,1}W_{l,1}^{n}) - (A_{l+1,1}W_{l,1}^{n} - A_{l,1}W_{l-1,1}^{n}) \\
&+ (A_{l,1}W_{l+1,1}^{n} - A_{l-1,1}W_{l,1}^{n}) - (A_{l,1}W_{l,1}^{n} - A_{l-1,1}W_{l-1,1}^{n})\Big] \\
&+ \frac{1}{4} p^2 B_{l,1}\Big[(B_{l,2}W_{l,2}^{n} - B_{l,1}W_{l,1}^{n}) - (B_{l,2}W_{l,1}^{n} - B_{l,1}W_{l,0}^{n}) \\
&+ (B_{l,1}W_{l,2}^{n} - B_{l,0}W_{l,1}^{n}) - (B_{l,1}W_{l,1}^{n} - B_{l,0}W_{l,0}^{n})\Big] \\
&+ \frac{1}{8} p^2 (A_{l,1}l_{2,1} + B_{l,1}A_{l,1})(W_{l+1,2}^{n} - W_{l-1,2}^{n} - W_{l+1,0}^{n} + W_{l-1,0}^{n}).
\end{aligned} \tag{4.11}$$

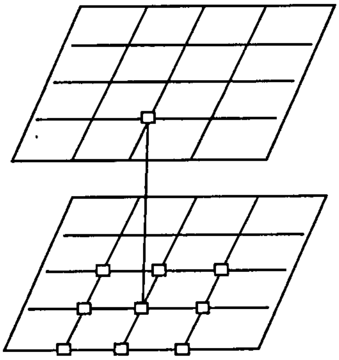


Fig.4.2: The stencil of case 2

From the boundary condition , we have $U = 0$ and $\dfrac{\partial V}{\partial y} = 0$. Then

$$U_{l,0} = 0, \tag{4.12}$$

and

$$\left(\frac{\partial V}{\partial y}\right)_{l,0} = 0. \tag{4.13}$$

From equation (4.13) are approximated by using the forward difference method, we obtain

$$\frac{V_{l,1} - V_{l,0}}{\Delta y} = 0. \tag{4.14}$$

Then

$$V_{l,0} = V_{l,1}. \tag{4.15}$$

From equations (4.12) and (4.15), we have

$$W_{l,0} = \begin{bmatrix} Z_{l,0} \\ 0 \\ V_{l,1} \end{bmatrix}. \tag{4.16}$$

Case3: If $l = L-1$ and $m = 1$, then

$$\begin{aligned}
W_{l,1}^{n+1} = {} & W_{l,1}^{n} + \frac{1}{2} pA_{l,1}(W_{l+1,1}^{n} - W_{l-1,1}^{n}) + \frac{1}{2} pB_{l,1}(W_{l,2}^{n} - W_{l,0}^{n}) \\
& + \frac{1}{4} p^2 A_{l,1}\Big[(A_{l+1,1}W_{l+1,1}^{n} - A_{l,1}W_{l,1}^{n}) - (A_{l+1,1}W_{l,1}^{n} - A_{l,1}W_{l-1,1}^{n}) \\
& + (A_{l,1}W_{l+1,1}^{n} - A_{l-1,1}W_{l,1}^{n}) - (A_{l,1}W_{l,1}^{n} - A_{l-1,1}W_{l-1,1}^{n})\Big] \\
& + \frac{1}{4} p^2 B_{l,1}\Big[(B_{l,2}W_{l,2}^{n} - B_{l,1}W_{l,1}^{n}) - (B_{l,2}W_{l,1}^{n} - B_{l,1}W_{l,0}^{n}) \\
& + (B_{l,1}W_{l,2}^{n} - B_{l,0}W_{l,1}^{n}) - (B_{l,1}W_{l,1}^{n} - B_{l,0}W_{l,0}^{n})\Big] \\
& + \frac{1}{8} p^2 (A_{l,1}B_{l,1} + B_{l,1}A_{l,1})(W_{l+1,2}^{n} - W_{l-1,2}^{n} - W_{l+1,0}^{n} + W_{l-1,0}^{n}).
\end{aligned} \tag{4.17}$$

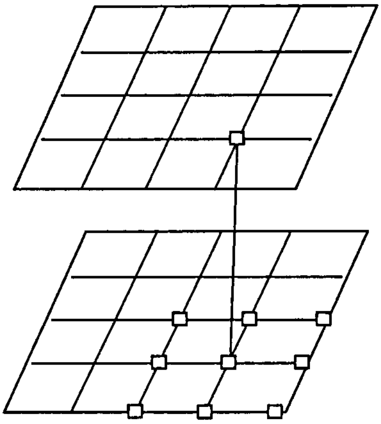


Fig.4.3: The stencil of case 3

From the boundary condition , we have $U = 0$ and $\frac{\partial V}{\partial y} = 0$. Then

$$U_{l,0} = 0, \tag{4.18}$$

and

$$\left(\frac{\partial V}{\partial y}\right)_{l,0} = 0. \tag{4.19}$$

From equation (4.19) are approximated by using the forward difference method, we obtain

$$\frac{V_{l,1} - V_{l,0}}{\Delta y} = 0. \tag{4.20}$$

Then

$$V_{l,0} = V_{l,1}. \tag{4.21}$$

From equations (4.18) and (4.21), we have

$$W_{l,0} = \begin{bmatrix} Z_{l,0} \\ 0 \\ V_{l,1} \end{bmatrix}. \tag{4.22}$$

From the boundary condition , we have $V = 0$ and $\frac{\partial U}{\partial x} = 0$. Then

$$V_{L,1} = 0, \tag{4.23}$$

and

$$\left(\frac{\partial U}{\partial x}\right)_{L,1} = 0. \tag{4.24}$$

From equation (4.24) are approximated by using the forward difference method, we obtain

$$\frac{U_{L,1}-U_{L-1,1}}{\Delta x}=0. \tag{4.25}$$

Then
$$U_{L,1}=U_{L-1,1}. \tag{4.26}$$

From equations (4.23) and (4.26), we have

$$W_{L,1}=\begin{bmatrix} Z_{L,1} \\ U_{L-1,1} \\ 0 \end{bmatrix}. \tag{4.27}$$

Case4: If $l=1$ and $m=2,\ldots,M-2$, then

$$\begin{aligned}
W_{1,m}^{n+1} &= W_{1,m}^{n}+\frac{1}{2}pA_{1,m}(W_{2,m}^{n}-W_{0,m}^{n})+\frac{1}{2}pB_{1,m}(W_{1,m+1}^{n}-W_{1,m-1}^{n}) \\
&+\frac{1}{4}p^{2}A_{1,m}\Big[(A_{2,m}W_{2,m}^{n}-A_{1,m}W_{1,m}^{n})-(A_{2m}W_{1,m}^{n}-A_{1,m}W_{0,m}^{n}) \\
&+(A_{1,m}W_{2,m}^{n}-A_{0,m}W_{1,m}^{n})-(A_{1,m}W_{1,m}^{n}-A_{0,m}W_{0,m}^{n})\Big] \\
&+\frac{1}{4}p^{2}B_{1,m}\Big[(B_{1,m+1}W_{1,m+1}^{n}-B_{1,m}W_{1,m}^{n})-(B_{1,m+1}W_{1,m}^{n}-B_{1,m}W_{1,m-1}^{n}) \\
&+(B_{1,m}W_{1,m+1}^{n}-B_{1,m-1}W_{1,m}^{n})-(B_{1,m}W_{1,m}^{n}-B_{1,m-1}W_{1,m-1}^{n})\Big] \\
&+\frac{1}{8}p^{2}(A_{1,m}B_{1,m}+B_{1,m}A_{1,m})(W_{2,m+1}^{n}-W_{0,m+1}^{n}-W_{2,m-1}^{n}+W_{0,m-1}^{n}).
\end{aligned} \tag{4.28}$$

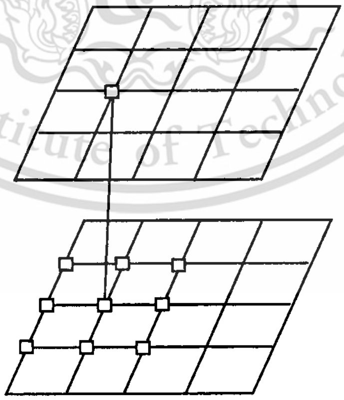


Fig.4.4: The stencil of case 4

From the boundary condition , we have $V=0$ and $\frac{\partial U}{\partial x}=0$. Then

$$V_{0,m}=0, \tag{4.29}$$

and

$$\left(\frac{\partial U}{\partial x}\right)_{0,m} = 0\,. \tag{4.30}$$

From equation (4.30) are approximate by using the forward difference method, we obtain

$$\frac{U_{1,m} - U_{0,m}}{\Delta x} = 0\,. \tag{4.31}$$

Then

$$U_{0,m} = U_{1,m}\,. \tag{4.32}$$

From equations (4.29) and (4.32), we have

$$W_{0,m} = \begin{bmatrix} Z_{0,m} \\ U_{1,m} \\ 0 \end{bmatrix}. \tag{4.33}$$

Case5: If $l = 2,\dots,L-2$ and $m = 2,\dots,M-2$, then

$$\begin{aligned} W_{l,m}^{n+1} = {} & W_{l,m}^{1} + \frac{1}{2}pA_{l,m}(W_{l+1,m}^{n} - W_{l-1,m}^{n}) + \frac{1}{2}pB_{l,m}(W_{l,m+1}^{n} - W_{l,m-1}^{n}) \\ & + \frac{1}{4}p^2 A_{l,m}\Big[(A_{l+1,m}W_{l+1,m}^{n} - A_{l,m}W_{l,m}^{n}) - (A_{l+1,m}W_{l,m}^{n} - A_{l,m}W_{l-1,m}^{n}) \\ & + (A_{l,m}W_{l+1,m}^{n} - A_{l-1,m}W_{l,m}^{n}) - (A_{l,m}W_{l,m}^{n} - A_{l-1,m}W_{l-1,m}^{n})\Big] \\ & + \frac{1}{4}p^2 B_{l,m}\Big[(B_{l,m+1}W_{l,m+1}^{n} - B_{l,m}W_{l,m}^{n}) - (B_{l,m+1}W_{l,m}^{n} - B_{l,m}W_{l,m-1}^{n}) \\ & + (B_{l,m}W_{l,m+1}^{n} - B_{l,m-1}W_{l,m}^{n}) - (B_{l,m}W_{l,m}^{n} - B_{l,m-1}W_{l,m-1}^{n})\Big] \\ & + \frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(W_{l+1,m+1}^{n} - W_{l-1,m+1}^{n} - W_{l+1,m-1}^{n} + W_{l-1,m-1}^{n}). \end{aligned} \tag{4.34}$$

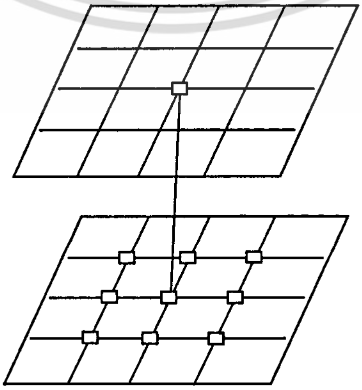


Fig.4.5: The stencil of case 5

Case6: If $l = L-1$ and $m = 2,\dots,M-2$, then

$$\begin{aligned}W_{l,m}^{n+1} &= W_{l,m}^{n} + \frac{1}{2}pA_{l,m}(W_{l+1,m}^{n} - W_{l-1,m}^{n}) + \frac{1}{2}pB_{l,m}(W_{l,m+1}^{n} - W_{l,m-1}^{n}) \\ &+\frac{1}{4}p^2 A_{l,m}\Big[(A_{l+1,m}W_{l+1,m}^{n} - A_{l,m}W_{l,m}^{n}) - (A_{l+1,m}W_{l,m}^{n} - A_{l,m}W_{l-1,m}^{n}) \\ &+(A_{l,m}W_{l+1,m}^{n} - A_{l-1,m}W_{l,m}^{n}) - (A_{l,m}W_{l,m}^{n} - A_{l-1,m}W_{l-1,m}^{n})\Big] \\ &+\frac{1}{4}p^2 B_{l,m}\Big[(B_{l,m+1}W_{l,m+1}^{n} - B_{l,m}W_{l,m}^{n}) - (B_{l,m+1}W_{l,m}^{n} - B_{l,m}W_{l,m-1}^{n}) \\ &+(B_{l,m}W_{l,m+1}^{n} - B_{l,m-1}W_{l,m}^{n}) - (B_{l,m}W_{l,m}^{n} - B_{l,m-1}W_{l,m-1}^{n})\Big] \\ &+\frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(W_{l+1,m+1}^{n} - W_{l-1,m+1}^{n} - W_{l+1,m-1}^{n} + W_{l-1,m-1}^{n}). \qquad (4.35)\end{aligned}$$

