SOME INEQUALITIES IN THE LÖWNER PARTIAL ORDER FOR POSITIVE DEFINITE SYMMETRIC MATRICES

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หัวข้อวิทยานิพนธ์	อสมการในอันดับบางส่วนโลว์เนอร์สำหรับเมทริกซ์ สมมาตรที่เป็นบวกแน่นอน
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บทคัดย่อ

งานวิจัยนี้เป็นการศึกษาอสมการในอันดับบางส่วนโลว์เนอร์ (the Löwner partial order) สำหรับเมทริกซ์แอร์มีตเชียน (Hermitian matrix) ที่มีคุณสมบัติพิเศษซึ่งเราเรียกว่า เมทริกซ์สมมาตรที่เป็นบวกแน่นอน (positive definite symmetric matrix) โดยมีจุดมุ่งหมายเพื่อแสดงความสัมพันธ์ในรูปของอสมการในอันดับบางส่วนโลว์เนอร์ ระหว่างรูปแบบผลรวมเชิงเส้นของเมทริกซ์ดังต่อไปนี้

1. $(\alpha A+\beta B)^2$ กับ $\alpha A^2+\beta B^2$
2. $(\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s$ กับ $\alpha(A^r \otimes C^s)+\beta(B^r \otimes D^s)$
3. $(\alpha A+\beta B)^r \circ (\alpha C+\beta D)^s$ กับ $\alpha(A^r \circ C^s)+\beta(B^r \circ D^s)$

เมื่อ α, β, r, s เป็นจำนวนจริงบวกซึ่ง $r+s=1$ โดย $\otimes$ แทน ผลคูณโครเนคเคอร์ (the Kronecker product) และ $\circ$ แทนผลคูณอาดามาร์ (the Hadamard product)

เมื่อประยุกต์ใช้ทฤษฎีบทในอันดับบางส่วนโลว์เนอร์กับอสมการที่ได้ เราจะได้อสมการที่เกี่ยวกับรอย ตัวกำหนด และ ค่าลักษณะเฉพาะของเมทริกซ์สมมาตรที่เป็นบวกแน่นอนในรูปแบบเดียวกัน

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ABSTRACT

This research is a study of inequalities in the Löwner partial order for the special kind of Hermitian matrices, namely, positive definite symmetric matrices. The aim is to show the relationship in the form of inequalities in the Löwner partial order between the following linear combination forms of those matrices:

1. $(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$;
2. $(\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s$ and $\alpha(A^r \otimes C^s)+\beta(B^r \otimes D^s)$; and
3. $(\alpha A+\beta B)^r \circ (\alpha C+\beta D)^s$ and $\alpha(A^r \circ C^s)+\beta(B^r \circ D^s)$

where α, β, r, s are positive real numbers such that $r+s=1$, $\otimes$ denotes the Kronecker product and $\circ$ denotes the Hadamard product.

By applying some theorems in the Löwner partial order to these inequalities, we get inequalities involving trace, determinant and eigenvalue of those matrices in the same linear combination forms.

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CHAPTER 1

INTRODUCTION

1.1 Importance and Inception

Matrix theory is a field of basic interest and has many applications in scientific computing, differential equation, probability and statistics, control and system theory, operations research, optimization, mathematical physics, economics, and engineering disciplines. A lot of theorems in matrix theory appear in the form of inequalities. These inequalities show relations or properties of matrices. Here is an example of the applications of matrix inequalities.

Application in control and system theory:

We can reduce a very wide variety of problems arising in control and system theory to a few standard optimization problems involving linear matrix inequalities (LMIs) which is of the form

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i \quad > \quad \mathbf{0},$$

where $x \in \mathbb{R}^m$ is the variable and the symmetric matrices $F_i = F_i^{\mathrm{T}} \in \mathbb{M}_n(\mathbb{R})$, $i = 0, \ldots, m$ are given. The symbol $F(x) > \mathbf{0}$ means that $F(x)$ is positive definite. These LMIs are the constraints of the optimization problems, for example,

$$\begin{array}{ll} \text{find} & P > \mathbf{0} \\ \text{subject to} & A^{\mathrm{T}}P + PA < \mathbf{0},\ \operatorname{tr}(P) = 1, \end{array}$$

where $P \in \mathbb{M}_n(\mathbb{R})$ is the variable. For more details about LMIs in control and system theory see [1].

Many inequalities of matrices are appeared in the form of $A \leq B$ where A, B are Hermitian matrices (A is Hermitian if $A^* = A$). So what is the meaning of the relation " $\leq$ " for matrices ? $A \leq B$ means that $B - A$ is a positive semidefinite matrix ($A \in \mathbb{M}_n$ is positive semidefinite if for all vector $x \in \mathbb{C}^n$. $x^*Ax \geq 0$). This relation is called the Löwner partial order. The inequalities in the Löwner partial order for positive definite symmetric matrices arise in various fields, for examples, semidefinite programming and cone programming (see [2, 3]).

In matrix theory, there are three important products of matrices. The first is the ordinary matrix multiplication which is familiar to ours. The second is the Kronecker product which is the block matrix defined in such a way that, given $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$ and $B = [b_{ij}] \in \mathbb{M}_{p,q}(\mathbb{F})$, the Kronecker product of A and B, denoted by $A \otimes B$, is defined by $A \otimes B := [a_{ij}B] \in \mathbb{M}_{mp,nq}(\mathbb{F})$. This notion is very useful in the study of matrix equations and others applications, and is of interest in its own right. For more details of its applications see [1, 4]. The last is the Hadamard product which is the entrywise product. The Hadamard product of $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$ and $B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$, denoted by $A \circ B$, is defined by $A \circ B := [a_{ij}b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$. The Hadamard product arises in a wide variety of ways of applications, for more details see [4, 5].

Also the trace, the determinant and eigenvalues of positive definite symmetric matrices play the important roles in matrix theory and have many applications. There are many studies about them, for example, Pranayanuntana [6] studied the determinant of the linear combination of positive definite symmetric matrices. The fact is that if we have the inequalities in the Löwner partial order for positive definite symmetric matrices, we can get the inequalities involving the trace, the determinant and eigenvalues of matrices by using some theorems in the Löwner partial order.

In this research we discuss inequalities in the Löwner partial order for positive definite symmetric matrices in some linear combination forms of matrices which are involving either the ordinary matrix multiplication, the Kronecker product or the Hadamard product of matrices and inequalities involving trace, determinant and eigenvalue of those matrices which are come from inequalities in the Löwner partial order.

1.2 Objectives

The objectives of the research are as follows:

1. To show the relationship between some linear combination forms of positive definite symmetric matrices in the form of inequalities in the Löwner partial order.
2. To show the relationship involving trace, determinant and eigenvalue of some linear combination forms of positive definite symmetric matrices in the form of inequalities.

1.3 Scope of the study

We study inequalities in the Löwner partial order for positive definite symmetric matrices in some linear combination forms (as below). Inequalities involving trace, determinant and eigenvalue are studied for those matrices in the inequalities. The product of the matrices in the inequalities are either ordinary matrix multiplication, the Kronecker product or the Hadamard product. We are not interested in inequalities of the form of block matrix but we may use some techniques of them in order to obtain results. The scope of this research is as follows:

1. We will study the relationship between the square of linear combination of two matrices and the linear combination of squares of two matrices as in this form: $(\alpha A + \beta B)^2$ and $\alpha A^2 + \beta B^2$ where $\alpha, \beta > 0$, respectively.
2. We will study the relationship between $(\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s$ and $\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)$ where $\alpha, \beta, r, s > 0$ such that $r + s = 1$.
3. We will study the relationship between $(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s$ and $\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)$ for $\alpha, \beta, r, s > 0$ such that $r + s = 1$.
4. We will study the inequalities involving trace, determinant and eigenvalue of matrices in the same forms as 1-3.

1.4 Benefits

The expected benefits for this research are:

1. Obtaining inequalities in the Löwner partial order for positive definite symmetric matrices .
2. Obtaining inequalities of trace, determinant and eigenvalue for positive definite symmetric matrices.

1.5 Research Methodology

In this research we use the mathematical proofs to find the results.

1.6 Steps of Research and Research Schedule

This research is concluded in 6 steps as follows:

1. Study more about matrix theory such as positive definite matrix, the Kronecker product, the Hadamard product, inequalities in the Löwner partial order, maps on matrix spaces, etc., and then determine the objectives and the scope of the research.

2. Show the relationship between $(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$.
   - Prove an inequality in the Löwner partial order between $(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$ where $\alpha,\beta>0$ which is the main result in this step.
   - Study special cases of the main result.
   - Apply some theorems in the Löwner partial order to the main result in order to get inequalities of trace, determinant and eigenvalue.
   - Study special cases of the inequalities of trace, determinant and eigenvalue.

3. Show the relationship between $(\alpha A+\beta B)^r\otimes(\alpha C+\beta D)^s$ and $\alpha(A^r\otimes C^s)+\beta(B^r\otimes D^s)$.
   - Prove an inequality in the Löwner partial order between $(\alpha A+\beta B)^r\otimes(\alpha C+\beta D)^s$ and $\alpha(A^r\otimes C^s)+\beta(B^r\otimes D^s)$ where $\alpha,\beta,r,s>0$ such that $r+s=1$. which is the main result in this step.
   - Study special cases of the main result.
   - Apply some theorems in the Löwner partial order to the main result in order to get inequalities of trace, determinant and eigenvalue.
   - Study special cases of the inequalities of trace, determinant and eigenvalue.

4. Show the relationship between $(\alpha A+\beta B)^r\circ(\alpha C+\beta D)^s$ and $\alpha(A^r\circ C^s)+\beta(B^r\circ D^s)$.
   - Prove an inequality in the Löwner partial order between $(\alpha A+\beta B)^r\circ(\alpha C+\beta D)^s$ and $\alpha(A^r\circ C^s)+\beta(B^r\circ D^s)$ where $\alpha,\beta,r,s>0$ such that $r+s=1$.
   - Study special cases of the main result.
   - Apply some theorems in the Löwner partial order to the main result in order to get inequalities involving trace, determinant and eigenvalue.
   - Study special cases of the inequalities of trace, determinant and eigenvalue.

5. Give some numerical examples to guarantee the results and state related applications. To give the examples we use the mathematical program which is called Maple 9.5.

6. Finally, discuss and conclude the results, make suggestions for further works and write the thesis. To write the thesis we use the program for typesetting text and mathematical formulae which is called LaTex.

The schdule for each corresponding step of the research is shown in Table 1.1.

Table 1.1: The Research Schedule

Work Plan	Month									
	1	2	3	4	5	6	7	8	9	10
Step 1	⟵ ⟶									
Step 2					⟷					
Step 3						⟷				
Step 4							⟷			
Step 5									↔	
Step 6										↔

1.7 Convention

Throughout this research, we denote the set of all real numbers by $\mathbb{R}$, the set of all positive real numbers by $\mathbb{R}^+$ and the set of all complex numbers by $\mathbb{C}$. Denote an unspecified field by $\mathbb{F}$ and the n-tuples vector space which elements come from $\mathbb{F}$ by $\mathbb{F}^n$.

Upper case letters are used to denote matrices, while lower case letters are used for vectors. Scalars are usually denoted by Greek letters. We denote the set of all $m \times n$ matrices over the field $\mathbb{F}$ by $\mathbb{M}_{m,n}(\mathbb{F})$, and $\mathbb{M}_{n,n}(\mathbb{F})$ is abbreviated to $\mathbb{M}_n(\mathbb{F})$. In the most common sense in which $\mathbb{F} = \mathbb{C}$, $\mathbb{M}_n(\mathbb{C})$ is further abbreviated to $\mathbb{M}_n$, and $\mathbb{M}_{m,n}(\mathbb{C})$ to $\mathbb{M}_{m,n}$. The identity matrix in $\mathbb{M}_n$ is denoted by I_n. The zero matrix is denoted by $\mathbf{0}$ (which is bolded), while the zero scalar is denoted by 0 (which is unbolded). If the size (dimension) of a matrix is not stated, it has appropriate size which depends on the context. Unless stated otherwise, every matrix is positive definite symmetric and every scalar is positive (real number).

This thesis is consisted of 7 chapters. The details for each chapter are as follows.

In Chapter 2, we present the basic notations, the useful concepts and facts about material which are used in this research. The results of this research are shown in Chapter 3–Chapter 6. In Chapter 3, the relationship between the linear combination forms of positive definite symmetric matrices in the form of inequalities in the Löwner partial order are presented. We present the inequalities of trace, determinant and eigenvalue of the linear combination forms of those matrices in Chapter 4, Chapter 5 and Chapter 6, respectively. Chapter 7 is the conclusion of this research.

Some applications of this research are presented in Appendix A. We also state the useful concepts and facts which are related to this research in Appendix B–Appendix I. They are useful for someone who would like to study or to do the research in matrix theory or related fields.

CHAPTER 2

DEFINITIONS, THEOREMS AND LITERATURE REVIEWS

The purpose of this chapter is to catalog briefly, without proof, a number of useful concepts and facts about the material which is the main portion of this research. We can see them in the books of matrix theory [4, 5, 7, 8, 9]. Each section consists of definitions, theorems and literature reviews.

Basic definitions which are familiar to ours are omitted. Some basic notations which are used in this research are as follows:

Let $A \in \mathbb{M}_{m,n}(\mathbb{F})$, the (i, j)th entry of the matrix A is referred to by a_{ij} or $(A)_{ij}$. We denote by A^{T} its **transpose**, by $\overline{A}$ its **conjugate**, by A^* its **conjugate transpose** and by A^{-1} its **inverse** (if existent, i.e., A **nonsingular**). The diagonal matrix with main diagonal's entries $a_{11}, a_{22}\cdots, a_{nn}$ is denoted by $diag[a_{11}, a_{22}, \cdots, a_{nn}]$.

For index sets $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \cdots, n\}$, we denote the **submatrix** that lies in the rows of A indexed by α and the columns of A indexed by β as $A(\alpha, \beta)$. If $m = n$ and $\alpha = \beta$, the submatrix $A(\alpha, \beta)$ is called a **principal submatrix** of A and is abbreviated by $A(\alpha)$. A **leading principal submatrix** of A is the principal submatrix of A determined by deleting the first i rows and the first i columns for some $i = 1, 2, \ldots, n$, i.e., $m = n$ and $\alpha = \beta \subseteq \{1, \ldots, i\}$.

2.1 Trace, Determinant and Eigenvalue

Definition 2.1.1 ([7]). *The* **trace** *of a matrix* $A \in \mathbb{M}_n$, *denoted by* $\operatorname{tr}(A)$, *is the sum of all its main diagonal's entries*:

$$\operatorname{tr}(A) \quad := \quad \sum_{i=1}^{n} a_{ii}. \tag{2.1}$$

Proposition 2.1.2 ([7]). *Let $A, B \in \mathbb{M}_n$ and α, β be scalars. Then*

$$\begin{aligned} \operatorname{tr}(\alpha A + \beta B) &= \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B), && (2.2) \\ \operatorname{tr}(A^{\mathrm{T}}) &= \operatorname{tr}(A), && (2.3) \\ \operatorname{tr}(AB) &= \operatorname{tr}(BA). && (2.4) \end{aligned}$$

Definition 2.1.3 ([7]). *The* **determinant** *of a matrix $A \in \mathbb{M}_n$, denoted by $\det(A)$, is defined by*

$$\det(A) := \sum_{j} (-1)^{t(j)} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \tag{2.5}$$

where $t(j)$ is the number of inversions in the permutation $j = (j_1, j_2, \ldots, j_n)$ and j varies over all $n!$ permutations of $1, 2, \ldots, n$.

Proposition 2.1.4 ([7]). *Let $A, B \in \mathbb{M}_n$ and k be a scalar. Then*

$$\begin{aligned} \det(kA) &= k^n \det(A), && (2.6) \\ \det(A^{\mathrm{T}}) &= \det(A), && (2.7) \\ \det(AB) &= \det(A)\det(B), && (2.8) \\ \det(A^{-1}) &= \frac{1}{\det(A)} \text{ (in this case, } A \text{ is nonsingular).} && (2.9) \end{aligned}$$

Definition 2.1.5 ([7]). *Though of as a formal polynomial in t, the* **characteristic polynomial** *of $A \in \mathbb{M}_n$ is defined by $p_A(t) := \det(tI - A)$. The equation*

$$\det(tI - A) = 0 \tag{2.10}$$

is called the **characteristic equation** *of A.*

Theorem 2.1.6 (Cayley-Hamition Theorem [7]). *Let $p_A(t)$ be the characteristic polynomial of $A \in \mathbb{M}_n$. Then $p_A(A) = \mathbf{0}$.*

Definition 2.1.7 ([5]). *Let $A \in \mathbb{M}_n$. An* **eigenvalue** *of A, denoted by $\lambda(A)$, is a scalar $\lambda \in \mathbb{C}$ such that there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying*

$$Ax = \lambda x \tag{2.11}$$

and the vector x is called the corresponding **eigenvector** *of A associated with λ. The set of all eigenvalues of A is called the* **spectrum** *of A and is denoted by $\sigma(A)$.*

Theorem 2.1.8 ([5]). *Let $p(\cdot)$ be a given polynomial. If λ is an eigenvalue of $A \in \mathbb{M}_n$ and $x \in \mathbb{C}^n$ is the corresponding eigenvector of A. Then $p(\lambda)$ is an eigenvalue of $p(A)$ and x is the corresponding eigenvector of $p(A)$.*

Corollary 2.1.9 ([5]). *Let $A \in \mathbb{M}_n$ be nonsingular. Then $0 \notin \sigma(A)$ and if $\lambda \in \sigma(A)$, then $\lambda^{-1} \in \sigma(A^{-1})$.*

Corollary 2.1.10 ([5]). *Let $A \in \mathbb{M}_n$, $\lambda \in \sigma(A)$ and let k be a real number and p be a nonnegative integer. Then $k\lambda \in \sigma(kA)$ and $\lambda^p \in \sigma(A^p)$.*

2.2 Hermitian Matrix

Definition 2.2.1 ([5, 7]). *A matrix $A \in \mathbb{M}_n$ is called*

- **symmetric** *if* $A^{\mathrm{T}} = A$,
- **Hermitian** *if* $A^* = A$,
- **orthogonal** *if* $AA^{\mathrm{T}} = I$,
- **unitary** *if* $AA^* = I$,
- **normal** *if* $AA^* = A^*A$,
- **real orthogonal** *if A is real and unitary.*

Theorem 2.2.2 ([5]). *Let $A \in \mathbb{M}_n$ be given. Then A is Hermitian if and only if at least one of the following holds:*

1. *x^*Ax is real for all $x \in \mathbb{C}^n$;*
2. *A is normal and all eigenvalues of A are real; or*
3. *S^*AS is Hermitian for all $S \in \mathbb{M}_n$.*

Theorem 2.2.3 (The Spectral Theorem for Hermitian Matrices [5]). *Let $A \in \mathbb{M}_n$. A is Hermitian if and only if there is a unitary matrix $U \in \mathbb{M}_n$ and a real diagonal matrix $\Lambda \in \mathbb{M}_n$ such that $A = U\Lambda U^*$. Moreover, A is symmetric if and only if there is a real orthogonal $P \in \mathbb{M}_n$ and a real diagonal matrix $\Lambda \in \mathbb{M}_n$ such that $A = P\Lambda P^{\mathrm{T}}$.*

The expression $A = U\Lambda U^*$ or $A = P\Lambda P^{\mathrm{T}}$ in the previous theorem is called **the spectral decomposition of** A.