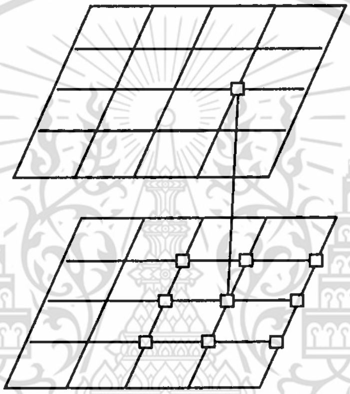


Fig.4.6: The stencil of case 6

From the boundary condition, we have $V = 0$ and $\frac{\partial U}{\partial x} = 0$. Then

$$V_{L,m} = 0, \qquad (4.36)$$

and

$$\left(\frac{\partial U}{\partial x}\right)_{L,m} = 0. \qquad (4.37)$$

From equation (4.37) are approximated by usingthe backward difference method, we obtain

$$\frac{U_{L,m} - U_{L-1,m}}{\Delta x} = 0. \qquad (4.38)$$

Then

$$U_{L,m} = U_{L-1,m}. \qquad (4.39)$$

From equations (4.36) and (4.39), we have

$$W_{L,m} = \begin{bmatrix} Z_{L,m} \\ U_{L-1,m} \\ 0 \end{bmatrix}. \qquad (4.40)$$

Case7: If $l = 1$ and $m = M - 1$, then

$$\begin{aligned} W_{1,m}^{n+1} &= W_{1,m}^{n} + \frac{1}{2} p A_{1,m} (W_{2,m}^{n} - W_{0,m}^{n}) + \frac{1}{2} p B_{1,m} (W_{1,m+1}^{n} - W_{1,m-1}^{n}) \\ &+ \frac{1}{4} p^2 A_{1,m} \Big[(A_{2,m} W_{2,m}^{n} - A_{1,m} W_{1,m}^{n}) - (A_{2,m} W_{1,m}^{n} - A_{1,m} W_{0,m}^{n}) \\ &+ (A_{1,m} W_{2,m}^{n} - A_{0,m} W_{1,m}^{n}) - (A_{1,m} W_{1,m}^{n} - A_{0,m} W_{0,m}^{n}) \Big] \\ &+ \frac{1}{4} p^2 B_{1,m} \Big[(B_{1,m+1} W_{1,m+1}^{n} - B_{1,m} W_{1,m}^{n}) - (B_{1,m+1} W_{1,m}^{n} - B_{1,m} W_{1,m-1}^{n}) \\ &+ (B_{1,m} W_{1,m+1}^{n} - B_{1,m-1} W_{1,m}^{n}) - (B_{1,m} W_{1,m}^{n} - B_{1,m-1} W_{1,m-1}^{n}) \Big] \\ &+ \frac{1}{8} p^2 (A_{1,m} B_{1,m} + B_{1,m} A_{1,m}) (W_{2,m+1}^{n} - W_{0,m+1}^{n} - W_{2,m-1}^{n} + W_{0,m-1}^{n}). \end{aligned} \tag{4.41}$$

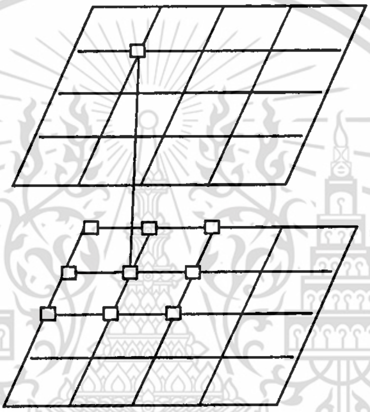


Fig.4.7: The stencil of case 7

From the boundary condition , we have $V = 0$ and $\frac{\partial U}{\partial x} = 0$. Then

$$V_{0,M-1} = 0, \tag{4.42}$$

and

$$\left(\frac{\partial U}{\partial x} \right)_{0,M-1} = 0. \tag{4.43}$$

From equation (4.43) are approximated by using the forward difference method, we obtain

$$\frac{U_{1,M-1} - U_{0,M-1}}{\Delta x} = 0. \tag{4.44}$$

Then
$$U_{0,M-1} = U_{1,M-1}. \tag{4.45}$$

From equations (4.42) and (4.45), we have

$$W_{0,M-1}=\begin{bmatrix} Z_{0,M-1} \\ U_{1,M-1} \\ 0 \end{bmatrix}. \tag{4.46}$$

From the boundary condition , we have $U=0$ and $\dfrac{\partial V}{\partial y}=0$. Then

$$U_{1,M}=0, \tag{4.47}$$

and

$$\left(\frac{\partial V}{\partial y}\right)_{1,M}=0. \tag{4.48}$$

From equation (4.48) are approximated by using the forward difference method, we obtain

$$\frac{V_{1,M}-V_{1,M-1}}{\Delta y}=0. \tag{4.49}$$

Then

$$V_{1,M}=V_{1,M-1}. \tag{4.50}$$

From equations (4.47) and (4.50), we have

$$W_{1,M}=\begin{bmatrix} Z_{1,M} \\ 0 \\ V_{1,M-1} \end{bmatrix}. \tag{4.51}$$

Case8: If $l=2,\ldots,L-2$ and $m=M-1$, then

$$\begin{aligned} W_{l,m}^{n+1} &= W_{l,m}^{n}+\frac{1}{2}pA_{l,m}(W_{l+1,m}^{n}-W_{l-1,m}^{n})+\frac{1}{2}pB_{l,m}(W_{l,m+1}^{n}-W_{l,m-1}^{n}) \\ &\quad +\frac{1}{4}p^{2}A_{l,m}\Big[(A_{l+1,m}W_{l+1,m}^{n}-A_{l,m}W_{l,m}^{n})-(A_{l+1,m}W_{l,m}^{n}-A_{l,m}W_{l-1,m}^{n}) \\ &\quad +(A_{l,m}W_{l+1,m}^{n}-A_{l-1,m}W_{l,m}^{n})-(A_{l,m}W_{l,m}^{n}-A_{l-1,m}W_{l-1,m}^{n})\Big] \\ &\quad +\frac{1}{4}p^{2}B_{l,m}\Big[(B_{l,m+1}W_{l,m+1}^{n}-B_{l,m}W_{l,m}^{n})-(B_{l,m+1}W_{l,m}^{n}-B_{l,m}W_{l,m-1}^{n}) \\ &\quad +(B_{l,m}W_{l,m+1}^{n}-B_{l,m-1}W_{l,m}^{n})-(B_{l,m}W_{l,m}^{n}-B_{l,m-1}W_{l,m-1}^{n})\Big] \\ &\quad +\frac{1}{8}p^{2}(A_{l,m}B_{l,m}+B_{l,m}A_{l,m})(W_{l+1,m+1}^{n}-W_{l-1,m+1}^{n}-W_{l+1,m-1}^{n}+W_{l-1,m-1}^{n}). \end{aligned} \tag{4.52}$$

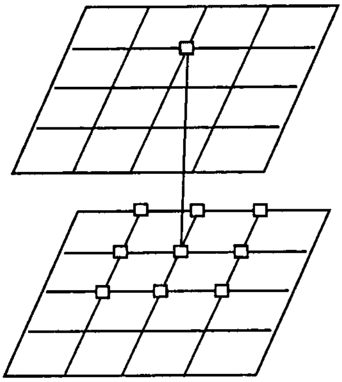


Fig.4.8: The stencil of case 8

From the boundary condition , we have $U=0$ and $\frac{\partial V}{\partial y}=0$. Then

$$U_{l,M}=0, \tag{4.53}$$

and

$$\left(\frac{\partial V}{\partial y}\right)_{l,M}=0. \tag{4.54}$$

From equation (4.54) are approximated by using the backward difference method, we obtain

$$\frac{V_{l,M}-V_{l,M-1}}{\Delta y}=0. \tag{4.55}$$

Then

$$V_{l,M}=V_{l,M-1}. \tag{4.56}$$

From equations (4.53) and (4.56), we have

$$W_{l,M}=\begin{bmatrix} Z_{l,M} \\ 0 \\ V_{l,M-1} \end{bmatrix}. \tag{4.57}$$

Case9: If $l=L-1$ and $m=M-1$, then

$$\begin{aligned} W_{l,m}^{n+1} &= W_{l,m}^{n}+\frac{1}{2}pA_{l,m}(W_{l+1,m}^{n}-W_{l-1,m}^{n})+\frac{1}{2}pB_{l,m}(W_{l,m+1}^{n}-W_{l,m-1}^{n}) \\ &+\frac{1}{4}p^{2}A_{l,m}\Big[(A_{l+1,m}W_{l+1,m}^{n}-A_{l,m}W_{l,m}^{n})-(A_{l+1,m}W_{l,m}^{n}-A_{l,m}W_{l-1,m}^{n}) \\ &+(A_{l,m}W_{l+1,m}^{n}-A_{l-1,m}W_{l,m}^{n})-(A_{l,m}W_{l,m}^{n}-A_{l-1,m}W_{l-1,m}^{n})\Big] \\ &+\frac{1}{4}p^{2}B_{l,m}\Big[(B_{l,m+1}W_{l,m+1}^{n}-B_{l,m}W_{l,m}^{n})-(B_{l,m+1}W_{l,m}^{n}-B_{l,m}W_{l,m-1}^{n}) \\ &+(B_{l,m}W_{l,m+1}^{n}-B_{l,m-1}W_{l,m}^{n})-(B_{l,m}W_{l,m}^{n}-B_{l,m-1}W_{l,m-1}^{n})\Big] \\ &+\frac{1}{8}p^{2}(A_{l,m}B_{l,m}+B_{l,m}A_{l,m})(W_{l+1,m+1}^{n}-W_{l-1,m+1}^{n}-W_{l+1,m-1}^{n}+W_{l-1,m-1}^{n}). \end{aligned} \tag{4.58}$$

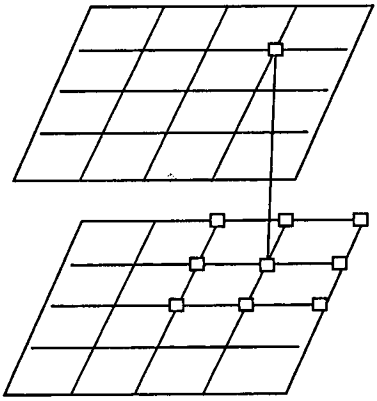


Fig.4.9: The stencil of case 9

From the boundary condition , we have $V = 0$ and $\frac{\partial U}{\partial x} = 0$. Then

$$V_{L,M-1} = 0, \tag{4.59}$$

and

$$\left(\frac{\partial U}{\partial x}\right)_{L,M-1} = 0. \tag{4.60}$$

From equation (4.60) are approximated by using the backward difference method, we obtain

$$\frac{U_{L,M-1} - U_{L-1,M-1}}{\Delta x} = 0. \tag{4.61}$$

Then

$$U_{L,M-1} = U_{L-1,M-1}. \tag{4.62}$$

From equations (4.59) and (4.62), we have

$$W_{L,M-1} = \begin{bmatrix} Z_{L,M-1} \\ U_{L-1,M-1} \\ 0 \end{bmatrix}. \tag{4.63}$$

From the boundary condition , we have $U = 0$ and $\frac{\partial V}{\partial y} = 0$. Then

$$U_{L-1,M} = 0, \tag{4.64}$$

and

$$\left(\frac{\partial V}{\partial y}\right)_{L-1,M} = 0. \tag{4.65}$$

From equation (4.65) are approximated by using the backward difference, we obtain

$$\frac{V_{L-1,M} - V_{L-1,M-1}}{\Delta y} = 0\,. \tag{4.66}$$

Then
$$V_{L-1,M} = V_{L-1,M-1}\,. \tag{4.67}$$

From equations (4.64) and (4.67), we have

$$W_{L-1,M} = \begin{bmatrix} Z_{L-1,M} \\ 0 \\ V_{L-1,M-1} \end{bmatrix}. \tag{4.68}$$

4.2 Numerical treatment of a dispersion model

We now introduces calculation the forward difference in time and backward difference in space method, it obtained the general form into 9 cases as below.