2.3 Positive Definite Matrix

In this section, we introduce the notion of positive definite matrix which plays an important role in this research.

Definition 2.3.1 ([4, 5, 8]). *A matrix $A \in \mathbb{M}_n$ is said to be* **positive definite** *if for all nonzero vector $x \in \mathbb{C}^n$*

$$x^*Ax \quad > \quad 0. \tag{2.12}$$

If the strict inequality required in (2.12) *is weaken to $x^*Ax \geq 0$, then A is said to be* **positive semidefinite** *or* **nonnegative definite.**

Proposition 2.3.2 ([5]). *Let $A, B \in \mathbb{M}_n$ be positive definite. Then*

1. *A is Hermitian,*
2. *A is positive semidefinite,*
3. *$A^{\mathrm{T}}, \overline{A}$ and A^* are positive definite,*
4. *any positive linear combination of A and B is positive definite,*
5. *A^k is positive definite for $k = 0, 1, 2, \ldots,$*
6. *any principal submatrix of A is positive definite,*
7. *all eigenvalues of A are positive real numbers,*
8. *the determinant, the trace, all minors and the main diagonal's entries of A are positive real numbers,*
9. *A is nonsingular, A^{-1} is also positive definite and*
10. *A is of full rank (i.e., rank$(A) = n$).*

Theorem 2.3.3 ([5]). *Let $A \in \mathbb{M}_n$. The following statements are equivalent:*

1. *A is positive definite;*
2. *$x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$;*
3. *$H(A) := \frac{1}{2}(A + A^*)$ is positive definite;*
4. *A is positive semidefinite and nonsingular;*
5. *A is positive semidefinite and of full rank;*
6. *A is symmetric and $x^{\mathrm{T}}Ax > 0$ for all nonzero vector $x \in \mathbb{R}^n$ (in this case, A is real);*
7. *A is Hermitian and have all eigenvalues positive;*
8. *A is Hermitian and the determinant of any leading principal submatrix of A is positive;*

9. *A is Hermitian and there exists a nonsingular matrix* $C \in \mathbb{M}_n$ *such that* C^*AC *is positive definite*;
10. *there exists a nonsingular matrix* $B \in \mathbb{M}_n$ *such that* $A = B^*B$; *and*
11. *there exists a nonsingular upper triangular matrix* $L \in \mathbb{M}_n$ *with positive diagonal entries such that* $A = L^*L$. *If A is real, L may be taken to be real.*

Theorem 2.3.4 ([5]). *Let* $A \in \mathbb{M}_n$ *be positive semidefinite and let* $k \geq 1$ *be a given integer. Then there exists a unique positive semidefinite Hermitian matrix X such that* $X^k = A$. *We also have*

1. $AX = XA$ *and there is a polynomial* $p(t)$ *such that* $X = p(A)$;
2. *rank* X = *rank* A, *so* X *is positive definite if* A *is*; *and*
3. X *is real if* A *is real.*

The most useful case of the preceding Theorem is for $k = 2$. The unique **positive (semi)definite square root** of the positive (semi)definite matrix A is usually denoted by $A^{\frac{1}{2}}$. Similarly, $A^{\frac{1}{k}}$ denote the unique positive (semi)definite kth root of A for each case $k = 1, 2, ..., n$.

Theorem 2.3.5 (Hadamard's Inequality [5]). *If* $A = [a_{ij}] \in \mathbb{M}_n$ *are positive semidefinite, then*

$$\det(A) \quad \leq \quad \prod_{i=1}^{n} a_{ii}. \tag{2.13}$$

Further, when A *is positive definite, then equality holds if and only if* A *is diagonal.*

Donna [10] proved the following theorem:

Theorem 2.3.6 ([10]). *Let* $A, B \in \mathbb{M}_n$ *be positive definite. Then for any integers* $m, n > 1$

$$0 \; < \; \operatorname{tr}\left((AB)^m\right) \; < \; \left(\operatorname{tr}(AB)\right)^m. \tag{2.14}$$

2.4 The Kronecker Product

In this section, we give the definition and state properties of the Kronecker product which is a useful notion in the study of matrix equations and others applications, and is of interest in its own right.

Definition 2.4.1 ([4]). **The Kronecker product** *of* $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$ *and* $B = [b_{ij}] \in \mathbb{M}_{p,q}(\mathbb{F})$ *is denoted by* $A \otimes B$ *is defined to be a block matrix*

$$A \otimes B \quad := \quad \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \quad \in \quad \mathbb{M}_{mp,nq}(\mathbb{F}). \tag{2.15}$$

Definition 2.4.2 ([4]). *Let* $A \in \mathbb{M}_n(\mathbb{F})$. **The** k**th Kronecker power** $A^{\otimes k}$ *is defined inductively for all positive integer* k *by* $A^{\otimes 1} := A$ *and*

$$A^{\otimes k} \quad := \quad A \otimes A^{\otimes(k-1)}, \quad k = 2, 3, \ldots \tag{2.16}$$

Proposition 2.4.3 ([4]). *In general,* $A \otimes B \neq B \otimes A$, *that is, the Kronecker product is not commutative. Some very basic properties of the Kronecker product include:*

1. $(\alpha A)\otimes B = \alpha(A\otimes B) = A\otimes(\alpha B)$, *for all* $\alpha \in \mathbb{F}, A \in \mathbb{M}_{m,n}(\mathbb{F})$ *and* $B \in \mathbb{M}_{p,q}(\mathbb{F})$;
2. $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$, *for* $A \in \mathbb{M}_{m,n}(\mathbb{F})$ *and* $B \in \mathbb{M}_{p,q}(\mathbb{F})$;
3. $(A \otimes B)^* = A^* \otimes B^*$, *for* $A \in \mathbb{M}_{m,n}$ *and* $B \in \mathbb{M}_{p,q}$;
4. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, *for* $A \in \mathbb{M}_{m,n}(\mathbb{F}), B \in \mathbb{M}_{p,q}(\mathbb{F})$ *and* $C \in \mathbb{M}_{r,s}(\mathbb{F})$;
5. $(A + B) \otimes C = (A \otimes C) + (A \otimes C)$, *for* $A, B \in \mathbb{M}_{m,n}(\mathbb{F})$ *and* $C \in \mathbb{M}_{p,q}(\mathbb{F})$;
6. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$, *for* $A \in \mathbb{M}_{m,n}\mathbb{F}), B, C \in \mathbb{M}_{p,q}(\mathbb{F})$.

Lemma 2.4.4 (Mixed-Product Property [4]). *Let* $A \in \mathbb{M}_{m,n}(\mathbb{F}), B \in \mathbb{M}_{p,q}(\mathbb{F}), C \in \mathbb{M}_{n,k}(\mathbb{F})$ *and* $D \in \mathbb{M}_{q,r}(\mathbb{F})$. *Then*

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \tag{2.17}$$

Corollary 2.4.5 ([4]). *If* $A \in \mathbb{M}_m(\mathbb{F})$ *and* $B \in \mathbb{M}_n(\mathbb{F})$ *are nonsingular, then so are* $A \otimes B$ *and*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{2.18}$$

Theorem 2.4.6 ([4]). *Let* $A \in \mathbb{M}_m$ *and* $B \in \mathbb{M}_n$. *If* $\lambda \in \sigma(A)$ *and* $x \in \mathbb{C}^m$ *is the corresponding eigenvector of* A, *and if* $\mu \in \sigma(B)$ *and* $y \in \mathbb{C}^n$ *is the corresponding eigenvector of* B. *Then* $\lambda\mu \in \sigma(A\otimes B)$ *and* $x\otimes y \in \mathbb{C}^{mn}$ *is the corresponding eigenvector of* $A \otimes B$. *Every eigenvalue of* $A \otimes B$ *arises as the product of the eigenvalues of* A *and* B. *If* $\sigma(A) = \{\lambda_1, \ldots, \lambda_m\}$ *and* $\sigma(B) = \{\mu_1, \ldots, \mu_n\}$, *then* $\sigma(A \otimes B) = \{\lambda_i\mu_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. *In particular,* $\sigma(A \otimes B) = \sigma(B \otimes A)$.

Corollary 2.4.7 ([4]). *Let* $A \in \mathbb{M}_{m,n}(\mathbb{F}), B \in \mathbb{M}_{p,q}(\mathbb{F})$. *If* A *and* B *are positive (semi)definite matrices, then* $A \otimes B$ *is also positive (semi)definite matrix.*

Proposition 2.4.8 ([4]). *If* $A \in \mathbb{M}_n, B \in \mathbb{M}_m$, *then*

1. $\det(A \otimes B) = \det(B \otimes A) = (\det(A))^m(\det(B))^n$,
2. $\operatorname{tr}(A \otimes B) = \operatorname{tr}(B \otimes A) = \operatorname{tr}(A)\operatorname{tr}(B)$.

2.5 The Hadamard Product

In this section, we present a matrix product that is much simpler than the ordinary product, but is much less widely understood. It arises naturally in a variety of ways and enjoys considerable rich structure.

Definition 2.5.1 ([4]). **The Hadamard product** *of* $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$ *and* $B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$ *is defined by* $A \circ B := [a_{ij}b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{F})$.

Definition 2.5.2 ([8]). *Given a positive integer* k, **the** k**th Hadamard power** *of* $A = [a_{ij}] \in \mathbb{M}_n$ *is defined by* $A^{(k)} := [a_{ij}^k] \in \mathbb{M}_n$.

Proposition 2.5.3 ([4]). *Some very basic properties of the Hadamard product include*:

1. $A \circ B = B \circ A$, *for* $A, B \in \mathbb{M}_{m,n}(\mathbb{F})$;
2. $(\alpha A) \circ B = \alpha(A \circ B) = A \circ (\alpha B)$, *for all* $\alpha \in \mathbb{F}, A, B \in \mathbb{M}_{m,n}(\mathbb{F})$;
3. $(A \circ B)^{\mathrm{T}} = A^{\mathrm{T}} \circ B^{\mathrm{T}}$, *for* $A, B \in \mathbb{M}_{m,n}(\mathbb{F})$;
4. $(A \circ B)^* = A^* \circ B^*$, *for* $A, B \in \mathbb{M}_{m,n}$;
5. $(A \circ B) \circ C = A \circ (B \circ C)$, *for* $A, B, C \in \mathbb{M}_{m,n}(\mathbb{F})$;
6. $(A + B) \circ C = (A \circ C) + (B \circ C)$, *for* $A, B, C \in \mathbb{M}_{m,n}(\mathbb{F})$;
7. $A \circ (B + C) = (A \circ B) + (A \circ C)$, *for*$A, B, C \in \mathbb{M}_{m,n}(\mathbb{F})$.

Lemma 2.5.4 ([4, 8]). *For any* $A, B \in \mathbb{M}_n$, $A \circ B = (A \otimes B)(\alpha)$ *where* $\alpha = \{1, n + 1, 2n + 3, \ldots, n^2\}$.

Theorem 2.5.5 (Schur Product Theorem [4, 8]). *If* $A, B \in \mathbb{M}_n$ *are positive semidefinite, then so is* $A \circ B$. *If, in addition,* B *is positive definite and* A *has no diagonal entry equal to 0, then* $A \circ B$ *is positive definite. In particular, if both* A *and* B *are positive definite, then so is* $A \circ B$.

Theorem 2.5.6 ([4]). *Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Any eigenvalue $\lambda(A \circ B)$ of $A \circ B$ satisfies*

$$\begin{aligned} \lambda_{min}(A)\lambda_{min}(B) &\leq [min\ a_{ii}]\ \lambda_{min}(B) \\ &\leq \lambda(A \circ B) \\ &\leq [max\ a_{ii}]\ \lambda_{max}(B) \\ &\leq \lambda_{max}(A)\ \lambda_{max}(B). \end{aligned}$$

Theorem 2.5.7 (Oppenheim's Inequality [5]). *If $A, B \in \mathbb{M}_n$ are positive semidefinite, then*

$$\det(A) \prod_{i=1}^{n} b_{ii} \quad \leq \quad \det(A \circ B). \tag{2.19}$$

Theorem 2.5.8 ([11]). *Let $A, B \in \mathbb{M}_n$ be positive semidefinite and α, β are real numbers. Then*

$$\operatorname{tr}\left((A \circ B)^{\alpha}\right) \leq \operatorname{tr}(A^{\alpha} \circ B^{\alpha}), \quad \alpha \leq 0 \text{ or } \alpha \geq 1, \tag{2.20}$$

$$\operatorname{tr}\left((A \circ B)^{\alpha}\right) \geq \operatorname{tr}(A^{\alpha} \circ B^{\alpha}), \quad 0 \leq \alpha \leq 1. \tag{2.21}$$

Equality occurs if and only if $\alpha = 0$ or $\alpha = 1$ or $(A \circ B)^{\alpha} = A^{\alpha} \circ B^{\alpha}$.

2.6 The Löwner Partial Order

Definition 2.6.1 ([12]). **A partially ordered set** *is a nonempty set P, together with a partial order defined on P.* **A partial order** *is a binary relation, denoted by $\leq$ and read "less than or equal to," with the following properties:*

1. **Reflexivity:** *for all $a \in P$, $a \leq a$.*
2. **Antisymmetry:** *for all $a, b \in P$, $a \leq b$ and $b \leq a$ implies $a = b$.*
3. **Transitivity:** *for all $a, b, c \in P$, $a \leq b$ and $b \leq c$ implies $a \leq c$.*

For Hermitian matrices $A, B \in \mathbb{M}_n$ we write $B \leq A$ or $A \geq B$ to mean that $A - B$ is positive semidefinite and we write $B < A$ or $A > B$ to mean that $A - B$ is positive definite. In particular, $A \geq \mathbf{0}$ indicates that A is positive semidefinite and $A > \mathbf{0}$ indicates that A is positive definite. The notion $\geq$ is reflexive, antisymmetric and transitive, but that it's not total order; that is there exists the Hermitian matrix A, B such that neither $A \geq B$ nor $B \geq A$. This is known as the **Löwner partial order** [13], it is induced in the real space of (complex) Hermitian matrix by the cone of positive semidefinite matrices.

A partial order on the real vector space is often defined by identifying some special closed convex cone and saying that one element is greater than or equal to the other if their difference lies in the special cone. In this case, the set of Hermitian matrices is the real linear space and the set of positive semidefinite matrices is the closed convex cone. This is clearly a generalization of the familiar case in which $\mathbb{R}$ itself is the real linear space and the nonnegative real numbers are the closed convex cone. This give the "usual" (total) order (not just a partial order) on $\mathbb{R}$.

Proposition 2.6.2 ([5]). *Let $A, B, C, D \in \mathbb{M}_n$ be Hermitian. Then*

1. $A \geq A$,
2. *If* $A \geq B$ *and* $B \geq C$, *then* $A \geq C$,
3. *If* $A \geq B$ *and* $B \geq A$, *then* $A = B$,
4. *If* $A \geq B$ *and* $C \geq D$, *then* $A + C \geq B + D$,
5. *If* $A \geq B$ *and* $C \geq 0$, *then* $A \circ C \geq B \circ C$,
6. *If* $A \geq B \geq 0$ *and* $C \geq D \geq 0$, *then* $A \circ C \geq B \circ D \geq 0$.

There exists A, B such that neither $A \geq B$ nor $B \geq A$. The condition $A \geq 0, B \geq 0$ does not imply $AB \geq 0$. It is not true that if $A \geq B$ and $A \neq B$, then $A > B$, that is, $A \geq B$ does not mean that $A > B$ or $A = B$.

Proposition 2.6.3 ([5]). *Let $A, B \in \mathbb{M}_n$ be Hermitian. Then*

$A \geq B$ implies $T^*AT \geq T^*BT$ for all $T \in \mathbb{M}_{n,m}$; we also have

$A > B$ implies $T^*AT > T^*BT$ whenever $T \in \mathbb{M}_{n,m}$ has rank m.

Theorem 2.6.4 ([5]). *If $A, B \in \mathbb{M}_n$ are positive definite. Then*

1. $A \geq B$ *if and only if* $B^{-1} \geq A^{-1}$,
2. *If* $A \geq B$, *then* $\det(A) \geq \det(B)$ *and* $\operatorname{tr}(A) \geq \operatorname{tr}(B)$,
3. *If* $A > B$, *then* $\det(A) > \det(B)$ *and* $\operatorname{tr}(A) > \operatorname{tr}(B)$, *and*
4. *More generally, if* $A \geq B$, *then* $\lambda_i(A) \geq \lambda_i(B)$ *for all* $i = 1, 2, \ldots, n$ *if the respective eigenvalues of A and B are arranged in the same order.*

Theorem 2.6.5 ([5, 8, 14, 15]). *Suppose that a Hermitian matrix is partitioned as*

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

where A and C are square. This matrix is positive (semi)definite if and only if A and C are positive definite and the **Schur's complement** *$C - B^*A^{-1}B$ is positive (semi)definite.*

2.7 Function of Matrices

Definition 2.7.1 ([8, 14]). *Let f be a continuous real-valued function on a real interval Ω and $A \in \mathbb{M}_n$ be a Hermitian matrix which its eigenvalues $\lambda_1, ..., \lambda_n$ contained in Ω. Let $A = Udiag[\lambda_1, ..., \lambda_n]U^*$ be the spectral decomposition of A with U unitary. Then the* **functional calculus** *for A defined as*

$$f(A) \quad := \quad Udiag[f(\lambda_1), ..., f(\lambda_n)]U^*. \tag{2.22}$$

Proposition 2.7.2 ([14]). *For any such function f, g and Hermitian matrix A in Definition 2.7.1, the matrices $f(A)$ and $g(A)$ commute and*

$$(f+g)(A) \quad = \quad f(A) + g(A) \tag{2.23}$$

$$(fg)(A) \quad = \quad f(A)\, g(A). \tag{2.24}$$

Further, if $f(t) \leq g(t)$ for all t, then

$$f(A) \quad \leq \quad g(A).$$

Theorem 2.7.3 ([7]). *Given a complex-valued function f of complex variable t. Let $A \in \mathbb{M}_n$, $\lambda_1, \ldots, \lambda_n \in \sigma(A)$ and $p(t)$ be the characteristic polynomial of A,*

$$p(t) = \prod_{i=1}^{m}(t-\lambda_i)^{n_i}, n = \sum_{i=1}^{m} n_i.$$

Define $h(t) := \beta_0 + \beta_1 t + \cdots + \beta_{n-1}t^{n-1}$.
$\beta_0, \beta_1, \ldots, \beta_{n-1}$ are to be solved from the following set of equations:

$$f^{(l)}(\lambda_i) = h^{(l)}(\lambda_i),$$

for all $l = 0, \ldots, n-1$, for all $i = 0, \ldots, m$.
Then $f(A) = h(A)$.