Case1: If $l = 1$ and $m = 1$, then

$$C_{1,1}^{n+1} = \left(\lambda D + \frac{\Delta t}{\Delta x\left(1 + x_{1,1}y_{1,1}\right)} u_{1,1}^{n} \right) C_{0,1}^{n} + \left(-4\lambda D - \frac{\Delta t}{\Delta x\left(1 + x_{1,1}y_{1,1}\right)} u_{1,1}^{n} - \frac{\Delta t}{\Delta y\left(1 + x_{1,1}y_{1,1}\right)} v_{1,1}^{n} + 1 \right) C_{1,1}^{n}$$
$$+ \lambda D C_{2,1}^{n} + \left(\lambda D + \frac{\Delta t}{\Delta y\left(1 + x_{1,1}y_{1,1}\right)} V_{1,1}^{n} \right) C_{1,0}^{n} + \lambda D C_{1,2}^{n}\,. \tag{4.69}$$

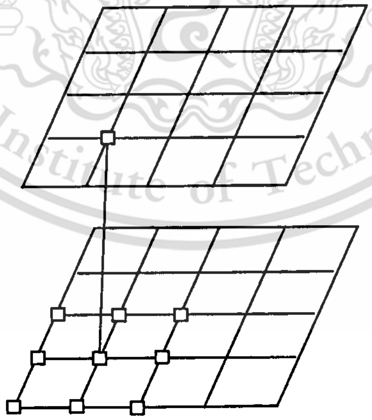


Fig.4.10: The stencil of case 1

From the boundary condition , we have $\frac{\partial C}{\partial x} = 0$. Then

$$\left(\frac{\partial C}{\partial x} \right)_{0,1} = 0\,. \tag{4.70}$$

From equation (4.70) are approximated by using the forward difference method, we obtain

$$\frac{C_{1,1}-C_{0,1}}{\Delta x}=0. \tag{4.71}$$

Then $$C_{0,1}=C_{1,1}. \tag{4.72}$$

From the boundary condition , we have $\frac{\partial C}{\partial y}=0$. Then

$$\left(\frac{\partial C}{\partial y}\right)_{1,0}=0. \tag{4.73}$$

From equation (4.73) are approximated by using the forward difference method, we obtain

$$\frac{C_{1,1}-C_{1,0}}{\Delta y}=0. \tag{4.74}$$

Then $$C_{1,0}=C_{1,1}. \tag{4.74}$$

Case2: If $l=2,\ldots,L-2$ and $m=1$, then

$$C_{l,1}^{n}=\left(\lambda D+\frac{\Delta t}{\Delta x\left(1+x_{l,1}y_{l,1}\right)}u_{l,1}^{n}\right)C_{l-1,1}^{n}+\left(-4\lambda D-\frac{\Delta t}{\Delta x\left(1+x_{l,1}y_{l,1}\right)}u_{l,1}^{n}-\frac{\Delta t}{\Delta y\left(1+x_{l,1}y_{l,1}\right)}v_{l,1}^{n}+1\right)C_{l,1}^{n}$$
$$+\lambda DC_{l+1,1}^{n}+\left(\lambda D+\frac{\Delta t}{\Delta y\left(1+x_{l,1}y_{l,1}\right)}v_{l,1}^{n}\right)C_{l,0}^{n}+\lambda DC_{l,2}^{n}. \tag{4.76}$$

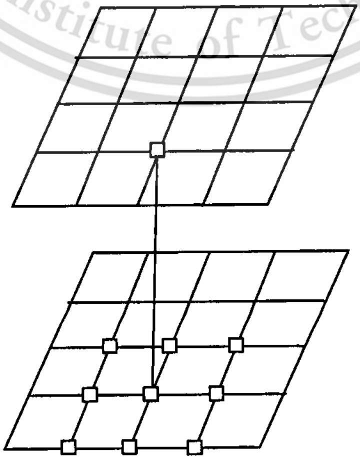


Fig.4.11: The stencil of case 2

From the boundary condition , we have $\dfrac{\partial C}{\partial y} = 0$. Then

$$\left(\frac{\partial C}{\partial y}\right)_{l,0} = 0\,. \tag{4.77}$$

From equation (4.77) are approximated by usingthe forward difference method, we obtain

$$\frac{C_{l,1} - C_{l,0}}{\Delta y} = 0\,. \tag{4.78}$$

Then
$$C_{l,0} = C_{l,1}\,. \tag{4.79}$$

Case3: If $l = L-1$ and $m = 1$, then

$$C^{n}_{l,1} = \left(\lambda D + \frac{\Delta t}{\Delta x\left(1 + x_{l,1}y_{l,1}\right)}u^{n}_{l,1}\right)C^{n}_{l-1,1} + \left(-4\lambda D - \frac{\Delta t}{\Delta x\left(1 + x_{l,1}y_{l,1}\right)}u^{n}_{l,1} - \frac{\Delta t}{\Delta y\left(1 + x_{l,1}y_{l,1}\right)}v^{n}_{l,1} + 1\right)C^{n}_{l,1}$$
$$+\lambda DC^{n}_{l+1,1} + \left(\lambda D + \frac{\Delta t}{\Delta y\left(1 + x_{l,1}y_{l,1}\right)}v^{n}_{l,1}\right)C^{n}_{l,0} + \lambda DC^{n}_{l,2}\,. \tag{4.80}$$

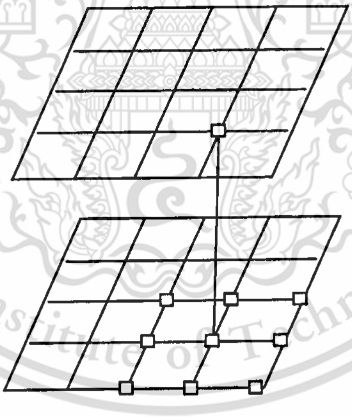


Fig.4.12: The stencil of case 3

From the boundary condition , we have $\dfrac{\partial C}{\partial y} = 0$. Then

$$\left(\frac{\partial C}{\partial y}\right)_{l,0} = 0\,. \tag{4.81}$$

From equation (4.81) are approximated by using the forward difference method, we obtain

$$\frac{C_{l,1} - C_{l,0}}{\Delta y} = 0\,. \tag{4.82}$$

Then
$$C_{l,0} = C_{l,1}. \tag{4.83}$$

From the boundary condition , we have $\dfrac{\partial C}{\partial x} = 0$. Then

$$\left(\frac{\partial C}{\partial x}\right)_{L,1} = 0. \tag{4.84}$$

From equation (4.84) are approximated by using the forward difference method, we obtain

$$\frac{C_{L,1} - C_{L-1,1}}{\Delta x} = 0. \tag{4.85}$$

Then
$$C_{L,1} = C_{L-1,1}. \tag{4.86}$$

Case4: If $l = 1$ and $m = 2,\ldots,M-2$, then

$$C_{1,m}^{n+1} = \left(\lambda D + \frac{\Delta t}{\Delta x\left(1 + x_{1,m}y_{1,m}\right)}u_{1,m}^{n}\right)C_{0,m}^{n} + \lambda D C_{2,m}^{n} + \left(\lambda D + \frac{\Delta t}{\Delta y\left(1 + x_{1,m}y_{1,m}\right)}v_{1,m}^{n}\right)C_{1,m-1}^{n} + \lambda D C_{1,m+1}^{n}$$
$$+\left(-4\lambda D - \frac{\Delta t}{\Delta x\left(1 + x_{1,m}y_{1,m}\right)}u_{1,m}^{n} - \frac{\Delta t}{\Delta y\left(1 + x_{1,m}y_{1,m}\right)}v_{1,m}^{n} + 1\right)C_{1,m}^{n}. \tag{4.87}$$

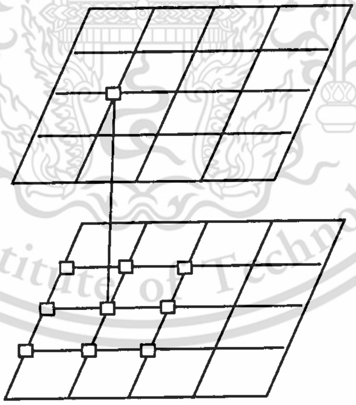


Fig.4.13: The stencil of case 4

From the boundary condition , we have $\dfrac{\partial C}{\partial x} = 0$. Then

$$\left(\frac{\partial C}{\partial x}\right)_{0,m} = 0. \tag{4.88}$$

From equation (4.88) are approximated by using the forward difference method, we obtain

$$\frac{C_{1,m}-C_{0,m}}{\Delta x}=0. \qquad (4.89)$$

Then
$$C_{0,m}=C_{1,m}. \qquad (4.90)$$

Case5:If l=2,…,L-2 and m=2,…,M-2 , then

$$C_{l,m}^{n+1}=\left(\lambda D+\frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n}\right)C_{l-1,m}^{n}+\lambda DC_{l+1,m}^{1}+\left(\lambda D+\frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n}\right)C_{l,m-1}^{n}+\lambda DC_{l,m+1}^{n}$$
$$+\left(-4\lambda D-\frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n}-\frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n}+1\right)C_{l,m}^{n}. \qquad (4.91)$$

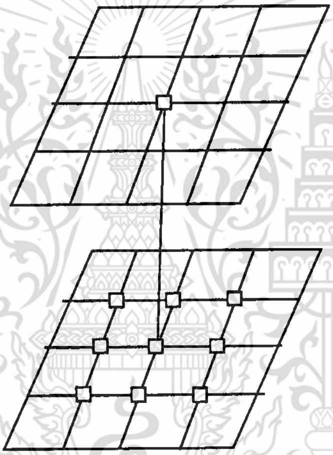


Fig.4.14: The stencil of case 5

Case6: If $l=L-1$ and $m=2,\ldots,M-2$, then

$$C_{l,m}^{n+1}=\left(\lambda D+\frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n}\right)C_{l-1,m}^{n}+\lambda DC_{l+1,m}^{1}+\left(\lambda D+\frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n}\right)C_{l,m-1}^{n}+\lambda DC_{l,m+1}^{n}$$
$$+\left(-4\lambda D-\frac{\Delta t}{\Delta x\left(1+x_{l,m}y_{l,m}\right)}u_{l,m}^{n}-\frac{\Delta t}{\Delta y\left(1+x_{l,m}y_{l,m}\right)}v_{l,m}^{n}+1\right)C_{l,m}^{n}. \qquad (4.92)$$

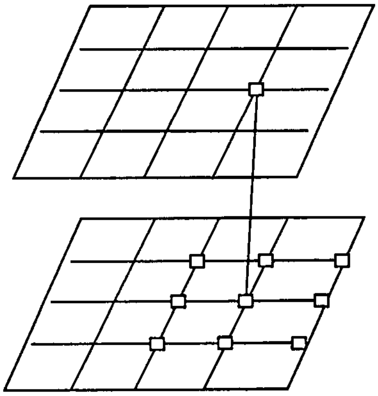


Fig.4.15: The stencil of case 6

From the boundary condition , we have $\frac{\partial C}{\partial x}=0$. Then

$$\left(\frac{\partial C}{\partial x}\right)_{L,m}=0. \tag{4.93}$$

From equation (4.93) are approximate by using the backward difference method, we obtain

$$\frac{C_{L,m}-C_{L-1,m}}{\Delta x}=0. \tag{4.94}$$

Then
$$C_{L,m}=C_{L-1,m}. \tag{4.95}$$

Case7: If $l=1$ and $m=M-1$, then

$$C_{1,m}^{n+1}=\left(\lambda D+\frac{\Delta t}{\Delta x\left(1+x_{1,m}y_{1,m}\right)}u_{1,m}^{n}\right)C_{0,m}^{n}+\lambda DC_{2,m}^{n}+\left(\lambda D+\frac{\Delta t}{\Delta y\left(1+x_{1,m}y_{1,m}\right)}v_{1,m}^{n}\right)C_{1,m-1}^{n}+\lambda DC_{1,m+1}^{n}$$
$$+\left(-4\lambda D-\frac{\Delta t}{\Delta x\left(1+x_{1,m}y_{1,m}\right)}u_{1,m}^{n}-\frac{\Delta t}{\Delta y\left(1+x_{1,m}y_{1,m}\right)}v_{1,m}^{n}+1\right)C_{1,m}^{n}. \tag{4.96}$$