2.8 Maps on Matrix Spaces

Definition 2.8.1 ([16, 17]). *A function f defined on a real interval Ω is said to be* **matrix monotone of order** *n if*

$$A \leq B \implies f(A) \leq f(B)$$

for all $n \times n$ Hermitian matrices A, B whose eigenvalues are contained in Ω. If f is matrix monotone of order n for all n we say f is **matrix monotone** *or* **operator monotone**. *f is called* **matrix convex** *or* **operator convex** *if for any $0 < \epsilon < 1$,*

$$f(\epsilon A + (1-\epsilon)B) \quad \leq \quad \epsilon f(A) + (1-\epsilon) f(B) \tag{2.25}$$

holds for all Hermitian matrices A, B of all orders with eigenvalues in Ω. f is called **operator concave** *if $-f$ is operator convex.*

For such a function f and matrices A, B in Definition 2.8.1, if f is continuous, the condition (2.25) can be replaced by

$$f\Big(\frac{1}{2}(A+B)\Big) \quad \leq \quad \frac{1}{2}(f(A)+f(B)). \tag{2.26}$$

The set of operator monotone functions and the set of operator convex functions are both closed under positive linear combination and also under pointwise limits. The composition of two operator monotone functions are operator monotone. The same is true for operator convex function.

Operator monotone functions were first studied in detail by K. Löwner in the seminal paper [13]. In this paper, he established the connection between operator monotonicity, the positivity of the matrix of divided differences. Operator convex functions were studied, soon afterwards, by F. Kraus [16].

In another well-known paper [17] E. Heinz used the theory of operator monotone functions to study several problems of perturbation theory for bounded and unbounded operators. The operator monotonicity of the map $A \to A^r$ for $0 \leq r \leq 1$ is sometimes called "Löwner-Heinz inequality", although it was discovered by Löwner.

Theorem 2.8.2 (Löwner-Heinz Inequality [13, 17]). *If $A \geq B \geq \mathbf{0}$ and $0 \leq r \leq 1$. Then*

$$A^r \quad \geq \quad B^r. \tag{2.27}$$

Proposition 2.8.3 ([8]). *A real-valued function f defined by $f(t) = t^r$ for $0 < r < 1$ has an integral representation*:

$$t^r \quad = \quad \frac{\sin r\pi}{\pi}\int_0^\infty \frac{s^{r-1}t}{s+t}ds. \tag{2.28}$$

Definition 2.8.4 ([8, 14, 15]). *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ be a map. Φ is called* **unital** *if $\Phi(I_n) = I_m$. Φ is called* **positive** *if it maps positive semidefinite matrices to positive semidefinite matrices*: $A \geq \mathbf{0} \implies \Phi(A) \geq \mathbf{0}$. *$\Phi$ is called* **linear** *if for all A and B in $\mathbb{M}_n$ and every scalars α, β*

$$\Phi(\alpha A + \beta B) \quad = \quad \alpha\Phi(A) + \beta\Phi(B). \tag{2.29}$$

Definition 2.8.5 ([8, 9, 14]). *A map $\psi : \mathbb{M}_m \times \mathbb{M}_n \to \mathbb{M}_p$ is* **jointly concave** *if for all Hermitian matrices $A, B \in \mathbb{M}_m$ and $C, D \in \mathbb{M}_n$ and for all $0 < \epsilon < 1$,*

$$\psi(\epsilon A + (1-\epsilon)B, \epsilon C + (1-\epsilon)D)) \geq \epsilon\,\psi(A, C) + (1-\epsilon)\psi(B, D). \qquad (2.30)$$

Lemma 2.8.6 ([8, 14]). *There exists a unital positive linear map from $\mathbb{M}_{n^2}$ to $\mathbb{M}_n$ such that $\Phi(A \otimes B) = A \circ B$ for all $A, B \in \mathbb{M}_n$.*

Ando [14] proved the followings:

Theorem 2.8.7 ([14]). *Let $A \in \mathbb{M}_n$ be positive definite. A map Φ defined by $\Phi(A) = A^p$ is concave if $0 < p \leq 1$ and is convex if $1 \leq p \leq 2$ or $-1 \leq p < 0$.*

Corollary 2.8.8 ([14]). *Let $A, B \in \mathbb{M}_n$. Then*

$$\begin{aligned} A^r \circ B^r &\leq (A \circ B)^r &&, A, B \geq 0,\ 0 < r \leq 1; \\ A^r \circ B^r &\geq (A \circ B)^r &&, A, B > 0,\ -1 \leq r \leq 0 \text{ or } 1 \leq r \leq 2. \end{aligned}$$

Lemma 2.8.9 ([14]). *Let $A, B \in \mathbb{M}_n$ be positive definite. Then a map Φ defined by*

$$\Phi(A, B) = (A^{-1} + B^{-1})^{-1} \qquad (2.31)$$

is jointly concave.

CHAPTER 3

INEQUALITIES IN THE LÖWNER PARTIAL ORDER

The purpose of this chapter is to present our results which are the relationship between some linear combination forms of positive definite symmetric matrices in the form of inequalities in the Löwner partial order. These inequalities are shown in Section 3.1–Section 3.3. We also give some numerical examples to guarantee our results in Section 3.4.

Before that we have some observations. First, since a positive definite matrix A is necessarily Hermitian, hence if A is symmetric, then

$$A = A^* = \overline{A^{\mathrm{T}}} = \overline{A}.$$

This means that a positive definite symmetric matrix is a real matrix. So in this research we will consider only real matrices.

The second is when we "compare" two matrices, we must sure that they are Hermitian. The third is some basic observations about inequalities in the Löwner partial order: for Hermitian $A, B, C \in \mathbb{M}_n$ if $A \geq B$, then

1. $A + C \geq B + C$,
2. $\alpha A \geq \alpha B$ for $\alpha \geq 0$,
3. $\alpha A \leq \alpha B$ for $\alpha \leq 0$.

It is not true that if $C \geq \mathbf{0}$, then $AC \geq BC$.

3.1 Inequalities of the 2nd Power of Linear Combination of Two Matrices

In this section, we present the relationship between the 2nd power of linear combination and the linear combination of the 2nd power of two positive definite symmetric matrices in the form of inequalities in the Löwner partial order. First we prove an inequality between $(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$ which is Theorem 3.1.1.1 in Subsection 3.1.1. Results from some special cases of this theorem are stated in Subsection 3.1.2. In Subsection 3.1.3, we give some interesting observations which are related to Subsection 3.1.1 and Subsection 3.1.2. We also give some interesting remarks and counter examples.

3.1.1 Main Results

Theorem 3.1.1.1. *Let* $A, B \in \mathbb{M}_n$ *be positive definite symmetric and let* $\alpha, \beta > 0$ *be scalars. Then*

$$\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2 \quad \leq \quad \alpha A^2+\beta B^2 \tag{3.1}$$

with equality holds if and only if $A = B$.

Proof. Consider two block matrices partitioned as

$$\begin{bmatrix} I & A \\ A & A^2 \end{bmatrix} \text{ and } \begin{bmatrix} I & B \\ B & B^2 \end{bmatrix}.$$

Notice that $A = A^*$, $B = B^*$ and the identity matrix is positive definite.
Since $A^2 - AI^{-1}A = 0$ and $B^2 - BI^{-1}B = 0$, the Schur's complements of these matrices are positive semidefinite.
By Theorem 2.6.5 both matrices are also positive semidefinite.
Since $\alpha, \beta > 0$, we get $\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta} > 0$. By Proposition 2.3.2, we have

$$\begin{aligned} 0 &\leq \frac{\alpha}{\alpha+\beta}\begin{bmatrix} I & A \\ A & A^2 \end{bmatrix} + \frac{\beta}{\alpha+\beta}\begin{bmatrix} I & B \\ B & B^2 \end{bmatrix} \\ &= \begin{bmatrix} I & \frac{\alpha}{\alpha+\beta}A+\frac{\beta}{\alpha+\beta}B \\ \frac{\alpha}{\alpha+\beta}A+\frac{\beta}{\alpha+\beta}B & \frac{\alpha}{\alpha+\beta}A^2+\frac{\beta}{\alpha+\beta}B^2 \end{bmatrix}. \end{aligned}$$

Since the partitioned matrix in the right hand side of this inequality is Hermitian and I is positive definite, by using Theorem 2.6.5 again, we obtain

$$\begin{aligned}
\frac{\alpha}{\alpha+\beta}A^2+\frac{\beta}{\alpha+\beta}B^2 &\geq \Big(\frac{\alpha}{\alpha+\beta}A+\frac{\beta}{\alpha+\beta}B\Big)I^{-1}\Big(\frac{\alpha}{\alpha+\beta}A+\frac{\beta}{\alpha+\beta}B\Big),\\
\Big(\frac{\alpha}{\alpha+\beta}A+\frac{\beta}{\alpha+\beta}B\Big)^2 &\leq \frac{\alpha}{\alpha+\beta}A^2+\frac{\beta}{\alpha+\beta}B^2,\\
\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2 &\leq \alpha A^2+\beta B^2.
\end{aligned}$$

For the case of equality, first assume that $A = B$.

We easily see that two sides of the inequality (3.1) are equal.

Now assume that

$$\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2 = \alpha A^2+\beta B^2.$$

It follows that

$$\begin{aligned}
\frac{\alpha^2}{\alpha+\beta}A^2+\frac{\alpha\beta}{\alpha+\beta}(AB+BA)+\frac{\beta^2}{\alpha+\beta}B^2 &= \alpha A^2+\beta B^2,\\
-\frac{\alpha\beta}{\alpha+\beta}A^2+\frac{\alpha\beta}{\alpha+\beta}(AB+BA)-\frac{\alpha\beta}{\alpha+\beta}B^2 &= 0,\\
AB+BA &= A^2+B^2,\\
(A-B)^2 &= 0.
\end{aligned}$$

Since A and B are symmetric, $A-B$ is also symmetric. Write $A-B=[x_{ij}]$.

Suppose that $A-B\neq 0$.

There is a positive integer k, $1\leq k\leq n$, such that $x_{ik}=x_{ki}\neq 0$.

Consider the (i,i)th entry of $(A-B)^2$.

$$\begin{aligned}
((A-B)^2)_{ii} &= \sum_{p=1}^{n} x_{ip}x_{pi}\\
&= x_{i1}x_{1i}+x_{i2}x_{2i}+\cdots+x_{ik}x_{ki}+\cdots+x_{in}x_{ni}\\
&= (x_{i1})^2+(x_{i2})^2+\cdots+(x_{ik})^2+\cdots+(x_{in})^2\\
&> 0.
\end{aligned}$$

But $(A-B)^2=0$, this leads to a contradiction.

Hence, $A-B=0$ or $A=B$ and the theorem is established. □

Counter Example 3.1.1.2.

If A and B are positive definite and $A \neq B$, it does not imply that

$$\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2 \quad < \quad \alpha A^2+\beta B^2.$$

A counter example is $A = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \alpha = 1, \beta = 1.$

The reason is $A \leq B$ does not mean that $A < B$ or $A = B$.

Remark 3.1.1.3.

1. Theorem 3.1.1.1 shows the relationship between $(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$.
2. The case of equality in this theorem does not depend on α or β.
3. When we compare two matrices we must sure that they are Hermitian. In this theorem, $\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$ are Hermitian since they are positive definite.

Corollary 3.1.1.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\frac{1}{\alpha+\beta}(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2 \quad \leq \quad \alpha A+\beta B \quad \leq \quad (\alpha+\beta)^{\frac{1}{2}}(\alpha A^2+\beta B^2)^{\frac{1}{2}} \tag{3.2}$$

with equalities hold if and only if $A = B$.

Proof. By replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in Theorem 3.1.1.1, we get

$$\frac{1}{\alpha+\beta}(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2 \quad \leq \quad \alpha A+\beta B \tag{3.3}$$

with equality holds if and only if $A = B$.
Since $\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$ are positive definite, by applying Löwner–Heinz inequality (Theorem 2.8.2) with $r = 1/2$ to Theorem 3.1.1.1, we get

$$\begin{aligned} \left(\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2\right)^{\frac{1}{2}} \quad &\leq \quad (\alpha A+\beta B)^{\frac{1}{2}}, \\ \frac{1}{(\alpha+\beta)^{\frac{1}{2}}}(\alpha A+\beta B) \quad &\leq \quad (\alpha A+\beta B)^{\frac{1}{2}}, \\ \alpha A+\beta B \quad &\leq \quad (\alpha+\beta)^{\frac{1}{2}}(\alpha A+\beta B)^{\frac{1}{2}} \end{aligned}$$

with equality holds if and only if $A = B$.
Combining (3.3) and the last inequality, we get (3.2) with equalities hold if and only if $A = B$. □

Remark 3.1.1.5.

1. Corollary 3.1.1.4 shows the relations among $(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2$, $\alpha A+\beta B$ and $(\alpha A^2+\beta B^2)^{\frac{1}{2}}$ or, in another point of view, a lower bound and an upper bound of $\alpha A+\beta B$.
2. In this Corollary, $\frac{1}{\alpha+\beta}(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2$, $\alpha A+\beta B$ and $(\alpha+\beta)^{\frac{1}{2}}(\alpha A^2+\beta B^2)^{\frac{1}{2}}$ are Hermitian since they are positive definite.

3.1.2 Special Cases

From the main result (Theorem 3.1.1.1), there are some interesting special cases. The first case is when $\alpha+\beta=1$ which yields a following corollary:

Corollary 3.1.2.1. *Let $A \in \mathbb{M}_n$ be positive definite symmetric. Then a map Φ defined by $\Phi(A)=A^2$ is convex.*

Proof. Let $0<\epsilon<1$. By specializing $\alpha=\epsilon$ and $\beta=1-\epsilon$ in Theorem 3.1.1.1, we obtain that for any Hermitian matrices A and B, and for any scalar $0<\epsilon<1$,

$$(\epsilon A+(1-\epsilon)B)^2 \quad \leq \quad \epsilon A^2+(1-\epsilon)B^2. \tag{3.4}$$

Thus, a map Φ defined by $\Phi(A)=A^2$ is convex from Definition 2.8.1 of operator convex function. □

Remark 3.1.2.2.

1. We knew that a real-valued function f defined by $f(t) := t^2$ is convex. Corollary 3.1.2.1 is a matrix version of this fact.
2. Ando [14] proved that for any positive definite matrix A, a map Φ defined by $\Phi(A)=A^p$ is convex if $1\leq p\leq 2$. This corollary is a special case of his work ($p=2$).

The second interesting special case is when $\alpha=\beta$ which yields two following corollaries:

Corollary 3.1.2.3. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$AB+BA \;\leq\; A^2+B^2, \tag{3.5}$$

$$2(AB+BA) \;\leq\; (A+B)^2 \;\leq\; 2(A^2+B^2) \tag{3.6}$$

with equalities hold if and only if $A=B$.

Proof. By setting $\alpha = \beta$ in Theorem 3.1.1.1, we get

$$(A+B)^2 \quad \leq \quad 2(A^2+B^2). \tag{3.7}$$

This implies

$$AB+BA \quad \leq \quad A^2+B^2$$

which proved (3.5). It follows that

$$\begin{aligned} 2(AB+BA) \quad &\leq \quad A^2+B^2+AB+BA \\ &= \quad (A+B)^2. \end{aligned}$$

Combine (3.7) and the last inequality, we get (3.6). The inequality (3.5) can be alternatively proved by using Theorem 3.1.1.1, in this case we get

$$\begin{aligned} \frac{\alpha^2}{\alpha+\beta}A^2 + \frac{\alpha\beta}{\alpha+\beta}(AB+BA) + \frac{\beta^2}{\alpha+\beta}B^2 \quad &\leq \quad \alpha A^2 + \beta B^2, \\ -\frac{\alpha\beta}{\alpha+\beta}A^2 + \frac{\alpha\beta}{\alpha+\beta}(AB+BA) - \frac{\alpha\beta}{\alpha+\beta}B^2 \quad &\leq \quad \mathbf{0}, \\ AB+BA \quad &\leq \quad A^2+B^2. \end{aligned}$$

The last inequality follows from the positivity of $\alpha\beta/(\alpha+\beta)$.
The case of equalities is same as Theorem 3.1.1.1. □

Remark 3.1.2.4.

1. This corollary shows the relationship among $AB+BA$, $(A+B)^2$ and A^2+B^2 or, in another point of view, a lower bound and an upper bound of $(A+B)^2$.

2. $AB+BA$ is Hermitian since

   $$(AB+BA)^* = (AB)^* + (BA)^* = B^*A^* + A^*B^* = BA+AB = AB+BA.$$

   $(A+B)^2, 2(A^2+B^2)$ and A^2+B^2 are also Hermitian since they are positive definite.

3. Note that $AB+BA$ is not necessarily positive definite while A^2+B^2 is always positive definite. Thus, the inequality (3.5) is make sense.

Corollary 3.1.2.5. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. If AB is positive definite, then*

$$(AB)^{\frac{1}{2}} \quad \leq \quad \frac{1}{2}(A+B) \tag{3.8}$$

with equality holds if and only if $A = B$.

Proof. Since AB is positive definite, AB is Hermitian. We get

$$AB = (AB)^* = B^*A^* = BA.$$

Since $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ are positive definite, by replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in the inequality (3.5) in Corollary 3.1.2.3, we obtain from the commutativity of A and B that

$$\begin{aligned} A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}} &\leq A + B, \\ A^{\frac{1}{2}}B^{\frac{1}{2}} &\leq \frac{1}{2}(A + B). \end{aligned}$$

Since $A^{\frac{1}{2}}B^{\frac{1}{2}} = B^{\frac{1}{2}}A^{\frac{1}{2}}$, we get

$$(A^{\frac{1}{2}}B^{\frac{1}{2}})^2 = A^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}} = A^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}} = AB.$$

Again, the positive definiteness of AB implies

$$(AB)^{\frac{1}{2}} = A^{\frac{1}{2}}B^{\frac{1}{2}}.$$

So we get (3.8). The case of equality is same as Corollary 3.1.2.3. □

Counter Example 3.1.2.6.