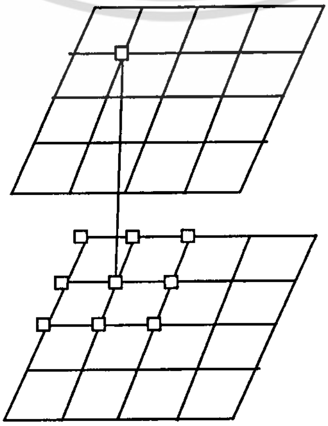


Fig.4.16: The stencil of case 7

From the boundary condition , we have $\frac{\partial C}{\partial x} = 0$. Then

$$\left(\frac{\partial C}{\partial x}\right)_{0,M-1} = 0. \tag{4.97}$$

From equation (4.97) are approximated by using the forward difference method, we obtain

$$\frac{C_{1,M-1} - C_{0,M-1}}{\Delta x} = 0. \tag{4.98}$$

Then $$C_{0,M-1} = C_{1,M-1}. \tag{4.99}$$

From the boundary condition , we have $\frac{\partial C}{\partial y} = 0$. Then

$$\left(\frac{\partial C}{\partial y}\right)_{1,M} = 0. \tag{4.100}$$

From equation (4.100) are approximated by using the forward difference method, we obtain

$$\frac{C_{1,M} - C_{1,M-1}}{\Delta y} = 0. \tag{4.101}$$

Then $$C_{1,M} = C_{1,M-1}. \tag{4.102}$$

Case8: If $l = 2,\ldots,L-2$ and $m = M-1$, then

$$C_{l,m}^{n+1} = \left(\lambda D + \frac{\Delta t}{\Delta x\left(1 + x_{l,m}y_{l,m}\right)}u_{l,m}^{n}\right)C_{l-1,m}^{n} + \lambda D C_{l+1,m}^{1} + \left(\lambda D + \frac{\Delta t}{\Delta y\left(1 + x_{l,m}y_{l,m}\right)}v_{l,m}^{n}\right)C_{l,m-1}^{n} + \lambda D C_{l,m+1}^{n}$$
$$+\left(-4\lambda D - \frac{\Delta t}{\Delta x\left(1 + x_{l,m}y_{l,m}\right)}u_{l,m}^{n} - \frac{\Delta t}{\Delta y\left(1 + x_{l,m}y_{l,m}\right)}v_{l,m}^{n} + 1\right)C_{l,m}^{n}. \tag{4.103}$$

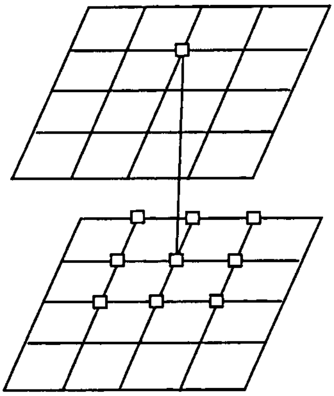


Fig.4.17: The stencil of case 8

From the boundary condition , we have $\frac{\partial C}{\partial y} = 0$. Then

$$\left(\frac{\partial C}{\partial y}\right)_{l,M} = 0. \tag{4.104}$$

From equation (4.104) are approximated by using the backward difference method, we obtain

$$\frac{C_{l,M} - C_{l,M-1}}{\Delta y} = 0. \tag{4.105}$$

Then
$$C_{l,M} = C_{l,M-1}. \tag{4.106}$$

Case9: If $l = L-1$ and $m = M-1$, then

$$C_{l,m}^{n+1} = \left(\lambda D + \frac{\Delta t}{\Delta x\left(1 + x_{l,m}y_{l,m}\right)}u_{l,m}^{n}\right)C_{l-1,m}^{n} + \lambda D C_{l+1,m}^{1} + \left(\lambda D + \frac{\Delta t}{\Delta y\left(1 + x_{l,m}y_{l,m}\right)}v_{l,m}^{n}\right)C_{l,m-1}^{n} + \lambda D C_{l,m+1}^{n}$$
$$+\left(-4\lambda D - \frac{\Delta t}{\Delta x\left(1 + x_{l,m}y_{l,m}\right)}u_{l,m}^{n} - \frac{\Delta t}{\Delta y\left(1 + x_{l,m}y_{l,m}\right)}v_{l,m}^{n} + 1\right)C_{l,m}^{n}. \tag{4.107}$$

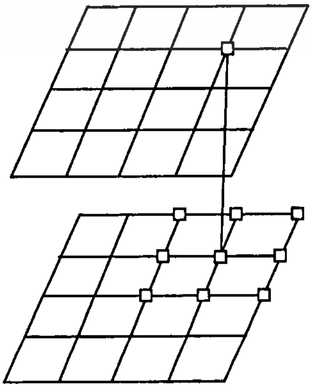


Fig.4.18: The stencil of case 9

From the boundary condition , we have $\frac{\partial C}{\partial x} = 0$. Then

$$\left(\frac{\partial C}{\partial x}\right)_{L,M-1} = 0. \tag{4.108}$$

From equation (4.108) are approximated by using the backward difference method, we obtain

$$\frac{C_{L,M-1} - C_{L-1,M-1}}{\Delta x} = 0. \tag{4.109}$$

Then $$C_{L,M-1} = C_{L-1,M-1}. \tag{4.110}$$

From the boundary condition , we have $\frac{\partial C}{\partial y} = 0$. Then

$$\left(\frac{\partial C}{\partial y}\right)_{L-1,M} = 0. \tag{4.111}$$

From equation (4.111) are approximated by using the backward difference method, we obtain

$$\frac{C_{L-1,M} - C_{L-1,M-1}}{\Delta y} = 0. \tag{4.112}$$

Then $$C_{L-1,M} = C_{L-1,M-1}. \tag{4.113}$$

4.3. Numerical Experiment

Example 1: We consider the uniform reservoir with dimension 3.2×3.2 km. and the constant depth $h = 1$ m. The reservoir is meshed with 400 grids points with $\Delta x = \Delta y = 320$ m. and taking time interval $\Delta t = 10.222$ sec. The diffusion coefficient is given by $\mathrm{D} = 0.009876\, m^2$ /sec.In order to simulate the problem, we change the variables into dimensionless form of equations (2.19)-(2.21). Initially the water in the reservoir is assumed to be motionless $u = 0$ and $v = 0$, and the water elevation is assumed to be a function that satisfied the initial and boundary conditions as $\mathrm{z}(x, \mathrm{y}, 0) = f(x, \mathrm{y}) = x(1-x)\,\mathrm{y}(1-\mathrm{y})$. The pollutant concentration in the reservoir is assumed to be a function that satisfied the initial and boundary conditions as $\mathrm{c}(x, \mathrm{y}, 0) = g(x, \mathrm{y}) = x(1-x)\,\mathrm{y}(1-\mathrm{y})$ and $\frac{\partial C}{\partial n} = 0$, non-absorbing boundary of the reservoir, respectively.

Table 4.1: The elevation when $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta t = 0.01$ at $T = 1$

x/y	0	320	960	1600	2240	2880	3200
0	0	0	0	0	0	0	0
320	0	-0.0167	-0.0082	-0.0001	-0.0101	-0.0166	0
960	0	-0.0082	0.0041	0.0087	0.0020	-0.0035	0
1600	0	-0.0001	0.0087	0.0094	0.0055	-0.0001	0
2240	0	-0.0101	0.0020	0.0055	0.0011	-0.0049	0
2880	0	-0.0166	-0.0035	-0.0001	-0.0049	-0.0100	0
3200	0	0	0	0	0	0	0

Table 4.2: The velocity in the direction of the axis x when $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta t = 0.01$ at $T = 1$

x/y	0	320	960	1600	2240	2880	3200
0	0	0	0	0	0	0	0
320	0	0.0163	0.0262	0.0227	0.0306	0.0269	0
960	0	0.0052	-0.0012	-0.0042	-0.0007	-0.0040	0
1600	0	-0.0028	-0.0064	-0.0039	-0.0077	-0.0102	0
2240	0	-0.0055	-0.0059	-0.0043	-0.0119	-0.0140	0
2880	0	-0.0141	-0.0182	-0.0188	-0.0276	-0.0294	0
3200	0	0	0	0	0	0	0

Table 4.3: The velocity in the direction of the axis y when $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta t = 0.01$ at $T = 1$

x/y	0	320	960	1600	2240	2880	3200
0	0	0	0	0	0	0	0
320	0	0.0163	0.0052	-0.0028	-0.0055	-0.0141	0
960	0	0.0262	-0.0012	-0.0064	-0.0059	-0.0182	0
1600	0	0.0227	-0.0042	-0.0039	-0.0043	-0.0188	0
2240	0	0.0306	-0.0007	-0.0077	-0.0119	-0.0276	0
2880	0	0.0269	-0.0040	-0.0102	-0.0140	-0.0294	0
3200	0	0	0	0	0	0	0

Table 4.4: The concentrations $C(\mathrm{x}, \mathrm{t})$ of water pollutant when $\Delta x = 0.1$, $\Delta y = 0.1$ $\Delta t = 0.01$ at $T = 1$ $\left(kg/m^3\right)$

x/y	0	320	960	1600	2240	2880	3200
0	0.0005	0.0005	0.0009	0.0011	0.0009	0.0003	0.0003
320	0.0005	0.0005	0.0009	0.0011	0.0009	0.0003	0.0003
960	0.0009	0.0009	0.0023	0.0027	0.0021	0.0008	0.0008
1600	0.0011	0.0011	0.0027	0.0032	0.0026	0.0010	0.0010
2240	0.0009	0.0009	0.0021	0.0026	0.0021	0.0008	0.0008
2880	0.0003	0.0003	0.0008	0.0010	0.0008	0.0003	0.0003
3200	0.0003	0.0003	0.0008	0.0010	0.0008	0.0003	0.0003

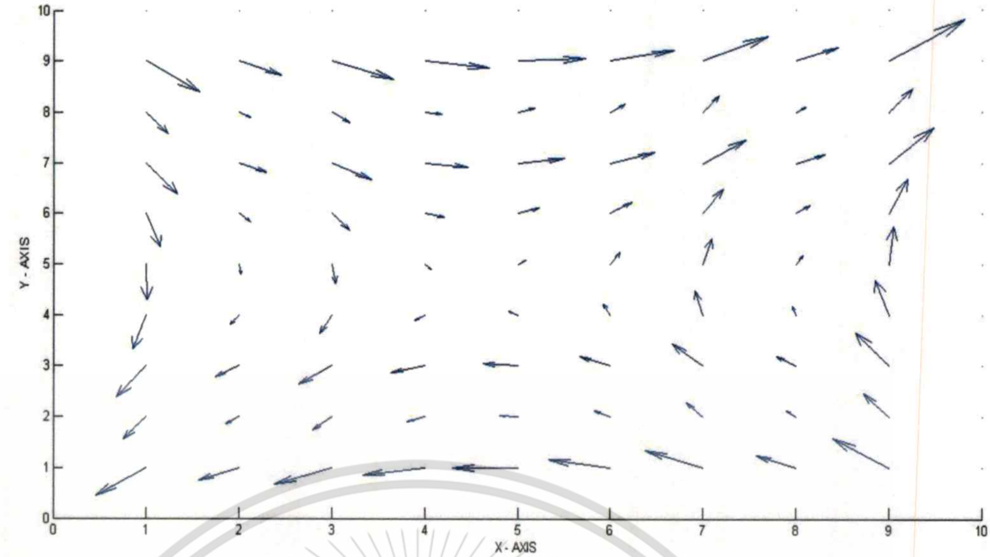


Fig.4.19: The velocity vector when $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta t = 0.01$ at $T = 1$

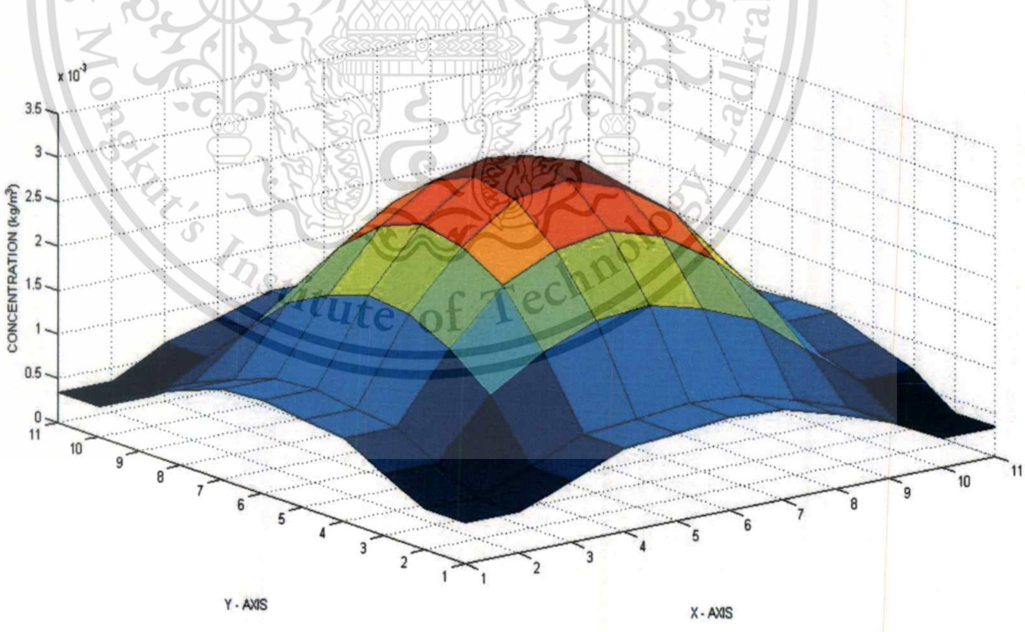


Fig.4.20: The surface of the water pollutant $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta t = 0.01$ at $T = 1$ $\left(kg/m^3\right)$

Example 2: We consider the uniform reservoir with dimension 3.2×3.2 km. and the constant depth $h = 1$ m. The reservoir is meshed with 100 grids points with $\Delta x = \Delta y = 160$ m. and taking time interval $\Delta t = 5.111$ sec. The diffusion coefficient is given by $\mathrm{D} = 0.009876\, m^2$ /sec.In order to simulate the problem, we change the variables into dimensionless form of equations (2.19)-(2.21). Initially the water in the reservoir is assumed to be motionless $u = 0$ and $v = 0$, and the water elevation is assumed to be a function that satisfied the initial and boundary conditions as $\mathrm{z}(x, \mathrm{y}, 0) = f(x, \mathrm{y}) = x(1-x)\,\mathrm{y}(1-\mathrm{y})$. The pollutant concentration in the reservoir is assumed to be a function that satisfied the initial and boundary conditions as $\mathrm{c}(x, \mathrm{y}, 0) = g(x, \mathrm{y}) = x(1-x)\,\mathrm{y}(1-\mathrm{y})$ and $\dfrac{\partial C}{\partial n} = 0$, non-absorbing boundary of the reservoir, respectively.

Table 4.5:The elevation when $\Delta x = 0.05,\ \Delta y = 0.05, \Delta t = 0.005$ at $T = 1$

x,y	0	480	960	1760	2080	2880	3200
0	0	0	0	0	0	0	0
480	0	0.0297	0.0118	0.0079	0.0135	0.0064	0
960	0	0.0118	-0.0128	-0.0210	-0.0104	-0.0041	0
1760	0	0.0079	-0.0210	-0.0247	-0.0124	-0.0045	0
2080	0	0.0135	-0.0104	-0.0124	-0.0033	-0.0012	0
2880	0	0.0064	-0.0041	-0.0045	-0.0012	-0.0002	0
3200	0	0	0	0	0	0	0

Table 4.6: The velocity in the direction of the axis x when $\Delta x = 0.05,\ \Delta y = 0.05, \Delta t = 0.005$ at $T = 1$

x,y	0	480	960	1760	2080	2880	3200
0	0	0	0	0	0	0	0
480	0	-0.0872	-0.0646	-0.0648	-0.0669	-0.0319	0
960	0	-0.0296	-0.0006	0.0149	0.0084	0.0011	0
1760	0	0.0195	0.0171	0.0175	0.0159	0.0070	0
2080	0	0.0424	0.0257	0.0221	0.0252	0.0133	0
2880	0	0.0200	0.0056	0.0051	0.0102	0.0077	0
3200	0	0	0	0	0	0	0

Table 4.7: The velocity in the direction of the axis y when $\Delta x = 0.05,\ \Delta y = 0.05, \Delta t = 0.005$ at $T = 1$

x,y	0	480	960	1760	2080	2880	3200
0	0	0	0	0	0	0	0
480	0	-0.0872	-0.0296	0.0195	0.0424	0.0200	0
960	0	-0.0646	-0.0006	0.0171	0.0257	0.0056	0
1760	0	-0.0648	0.0149	0.0175	0.0221	0.0051	0
2080	0	-0.0669	0.0084	0.0159	0.0252	0.0102	0
2880	0	-0.0319	0.0011	0.0070	0.0133	0.0077	0
3200	0	0	0	0	0	0	0

Table 4.8: The concentrations $C(\mathrm{x},\mathrm{t})$ of water pollutant when $\Delta x = 0.05,\ \Delta y = 0.05$ $\Delta t = 0.005$ at $T = 1\ \left(kg / m^3\right)$

x,y	0	480	960	1760	2080	2880	3200
0	0.0008	0.0015	0.0016	0.0007	0.0005	0.0001	0.0000
480	0.0015	0.0030	0.0037	0.0023	0.0017	0.0004	0.0001
960	0.0016	0.0037	0.0054	0.0046	0.0036	0.0010	0.0004
1760	0.0007	0.0023	0.0046	0.0062	0.0057	0.0021	0.0011
2080	0.0005	0.0017	0.0036	0.0057	0.0056	0.0023	0.0013
2880	0.0001	0.0004	0.0010	0.0021	0.0023	0.0011	0.0006
3200	0.0000	0.0001	0.0004	0.0011	0.0013	0.0006	0.0004

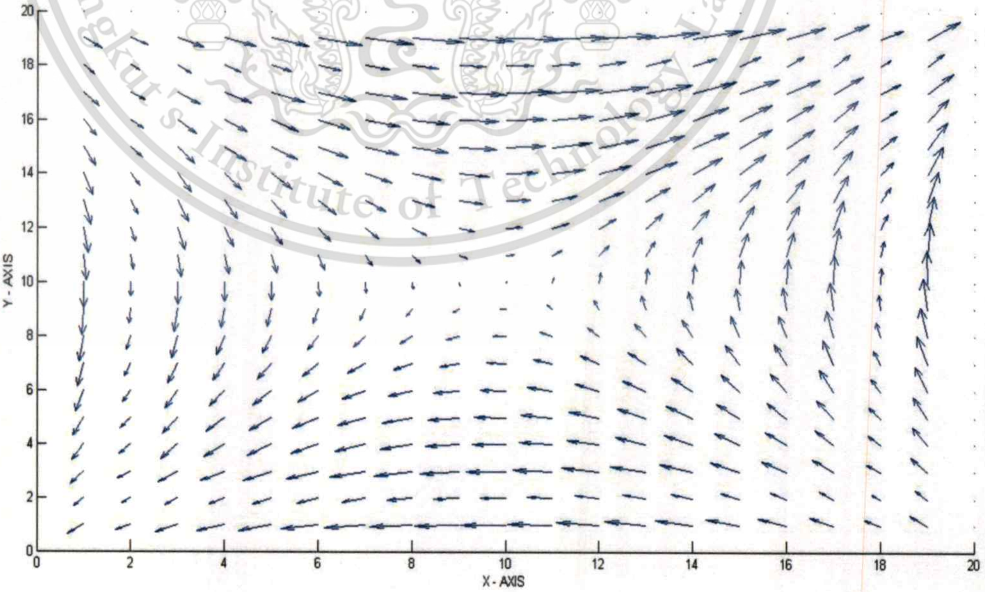


Fig.4.21: The velocity vector when $\Delta x = 0.05,\ \Delta y = 0.05, \Delta t = 0.005$ at $T = 1$

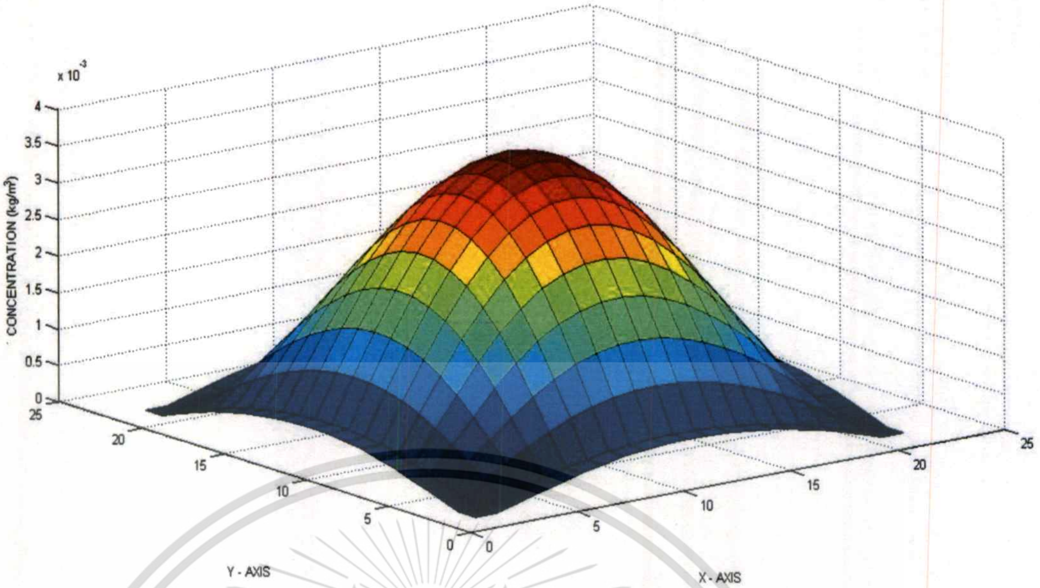


Fig.4.22: The surface of the water pollutant $\Delta x = 0.05,\ \Delta y = 0.05, \Delta t = 0.01$ at $T = 1\ \left(kg / m^3\right)$

Example3: We consider the uniform reservoir with dimension 3.2×3.2 km. and the constant depth $h = 1$ m. The reservoir is meshed with 1024 grids points with $\Delta x = \Delta y = 100$ m. and taking time interval $\Delta t = 3.190$ sec. The diffusion coefficient is given by $\mathrm{D} = 0.009876\ m^2$ /sec.In order to simulate the problem, we change the variables into dimensionless form of equations (2.19)-(2.21). Initially the water in the reservoir is assumed to be motionless $u = 0$ and $v = 0$, and the water elevation is assumed to be a function that satisfied the initial and boundary conditions as $z(x, y, 0) = f(x, y) = x(1-x)\,y(1-y)$. The pollutant concentration in the reservoir is assumed to be a function that satisfied the initial and boundary conditions as

$c(x, y, 0) = g(x, y) = x(1-x)\,y(1-y)$ and $\dfrac{\partial C}{\partial n} = 0$, non-absorbing boundary of the reservoir, respectively.

Table 4.9:The elevation when $\Delta x = 0.03125,\ \Delta y = 0.03125, \Delta t = 0.003125$ at $T = 1$

x,y	0	400	1000	1700	2100	2700	3200
0	0	0	0	0	0	0	0
400	0	0.0084	0.0108	0.0125	0.0106	0.0148	0
1000	0	0.0108	0.0090	0.0084	0.0061	0.0198	0
1700	0	0.0125	0.0084	0.0044	0.0022	0.0191	0
2100	0	0.0106	0.0061	0.0022	0.0015	0.0166	0
2700	0	0.0148	0.0198	0.0191	0.0166	0.0201	0
3200	0	0	0	0	0	0	0

Table 4.10: The velocity in the direction of the axis x when $\Delta x = 0.03125,\ \Delta y = 0.03125$, $\Delta t = 0.003125$ at $T = 1$

x,y	0	400	1000	2100	2700	3200
0	0	0	0	0	0	0
400	0	-0.0271	-0.0462	-0.0472	-0.0399	0
1000	0	-0.0344	-0.0612	-0.0593	-0.0466	0
1700	0	0.0067	0.0185	0.0305	0.0194	0
2100	0	0.0319	0.0656	0.0772	0.0565	0
2700	0	0.0477	0.0890	0.0967	0.0810	0
3200	0	0	0	0	0	0

Table 4.11: The velocity in the direction of the axis y when $\Delta x = 0.03125,\ \Delta y = 0.03125$, $\Delta t = 0.003125$ at $T = 1$

x,y	0	400	1000	1700	2100	2700	3200
0	0	0	0	0	0	0	0
400	0	-0.0271	-0.0344	0.0067	0.0319	0.0477	0
1000	0	-0.0462	-0.0612	0.0185	0.0656	0.0890	0
1700	0	-0.0542	-0.0689	0.0300	0.0844	0.1078	0
2100	0	-0.0472	-0.0593	0.0305	0.0772	0.0967	0
2700	0	-0.0399	-0.0466	0.0194	0.0565	0.0810	0
3200	0	0	0	0	0	0	0

Table 4.12: The concentrations $C(\mathrm{x},\mathrm{t})$ of water pollutant when $\Delta x = 0.03125,\ \Delta y = 0.03125$ $\Delta t = 0.003125$ at $T = 1\ \left(kg/m^3\right)$

x,y	0	400	1000	1700	2100	2700	3200
0	0.0002	0.0007	0.0011	0.0007	0.0005	0.0002	0.0000
400	0.0007	0.0022	0.0037	0.0029	0.0021	0.0009	0.0002
1000	0.0011	0.0037	0.0069	0.0065	0.0051	0.0024	0.0004
1700	0.0007	0.0029	0.0065	0.0082	0.0073	0.0040	0.0009
2100	0.0005	0.0021	0.0051	0.0073	0.0071	0.0043	0.0010
2700	0.0002	0.0009	0.0024	0.0040	0.0043	0.0028	0.0007
3200	0.0000	0.0002	0.0004	0.0009	0.0010	0.0007	0.0002