Corollary 3.1.2.5 is not true if the condition "AB is positive definite" is omitted. A counter example is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

The reason is that there are two positive definite matrices such that the product of them is not positive definite.

Remark 3.1.2.7.

1. In Corollary 3.1.2.5, AB is also symmetric since AB is real and Hermitian.
2. Compare Corollary 3.1.2.5 with the arithmetic mean–geometric mean (AM–GM) inequality for positive real numbers which is stated that for any positive real numbers A and B,

   $$\sqrt{ab} = (ab)^{\frac{1}{2}} \leq \frac{1}{2}(a + b) \tag{3.9}$$

   with equality holds if and only if $a = b$.
   This means that the geometric mean of positive definite matrices less than or equal to their arithmetic mean. This is a matrix version of the AM–GM inequality for positive real numbers.

3. The condition "AB is positive definite" in this corollary can be replaced by "A commutes with B" since they can imply each other.
   Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. If A commutes with B, then we have
   $$(AB)^* = B^*A^* = BA = AB$$
   and
   $$\lambda(AB) = \lambda(AB^{\frac{1}{2}}B^{\frac{1}{2}}) = \lambda(B^{\frac{1}{2}}AB^{\frac{1}{2}}) > 0.$$
   So AB is Hermitian and all its eigenvalues are positive.
   By Theorem 2.3.3, AB is positive definite.

A next corollary is a special case of Corollary 3.1.1.4.

Corollary 3.1.2.8. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\frac{1}{2}(A^{\frac{1}{2}} + B^{\frac{1}{2}})^2 \quad \leq \quad A + B \quad \leq \quad \sqrt{2}(A^2 + B^2)^{\frac{1}{2}} \tag{3.10}$$

with equalities hold if and only if $A = B$.

Proof. By setting $\alpha = \beta$ in Corollary 3.1.1.4, we obtain

$$\begin{aligned}
\frac{1}{\alpha + \alpha}(\alpha A^{\frac{1}{2}} + \alpha B^{\frac{1}{2}})^2 \quad &\leq \quad \alpha A + \alpha B \quad \leq \quad (\alpha + \alpha)^{\frac{1}{2}}(\alpha A^2 + \alpha B^2)^{\frac{1}{2}}\,. \\
\frac{\alpha^2}{2\alpha}(A^{\frac{1}{2}} + B^{\frac{1}{2}})^2 \quad &\leq \quad \alpha(A + B) \quad \leq \quad (2\alpha)^{\frac{1}{2}}\alpha^{\frac{1}{2}}(A^2 + B^2)^{\frac{1}{2}}\,. \\
\frac{1}{2}(A^{\frac{1}{2}} + B^{\frac{1}{2}})^2 \quad &\leq \quad A + B \quad \leq \quad \sqrt{2}(A^2 + B^2)^{\frac{1}{2}}.
\end{aligned}$$

The case of equalities is same as Corollary 3.1.1.4. □

3.1.3 Observations

Remark 3.1.2.7 yields a following observation:

Observation 3.1.3.1. *Let $A, B \in \mathbb{M}_n$ be positive (semi)definite. Then AB is positive (semi)definite if and only if A commutes with B.*

Proof. It suffices to prove only when A and B are positive definite.
First assume that AB is positive definite. We get

$$AB = (AB)^* = B^*A^* = BA.$$

Now assume that A commutes with B. We get

$$(AB)^* = B^*A^* = BA = AB.$$

By Proposition 2.6.3, we have

$$\begin{aligned}\lambda(AB) &= \lambda(AB^{\frac{1}{2}}B^{\frac{1}{2}}) \\ &= \lambda(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \\ &> 0.\end{aligned}$$

So AB is Hermitian and all its eigenvalues are positive.
By Theorem 2.3.3, AB is positive definite and the proof is completed. □

Observation 3.1.3.2. *Let $A \in \mathbb{M}_n$ be positive definite. Then for any real number r, A^r is positive definite.*

Proof. Assume that A is positive definite. From the spectral theorem for Hermitian matrices, there exists a unitary matrix U such that

$$A = U\,diag[\lambda_1, \lambda_2, \ldots, \lambda_n]\,U^*$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A. Since A is positive definite, all eigenvalues of A are positive. Define a real-valued function f by $f(t) = t^r$ for $r \in \mathbb{R}$. Hence, the functional calculus for A is

$$A^r = U\,diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\,U^*.$$

It follows that

$$\begin{aligned}(A^r)^* &= \left(U\,diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\,U^*\right)^* \\ &= (U^*)^*\left(diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\right)^* U^* \\ &= U\,diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\,U^* \\ &= A^r \\ \text{and} \quad \lambda(A^r) &= (\lambda(A))^r > 0.\end{aligned}$$

Hence, A^r is Hermitian and have all eigenvalues positive.
Thus, A^r is positive definite by Theorem 2.3.3. □

The stronger implications of Observation 3.1.3.1 are two following observations:

Observation 3.1.3.3. *Let $A, B \in \mathbb{M}_n$ be positive definite.*

1. *If AB is positive definite, then for any real numbers r and s, A^r commutes with B^s.*

2. *If there exist nonzero real numbers r and s such that A^r commutes with B^s, then AB is positive definite.*

Proof. First assume that AB is positive definite.
Let r and s be arbitrary real numbers. We have that A^r and B^s are also positive definite. Since the positive definiteness of AB is held for arbitrary positive definite matrices A and B, this implies that A^rB^s is positive definite. Hence,

$$A^rB^s = (A^rB^s)^* = (B^s)^*(A^r)^* = B^sA^r.$$

Now assume that there exist nonzero real numbers r and s such that A^r commutes with B^s. By Observation 3.1.3.2, $A^{\frac{1}{r}}$ and $B^{\frac{1}{s}}$ are positive definite.
Since the commutativity of A^r and B^s is held for arbitrary positive definite matrices A and B. This implies that

$$\begin{aligned}(A^{\frac{1}{r}})^r(B^{\frac{1}{s}})^s &= (B^{\frac{1}{s}})^s(A^{\frac{1}{r}})^r,\\ AB &= BA.\end{aligned}$$

So A commutes with B. Using Observation 3.1.3.1, we obtain that AB is positive definite and this completes the proof. □

Observation 3.1.3.4. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite.*

1. *If AB is positive semidefinite, then for any positive real numbers r and s, A^r commutes with B^s.*
2. *If there exists positive real numbers r and s such that A^r commutes with B^s, then AB is positive semidefinite.*

Proof. A proof is similar to the previous observation. The difference is that for positive semidefinite matrices A and B, A^r and B^s exist if r and s are positive real numbers. But for positive definite matrices X and Y, X^p and Y^q exist for all real numbers p, q. □

In the proof of Theorem 3.1.1.1, we see that if A is symmetric, then $A^2 = 0$ if and only if $A = 0$. Let's generalize this statement. We begin with the following observation.

Observation 3.1.3.5. *Let $A \in \mathbb{M}_n$ be Hermitian. Then $\sigma(A) = \{0\}$ if and only if $A = 0$.*

Proof. Obviously, if $A = 0$, then $\sigma(A) = \{0\}$. Now assume that $\sigma(A) = \{0\}$.
Since A is Hermitian, there is a unitary matrix U such that

$$A = U diag[0, 0, \ldots, 0]U^*.$$

Hence,

$$A \quad = \quad 0.$$

That is $A = 0$ if and only if $\sigma(A) = \{0\}$. □

Observation 3.1.3.6. *Let $A \in \mathbb{M}_n$ be Hermitian. Then for any even positive integer k, $A^k = 0$ if and only if $A = 0$.*

Proof. Assume that $A^k = 0$. Since A is Hermitian, there is a unitary matrix $U = [u_{ij}]$ such that

$$A = U\, diag[\lambda_1, \lambda_2, \ldots, \lambda_n]\, U^*$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A.

Since λ_i are all real for all i, we have

$$A^k = U\, diag[\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k]\, U^*.$$

Note that $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are all nonnegative because k is even.

Suppose that $A \neq 0$.

From the previous observation, we may assume that A has a nonzero eigenvalue, i.e., there exists an integer p, $1 \leq p \leq n$, such that $\lambda_p \neq 0$.

Consider the (i,i)th entry of A^k $(1 \leq i \leq n)$.

$$\begin{aligned} (A^k)_{ii} &= \sum_{m=1}^{n} u_{im} \lambda_i^k \overline{u_{im}} \\ &= \sum_{m=1}^{n} \lambda_i^k |u_{im}|^2 \\ &= \lambda_1^k |u_{i1}|^2 + \lambda_2^k |u_{i2}|^2 + \cdots + \lambda_p^k |u_{ip}|^2 + \cdots + \lambda_n^k |u_{in}|^2. \end{aligned}$$

Since $A^k = 0$, so $u_{ip} = 0$ for all $i = 1, 2, \ldots, n$.

This mean that all entries in the pth row of U are all zero, i.e., $\det(U) = 0$.

But U is unitary, we have $UU^* = I$ and

$$\begin{aligned} \det(I) &= \det(UU^*) \\ &= \det(U)\ \det(U^*) \\ &= \det(U)\, \overline{\det(U)} \\ &= |\det(U)|^2. \end{aligned}$$

Since $\det(I) = 1$, so $\det(U) \neq 0$. This is a contradiction.

Thus, $A = 0$. The converse statement is obvious. □

Observation 3.1.3.7. *Let $A \in \mathbb{M}_n$ be positive semidefinite and r be a positive real number. Then $A^r = 0$ if and only if $A = 0$.*

Proof. A proof is similar to the previous observation. The difference is that in this case A^r exists if r is a positive real number. □

3.2 Inequalities Involving the Kronecker Product of Power of Linear Combination of Matrices

The purpose of this section is to show the relationship between $(\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s$ and $\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)$ in the form of inequalities in the Löwner partial order. An inequality between them is shown in Theorem 3.2.1.5 in Subsection 3.2.1. New matrix inequalities which are followed from special cases of this theorem are stated in Subsection 3.2.2. In Subsection 3.2.3, we give some observations which are related to Subsection 3.2.1 and Subsection 3.2.2. We also give some interesting remarks.

3.2.1 Main Results

To prove the main theorem, we need some lemmas.

Lemma 3.2.1.1. *Let $A, B \in \mathbb{M}_n$ be positive definite. Then $A^{-1} + B^{-1}$ is nonsingular and*

$$(A^{-1} + B^{-1})^{-1} = A - A(A+B)^{-1}A. \tag{3.11}$$

Proof. Since A and B are positive definite, so A and B are nonsingular. Consequently, $A+B, A^{-1}, B^{-1}$ and $A^{-1}+B^{-1}$ are existent, positive definite and nonsingular by Theorem 2.3.3. Consider $(A^{-1} + B^{-1})(A - A(A+B)^{-1}A)$, we obtain

$$\begin{aligned}
(A^{-1} + B^{-1})(A - A(A+B)^{-1}A) &= (A^{-1} + B^{-1})A(I - (A+B)^{-1}A) \\
&= (I + B^{-1}A)(I - (A+B)^{-1}A) \\
&= I + B^{-1}A - (A+B)^{-1}A - B^{-1}A(A+B)^{-1}A \\
&= I + \left(B^{-1} - (A+B)^{-1} - B^{-1}A(A+B)^{-1}\right) A \\
&= I + \left(B^{-1} - (I + B^{-1}A)(A+B)^{-1}\right) A \\
&= I + \left(B^{-1} - B^{-1}(A+B)(A+B)^{-1}\right) A \\
&= I.
\end{aligned}$$

This means that $(A^{-1} + B^{-1})^{-1} = A - A(A+B)^{-1}A$.
The proof is completed. □

Lemma 3.2.1.2. *Let $A, B \in \mathbb{M}_n$ be positive definite and let s be any positive real number. Then*

$$((s^{-1}A \otimes I)^{-1} + (I \otimes B)^{-1})^{-1} = (A \otimes B)^{-1}\Big((A \otimes B^{-1}) + (sI \otimes I)\Big)^{-1}(I \otimes B) \quad (3.12)$$

Proof. Since A and B are positive definite and s is positive, so all terms in the equality (3.12) are existent, positive definite and nonsingular. We get

$$\begin{aligned}
&((A \otimes B^{-1}) + (sI \otimes I))\Big((s^{-1}I \otimes I) - (s^{-1}I \otimes B)((s^{-1}A \otimes I) + (I \otimes B))^{-1}(s^{-1}A \otimes B^{-1})\Big) \\
&\quad = (A \otimes B^{-1})(s^{-1}I \otimes I) + (sI \otimes I)(s^{-1}I \otimes I) \\
&\qquad -(A \otimes B^{-1})(s^{-1}I \otimes B)\Big((s^{-1}A \otimes I) + (I \otimes B)\Big)^{-1}(s^{-1}A \otimes B^{-1}) \\
&\qquad -(sI \otimes I)(s^{-1}I \otimes B)((s^{-1}A \otimes I) + (I \otimes B))^{-1}(s^{-1}A \otimes B^{-1}) \\
&\quad = (s^{-1}A \otimes B^{-1}) + (I \otimes I) - (I \otimes B)\Big((s^{-1}A \otimes I) + (I \otimes B)\Big)^{-1}(s^{-1}A \otimes B^{-1}) \\
&\qquad -(s^{-1}A \otimes I)((s^{-1}A \otimes I) + (I \otimes B^{-1}))^{-1}(s^{-1}A \otimes B^{-1}) \\
&\quad = I_{n^2} - ((s^{-1}A \otimes I) + (I \otimes B))\Big((s^{-1}A \otimes I) + (I \otimes B)\Big)^{-1}(s^{-1}A \otimes B^{-1}) \\
&\qquad +(s^{-1}A \otimes B^{-1}) \\
&\quad = I_{n^2}.
\end{aligned}$$

This means that

$$((A \otimes B^{-1}) + (sI \otimes I))^{-1} = (s^{-1}I \otimes I) - (s^{-1}I \otimes B)((s^{-1}A \otimes I) + (I \otimes B))^{-1}(s^{-1}A \otimes B^{-1}).$$

It follows that

$$\begin{aligned}
&(A \otimes B^{-1})^{-1}\Big((s^{-1}A \otimes I) - (s^{-1}A \otimes I)((s^{-1}A \otimes I) + (I \otimes B))^{-1}(s^{-1}A \otimes I)\Big)(I \otimes B)^{-1} \\
&\quad = (A^{-1} \otimes B)\Big((s^{-1}A \otimes I) - (s^{-1}A \otimes I)((s^{-1}A \otimes I) + (I \otimes B))^{-1}(s^{-1}A \otimes I)\Big) \\
&\qquad (I \otimes B^{-1}) \\
&\quad = (A^{-1} \otimes B)(s^{-1}A \otimes I)(I \otimes B^{-1}) \\
&\qquad -(A^{-1} \otimes B)(s^{-1}A \otimes I)((s^{-1}A \otimes I) + (I \otimes B))^{-1}(s^{-1}A \otimes I)(I \otimes B^{-1}) \\
&\quad = (s^{-1}I \otimes I) - (s^{-1}I \otimes B)((s^{-1}A \otimes I) + (I \otimes B))^{-1}(s^{-1}A \otimes B^{-1}) \\
&\quad = ((A \otimes B^{-1}) + (sI \otimes I))^{-1}.
\end{aligned}$$

From Lemma 3.2.1.1, we have

$$((A \otimes B^{-1}) + (sI \otimes I))^{-1} = (A \otimes B^{-1})^{-1}((s^{-1}A \otimes I)^{-1} + (I \otimes B)^{-1})^{-1}(I \otimes B^{-1}).$$

Thus,

$$((s^{-1}A \otimes I)^{-1} + (I \otimes B)^{-1})^{-1} = (A \otimes B^{-1})((A \otimes B^{-1}) + (sI \otimes I))^{-1}(I \otimes B).$$

The proof is completed. □

Lemma 3.2.1.3. *Let $A, B \in \mathbb{M}_n$ be positive definite, then for any real number r,*

$$(A \otimes B)^r = A^r \otimes B^r. \tag{3.13}$$

Proof. Assume that A and B are positive definite.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of A and let $\mu_1, \mu_2, \ldots, \mu_n$ be eigenvalues of B.
Since A and B are Hermitian, there exist unitary matrices U_A and U_B such that

$$\begin{aligned} A &= U_A \Lambda_A U_A^* \\ \text{and} \qquad B &= U_B \Lambda_B U_B^* \\ \text{where} \qquad \Lambda_A &:= diag[\lambda_1, \lambda_2, \ldots, \lambda_n] \\ \text{and} \qquad \Lambda_B &:= diag[\mu_1, \mu_2, \ldots, \mu_n]. \end{aligned}$$

Hence,

$$\begin{aligned} A \otimes B &= (U_A \Lambda_A U_A^*) \otimes (U_B \Lambda_B U_B^*) \\ &= (U_A \otimes U_B)(\Lambda_A \otimes \Lambda_B)(U_A^* \otimes U_B^*) \\ &= (U_A \otimes U_B)\big(diag[\lambda_1, \lambda_2, \ldots, \lambda_n] \otimes diag[\mu_1, \mu_2, \ldots, \mu_n]\big)(U_A \otimes U_B)^* \\ &= V \Sigma V^* \end{aligned}$$

where $V := U_A \otimes U_B$ and $\Sigma := diag[\lambda_1\mu_1, \lambda_1\mu_2, \ldots, \lambda_1\mu_n, \lambda_2\mu_1, \lambda_2\mu_2, \ldots, \lambda_n\mu_n]$.
Since

$$\begin{aligned} VV^* &= (U_A \otimes U_B)(U_A \otimes U_B)^* = (U_A \otimes U_B)(U_A^* \otimes U_B^*) \\ &= (U_A U_A^*) \otimes (U_B U_B^*) \\ &= I_n \otimes I_n = I_{n^2}, \end{aligned}$$

so V is unitary.
From Theorem 2.4.6, the eigenvalues of $A \otimes B$ are $\lambda_i\mu_j$ for each $i, j = 1, 2, \ldots, n$.
This implies that $A \otimes B = V\Sigma V^*$ is the spectral decomposition of $A \otimes B$.
Let f be a real-valued function on $\mathbb{R}^+$ defined by

$$f(t) = t^r \qquad \text{for } t > 0,\ r \in \mathbb{R}.$$

Clearly, f is continuous.
Since all eigenvalues of A, B and $A \otimes B$ are positive real numbers, the functional calculus for A, B and $A \otimes B$ are respectively defined by

$$\begin{aligned} f(A) &= A^r = U_A\, diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\, U_A^*, \\ f(B) &= B^r = U_B\, diag[\mu_1^r, \mu_2^r, \ldots, \mu_n^r]\, U_B^* \text{ and} \\ f(A \otimes B) &= (A \otimes B)^r \\ &= V diag[(\lambda_1\mu_1)^r, (\lambda_1\mu_2)^r, \ldots, (\lambda_1\mu_n)^r, (\lambda_2\mu_1)^r, (\lambda_2\mu_2)^r, \ldots, (\lambda_n\mu_n)^r] V^*. \end{aligned}$$