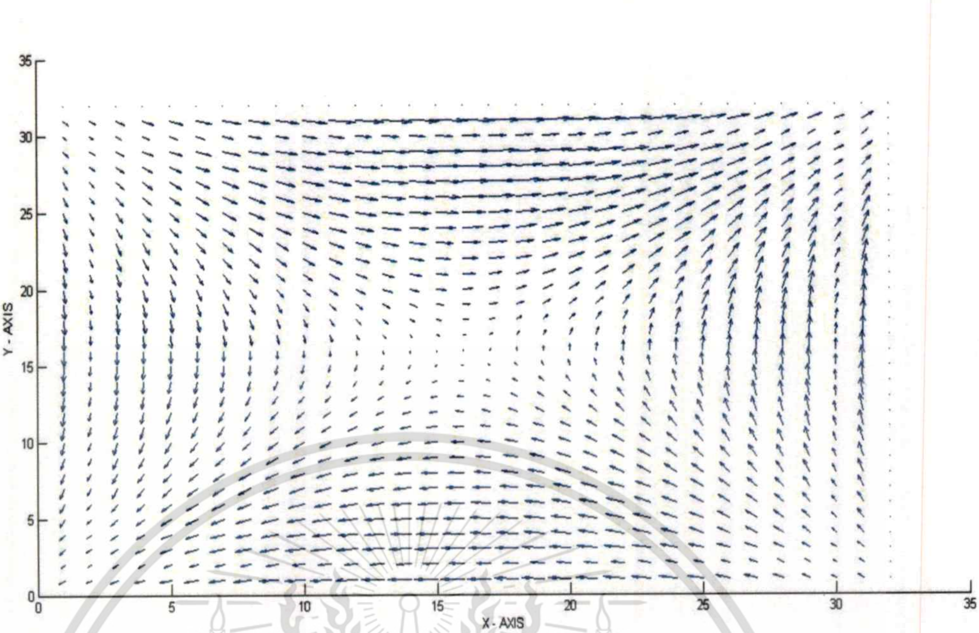


Fig.4.23: The velocity vector when $\Delta x = 0.03125,\ \Delta y = 0.03125, \Delta t = 0.003125$ at $T = 1$

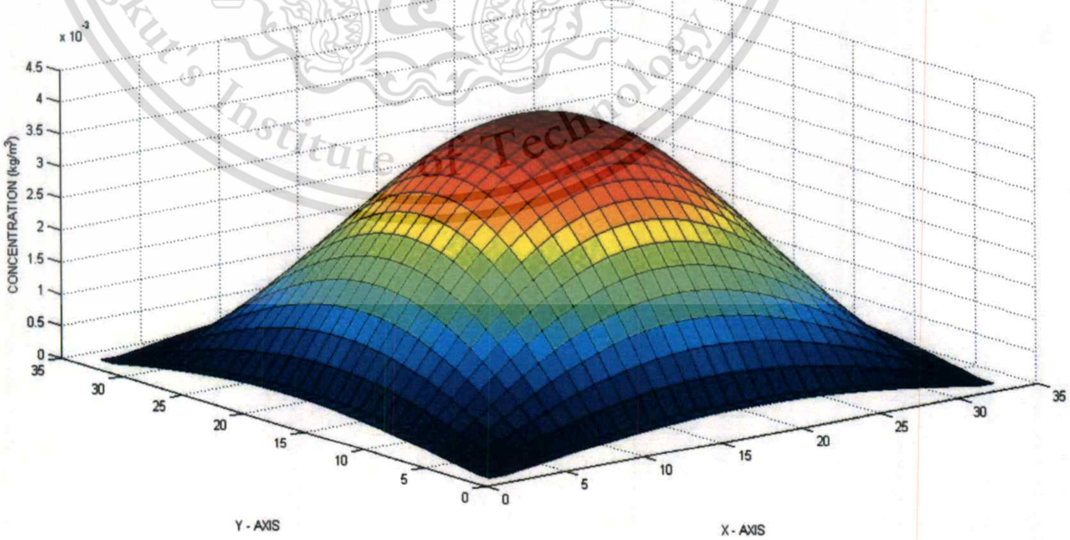


Fig.4.24: The surface of the water pollutant $\Delta x = 0.03125,\ \Delta y = 0.03125, \Delta t = 0.003125$ at $T = 1\left(kg/m^3\right)$

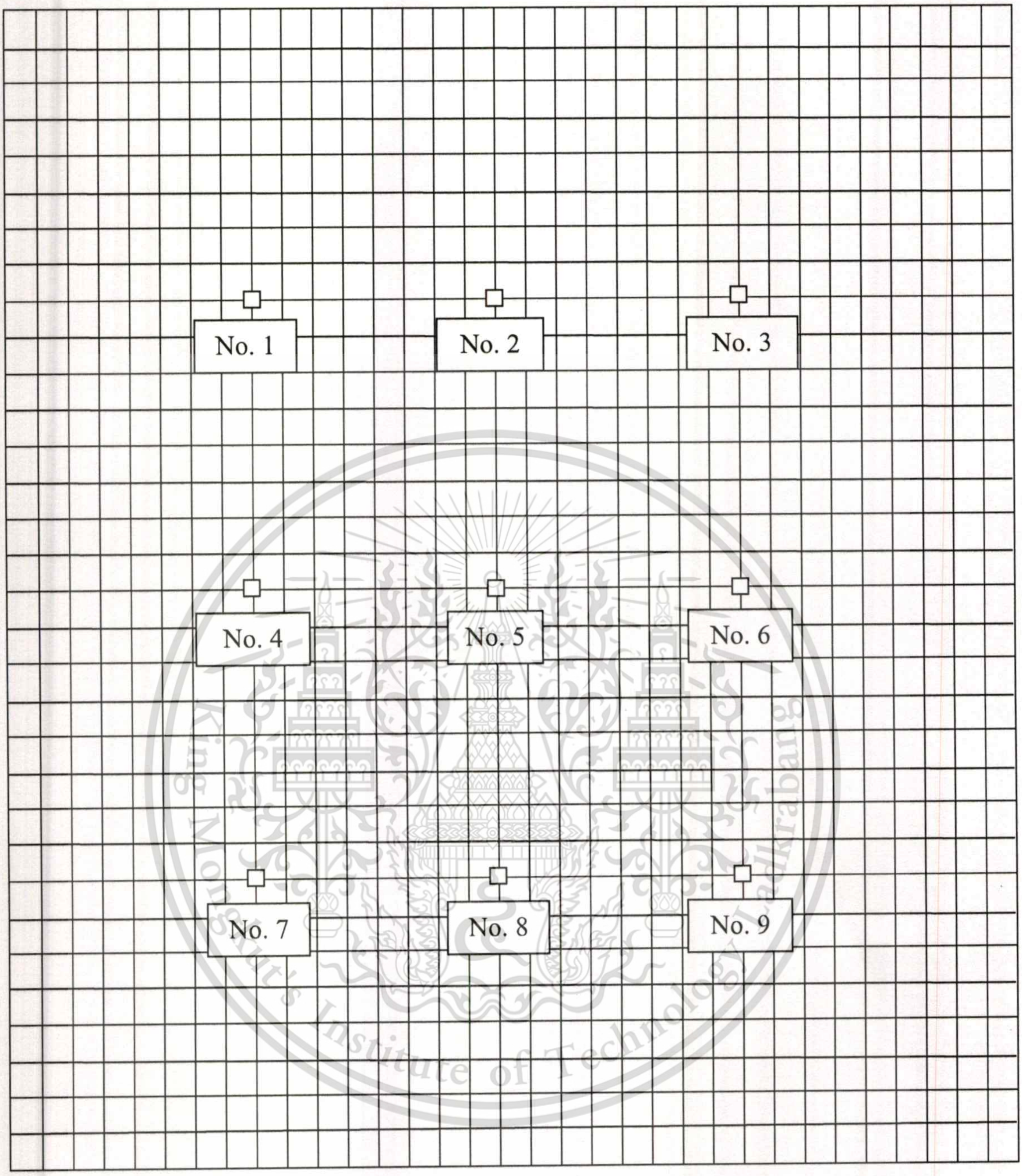


Fig4.25: Simple point

Concentration $C(x,t)$ at 9 simple points

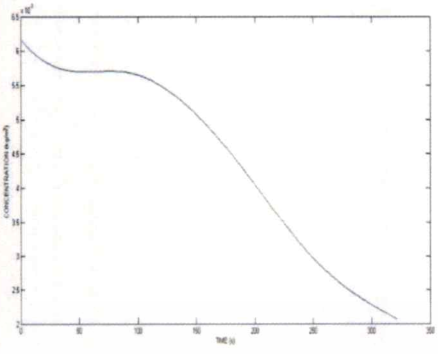
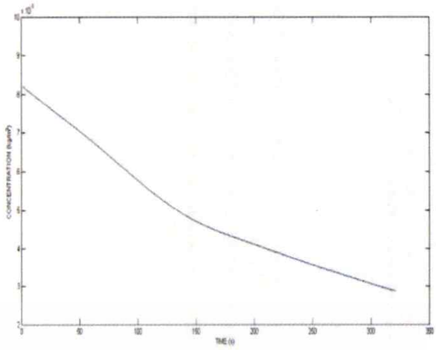


Fig.4.26 No.1



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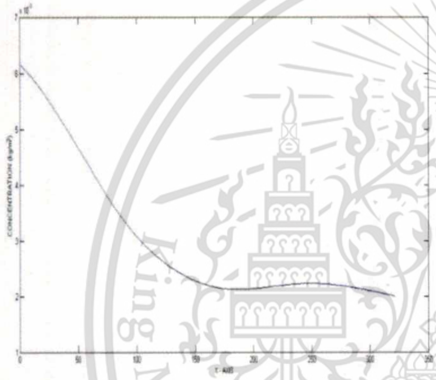

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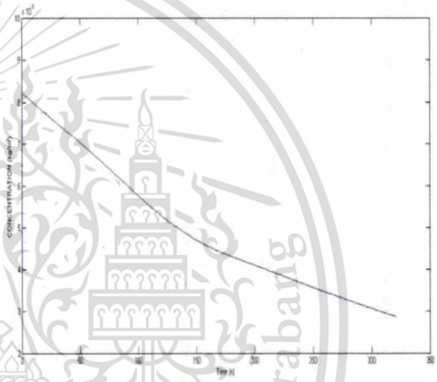

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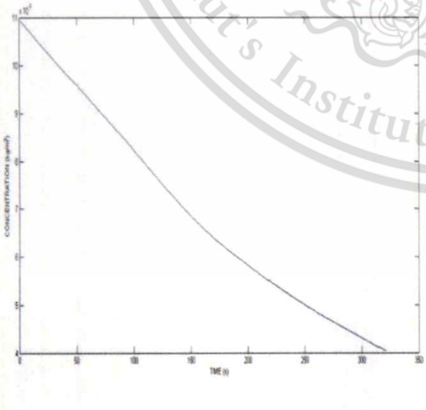

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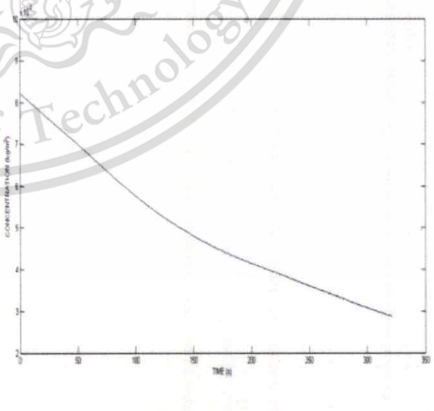

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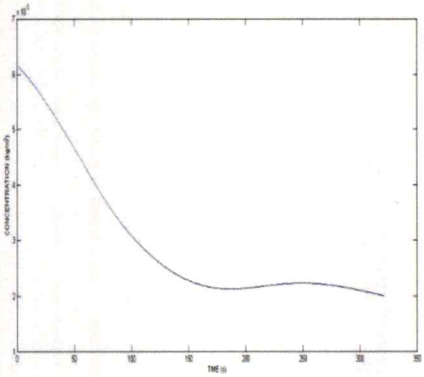

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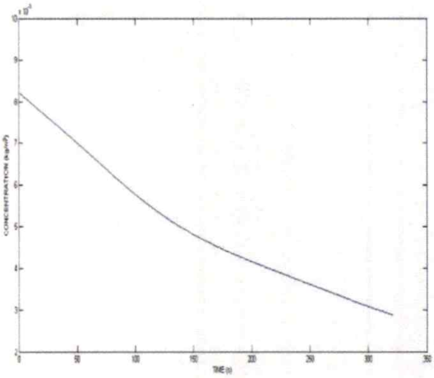

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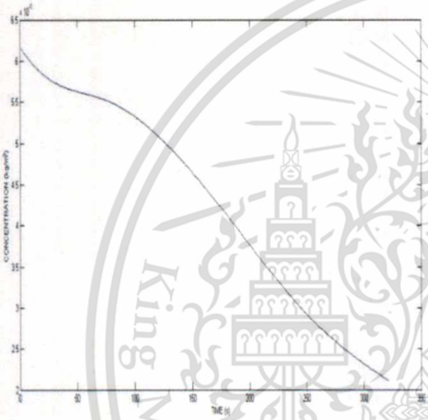

Fig.4.34 No.9

CHAPTER 5

Discussion and Conclusion

5.1.Discussion

By consider concentration of water pollutant in figure 4.26-4.34, we obtain the following results. In figure 4.26 and 4.34, the concentration is slowly decreasing at the beginning, being constant and decreasing again in the end. From figure 4.27, 4.29, 4.30, 4.31 and 4.33, the concentration is continually decreasing. Moreover, in figure 4.28 and 4.32, the concentration is decreasing at the beginning and being constant at the end.

5.2. Conclusion

The aim of the thesis is to determine the concentration of water pollution in arectangular reservoir.The hydrodynamic model provides the velocity field and elevation of the water. The calculated results of the first model will input into the second model to find a concentration.

The first model is the hydrodynamic model.The Lax-Wendroff method is used to find the velocity field and elevation of the water flow from the model.The Lax-Wendroff method is easy to implement and economical to used. However, the Lax-Wendroff method has some limitation to used in aspect of stability calculation. The stability condition is p $< \frac{1}{2\sqrt{2}}$ where $p = \frac{\Delta t}{\Delta x}$.

Consequently we have to use very small increment of t.

The second model is the dispersion model. The forward difference in time and backward difference in space method to find the pollutant concentration field. The is use of Forward in Time Backward in Space Method is an explicit method. It is not need to solve the large linear system of equations. Forward in Time Backward in Space is easy to implement and calculation. But there are some limitation of meshing since The strictly stability condition

For the future work, the thesis can be apply to use in the realistic case with be there is the drag flow of water in the irregular shape reservoir.

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APPENDIX

The research paper

การคำนวณเชิงตัวเลขของตัวแบบเชิงอุทกพลศาสตร์ในรูปอ่างเก็บน้ำ
A Numerical Simulation for a Hydrodynamic Model in a Uniform Reservoir.

อาภารวี อินวงศ์[1]*, และ นพรัตน์ โพธิ์ชัย[1]
Arparavee Ainvong[1]* and Nopparat Pochai[1]
[1] สาขาวิชา คณิตศาสตร์ คณะวิทยาศาสตร์ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

บทคัดย่อ

การควบคุมมลพิษจึงมีความจำเป็นต่อการรักษาสิ่งแวดล้อม เช่น แม่น้ำ ทะเลสาบ อ่างเก็บน้ำ ลำธาร บริเวณปากแม่น้ำ มหาสมุทรและน้ำใต้ดิน ในงานวิจัยนี้กล่าวถึงเราการคำนวณเชิงตัวเลขใช้แบบจำลองทางคณิตศาสตร์คำนวณคุณภาพน้ำในอ่างเก็บน้ำเอกรูป โดยจะพิจารณาในกรณีที่มีการไหลไม่คงตัวในบริเวณที่มีขอบเขตปกติแล้วจะใช้ตัวแบบอุทกพลศาสตร์ที่มีสัมประสิทธิ์เป็นตัวแปรที่จะให้ผลลัพธ์เป็นความเร็วและระดับการยกตัวของน้ำในแต่ละช่วงเวลาตัวแบบอุทกพลศาสตร์เราจะใช้วิธี Lax-Wendroff ในการหาผลเฉลย

คำสำคัญ : ตัวแบบเชิงอุทกพลศาสตร์ในรูปอ่างเก็บน้ำ

Abstract

The water current simulations are required for the water pollution model in rivers, lakes, reservoirs, streams, and estuaries. In this paper, a numerical simulation of a hydrodynamic model in a uniform reservoir is proposed. The case of unsteady flows with regular boundary is considered. The mathematical model is used to simulate water current to wave maker in the uniform reservoir. The hydrodynamic model with variable coefficients is provides the velocity in X , Y -directions and the elevation of the water flow. In the simulating processes, we will use the Lax-Wendroff method to approximate the solutions of the hydrodynamic model.

Keywords : Hydrodynamic Model in a Uniform Reservoir.

*Corresponding author. E-mail : bring_na@hotmail.com

1. INTRODUCTION

The methods to detect the amount of pollutant both in the air and water mostly are conducted by a field measurement and a mathematical simulation. For the shallow water mass transport problems that presented in [1], the method of characteristics has been reported as being applied with success, but it presents in real cases some difficulties. In [4] and [17], the finite element method for solving the water pollution models in one and two-dimensional water areas are presented respectively. The most of mathematical model require data concerning with velocity of the current at any point in the domain. The hydrodynamic model provides the velocity field and tidal elevation of the water. Those results are data for the dispersion model. In [5]-[16], they used the finite difference method to the hydrodynamic model with constant coefficients in the uniform reservoir.

Averaging the equation over the depth, discarding the term due to Coriolis force, shearing stresses and surface wind, it follows that the two-dimensional linear shallow water equation is applicable [3]. In this research, we use the Lax-Wendroff method to approximate the velocity and the tidal elevation with non-linear terms.

2. THE HYDRODYNAMIC MODEL

The continuity and momentum equations are govern the hydrodynamic behavior of the reservoir. The model is focused on the averaging the equations over the depth, discarding the term due to Coriolis parameter, shearing stresses and surface wind. We now introduce the well-known two-dimensional shallow water equations.

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}[(h+\zeta)u]+\frac{\partial}{\partial y}[h+\zeta)v]=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial \zeta}{\partial y}=0, \tag{3}$$

where $h(x,y)$ be the depth measured from the mean water level to the bed of the reservoir, $\zeta(x,y,t)$ is the elevation from the mean water level to the temporary water surface or the tidal elevation, g is the acceleration due to gravity, and $u(x,y,t)$ and $v(x,y,t)$ are the velocity components, for all $(x,y)\in[0,l]\times[0,l]$. We now introduce the two-dimensional non-linear shallow water equations with dimensionless from $U=u/\sqrt{gh}$, $V=v/\sqrt{gh}$, $X=x/l$, $Y=y/l$, $Z=\zeta/h$ and $T=t\sqrt{gh}/l$

$$\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{4}$$

$$\frac{\partial U}{\partial T}+(1+\mathrm{XY})\frac{\partial Z}{\partial X}=0, \tag{5}$$

$$\frac{\partial V}{\partial T}+(1+\mathrm{XY})\frac{\partial Z}{\partial Y}=0, \tag{6}$$

In $\Omega\times[0,T]$ where $\Omega=(0,1)\times(0,1)$ with the initial conditions $Z(\mathrm{X},0,T)=f(\mathrm{X},\mathrm{Y})$ and $\mathrm{U}(\mathrm{X},\mathrm{Y},0)=V(\mathrm{X},\mathrm{Y},0)=0$. The boundary conditions $\mathrm{Z}(0,\mathrm{Y},\mathrm{T})=\mathrm{Z}(1,\mathrm{Y},\mathrm{T})=Z(\mathrm{X},0,\mathrm{T})=\mathrm{Z}(\mathrm{X},1,\mathrm{T})=0$ at $\partial\Omega$.

3. NUMERICAL SOLUTIONS OF THE HYDRODYNAMIC MODEL

The equation (4)-(6) can be written in the matrix form

$$\frac{\partial W}{\partial T}=A\frac{\partial W}{\partial X}+B\frac{\partial W}{\partial Y}=0, \tag{7}$$

where

$$W=\begin{pmatrix}W_1\\W_2\\W_3\end{pmatrix},A=\begin{bmatrix}0&-1&0\\-(1+\mathrm{XY})&0&0\\0&0&0\end{bmatrix},\mathrm{B}=\begin{bmatrix}0&0&-1\\0&0&0\\-(1+\mathrm{XY})&0&0\end{bmatrix}$$

$W_1=Z$, $W_2=U$ and $W_3=V$. We now discretize Eq. (7) by diving the interval $[0,1]$ into L and M subintervals such that $L\Delta x=1$ and $M\Delta y=1$, and the interval $[0,T]$ into N subintervals such that $N\Delta t=T$. We can then approximate $W_1(x_l,y_m,t_n)$ by $W^n_{1l,m}$ value of the difference approximation of $W_1(x,y,t)$ at point $X=l\Delta x$, $Y=m\Delta y$ and $T=n\Delta t$,where $0\le l\le L$, $0\le m\le M$ and $0\le n\le N$,and similarly defined for $W^n_{2l,m}$ and $W^n_{3l,m}$. The grid point $(X_l,\mathrm{Y}_m,\mathrm{T}_n)$ are defined by $X_l=l\Delta x$ for all $l=0,1,2,...,L$, $Y_m=m\Delta y$ for all $m=0,1,2,...,M$ and $T_n=n\Delta t$ for all $n=0,1,2,...,N$ in which L , M and N are positive integers. Using Lax-Wendroff method [2] to Eq.(7) , we can simplified the following finite difference equation

$$\begin{aligned}W^{n+1}_{l,m}=&W^n_{l,m}+\frac{1}{2}pA_{l,m}(W^n_{l+1,m}-W^n_{l-1,m})+\frac{1}{2}pB_{l,m}(W^n_{l,m+1}-W^n_{l,m-1})\\&+\frac{1}{4}p^2A_{l,m}[(A_{l+1,m}\,\mathrm{W}^n_{l+1,m}-A_{l,m}\,\mathrm{W}^n_{l,m})-(A_{l+1,m}\,\mathrm{W}^n_{l,m}-A_{l,m}\,\mathrm{W}^n_{l-1,m})\\&+(A_{l,m}\,\mathrm{W}^n_{l+1,m}-A_{l-1,m}\,\mathrm{W}^n_{l,m})-(A_{l,m}\,\mathrm{W}^n_{l,m}-A_{l-1,m}\,\mathrm{W}^n_{l-1,m})]\\&+\frac{1}{4}p^2B_{l,m}[(\mathrm{B}_{l,m+1}\mathrm{W}^n_{l,m+1}-B_{l,m}\,\mathrm{W}^n_{l,m})-(\mathrm{B}_{l,m}\mathrm{W}^n_{l,m}-B_{l,m}\,\mathrm{W}^n_{l,m-1})\\&+(\mathrm{B}_{l,m}\mathrm{W}^n_{l,m+1}-B_{l,m-1}\,\mathrm{W}^n_{l,m})-(\mathrm{B}_{l,m}\mathrm{W}^n_{l,m}-B_{l,m-1}\,\mathrm{W}^n_{l,m-1})]\\&+\frac{1}{8}p^2\left(A_{l,m}B_{l,m}+B_{l,m}A_{l,m}\right)(W^n_{l+1,m+1}-W^n_{l-1,m+1}-W^n_{l+1,m-1}+W^n_{l-1,m-1})\end{aligned} \tag{8}$$

where

$$W^n_{l,m}=\begin{pmatrix}W^n_{1l,m}\\W^n_{2l,m}\\W^n_{3l,m}\end{pmatrix},\Delta xW^n_{\mathrm{i},m}=W^n_{\mathrm{i}+1}+W^n_{\mathrm{i}},\nabla xW^n_{\mathrm{i}}=W^n_{\mathrm{i}}-W^n_{\mathrm{i}-1}$$

and $p = \Delta t / \Delta x$. A stability analysis of Lax-Wendroff scheme (11) with matrices A and B has shown in [2]. The Lax-Wendroff scheme is stable if $p\left|\lambda_0\right| \le \frac{1}{2\sqrt{2}}$ where $\left|\lambda_0\right| = \max\left\{\left|\lambda_{Al,m}\right|,\left|\lambda_{Bl,m}\right|\right\}$ where $\lambda_{Al,m}, \lambda_{Bl,m}$ are eigenvalues of $\lambda_{Al,m}$ and $\lambda_{Bl,m}$ respectively.

4. CONCLUSION

The model for elevation of the velocity and elevation of the reservoir is approximated by using the Lax-Wendroff method with variable coefficients. The results can be the input data for the advection-diffusion equation of the water quality model. This model can be applied to the real cases for current in the regular shape reservoir with flat bottom.