Since λ_i and μ_j are positive for all i, j, we obtain

$$\begin{aligned}(A\otimes B)^r &= V diag[\lambda_1^r\mu_1^r, \lambda_1^r\mu_2^r, \ldots, \lambda_1^r\mu_n^r, \lambda_2^r\mu_1^r, \lambda_2^r\mu_2^r, \ldots, \lambda_n^r\mu_n^r]V^* \\ &= (U_A\otimes U_B)\Big(diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\otimes diag[\mu_1^r, \mu_2^r, \ldots, \mu_n^r]\Big)(U_A\otimes U_B)^* \\ &= \Big(U_A diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]U_A^*\Big)\otimes\Big(U_B diag[\mu_1^r, \mu_2^r, \ldots, \mu_n^r]U_B^*\Big) \\ &= A^r\otimes B^r.\end{aligned}$$

The proof is completed. □

Lemma 3.2.1.4. *Let $\Phi_1, \Phi_2, \ldots, \Phi_k$ be jointly concave maps from $\mathbb{M}_m\times\mathbb{M}_n$ into $\mathbb{M}_p$ and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be positive scalars. Then $\alpha_1\Phi_1+\alpha_2\Phi_2+\cdots+\alpha_k\Phi_k$ is jointly concave. That is the positive linear combination of jointly concave maps is jointly concave.*

Proof. Let $A, B\in\mathbb{M}_m$ be Hermitian and $C, D\in\mathbb{M}_n$ be Hermitian and let $0<\epsilon<1$ be a scalar. It follows straight forward from the definition of jointly concave map that

$$\begin{aligned}&(\alpha_1\Phi_1+\alpha_2\Phi_2)(\epsilon A+(1-\epsilon)B, \epsilon C+(1-\epsilon)D) \\ &\quad= (\alpha_1\Phi_1)(\epsilon A+(1-\epsilon)B, \epsilon C+(1-\epsilon)D)+(\alpha_2\Phi_2)(\epsilon A+(1-\epsilon)B, \epsilon C+(1-\epsilon)D) \\ &\quad= \alpha_1\Phi_1(\epsilon A+(1-\epsilon)B, \epsilon C+(1-\epsilon)D)+\alpha_2\Phi_2(\epsilon A+(1-\epsilon)B, \epsilon C+(1-\epsilon)D) \\ &\quad\geq \alpha_1(\epsilon\Phi_1(A, C)+(1-\epsilon)\Phi_1(B, D))+\alpha_2(\epsilon\Phi_2(A, C)+(1-\epsilon)\Phi_2(B, D)) \\ &\quad= \epsilon\alpha_1\Phi_1(A, C)+(1-\epsilon)\alpha_1\Phi_1(B, D)+\epsilon\alpha_2\Phi_2(A, C)+(1-\epsilon)\alpha_2\Phi_2(B, D) \\ &\quad= \epsilon(\alpha_1\Phi_1)(A, C)+(1-\epsilon)(\alpha_1\Phi_1)(B, D)+\epsilon(\alpha_2\Phi_2)(A, C)+(1-\epsilon)(\alpha_2\Phi_2)(B, D) \\ &\quad= \epsilon(\alpha_1\Phi_1+\alpha_2\Phi_2)(A, C)+(1-\epsilon)(\alpha_1\Phi_1+\alpha_2\Phi_2)(B, D).\end{aligned}$$

Hence, $\alpha_1\Phi_1+\alpha_2\Phi_2$ is jointly concave.
Analogously, $\alpha_1\Phi_1+\alpha_2\Phi_2+\cdots+\alpha_k\Phi_k$ is jointly concave. □

Theorem 3.2.1.5. *Let $A, B, C, D\in\mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s>0$ be scalars such that $r+s=1$. Then*

$$(\alpha A+\beta B)^r\otimes(\alpha C+\beta D)^s \quad\geq\quad \alpha(A^r\otimes C^s)+\beta(B^r\otimes D^s) \tag{3.14}$$

with equality holds if $A=B, C=D$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of A and let $\mu_1, \mu_2, \ldots, \mu_n$ be eigenvalues of B. Let f be a real-valued function on $\mathbb{R}^+$ defined by

$$f(t) \quad=\quad t^r \qquad \text{for } t>0,\ 0<r<1.$$

Clearly, f is continuous.

Since $\lambda_i, \mu_j > 0$ for each i, j, so $\lambda_i \mu_j^{-1} \in \sigma(A \otimes B^{-1})$ for each i, j.

Since the function $f(t) = t^r$ has an integral representation:

$$t^r = \frac{\sin r\pi}{\pi} \int_0^\infty \frac{s^{r-1}t}{s+t} ds.$$

The functional calculus for $A \otimes B^{-1}$ is

$$\begin{aligned} f(A \otimes B^{-1}) &= (A \otimes B^{-1})^r \\ &= \frac{\sin r\pi}{\pi} \int_0^\infty (sI \otimes I)^{r-1} (A \otimes B^{-1}) \big((A \otimes B^{-1}) + (sI \otimes I)\big)^{-1} ds. \end{aligned}$$

It follows from Lemma 3.2.1.2 and Lemma 3.2.1.3 that

$$\begin{aligned} A^r \otimes B^{1-r} &= (A^r I) \otimes (B^{-r} B) \\ &= (A^r \otimes B^{-r})(I \otimes B) \\ &= (A \otimes B^{-1})^r (I \otimes B) \\ &= \frac{\sin r\pi}{\pi} \int_0^\infty (sI \otimes I)^{r-1} (A \otimes B^{-1}) \big((A \otimes B^{-1}) + (sI \otimes I)\big)^{-1} ds (I \otimes B) \\ &= \frac{\sin r\pi}{\pi} \int_0^\infty s^{r-1} (A \otimes B^{-1}) \big((A \otimes B^{-1}) + (sI \otimes I)\big)^{-1} ds (I \otimes B) \\ &= \frac{\sin r\pi}{\pi} \int_0^\infty s^{r-1} (A \otimes B^{-1}) \big((A \otimes B^{-1}) + (sI \otimes I)\big)^{-1} (I \otimes B) ds \\ &= \frac{\sin r\pi}{\pi} \int_0^\infty s^{r-1} \big((s^{-1} A \otimes I)^{-1} + (I \otimes B)^{-1}\big)^{-1} ds. \end{aligned}$$

Since $s^{-1} A \otimes I$ and $I \otimes B$ are positive definite, by Lemma 2.8.9 we have the map Φ defined by

$$\Phi(s^{-1} A \otimes I, I \otimes B) = \big((s^{-1} A \otimes I)^{-1} + (I \otimes B)^{-1}\big)^{-1}$$

is jointly concave.

Lemma 3.2.1.4 states that the positive linear combination of the jointly concave maps is jointly concave. Hence, from a view of the Riemann integral, the integrand is also jointly concave, and so is $A^r \otimes B^{1-r}$.

That is, for any Hermitian matrices A, B, C, D and $0 < \epsilon < 1$,

$$\begin{aligned} (\epsilon A + (1-\epsilon) B)^r \otimes (\epsilon C + (1-\epsilon) D)^{1-r} &\geq \epsilon (A^r \otimes C^{1-r}) + (1-\epsilon)(B^r \otimes D^{1-r}), \\ (\epsilon A + (1-\epsilon) B)^r \otimes (\epsilon C + (1-\epsilon) D)^{s} &\geq \epsilon (A^r \otimes C^{s}) + (1-\epsilon)(B^r \otimes D^{s}) \end{aligned}$$

where $s > 0$ and $r + s = 1$.

Since $0 < \frac{\alpha}{\alpha+\beta} < 1$, by setting $\epsilon = \frac{\alpha}{\alpha+\beta}$, we get

$$\begin{aligned}
&\frac{1}{\alpha+\beta}(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)) \\
&\quad = \frac{\alpha}{\alpha+\beta}(A^r \otimes C^s) + \frac{\beta}{\alpha+\beta}(B^r \otimes D^s) \\
&\quad \leq \left(\frac{\alpha}{\alpha+\beta}A + \frac{\beta}{\alpha+\beta}B\right)^r \otimes \left(\frac{\alpha}{\alpha+\beta}C + \frac{\beta}{\alpha+\beta}D\right)^s \\
&\quad = \frac{1}{(\alpha+\beta)^r}(\alpha A + \beta B)^r \otimes \frac{1}{(\alpha+\beta)^s}(\alpha C + \beta D)^s \\
&\quad = \frac{1}{(\alpha+\beta)^r(\alpha+\beta)^s}\left((\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s\right) \\
&\quad = \frac{1}{\alpha+\beta}\left((\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s\right).
\end{aligned}$$

Since $\alpha + \beta > 0$, we get

$$(\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s \quad \geq \quad \alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s).$$

For the case of equality, assume that $A = B$, $C = D$. We obtain

$$\begin{aligned}
(\alpha A + \beta A)^r \otimes (\alpha C + \beta C)^s &= ((\alpha+\beta)A)^r \otimes ((\alpha+\beta)C)^s \\
&= (\alpha+\beta)^r A^r \otimes (\alpha+\beta)^s C^s \\
&= (\alpha+\beta)^r(\alpha+\beta)^s(A^r \otimes C^s) \\
&= (\alpha+\beta)(A^r \otimes C^s) \\
&= \alpha(A^r \otimes C^s) + \beta(A^r \otimes C^s).
\end{aligned}$$

So the equality holds. The theorem is established. □

Remark 3.2.1.6.

$(\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s$ and $\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)$ are Hermitian since they are positive definite.

3.2.2 Special cases

Now we will consider some special cases of Theorem 3.2.1.5. The first case is when $\alpha + \beta = 1$.

Corollary 3.2.2.1. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then a map Φ defined by $\Phi(A, B) = A^r \otimes B^s$ is jointly concave.*

Proof. Let $0 < \epsilon < 1$. By specializing $\alpha = \epsilon$ and $\beta = 1 - \epsilon$ in Theorem 3.2.1.5, we obtain that for any Hermitian matrices A, B, C, D and any scalar $0 < \epsilon < 1$,

$$(\epsilon A + (1-\epsilon)B)^r \otimes (\epsilon C + (1-\epsilon)D)^s \quad \geq \quad \epsilon(A^r \otimes C^s) + (1-\epsilon)(B^r \otimes D^s).$$

Thus, a map Φ defined by $\Phi(A, B) = A^r \otimes B^s$ is jointly concave by Definition 2.8.5 of jointly concave map. □

Before going on, we prove some lemmas.

Lemma 3.2.2.2. *Let $A \in \mathbb{M}_n$ and let k be a positive integer. If A is positive definite, then for any real number r,*

$$(A^{\otimes k})^r \quad = \quad (A^r)^{\otimes k}. \tag{3.15}$$

Proof. We use an induction on k. Clearly, this statement is true for $k = 1$.
Now assume that this is true for positive integer $k = p$, i.e.,

$$(A^{\otimes p})^r \quad = \quad (A^r)^{\otimes p}.$$

Since $A^{\otimes p}$ is positive definite, by Lemma 3.2.1.3, we have

$$(A \otimes A^{\otimes p})^r \quad = \quad A^r \otimes (A^{\otimes p})^r.$$

Hence,

$$\begin{aligned}(A^{\otimes(p+1)})^r &= A^r \otimes (A^r)^{\otimes p} \\ &= (A^r)^{\otimes(p+1)}.\end{aligned}$$

Thus, this is also true for $k = p + 1$. The proof is completed. □

There are many others special cases. There are 3 types of them: the equality between the matrices; the equality between the coefficients (scalars α, β); and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C$, $B = D$;
2. the case when $A = D$, $B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

Combining all of them, there are 16 possible cases (which include the general case). The details for each case are shown in Table 3.1 (except for the general case).

Table 3.1: The Special Cases of Theorem 3.2.1.5

Conditions			Results	
A, B, C, D	r, s	α, β	Corollary	Inequality
-	-	$\alpha = \beta$	3.2.2.3	3.16
-	$r = s$	-	3.2.2.4	3.17
-	$r = s$	$\alpha = \beta$	3.2.2.6	3.20
$A = C, B = D$	-	-	3.2.2.5	3.18
$A = C, B = D$	-	$\alpha = \beta$	3.2.2.7	3.21
$A = C, B = D$	$r = s$	-	3.2.2.8	3.23
$A = C, B = D$	$r = s$	$\alpha = \beta$	3.2.2.9	3.25
$A = D, B = C$	-	-	3.2.2.5	3.19
$A = D, B = C$	-	$\alpha = \beta$	3.2.2.7	3.22
$A = D, B = C$	$r = s$	-	3.2.2.8	3.24
$A = D, B = C$	$r = s$	$\alpha = \beta$	3.2.2.9	3.26
$A = B = C = D$	-	-	Equality Case	
$A = B = C = D$	-	$\alpha = \beta$	Equality Case	
$A = B = C = D$	$r = s$	-	Equality Case	
$A = B = C = D$	$r = s$	$\alpha = \beta$	Equality Case	

We summarize these cases into 7 corollaries as follows (depend on the hypotheses).

Corollary 3.2.2.3. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ scalars such that $r + s = 1$. Then*

$$(A+B)^r \otimes (C+D)^s \quad \geq \quad (A^r \otimes C^s) + (B^r \otimes D^s) \tag{3.16}$$

with equality holds if $A = B$, $C = D$.

Proof. By setting $\alpha = \beta$ in Theorem 3.2.1.5, we obtain

$$\begin{aligned}
(\alpha A + \alpha B)^r \otimes (\alpha C + \alpha D)^s &\geq \alpha(A^r \otimes C^s) + \alpha(B^r \otimes D^s), \\
\alpha^r (A+B)^r \otimes \alpha^s (C+D)^s &\geq \alpha(A^r \otimes C^s) + \alpha(B^r \otimes D^s), \\
\alpha^r \alpha^s ((A+B)^r \otimes (C+D)^s) &\geq \alpha(A^r \otimes C^s) + \alpha(B^r \otimes D^s), \\
\alpha((A+B)^r \otimes (C+D)^s) &\geq \alpha((A^r \otimes C^s) + (B^r \otimes D^s)), \\
(A+B)^r \otimes (C+D)^s &\geq (A^r \otimes C^s) + (B^r \otimes D^s).
\end{aligned}$$

The case of equality is same as Theorem 3.2.1.5. The proof is completed. □

Corollary 3.2.2.4. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$((\alpha A + \beta B) \otimes (\alpha C + \beta D))^{\frac{1}{2}} \quad \geq \quad \alpha(A \otimes C)^{\frac{1}{2}} + \beta(B \otimes D)^{\frac{1}{2}} \tag{3.17}$$

with equality holds if $A = B$, $C = D$.

Proof. By setting $r = s$ in Theorem 3.2.1.5, we get $r = 1/2$ and

$$\begin{aligned}
(\alpha A + \beta B)^{\frac{1}{2}} \otimes (\alpha C + \beta D)^{\frac{1}{2}} &\geq \alpha(A^{\frac{1}{2}} \otimes C^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \otimes D^{\frac{1}{2}}), \\
((\alpha A + \beta B) \otimes (\alpha C + \beta D))^{\frac{1}{2}} &\geq \alpha(A \otimes C)^{\frac{1}{2}} + \beta(B \otimes D)^{\frac{1}{2}}.
\end{aligned}$$

The last inequality follows from Lemma 3.2.1.3.
The case of equality is same as Theorem 3.2.1.5. The proof is completed. □

Corollary 3.2.2.5. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\begin{aligned}
(\alpha A + \beta B)^r \otimes (\alpha A + \beta B)^s &\geq \alpha(A^r \otimes A^s) + \beta(B^r \otimes B^s), && (3.18) \\
(\alpha A + \beta B)^r \otimes (\beta A + \alpha B)^s &\geq \alpha(A^r \otimes B^s) + \beta(B^r \otimes A^s) && (3.19)
\end{aligned}$$

with equalities hold if $A = B$. The inequality (3.18) *also becomes an equality if $n = 1$.*

Proof. By setting $A = C, B = D$ in Theorem 3.2.1.5, we get (3.18). By setting $A = D, B = C$ in Theorem 3.2.1.5, we get (3.19).
For the inequality (3.18) if $A = B$, we obtain

$$\begin{aligned}(\alpha A + \beta A)^r \otimes (\alpha A + \beta A)^s &= (\alpha + \beta)^r A^r \otimes (\alpha + \beta)^s A^s\\ &= (\alpha + \beta)^r (\alpha + \beta)^s (A^r \otimes A^s)\\ &= (\alpha + \beta)(A^r \otimes A^s)\\ &= \alpha(A^r \otimes A^s) + \beta(A^r \otimes A^s).\end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the inequality (3.18) also becomes an equality if $n = 1$. □

Corollary 3.2.2.6. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$((A + B) \otimes (C + D))^{\frac{1}{2}} \geq (A \otimes C)^{\frac{1}{2}} + (B \otimes D)^{\frac{1}{2}} \tag{3.20}$$

with equality holds if $A = B$, $C = D$.

Proof. By setting $\alpha = \beta$ in Corollary 3.2.2.4, we get

$$\begin{aligned}((\alpha A + \alpha B) \otimes (\alpha C + \alpha D))^{\frac{1}{2}} &\geq \alpha(A \otimes C)^{\frac{1}{2}} + \alpha(B \otimes D)^{\frac{1}{2}},\\ (\alpha(A + B) \otimes \alpha(C + D))^{\frac{1}{2}} &\geq \alpha((A \otimes C)^{\frac{1}{2}} + (B \otimes D)^{\frac{1}{2}}),\\ (\alpha^2(A + B) \otimes (C + D))^{\frac{1}{2}} &\geq \alpha((A \otimes C)^{\frac{1}{2}} + (B \otimes D)^{\frac{1}{2}}),\\ ((A + B) \otimes (C + D))^{\frac{1}{2}} &\geq (A \otimes C)^{\frac{1}{2}} + (B \otimes D)^{\frac{1}{2}}.\end{aligned}$$

The case of equality is same as Corollary 3.2.2.4. □

Corollary 3.2.2.7. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$(A + B)^r \otimes (A + B)^s \geq (A^r \otimes A^s) + (B^r \otimes B^s), \tag{3.21}$$

$$(A + B)^r \otimes (A + B)^s \geq (A^r \otimes B^s) + (B^r \otimes A^s) \tag{3.22}$$

with equalities hold if $A = B$. The inequality (3.21) *also becomes an equality if $n = 1$.*

Proof. By setting $\alpha = \beta$ in (3.18) in Corollary 3.2.2.5, we get

$$\begin{aligned}(\alpha A + \alpha B)^r \otimes (\alpha A + \alpha B)^s &\geq \alpha(A^r \otimes A^s) + \alpha(B^r \otimes B^s),\\ \alpha^r (A + B)^r \otimes \alpha^s (A + B)^s &\geq \alpha(A^r \otimes A^s) + \alpha(B^r \otimes B^s),\\ \alpha^r \alpha^s ((A + B)^r \otimes (A + B)^s) &\geq \alpha((A^r \otimes A^s) + (B^r \otimes B^s)),\\ \alpha((A + B)^r \otimes (A + B)^s) &\geq \alpha((A^r \otimes A^s) + (B^r \otimes B^s)),\\ (A + B)^r \otimes (A + B)^s &\geq (A^r \otimes A^s) + (B^r \otimes B^s).\end{aligned}$$

Similarly, by setting $\alpha = \beta$ in (3.19) in Corollary 3.2.2.5, we get (3.22).
The case of equalities is same as Corollary 3.2.2.5. □

Corollary 3.2.2.8. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\begin{aligned} ((\alpha A + \beta B)^{\frac{1}{2}})^{\otimes 2} &\geq \alpha (A^{\frac{1}{2}})^{\otimes 2} + \beta (B^{\frac{1}{2}})^{\otimes 2}, && (3.23) \\ ((\alpha A + \beta B) \otimes (\beta A + \alpha B))^{\frac{1}{2}} &\geq \alpha (A \otimes B)^{\frac{1}{2}} + \beta (B \otimes A)^{\frac{1}{2}} && (3.24) \end{aligned}$$

with equalities hold if $A = B$. The inequality (3.23) *also becomes an equality if $n = 1$.*

Proof. By setting $r = s$ in the inequality (3.18), we get $r = 1/2$ and

$$\begin{aligned} ((\alpha A + \beta B)^{\frac{1}{2}})^{\otimes 2} &\geq \alpha (A^{\otimes 2})^{\frac{1}{2}} + \beta (B^{\otimes 2})^{\frac{1}{2}}, \\ &= \alpha (A^{\frac{1}{2}})^{\otimes 2} + \beta (B^{\frac{1}{2}})^{\otimes 2}. \end{aligned}$$

The last equality follows from Lemma 3.2.2.2.
By setting $r = s$ in the inequality (3.19), we get the inequality (3.24).
For the inequality (3.23), if $A = B$, we obtain

$$\begin{aligned} ((\alpha A + \beta A)^{\frac{1}{2}})^{\otimes 2} &= ((\alpha + \beta)^{\frac{1}{2}} A^{\frac{1}{2}})^{\otimes 2} \\ &= ((\alpha + \beta)^{\frac{1}{2}})^2 (A^{\frac{1}{2}})^{\otimes 2} \\ &= (\alpha + \beta)(A^{\frac{1}{2}})^{\otimes 2} \\ &= \alpha (A^{\frac{1}{2}})^{\otimes 2} + \beta (A^{\frac{1}{2}})^{\otimes 2}. \end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the inequality (3.23) becomes an equality when $n = 1$. □

Corollary 3.2.2.9. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\begin{aligned} ((A + B)^{\frac{1}{2}})^{\otimes 2} &\geq (A^{\frac{1}{2}})^{\otimes 2} + (B^{\frac{1}{2}})^{\otimes 2}, && (3.25) \\ ((A + B)^{\frac{1}{2}})^{\otimes 2} &\geq (A \otimes B)^{\frac{1}{2}} + (B \otimes A)^{\frac{1}{2}} && (3.26) \end{aligned}$$

with equalities hold if $A = B$. The inequality (3.25) *also becomes an equality if $n = 1$.*

Proof. By setting $r = s$ in (3.21) in Corollary 3.2.2.7, we get $r = 1/2$ and

$$\begin{aligned} (A + B)^{\frac{1}{2}} \otimes (A + B)^{\frac{1}{2}} &\geq (A^{\frac{1}{2}} \otimes A^{\frac{1}{2}}) + (B^{\frac{1}{2}} \otimes B^{\frac{1}{2}}), \\ ((A + B)^{\frac{1}{2}})^{\otimes 2} &\geq (A^{\frac{1}{2}})^{\otimes 2} + (B^{\frac{1}{2}})^{\otimes 2}. \end{aligned}$$

Similarly, by setting $r = s$ in the inequality (3.22) in Corollary 3.2.2.7, we get (3.26). The case of equalities is same as Corollary 3.2.2.7. Obviously, the inequality (3.25) becomes an equality when $n = 1$. □

Remark 3.2.2.10.

For the inequality (3.26) in Corollary 3.2.2.9, when $n = 1$, let $A = [a]$ and $B = [b]$, we get

$$\begin{aligned} (a+b)^{\frac{1}{2}}(a+b)^{\frac{1}{2}} &\geq (ab)^{\frac{1}{2}} + (ba)^{\frac{1}{2}}, \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a+b). \end{aligned}$$

This is the AM–GM inequality for real numbers.

3.2.3 Observations

Lemma 3.2.1.3 states that if A and B are positive definite, then for any real number r, $(A \otimes B)^r = A^r \otimes B^r$. We knew that for any positive integer p, $(A \otimes B)^p = A^p \otimes B^p$. What happen when A and B are positive semidefinite or nonsingular?

Observation 3.2.3.1. *Let $A, B \in \mathbb{M}_n$.*

1. *If A and B are positive semidefinite, then for any nonnegative real number r, $(A \otimes B)^r = A^r \otimes B^r$.*
2. *If A and B are nonsingular, then for any integer k, $(A \otimes B)^k = A^k \otimes B^k$.*

Proof. The proof of the first assertion is similar to Lemma 3.2.1.3. Observe that when A is positive semidefinite, A^r exists for any nonnegative real number r. For the second assertion, it suffices to prove only when $k < 0$. Since A and B are nonsingular, $A \otimes B$ is also nonsingular by Corollary 2.4.5. It follows that

$$\begin{aligned} (A \otimes B)^k &= ((A \otimes B)^{-1})^{-k} \\ &= (A^{-1} \otimes B^{-1})^{-k} \\ &= (A^{-1})^{-k} \otimes (B^{-1})^{-k} \\ &= A^k \otimes B^k. \end{aligned}$$

The proof is completed. □

Lemma 3.1.3.2 states that if A is positive definite, then for any real number r, A^r is also positive definite. What happen when A is positive semidefinite or Hermitian ?

Observation 3.2.3.2. *Let $A \in \mathbb{M}_n$.*

1. *If A is positive semidefinite, then for any nonnegative real number s, A^s is positive semidefinite.*

2. *If A is nonsingular Hermitian, then for any integer k,* A^k *is nonsingular Hermitian.*

3. *If A is Hermitian, then for any nonnegative integer p,* A^p *is Hermitian.*

Proof. The proof is similar to Lemma 3.1.3.2. Observe that when A is Hermitian, A^p exists for any nonnegative integer p; when A is nonsingular Hermitian, A^k is exist for any integer k; and when A is positive semidefinite, A^s exists for any nonnegative real number s. □

Lemma 3.2.2.2 states that if A is positive definite, then for any real number r and positive integer k, $(A^{\otimes k})^r = (A^r)^{\otimes k}$. In fact, we knew that for any square matrix A (not necessarily Hermitian) if k, l are positive integers, then $(A^{\otimes k})^l = (A^l)^{\otimes k}$. What happen when A is positive semidefinite or nonsingular?

Observation 3.2.3.3. *Let* $A \in \mathbb{M}_n$ *and let k be any positive integer.*

1. *If A is positive semidefinite, then for any nonnegative real number s,* $(A^{\otimes k})^s = (A^s)^{\otimes k}$.

2. *If A is nonsingular, then for any integer p,* $(A^{\otimes k})^p = (A^p)^{\otimes k}$.

Proof. The proof f the first assertion is similar to Lemma 3.2.2.2. Observe that when A is positive semidefinite, A^r exists for any nonnegative real number r. For the second assertion, it suffices to prove only when $p < 0$. Since A and B are nonsingular, from Corollary 2.4.5, we have $(A^{\otimes k})^{-1} = (A^{-1})^{\otimes k}$ and consequently

$$\begin{aligned} (A^{\otimes k})^p &= ((A^{\otimes k})^{-1})^{-p} \\ &= ((A^{-1})^{\otimes k})^{-p} \\ &= ((A^{-1})^{-p})^{\otimes k} \\ &= (A^p)^{\otimes k}. \end{aligned}$$

The proof is completed. □

3.3 Inequalities Involving the Hadamard Product of Power of Linear Combination of Matrices

In this section, inequalities involving the Hadamard product of power of linear combination of matrices are established. First we prove an inequality in the Löwner partial order between $(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s$ and $\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)$ in Theorem 3.3.1.3 in Subsection 3.3.1. This theorem yields another inequalities as its special cases which are stated in Subsection 3.3.2. In Subsection 3.3.3, we give some interesting observations which are related to Subsection 3.3.1 and Subsection 3.3.2.

3.3.1 Main Results

To prove the main theorem, we need some lemmas.

Lemma 3.3.1.1. *Let $\varphi : \mathbb{M}_n \to \mathbb{M}_m$. If φ is linear, then φ is an operator monotone if and only if φ is positive.*

Proof. Let $\varphi : \mathbb{M}_n \to \mathbb{M}_m$ be linear.
First assume that φ is positive.
Let $A, B \in \mathbb{M}_n$ such that $A \leq B$, i.e., $B - A \geq 0$.
The positivity of φ implies $\varphi(B - A) \geq 0$.
Then the linearity of φ implies $\varphi(B) - \varphi(A) \geq 0$, i.e., $\varphi(A) \leq \varphi(B)$.
Hence, φ is an operator monotone.
Now assume that φ is an operator monotone.
Let $A \in \mathbb{M}_n$ be positive semidefinite, i.e., $A \geq 0$.
The monotonicity of φ implies $\varphi(A) \geq \varphi(0)$ and so $\varphi(A) - \varphi(0) \geq 0$.
Then the linearity of φ implies $\varphi(A - 0) = \varphi(A) \geq 0$.
Hence, φ is positive.
The proof is completed. □

Lemma 3.3.1.2. *Let $\varphi : \mathbb{M}_n \to \mathbb{M}_m$ be positive linear. Let $\Phi : \mathbb{M}_p \times \mathbb{M}_p \to \mathbb{M}_n$ be jointly concave. Then $\varphi \circ \Phi$ is jointly concave.*

Proof. Let $A, B, C, D \in \mathbb{M}_p$ be Hermitian and let $0 < \epsilon < 1$ be a scalar.
By Lemma 3.3.1.1, φ is an operator monotone. It follows from the linearity and monotonicity of φ and the concavity of Φ that

$$\begin{aligned}(\varphi\circ\Phi)(\epsilon A+(1-\epsilon)B,\epsilon C+(1-\epsilon)D) &= \varphi\Big(\Phi(\epsilon A+(1-\epsilon)B,\epsilon C+(1-\epsilon)D)\Big)\\ &\geq \varphi(\epsilon\Phi(A,C)+(1-\epsilon)\Phi(B,D))\\ &= \epsilon\varphi(\Phi(A,C))+(1-\epsilon)\varphi(\Phi(B,D))\\ &= \epsilon(\varphi\circ\Phi)(A,C)+(1-\epsilon)(\varphi\circ\Phi)(B,D).\end{aligned}$$

Hence, $\varphi\circ\Phi$ is jointly concave. The proof is completed. □

Theorem 3.3.1.3. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r+s=1$. Then*

$$(\alpha A+\beta B)^r\circ(\alpha C+\beta D)^s \quad\geq\quad \alpha(A^r\circ C^s)+\beta(B^r\circ D^s) \tag{3.27}$$

with equality holds if $A=B, C=D$.

Proof. From Corollary 3.2.2.1, we have that a map Φ defined by $\Phi(A,B)=A^r\otimes B^s$ is jointly concave for $r,s>0$ such that $r+s=1$.
Since there is a unital positive linear map φ such that $\varphi(A\otimes B)=A\circ B$, this implies that for $r,s>0$ such that $r+s=1$,

$$(\varphi\circ\Phi)(A,B) = \varphi(\Phi(A,B)) = \varphi(A^r\otimes B^s) = A^r\circ B^s.$$

By Lemma 3.3.1.2, this map is jointly concave.
Hence, for any positive definite matrices A, B, C, D and $0<\epsilon<1$,

$$(\epsilon A+(1-\epsilon)B)^r\circ(\epsilon C+(1-\epsilon)D)^s \quad\geq\quad \epsilon(A^r\circ C^s)+(1-\epsilon)(B^r\circ D^s).$$

Since $0<\frac{\alpha}{\alpha+\beta}<1$, by replacing ϵ with $\frac{\alpha}{\alpha+\beta}$, we get

$$\begin{aligned}\Big(\frac{\alpha}{\alpha+\beta}A+\frac{\beta}{\alpha+\beta}B\Big)^r &\circ \Big(\frac{\alpha}{\alpha+\beta}C+\frac{\beta}{\alpha+\beta}D\Big)^s\\ &\geq \frac{\alpha}{\alpha+\beta}(A^r\circ C^s)+\frac{\beta}{\alpha+\beta}(B^r\circ D^s),\\ \frac{1}{(\alpha+\beta)^r}(\alpha A+\beta B)^r &\circ \frac{1}{(\alpha+\beta)^s}(\alpha C+\beta D)^s\\ &\geq \frac{1}{\alpha+\beta}\Big(\alpha(A^r\circ C^s)+\beta(B^r\circ D^s)\Big),\\ \frac{1}{\alpha+\beta}\Big((\alpha A+\beta B)^r &\circ (\alpha C+\beta D)^s\Big)\\ &\geq \frac{1}{\alpha+\beta}\Big(\alpha(A^r\circ C^s)+\beta(B^r\circ D^s)\Big),\\ (\alpha A+\beta B)^r\circ(\alpha C+\beta D)^s &\geq \alpha(A^r\circ C^s)+\beta(B^r\circ D^s).\end{aligned}$$

For the case of equality, assume that $A = B, C = D$. We obtain

$$\begin{aligned}(\alpha A + \beta A)^r \circ (\alpha C + \beta C)^s &= ((\alpha+\beta)A)^r \circ ((\alpha+\beta)C)^s \\ &= (\alpha+\beta)^r A^r \circ (\alpha+\beta)^s C^s \\ &= (\alpha+\beta)^r (\alpha+\beta)^s (A^r \circ C^s) \\ &= (\alpha+\beta)(A^r \circ C^s) \\ &= \alpha(A^r \circ C^s) + \beta(A^r \circ C^s).\end{aligned}$$

So the equality holds. The theorem is established. □

3.3.2 Special cases

Now, let us consider special cases of the main result (Theorem 3.3.1.3). There are 3 types of them: the equality between the matrices; the equality between the coefficients (scalars); and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C$, $B = D$;
2. the case when $A = D$, $B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

Combining all of them, there are 16 possible cases (which include the general case). The details for each case are shown in Table 3.1 (except for the general case).

Table 3.2: The Special Cases of Theorem 3.3.1.3

Conditions			Results	
A, B, C, D	r, s	α, β	Corollary	Inequality
-	-	$\alpha = \beta$	3.3.2.1	3.28
-	$r = s$	-	3.3.2.2	3.29
-	$r = s$	$\alpha = \beta$	3.3.2.4	3.32
$A = C. B = D$	-	-	3.3.2.3	3.30
$A = C. B = D$	-	$\alpha = \beta$	3.3.2.5	3.33
$A = C. B = D$	$r = s$	-	3.3.2.6	3.35
$A = C. B = D$	$r = s$	$\alpha = \beta$	3.3.2.7	3.37
$A = D, B = C$	-	-	3.3.2.3	3.31
$A = D, B = C$	-	$\alpha = \beta$	3.3.2.5	3.34
$A = D, B = C$	$r = s$	-	3.3.2.6	3.36
$A = D, B = C$	$r = s$	$\alpha = \beta$	3.3.2.7	3.38
$A = B = C = D$	-	-	Equality Case	
$A = B = C = D$	-	$\alpha = \beta$	Equality Case	
$A = B = C = D$	$r = s$	-	Equality Case	
$A = B = C = D$	$r = s$	$\alpha = \beta$	Equality Case	

These cases lead to 7 corollaries as follows (depend on the hypotheses).

Corollary 3.3.2.1. *Let $A, B, C, D \in \mathrm{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$(A+B)^r \circ (C+D)^s \quad \geq \quad (A^r \circ C^s) + (B^r \circ D^s) \tag{3.28}$$

with equality holds if $A = B, C = D$.

Proof. By setting $\alpha = \beta$ in Theorem 3.3.1.3, we obtain

$$\begin{aligned}
(\alpha A + \alpha B)^r \circ (\alpha C + \alpha D)^s &\geq \alpha(A^r \circ C^s) + \alpha(B^r \circ D^s), \\
\alpha^r (A+B)^r \circ \alpha^s (C+D)^s &\geq \alpha(A^r \circ C^s) + \alpha(B^r \circ D^s), \\
\alpha^r \alpha^s ((A+B)^r \circ (C+D)^s) &\geq \alpha((A^r \circ C^s) + (B^r \circ D^s)), \\
(A+B)^r \circ (C+D)^s &\geq (A^r \circ C^s) + (B^r \circ D^s).
\end{aligned}$$

The case of equality is same as Theorem 3.3.1.3. □

Corollary 3.3.2.2. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$(\alpha A + \beta B)^{\frac{1}{2}} \circ (\alpha C + \beta D)^{\frac{1}{2}} \quad \geq \quad \alpha(A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \circ D^{\frac{1}{2}}) \tag{3.29}$$

with equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Theorem 3.3.1.3, we get $r = 1/2$ and then (3.29). The case of equality is same as Theorem 3.3.1.3. □

Corollary 3.3.2.3. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\begin{aligned} (\alpha A + \beta B)^r \circ (\alpha A + \beta B)^s \quad &\geq \quad \alpha(A^r \circ A^s) + \beta(B^r \circ B^s)\,, && (3.30) \\ (\alpha A + \beta B)^r \circ (\beta A + \alpha B)^s \quad &\geq \quad \alpha(A^r \circ B^s) + \beta(A^s \circ B^r) && (3.31) \end{aligned}$$

with equalities hold if $A = B$. The inequality (3.30) *also becomes an equality if $n = 1$.*

Proof. By setting $A = C, B = D$ in Theorem 3.3.1.3, we get (3.30). By setting $A = D, B = C$ in Theorem 3.3.1.3, we get

$$\begin{aligned} (\alpha A + \beta B)^r \circ (\beta A + \alpha B)^s \quad &\geq \quad \alpha(A^r \circ B^s) + \beta(B^r \circ A^s) \\ &= \quad \alpha(A^r \circ B^s) + \beta(A^s \circ B^r). \end{aligned}$$

For the inequality (3.30), if $A = B$, we obtain

$$\begin{aligned} (\alpha A + \beta A)^r \circ (\alpha A + \beta A)^s \quad &= \quad (\alpha + \beta)^r A^r \circ (\alpha + \beta)^s A^s \\ &= \quad (\alpha + \beta)^r (\alpha + \beta)^s (A^r \circ A^s) \\ &= \quad (\alpha + \beta)(A^r \circ A^s) \\ &= \quad \alpha(A^r \circ A^s) + \beta(A^r \circ A^s). \end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the inequality (3.30) also becomes an equality when $n = 1$. □

Corollary 3.3.2.4. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$(A + B)^{\frac{1}{2}} \circ (C + D)^{\frac{1}{2}} \quad \geq \quad (A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + (B^{\frac{1}{2}} \circ D^{\frac{1}{2}}) \tag{3.32}$$

with equality holds if $A = B$, $C = D$.