TABLE I

TABLE OF WATER VELOCITY $u(x,y,t)$

IN THE X-DIRECTION WHEN

$\Delta x = \Delta y = 0.05$ AND $\Delta t = 0.005$

x,y	0	0.2	0.4	0.6	0.8	1
0	0	0	0	0	0	0
0.1	0	-0.0007	-0.0010	-0.0010	-0.0007	0
0.2	0	-0.0005	-0.0008	-0.0008	-0.0006	0
0.3	0	-0.0003	-0.0005	-0.0006	-0.0004	0
0.4	0	-0.0002	-0.0003	-0.0003	-0.0002	0
0.5	0	0.0000	0.0000	0.0000	0.0000	0
0.6	0	0.0002	0.0003	0.0003	0.0002	0
0.7	0	0.0004	0.0006	0.0007	0.0005	0
0.8	0	0.0006	0.0010	0.0011	0.0008	0
0.9	0	0.0008	0.0013	0.0015	0.0011	0
1	0	0	0	0	0	0

TABLE II

TABLE OF WATER VELOCITY $v(x,y,t)$

IN THE Y-DIRECTION WHEN

$\Delta x = \Delta y = 0.05$ AND $\Delta t = 0.005$

x,y	0	0.2	0.4	0.6	0.8	1
0	0	0	0	0	0	0
0.1	0	-0.0003	-0.0001	0.0001	0.0003	0
0.2	0	-0.0005	-0.0002	0.0002	0.0006	0
0.3	0	-0.0007	-0.0002	0.0002	0.0008	0
0.4	0	-0.0008	-0.0003	0.0003	0.0010	0
0.5	0	-0.0008	-0.0003	0.0003	0.0011	0
0.6	0	-0.0008	-0.0003	0.0003	0.0011	0
0.7	0	-0.0007	-0.0003	0.0003	0.0010	0
0.8	0	-0.0006	-0.0002	0.0002	0.0008	0
0.9	0	-0.0003	-0.0001	0.0001	0.0005	0
1	0	0	0	0	0	0

TABLE III

TABLE OF WATER $\zeta(x,y,t)$ WHEN

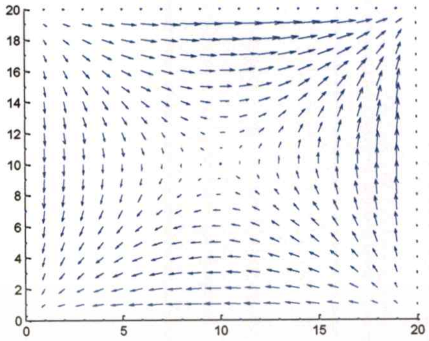
$\Delta x = \Delta y = 0.05$ AND $\Delta t = 0.005$

x, y	0	0.2	0.4	0.6	0.8	1
0	0	0	0	0	0	0
0.1	0	0.0144	0.0216	0.0216	0.0144	0
0.2	0	0.0256	0.0384	0.0384	0.0256	0
0.3	0	0.0336	0.0504	0.0504	0.0336	0
0.4	0	0.0384	0.0576	0.0576	0.0384	0
0.5	0	0.0400	0.0600	0.0600	0.0400	0
0.6	0	0.0384	0.0576	0.0576	0.0384	0
0.7	0	0.0336	0.0504	0.0504	0.0336	0
0.8	0	0.0256	0.0384	0.0384	0.0256	0
0.9	0	0.0144	0.0216	0.0216	0.0144	0
1	0	0	0	0	0	0

elevation $\Delta t = 0.005$.

Figure 1 The direction of water at

$\Delta x = \Delta y = 0.05$ AND $\Delta t = 0.005$



industry or urban, which we can change the inputs

5. ACKNOWLEDGEMENTS

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มหาวิทยาลัยบูรพา

ขอมอบเกียรติบัตรเพื่อรับรองว่าผลงานวิจัย

เรื่อง การคำนวณเชิงตัวเลขของตัวแบบเชิงอุทกพลศาสตร์ในรูปอ่างเก็บน้ำ

อาภารวี อินวงศ์

ได้ผ่านการพิจารณาจากคณะกรรมการผู้ทรงคุณวุฒิ ให้ได้รับรางวัลชมเชย
การนำเสนอผลงานวิจัยแบบบรรยาย กลุ่มสาขาคณิตศาสตร์และสถิติ
ในการประชุมวิชาการระดับชาติ "วิทยาศาสตร์วิจัย" ครั้งที่ ๖
ระหว่างวันที่ ๒๐ – ๒๑ มีนาคม พ.ศ. ๒๕๕๗

(ผู้ช่วยศาสตราจารย์ ดร.อุษาวดี ตันติวรานุรักษ์)
คณบดีคณะวิทยาศาสตร์ มหาวิทยาลัยบูรพา

APPENDIX

Matlab programming

```
clearall;
syms('x','y');
%step 1 : Input Value
s = input('x-stepsize : l = ');
l = s ;
s = input('y-stepsize : m = ');
m = s ;
s = input('Time : t = ');
tt = s ;
s = input('t-stepsize : t = ');
t = s ;
s = input('Diffusion coefficient : D = ');
D = s ;
s = 'x*(1-x)*y*(1-y)';
c1 = inline(s,'x','y');
s = 'x*(1-x)*y*(1-y)';
c2 = inline(s,'x','y');
s = '[0 -1 0; -1*(1 + x*y) 0 0; 0 0 0;]';
A = inline(s,'x','y');
s = '[0 0 -1; 0 0 0; -1*(1 + x*y) 0 0;]';
B = inline(s,'x','y');

%step 2 : given value
%step 2.1 basic value
g = 9.8 ;
h = 1;
M = 1/m;
L = 1/l;
p = t/l ;
LA = t/(l^2) ;
Z1 = zeros(L+1,M+1);
U1 = zeros(L+1,M+1);
V1 = zeros(L+1,M+1);
CS = zeros(L+1,M+1);
CST = zeros(L+1,M+1,(tt/t)+1);
ZZ = zeros(L+1,M+1,(tt/t)+1);
UU = zeros(L+1,M+1,(tt/t)+1);
VV = zeros(L+1,M+1,(tt/t)+1);
UUT = zeros(L+1,M+1,(tt/t)+1);
VVT = zeros(L+1,M+1,(tt/t)+1);
%step 2.2 initial condition
for i = 0 : M
for j = 0 : L
Z(i+1,j+1) = c1(j*l,i*m);
ZZ(i+1,j+1,1) = c1(j*l,i*m);
C(i+1,j+1) = c2(j*l,i*m);
U(i+1,j+1) = 0;
V(i+1,j+1) = 0;
CST(i+1,j+1,1) = C(i+1,j+1);
end;
end;

%step 3 solution
for k = 1 : (tt/t)
for i = 1 : M-1
for j = 1 : L-1
      a1 = l*(j-1);
      a2 = l*j;
      a3 = l*(j+1);
      b1 = m*(i-1);
      b2 = m*i;
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        b3 = m*(i+1);
        pp1 = (1/2)*p;
        pp2 = (1/4)*(p^2);
        pp3 = (1/8)*(p^2);

%Matrix W
w1(1,1) = Z(i,j);
w1(2,1) = U(i,j);
w1(3,1) = V(i,j);
%W1=W(l-1,m-1)
w2(1,1) = Z(i+1,j);
w2(2,1) = U(i+1,j);
w2(3,1) = V(i+1,j);
%W2=W(l,m-1)
w3(1,1) = Z(i+2,j);
w3(2,1) = U(i+2,j);
w3(3,1) = V(i+2,j);
%W3=W(l+1,m-1)
w4(1,1) = Z(i,j+1);
w4(2,1) = U(i,j+1);
w4(3,1) = V(i,j+1);
%W4=W(l-1,m)
w5(1,1) = Z(i+1,j+1);
w5(2,1) = U(i+1,j+1);
w5(3,1) = V(i+1,j+1);
%W5=W(l,m)
w6(1,1) = Z(i+2,j+1);
w6(2,1) = U(i+2,j+1);
w6(3,1) = V(i+2,j+1);
%W6=W(l+1,m)
w7(1,1) = Z(i,j+2);
w7(2,1) = U(i,j+2);
w7(3,1) = V(i,j+2);
%W7=W(l-1,m+1)
w8(1,1) = Z(i+1,j+2);
w8(2,1) = U(i+1,j+2);
w8(3,1) = V(i+1,j+2);
%W8=W(l,m+1)
w9(1,1) = Z(i+2,j+2);
w9(2,1) = U(i+2,j+2);
w9(3,1) = V(i+2,j+2);
%W9=W(l+1,m+1)
ww = w5 + pp1*A(a2,b2)*(w6-w4) + pp1*B(a2,b2)*(w8-w2);
ww = ww + pp2*A(a2,b2)*((A(a3,b2)*w6-A(a2,b2)*w5) - (A(a3,b2)*w5-
A(a2,b2)*w4) + (A(a2,b2)*w6-A(a1,b2)*w5) - (A(a2,b2)*w5-
A(a1,b2)*w4));
ww = ww + pp2*B(a2,b2)*((B(a2,b3)*w8-B(a2,b2)*w5) - (B(a2,b3)*w5-
B(a2,b2)*w2) + (B(a2,b2)*w8-B(a2,b1)*w5) - (B(a2,b2)*w5-
B(a2,b1)*w2));
ww = ww + pp3*(A(a2,b2)*B(a2,b2)+B(a2,b2)*A(a2,b2))*(w9-w7-w3+w1);
Z1(i+1,j+1) = ww(1,1);
U1(i+1,j+1) = ww(2,1);
V1(i+1,j+1) = ww(3,1);
ZZ(i+1,j+1,k+1) = Z1(i+1,j+1);
UU(i+1,j+1,k+1) = U1(i+1,j+1);
VV(i+1,j+1,k+1) = V1(i+1,j+1);
end;
end;
   Z = Z1 ;
   U = U1 ;
   V = V1 ;
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end;

ZZZ = ZZ*h;
UUU = UU*sqrt(g*h);
VVV = VV*sqrt(g*h);
Umax = max(max(max(abs(UUU)))) ;
Vmax = max(max(max(abs(VVV)))) ;
UVmax = max(Umax,Vmax) ;
Umin = min(min(min((UUU)))) ;
Vmin = min(min(min((VVV)))) ;
UVmin = min(Umin,Vmin) ;

for k = 1 : (tt/t)
for i = 1 : M-1
for j = 1 : L-1
xx = j*l;
yy = i*m;
kk = (1 + xx*yy);
CS(i+1,j+1) = (LA*(D/UVmax) +
p*(UUU(i+1,j+1,k+1)/UVmax)/kk)*C(i+1,j);
      CS(i+1,j+1) = CS(i+1,j+1) + (-4*LA*(D/UVmax) -
p*(UUU(i+1,j+1,k+1)/UVmax)/kk -
(p*(VVV(i+1,j+1,k+1)/UVmax)/kk)+1)*C(i+1,j+1);
CS(i+1,j+1) = CS(i+1,j+1) + (LA*(D/UVmax))*C(i+1,j+2);
CS(i+1,j+1) = CS(i+1,j+1) + (LA*(D/UVmax) +
p*(VVV(i+1,j+1,k+1)/UVmax)/kk)*C(i,j+1);
CS(i+1,j+1) = CS(i+1,j+1) + (LA*(D/UVmax))*C(i+2,j+1);
CST(i+1,j+1,k+1) = CS(i+1,j+1);
end;
end;
CST(1,:,k+1) = CST(2,:,k+1);
CST(M+1,:,k+1) = CST(M,:,k+1);
CST(:,1,k+1) = CST(:,2,k+1);
CST(:,L+1,k+1) = CST(:,L,k+1);
  CST(1,1,k+1) = (CST(1,2,k+1) + CST(2,1,k+1))/2 ;
  CST(1,L+1,k+1) = (CST(1,L,k+1) + CST(2,L+1,k+1))/2 ;
  CST(M+1,1,k+1) = (CST(M,1,k+1) + CST(M+1,2,k+1))/2 ;
  CST(M+1,L+1,k+1) = (CST(M+1,L,k+1) + CST(M,L+1,k+1))/2 ;
  C = CS ;
end;
CST = CST*UVmax ;

%step 4 Plot Graph
figure(1)
[aa,bb] = meshgrid(0:1:M);
holdon
quiver(aa,bb,UUU(:,:,40),VVV(:,:,40));
colormaphsv
holdoff
figure(2)
surf(CST(:,:,(tt/t)+1));
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BIOGRAPHY

NAME	Miss. Arparavee Invong
DATE OF BIRTH	August 18,1985
PLACE OF BIRTH	Nakornsawan, Thailand
ATTENDED INSTITUTION	2005-2008 Bachelor of Science in Applied Mathematics, King Mongkut's Institute of technology Ladkrabang, 2009-20013 Mater of Science in Applied Mathematics, King Mongkut's Institute of technology Ladkrabang, Applied Mathematics
SCHOLARSHIP	Centre of Excellence in Mathematics
HOME ADDRESS	129, Thambon Krokphra, Amphur Krokphra, Nakornsawan, 60170, Thailand
E-MAIL	bring_na@hotmail.com