Proof. By setting $\alpha = \beta$ in Corollary 3.3.2.2, we get

$$\begin{aligned}
(\alpha A+\alpha B)^{\frac{1}{2}} \circ (\alpha C+\alpha D)^{\frac{1}{2}} &\geq \alpha(A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + \alpha(B^{\frac{1}{2}} \circ D^{\frac{1}{2}}), \\
\alpha^{\frac{1}{2}}(A+B)^{\frac{1}{2}} \circ \alpha^{\frac{1}{2}}(C+D)^{\frac{1}{2}} &\geq \alpha((A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + (B^{\frac{1}{2}} \circ D^{\frac{1}{2}})), \\
\alpha(A+B)^{\frac{1}{2}} \circ (C+D)^{\frac{1}{2}} &\geq \alpha((A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + (B^{\frac{1}{2}} \circ D^{\frac{1}{2}})), \\
(A+B)^{\frac{1}{2}} \circ (C+D)^{\frac{1}{2}} &\geq (A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + (B^{\frac{1}{2}} \circ D^{\frac{1}{2}}).
\end{aligned}$$

The case of equality is same as Corollary 3.3.2.2. □

Corollary 3.3.2.5. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$(A+B)^r \circ (A+B)^s \geq (A^r \circ A^s) + (B^r \circ B^s), \tag{3.33}$$

$$(A+B)^r \circ (A+B)^s \geq (A^r \circ B^s) + (A^s \circ B^r) \tag{3.34}$$

with equalities hold if $A = B$. The inequality (3.33) *also becomes an equality when $n = 1$.*

Proof. By setting $\alpha = \beta$ in (3.30) in Corollary 3.3.2.3, we get

$$\begin{aligned}
(\alpha A+\alpha B)^r \circ (\alpha A+\alpha B)^s &\geq \alpha(A^r \circ A^s) + \alpha(B^r \circ B^s), \\
\alpha^r(A+B)^r \circ \alpha^s(A+B)^s &\geq \alpha(A^r \circ A^s) + \alpha(B^r \circ B^s), \\
\alpha^r\alpha^s((A+B)^r \circ (A+B)^s) &\geq \alpha((A^r \circ A^s) + (B^r \circ B^s)), \\
\alpha((A+B)^r \circ (A+B)^s) &\geq \alpha((A^r \circ A^s) + (B^r \circ B^s)), \\
(A+B)^r \circ (A+B)^s &\geq (A^r \circ A^s) + (B^r \circ B^s).
\end{aligned}$$

Similarly, by setting $\alpha = \beta$ in (3.31) in Corollary 3.3.2.3, we get (3.34).
The case of equalities is same as Corollary 3.3.2.3. □

Corollary 3.3.2.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$((\alpha A+\beta B)^{\frac{1}{2}})^{(2)} \geq \alpha(A^{\frac{1}{2}})^{(2)} + \beta(B^{\frac{1}{2}})^{(2)}, \tag{3.35}$$

$$(\alpha A+\beta B)^{\frac{1}{2}} \circ (\beta A+\alpha B)^{\frac{1}{2}} \geq (\alpha+\beta)(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) \tag{3.36}$$

with equalities hold if $A = B$. The inequality (3.35) *also becomes an equality if $n = 1$.*

Proof. By setting $r = s$ in the inequality (3.30), we get $r = 1/2$ and

$$\begin{aligned}
(\alpha A+\beta B)^{\frac{1}{2}} \circ (\alpha A+\beta B)^{\frac{1}{2}} &\geq \alpha(A^{\frac{1}{2}} \circ A^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \circ B^{\frac{1}{2}}), \\
((\alpha A+\beta B)^{\frac{1}{2}})^{(2)} &\geq \alpha(A^{\frac{1}{2}})^{(2)} + \beta(B^{\frac{1}{2}})^{(2)}.
\end{aligned}$$

By setting $r = s$ in the inequality (3.31), we get

$$\begin{aligned}((\alpha A + \beta B) \circ (\beta A + \alpha B))^{\frac{1}{2}} &\geq \alpha(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \circ A^{\frac{1}{2}}) \\ &= \alpha(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) + \beta(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) \\ &= (\alpha + \beta)(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}).\end{aligned}$$

For the inequality (3.35), if $A = B$, we obtain

$$\begin{aligned}((\alpha A + \beta A)^{\frac{1}{2}})^{(2)} &= ((\alpha + \beta)^{\frac{1}{2}} A^{\frac{1}{2}})^{(2)} \\ &= ((\alpha + \beta)^{\frac{1}{2}})^2 (A^{\frac{1}{2}})^{(2)} \\ &= (\alpha + \beta)(A^{\frac{1}{2}})^{(2)} \\ &= \alpha(A^{\frac{1}{2}})^{(2)} + \beta(A^{\frac{1}{2}})^{(2)}.\end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the inequality (3.35) becomes an equality when $n = 1$. □

Corollary 3.3.2.7. *Let $A, B \in \mathrm{M}_n$ be positive definite symmetric. Then*

$$\begin{aligned}((A + B)^{\frac{1}{2}})^{(2)} &\geq (A^{\frac{1}{2}})^{(2)} + (B^{\frac{1}{2}})^{(2)}, &(3.37)\\ ((A + B)^{\frac{1}{2}})^{(2)} &\geq 2(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) &(3.38)\end{aligned}$$

with equalities hold if $A = B$. The inequality (3.37) *also becomes an equality if* $n = 1$.

Proof. By setting $r = s$ in the inequality (3.33) in Corollary 3.3.2.5, we get $r = 1/2$ and

$$\begin{aligned}(A + B)^{\frac{1}{2}} \circ (A + B)^{\frac{1}{2}} &\geq (A^{\frac{1}{2}} \circ A^{\frac{1}{2}}) + (B^{\frac{1}{2}} \circ B^{\frac{1}{2}}), \\ ((A + B)^{\frac{1}{2}})^{(2)} &\geq (A^{\frac{1}{2}})^{(2)} + (B^{\frac{1}{2}})^{(2)}.\end{aligned}$$

Similarly, by setting $r = s$ in the inequality (3.34) in Corollary 3.3.2.5, we get (3.38). The case of equalities is same as Corollary 3.3.2.5. Obviously, the inequality (3.37) becomes an equality when $n = 1$. □

Remark 3.3.2.8.

For the inequality (3.38) in Corollary 3.3.2.7, when $n = 1$ let $A = [a]$ and $B = [b]$, we get

$$\begin{aligned}(a + b)^{\frac{1}{2}}(a + b)^{\frac{1}{2}} &\geq 2(a^{\frac{1}{2}} b^{\frac{1}{2}}), \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a + b).\end{aligned}$$

This is the AM–GM inequality for real numbers.

3.3.3 Observations

We knew that if $A \otimes A = 0$ then $A = 0$ and $A \circ A = 0$ then $A = 0$. Let us generalize these statements.

Observation 3.3.3.1. *Let $A \in \mathbb{M}_n$ (not necessarily Hermitian). Then for any positive integer k,*

1. $A^{\otimes k} = 0$ *if and only if* $A = 0$,
2. $A^{(k)} = 0$ *if and only if* $A = 0$.

Proof. Use the definition of the kth the Kronecker power and the definition of the kth Hadamard power. □

Observation 3.3.3.2. *Let $A \in \mathbb{M}_n$ be Hermitian. Then $A = I$ if and only if $\sigma(A) = \{1\}$.*

Proof. Obviously, if $A = I$, then $\sigma(A) = \{1\}$. Now assume that $\sigma(A) = \{1\}$. Since A is Hermitian, there is a unitary matrix U such that

$$A \quad = \quad U diag[1, 1, \ldots, 1]U^*.$$

Hence,

$$A \quad = \quad UU^* \quad = \quad I.$$

That is $A = I$ if and only if $\sigma(A) = \{1\}$. □

We finish this chapter with the question that does $A^r = A^s$ implies $r = s$ when A and B are positive definite. The answer is the following observation.

Observation 3.3.3.3. *Let $A \in \mathbb{M}_n$ be positive definite. Then for any real numbers r and s, $A^r = A^s$ if and only if $r = s$ or $A = I$.*

Proof. Obviously, if $r = s$ or $A = I$, then $A^r = A^s$. Now assume that $A^r = A^s$. Recall the spectral decomposition of A, there is a unitary matrix $U = [u_{ij}]$ such that

$$A \quad = \quad U\, diag[\lambda_1, \lambda_2, \ldots, \lambda_n]\, U^*$$

where $\lambda_1. \lambda_2, \ldots, \lambda_n$ are eigenvalues of A. It follows that

$$\begin{aligned} A^r - A^s &= U\, diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\, U^* - U\, diag[\lambda_1^s, \lambda_2^s, \ldots, \lambda_n^s]\, U^* \\ &= U\, diag[\lambda_1^r - \lambda_1^s, \lambda_2^r - \lambda_2^s, \ldots, \lambda_n^r - \lambda_n^s]\, U^*. \end{aligned}$$

Consider (i,i)th entry of $A^r - A^s$. We have

$$\begin{aligned}(A^r - A^s)_{ii} &= u_{i1}(\lambda_1^r - \lambda_1^s)\overline{u_{i1}} + u_{i2}(\lambda_2^r - \lambda_2^s)\overline{u_{i2}} + \cdots + u_{in}(\lambda_n^r - \lambda_n^s)\overline{u_{in}} \\ &= (\lambda_1^r - \lambda_1^s)|u_{i1}|^2 + (\lambda_2^r - \lambda_2^s)|u_{i2}|^2 + \cdots + (\lambda_n^r - \lambda_n^s)|u_{in}|^2 \\ &= \sum_{m=1}^{n} (\lambda_m^r - \lambda_m^s)|u_{im}|^2.\end{aligned}$$

Assume that $A \neq I$. Need to show that $r = s$.

Since A is Hermitian, from the previous observation this is equivalent to say that there is $\lambda_p \in \sigma(A)$, $1 \leq p \leq n$ such that $\lambda_p \neq 1$.

Suppose that $r \neq s$.

We have that $\lambda_m^r - \lambda_m^s$ are all positive or negative for $m = 1, 2, \ldots, n$ such that $\lambda_m \neq 1$.

This implies $u_{pm} = 0$ for all $m = 1, 2, \ldots, n$.

This yields that all entries in the pth row of U are all zero. Hence, $\det(U) = 0$.

This leads to a contradiction since U is unitary ($|\det(U)| = 1$).

So $r = s$.

Now assume that $r \neq s$. Need to show that $A = I$.

Suppose that $A \neq I$.

Since A is Hermitian, from the previous observation this is equivalent to say that there is $\lambda_l \in \sigma(A)$, $1 \leq l \leq n$ such that $\lambda_l \neq 1$.

We have that $\lambda_m^r - \lambda_m^s$ are all positive or negative for $m = 1, 2, \ldots, n$ such that $\lambda_m \neq 1$.

This implies $u_{lm} = 0$ for all $m = 1, 2, \ldots, n$.

This yields that all entries in the lth row of U are all zero. Hence, $\det(U) = 0$.

This leads to a contradiction since U is unitary ($|\det(U)| = 1$).

So $A = I$.

This completes the proof. □

The previous observation states that if A is positive definite, then for any real number r and s, $A^r = A^s$ if and only if $r = s$ or $A = I$. What happen when A is positive semidefinite or Hermitian ?

Observation 3.3.3.4. *Let $A \in \mathbb{M}_n$ be Hermitian.*

1. *Then for any nonnegative integers p and q, $A^p = A^q$ if and only if $p = q$ or $A = I$ or $A = 0$.*
2. *If A is nonsingular Hermitian, then for any integers h and k, $A^h = A^k$ if and only if $h = k$ or $A = I$.*

3. *If A is positive semidefinite, then for any nonnegative real numbers r and s, $A^r = A^s$ if and only if $r = s$ or $A = I$ or $A = 0$.*

Proof. The proof is similar to Observation 3.3.3.3. Observe that when A is Hermitian, A^p exists for any nonnegative integer p; when A is nonsingular Hermitian, A^k exists for any integer k; and when A is positive semidefinite, A^r exists for any nonnegative real number r. □

Remark 3.3.3.5.

Compare Observation 3.3.3.4 with a following fact for real numbers. Let $a \in \mathbb{R}$.

1. Then for any nonnegative integers p and q, $a^p = a^q$ if and only if $p = q$ or $a = 1$ or $a = 0$.
2. If a is nonzero, then for any integers h and k, $a^h = a^k$ if and only if $h = k$ or $a = 1$.
3. If a is nonnegative, then for any nonnegative real numbers r and s, $a^r = a^s$ if and only if $r = s$ or $a = 1$ or $a = 0$.

So the previous observation is a matrix version of this fact. We knew that the zero matrix plays a role like the real number 0 and the identity matrix plays a role like the real number 1. Actually, a Hermitian matrix plays a role like a real number, a nonsingular Hermitian matrix plays a role like a nonzero real number, a positive semidefinite matrix (sometimes called nonnegative definite matrix) plays a role like a nonnegative real number and a positive definite matrix plays a role like a positive real number.

3.4 Numerical Examples

In this section, we give some numerical examples to guarantee our results. The tool we use is the mathematical program which is called Maple 9.5.

Example 3.4.1.

Consider the inequality involving the 2nd power of linear combination of the given matrices

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

in the case when $\alpha = 4,\ \beta = 3$.

We see that A and B are positive definite symmetric and α and β satisfy the hypothesis of Theorem 3.1.1.1. We get

$$\begin{aligned} A^2 &= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \; B^2 = \begin{bmatrix} 8 & 14 \\ 14 & 29 \end{bmatrix} \\ \alpha A + \beta B &= \begin{bmatrix} 14 & 10 \\ 10 & 23 \end{bmatrix} \\ (\alpha A + \beta B)^2 &= \begin{bmatrix} 296 & 370 \\ 370 & 629 \end{bmatrix} \\ \alpha A^2 + \beta B^2 &= \begin{bmatrix} 44 & 58 \\ 58 & 107 \end{bmatrix} \\ (\alpha A^2 + \beta B^2) - \frac{1}{\alpha+\beta}(\alpha A + \beta B)^2 &= \frac{1}{7}\begin{bmatrix} 12 & 36 \\ 36 & 140 \end{bmatrix} \geq 0. \end{aligned}$$

Therefore,

$$\frac{1}{\alpha+\beta}(\alpha A + \beta B)^2 \quad \leq \quad \alpha A^2 + \beta B^2.$$

This example supports the result of Theorem 3.1.1.1. □

Example 3.4.2.

Consider the inequality involving the 2nd power of the given matrices

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix}.$$

We see that A and B are positive definite symmetric. We obtain

$$\begin{aligned} A^2 &= \begin{bmatrix} 13 & -21 \\ -21 & 34 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 73 & 30 \\ 30 & 13 \end{bmatrix}, \\ (A+B)^2 &= \begin{bmatrix} 100 & 0 \\ 0 & 49 \end{bmatrix}. \end{aligned}$$

Hence,

$$2(A^2+B^2) - (A+B)^2 = \begin{bmatrix} 76 & 18 \\ 18 & 45 \end{bmatrix} \geq \mathbf{0}.$$

Therefore,

$$(A+B)^2 \leq 2(A^2+B^2).$$

Next since

$$AB = \begin{bmatrix} 7 & 0 \\ -9 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 7 & -9 \\ 0 & 1 \end{bmatrix},$$

we get

$$\begin{aligned} (A^2+B^2) - (AB+BA) &= \begin{bmatrix} 72 & 18 \\ 18 & 45 \end{bmatrix} \geq \mathbf{0}, \\ (A+B)^2 - 2(AB+BA) &= \begin{bmatrix} 72 & 18 \\ 18 & 45 \end{bmatrix} \geq \mathbf{0}. \end{aligned}$$

Therefore,

$$2(AB+BA) \leq (A+B)^2 \leq 2(A^2+B^2).$$

This example supports the result of Corollary 3.1.2.3. □

Example 3.4.3.

Consider the inequality involving the Kronecker product of power of linear combination of the given matrices

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha = 1,\ \beta = 2,\ r = 1/3,\ s = 2/3.$

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 3.2.1.5. We obtain

$$\begin{aligned}
\alpha A + \beta B &= \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}, \alpha C + \beta D = \begin{bmatrix} 4 & -4 \\ -4 & 14 \end{bmatrix}, \\
(\alpha A + \beta B)^r &\approx \begin{bmatrix} 2.076 & 0.099 \\ 0.099 & 1.581 \end{bmatrix}, (\alpha C + \beta D)^s \approx \begin{bmatrix} 2.361 & -1.344 \\ -1.344 & 5.719 \end{bmatrix}, \\
A^r &\approx \begin{bmatrix} 1.520 & 0.614 \\ 0.614 & 0.906 \end{bmatrix}, C^s \approx \begin{bmatrix} 1.587 & 0 \\ 0 & 1.587 \end{bmatrix}, \\
B^r &\approx \begin{bmatrix} 1.198 & -0.292 \\ -0.292 & 0.906 \end{bmatrix}, D^s \approx \begin{bmatrix} 0.787 & -0.970 \\ -0.970 & 3.214 \end{bmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
A^r \otimes C^s &\approx \begin{bmatrix} 2.412 & 0 & 0.974 & 0 \\ 0 & 2.412 & 0 & 0.974 \\ 0.974 & 0 & 1.438 & 0 \\ 0 & 0.974 & 0 & 1.438 \end{bmatrix}, \\
B^r \otimes D^s &\approx \begin{bmatrix} 0.943 & -1.162 & -0.230 & 0.283 \\ -1.162 & 3.850 & 0.283 & 0.938 \\ -0.230 & 0.283 & 0.713 & -0.879 \\ 0.283 & -0.938 & -0.879 & 2.912 \end{bmatrix}.
\end{aligned}$$

Hence,

$$(\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s \approx \begin{bmatrix} 4.091 & -2.769 & 0.234 & -0.133 \\ -2.769 & 11.873 & -0.133 & 0.157 \\ 0.234 & -0.133 & 3.733 & -2.125 \\ -0.133 & 0.157 & -2.125 & 9.042 \end{bmatrix},$$

$$\alpha(A^r \otimes C^s)+\beta(B^r \otimes D^s) \approx \begin{bmatrix} 4.298 & -2.324 & 0.514 & 0.476 \\ -2.324 & 10.112 & 0.566 & -0.902 \\ 0.514 & 0.566 & 2.864 & -1.758 \\ 0.476 & -0.902 & -1.758 & 7.262 \end{bmatrix}.$$

Thus,

$$(\alpha(A^r \otimes C^s)+\beta(B^r \otimes D^s)) - ((\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s)$$

$$\approx \begin{bmatrix} 0.603 & -0.445 & -0.280 & -0.609 \\ -0.445 & 1.761 & -0.699 & 1.059 \\ -0.280 & -0.699 & 0.869 & -0.367 \\ -0.609 & 1.059 & -0.367 & 1.780 \end{bmatrix}.$$

The eigenvalues of this matrix are 0.019, 0.554, 1.190 and 3.249. So this matrix is Hermitian and its eigenvalues are all nonnegative. This means that this matrix is positive semidefinite, i.e.,

$$(\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s \quad \geq \quad \alpha(A^r \otimes C^s)+\beta(B^r \otimes D^s).$$

This example supports the result of Theorem 3.2.1.5. □

Example 3.4.4.

Consider the inequality involving the Hadamard product of power of linear combination of the given matrices

$$A=\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix},\ B=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},\ C=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ \text{and } D=\begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha=2,\ \beta=1,\ r=2/3,\ s=1/3$.

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 3.3.1.3. We obtain

$$\begin{aligned}
\alpha A+\beta B &= \begin{bmatrix} 12 & 5 \\ 5 & 5 \end{bmatrix}, \; \alpha C+\beta D = \begin{bmatrix} 5 & -2 \\ -2 & 10 \end{bmatrix}, \\
(\alpha A+\beta B)^r &\approx \begin{bmatrix} 5.082 & 1.714 \\ 1.714 & 2.683 \end{bmatrix}, \; (\alpha C+\beta D)^s \approx \begin{bmatrix} 1.689 & -0.180 \\ -0.180 & 2.140 \end{bmatrix}, \\
A^r &\approx \begin{bmatrix} 2.688 & 1.490 \\ 1.490 & 1.198 \end{bmatrix}, \; C^s \approx \begin{bmatrix} 1.260 & 0 \\ 0 & 1.260 \end{bmatrix}, \\
B^r &\approx \begin{bmatrix} 1.520 & -0.614 \\ -0.614 & 0.906 \end{bmatrix}, \; D^s \approx \begin{bmatrix} 0.802 & -0.380 \\ -0.380 & 1.752 \end{bmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
A^r \circ C^s &\approx \begin{bmatrix} 3.387 & 0 \\ 0 & 1.509 \end{bmatrix}, \\
B^r \circ D^s &\approx \begin{bmatrix} 1.219 & 0.238 \\ 0.238 & 1.587 \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\alpha A+\beta B)^r \circ (\alpha C+\beta D)^s &\approx \begin{bmatrix} 8.583 & -0.309 \\ -0.309 & 5.742 \end{bmatrix}, \\
\alpha(A^r \circ C^s)+\beta(B^r \circ D^s) &\approx \begin{bmatrix} 7.993 & 0.233 \\ 0.233 & 4.605 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\alpha(A^r \circ C^s)+\beta(B^r \circ D^s)) - ((\alpha A+\beta B)^r \circ (\alpha C+\beta D)^s) \\
&\approx \begin{bmatrix} 0.490 & -0.542 \\ -0.542 & 1.137 \end{bmatrix}.
\end{aligned}$$

We see that this matrix is positive semidefinite, i.e.,

$$(\alpha A+\beta B)^r \circ (\alpha C+\beta D)^s \geq \alpha(A^r \circ C^s)+\beta(B^r \circ D^s).$$

This example supports the result of Theorem 3.3.1.3. □

Example 3.4.5.

Consider the inequality involving the Hadamard product of power of linear combination of the given matrices

$$A = \begin{bmatrix} \sqrt{3} & 1/2 & -1 \\ 1/2 & \sqrt{7}-1 & 0.851 \\ -1 & 0.851 & 2\sqrt{2} \end{bmatrix}, B = \begin{bmatrix} \sqrt{7} & 17/4 & 1 \\ 17/4 & 6\sqrt{6}-4 & 3.851 \\ 1 & 3.851 & 3\sqrt{3} \end{bmatrix},$$

$$C = \begin{bmatrix} \pi & 2 & 0 \\ 2 & 5.219 & \sqrt[4]{29} \\ 0 & \sqrt[4]{29} & 8 \end{bmatrix}, D = \begin{bmatrix} e & -2 & -0.1 \\ -2 & 4.567 & \sqrt[3]{17} \\ -0.1 & \sqrt[3]{17} & 8 \end{bmatrix}$$

in the case when $\alpha = 3,\ \beta = 4,\ r = 3/4,\ s = 1/4.$

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 3.3.1.3. We obtain

$$\alpha A + \beta B = \begin{bmatrix} 3\sqrt{3}+4\sqrt{7} & 37/2 & 1 \\ 37/2 & 3\sqrt{7}+24\sqrt{6}-16 & 17.957 \\ 1 & 17.957 & 6\sqrt{2}+12\sqrt{3} \end{bmatrix},$$

$$\alpha C + \beta D = \begin{bmatrix} 3\pi + 4e & -2 & -0.4 \\ -2 & 33.925 & 3\sqrt[4]{29}+4\sqrt[3]{17} \\ -0.4 & 3\sqrt[4]{29}+4\sqrt[3]{17} & 56 \end{bmatrix},$$

$$(\alpha A + \beta B)^r \approx \begin{bmatrix} 7.172 & 6.194 & -0.200 \\ 6.194 & 17.371 & 5.596 \\ -0.200 & 5.596 & 12.170 \end{bmatrix},$$

$$(\alpha C + \beta D)^s \approx \begin{bmatrix} 2.121 & -0.043 & -0.0001 \\ -0.043 & 2.366 & 0.261 \\ -0.0001 & 0.261 & 2.701 \end{bmatrix},$$

$$A^r \approx \begin{bmatrix} 1.442 & 0.381 & -0.651 \\ 0.381 & 1.395 & 0.563 \\ -0.650 & 0.563 & 2.112 \end{bmatrix},$$

$$C^s \approx \begin{bmatrix} 1.286 & 0.193 & -0.029 \\ 0.193 & 1.453 & 0.154 \\ -0.029 & 0.154 & 1.662 \end{bmatrix},$$

$$\begin{aligned} B^r &\approx \begin{bmatrix} 1.759 & 2.103 & 0.352 \\ 2.103 & 5.649 & 1.754 \\ 0.352 & 1.754 & 3.301 \end{bmatrix}, \\ D^s &\approx \begin{bmatrix} 1.222 & -0.223 & 0.031 \\ -0.223 & 1.379 & 0.182 \\ 0.031 & 0.182 & 1.655 \end{bmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} A^r \circ C^s &\approx \begin{bmatrix} 1.854 & 0.074 & 0.019 \\ 0.074 & 2.027 & 0.087 \\ 0.019 & 0.087 & 3.512 \end{bmatrix}, \\ B^r \circ D^s &\approx \begin{bmatrix} 2.150 & -0.469 & 0.011 \\ -0.469 & 7.790 & 0.320 \\ 0.011 & 0.320 & 5.465 \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} (\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s &\approx \begin{bmatrix} 15.213 & -0.269 & 0.00003 \\ -0.269 & 41.103 & 1.461 \\ 0.00003 & 1.461 & 32.876 \end{bmatrix}, \\ \alpha(A^r \circ C^s) + \beta(B^r \circ D^s) &\approx \begin{bmatrix} 14.164 & -1.655 & 0.100 \\ -1.655 & 37.241 & 1.539 \\ 0.100 & 1.539 & 32.396 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} &(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)) - ((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s) \\ &\approx \begin{bmatrix} 1.049 & 1.386 & -0.100 \\ 1.386 & 3.862 & -0.078 \\ -0.100 & -0.078 & 0.480 \end{bmatrix}. \end{aligned}$$

The eigenvalues of this matrix are $0.415, 0.542$ and 4.432. So this matrix is Hermitian and its eigenvalues are all nonnegative. This means that this matrix is positive semidefinite, i.e.,

$$(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s \geq \alpha(A^r \circ C^s) + \beta(B^r \circ D^s).$$

This example supports the result of Theorem 3.3.1.3. □

CHAPTER 4

INEQUALITIES OF TRACE

In this chapter, inequalities of trace for positive definite symmetric matrices are established. These inequalities are shown in Section 4.1–Section 4.3. To guarantee our results, we give some numerical examples in Section 4.4.

The main tools we use are properties of trace in Proposition 2.1.2 and Theorem 2.6.4 which is stated that for any positive definite matrices $A, B \in \mathbb{M}_n$,

$$A \geq B \quad \Longrightarrow \quad \operatorname{tr}(A) \geq \operatorname{tr}(B).$$

Note that the traces of positive definite symmetric matrices are always positive real numbers, assuring that we can compare them.

4.1 Inequalities of Trace Involving the 2nd Power of Linear Combination of Two Matrices

In this section, we prove inequalities of trace involving the 2nd power of linear combination of two matrices. The main result is Theorem 4.1.1. The consequent results and special cases are also considered.

Theorem 4.1.1. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\frac{1}{\alpha+\beta} \operatorname{tr}\left((\alpha A+\beta B)^2\right) \quad \leq \quad \alpha \operatorname{tr}(A^2)+\beta \operatorname{tr}(B^2) \tag{4.1}$$

with equality holds if and only if $A = B$.

Proof. Since A and B are positive definite and α, β are positive, so $\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2$ and $\alpha A^2 + \beta B^2$ are also positive definite. By applying Theorem 2.6.4 to Theorem 3.1.1.1 and then using properties of trace, we get

$$\begin{aligned} \operatorname{tr}\left(\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2\right) &\leq \operatorname{tr}\left(\alpha A^2+\beta B^2\right), \\ \frac{1}{\alpha+\beta} \operatorname{tr}\left((\alpha A+\beta B)^2\right) &\leq \alpha \operatorname{tr}(A^2)+\beta \operatorname{tr}(B^2). \end{aligned}$$

For the case of equality, first assume that $A = B$.
We obtain from the linearity of trace that

$$\begin{aligned}\frac{1}{\alpha+\beta}\operatorname{tr}\left((\alpha A+\beta A)^2\right) &= \frac{1}{\alpha+\beta}\operatorname{tr}\left((\alpha+\beta)^2A^2\right)\\ &= (\alpha+\beta)\operatorname{tr}(A^2)\\ &= \alpha\operatorname{tr}(A^2)+\beta\operatorname{tr}(A^2).\end{aligned}$$

So the equality holds. Now assume that

$$\frac{1}{\alpha+\beta}\operatorname{tr}\left((\alpha A+\beta B)^2\right) = \alpha\operatorname{tr}(A^2)+\beta\operatorname{tr}(B^2).$$

It follows from the properties of trace and the symmetry of A and B that

$$\begin{aligned}\frac{1}{\alpha+\beta}\operatorname{tr}(\alpha^2A^2+\alpha\beta AB+\beta\alpha BA+\beta^2B^2) &= \alpha\operatorname{tr}(A^2)+\beta\operatorname{tr}(B^2),\\ \alpha^2\operatorname{tr}(A^2)+2\alpha\beta\operatorname{tr}(AB)+\beta^2\operatorname{tr}(B^2) &= (\alpha^2+\alpha\beta)\operatorname{tr}(A^2)+(\beta^2+\alpha\beta)\operatorname{tr}(B^2),\\ 2\alpha\beta\operatorname{tr}(AB) &= \alpha\beta\operatorname{tr}(A^2)+\alpha\beta\operatorname{tr}(B^2),\\ 2\operatorname{tr}(AB) &= \operatorname{tr}(A^2)+\operatorname{tr}(B^2),\\ 2\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}b_{ji} &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}+\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}b_{ji},\\ \sum_{i=1}^{n}\sum_{j=1}^{n}2a_{ij}b_{ij} &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}^2+b_{ij}^2),\\ \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}^2-2a_{ij}b_{ij}+b_{ij}^2) &= 0,\\ \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}-b_{ij})^2 &= 0.\end{aligned}$$

Hence, $a_{ij} = b_{ij}$ for all $i, j = 1, 2, \ldots, n$. That is $A = B$.
The proof is completed. $\square$

Corollary 4.1.2. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\frac{1}{\alpha+\beta}\operatorname{tr}\left((\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\right) \le \alpha\operatorname{tr}(A)+\beta\operatorname{tr}(B) \le (\alpha+\beta)^{\frac{1}{2}}\operatorname{tr}\left((\alpha A^2+\beta B^2)^{\frac{1}{2}}\right). \quad (4.2)$$

The left equality holds if and only if $A = B$. The right equality holds if $A = B$.

Proof. By replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in Theorem 4.1.1, we get

$$\frac{1}{\alpha+\beta}\operatorname{tr}\left((\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\right) \le \alpha\operatorname{tr}(A)+\beta\operatorname{tr}(B) \quad (4.3)$$

with equality holds if and only if $A = B$.

Then by using Löwner–Heinz inequality (Theorem 2.8.2) to Theorem 3.1.1.1, we get

$$\begin{aligned} \left(\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2\right)^{\frac{1}{2}} &\le (\alpha A+\beta B)^{\frac{1}{2}}, \\ \frac{1}{(\alpha+\beta)^{\frac{1}{2}}}(\alpha A+\beta B) &\le (\alpha A+\beta B)^{\frac{1}{2}}, \\ \alpha A+\beta B &\le (\alpha+\beta)^{\frac{1}{2}}(\alpha A+\beta B)^{\frac{1}{2}}. \end{aligned}$$

Since $\alpha A+\beta B$ and $(\alpha+\beta)^{\frac{1}{2}}(\alpha A+\beta B)^{\frac{1}{2}}$ are positive definite, it follows from Theorem 2.6.4 that

$$\operatorname{tr}(\alpha A+\beta B) \le \operatorname{tr}\left((\alpha+\beta)^{\frac{1}{2}}(\alpha A^2+\beta B^2)^{\frac{1}{2}}\right).$$

Hence,

$$\alpha \operatorname{tr}(A)+\beta \operatorname{tr}(B) \le (\alpha+\beta)^{\frac{1}{2}} \operatorname{tr}\left((\alpha A^2+\beta B^2)^{\frac{1}{2}}\right) \tag{4.4}$$

Combining (4.3) and (4.4), we get (4.2).

For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned} (\alpha+\beta)^{\frac{1}{2}} \operatorname{tr}\left((\alpha A^2+\beta A^2)^{\frac{1}{2}}\right) &= (\alpha+\beta)^{\frac{1}{2}} \operatorname{tr}\left((\alpha+\beta)^{\frac{1}{2}}A)\right) \\ &= (\alpha+\beta)^{\frac{1}{2}}(\alpha+\beta)^{\frac{1}{2}} \operatorname{tr}(A)) \\ &= \alpha \operatorname{tr}(A)+\beta \operatorname{tr}(A). \end{aligned}$$

So the equality holds.

This corollary can be alternatively proved as follows.

Since A and B are positive definite and α, β are positive, so $\frac{1}{\alpha+\beta}(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2, \alpha A+\beta B$ and $(\alpha+\beta)^{\frac{1}{2}}(\alpha A^2+\beta B^2)^{\frac{1}{2}}$ are also positive definite.

By applying Theorem 2.6.4 to Corollary 3.1.1.4 and then using properties of trace, we obtain

$$\begin{aligned} \operatorname{tr}\left(\frac{1}{\alpha+\beta}(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\right) &\le \operatorname{tr}(\alpha A+\beta B) \le \operatorname{tr}\left((\alpha+\beta)^{\frac{1}{2}}(\alpha A^2+\beta B^2)^{\frac{1}{2}}\right), \\ \frac{1}{\alpha+\beta} \operatorname{tr}\left((\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\right) &\le \alpha \operatorname{tr}(A)+\beta \operatorname{tr}(B) \le (\alpha+\beta)^{\frac{1}{2}} \operatorname{tr}\left((\alpha A^2+\beta B^2)^{\frac{1}{2}}\right). \end{aligned}$$

The proof is completed. □

Next we will show the special cases which are come from Theorem 4.1.1 and Corollary 4.1.2. But first, we prove a following lemma:

Lemma 4.1.3. *Let $A \in \mathrm{M}_n$ be positive definite. Then for any positive integer k,*

$$\operatorname{tr}(A^{\frac{1}{k}}) > (\operatorname{tr}(A))^{\frac{1}{k}} \tag{4.5}$$

with equality holds if $k = 1$ or $n = 1$.

Proof. Obviously, equality holds if $k = 1$ or $n = 1$.
From Theorem 2.3.6, we have

$$\operatorname{tr}(A^k) \quad < \quad (\operatorname{tr}(A))^k$$

holds for any positive integers $n, k > 1$. By replacing A with $A^{\frac{1}{k}}$, we get

$$\begin{aligned} \operatorname{tr}(A) \quad &< \quad \left(\operatorname{tr}(A)^{\frac{1}{k}}\right)^k . \\ \left(\operatorname{tr}(A)\right)^{\frac{1}{k}} \quad &< \quad (\operatorname{tr}(A))^{\frac{1}{k}} . \end{aligned}$$

This completes the proof. □

Corollary 4.1.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\operatorname{tr}\left((A+B)^2\right) \quad \leq \quad 2\operatorname{tr}(A^2) + 2\operatorname{tr}(B^2), \tag{4.6}$$

$$\operatorname{tr}(AB) \quad \leq \quad \frac{1}{2}\operatorname{tr}(A^2) + \frac{1}{2}\operatorname{tr}(B^2) \tag{4.7}$$

with equalities hold if and only if $A = B$.

Proof. By setting $\alpha = \beta$ in Theorem 4.1.1, we get

$$\begin{aligned} \frac{1}{\alpha+\alpha}\operatorname{tr}\left((\alpha A + \alpha B)^2\right) \; &\leq \; \alpha \operatorname{tr}(A^2) + \alpha \operatorname{tr}(B^2), \\ \frac{1}{2\alpha}\operatorname{tr}\left(\alpha^2 (A+B)^2\right) \; &\leq \; \alpha\left(\operatorname{tr}(A^2) + \operatorname{tr}(B^2)\right), \\ \operatorname{tr}\left((A+B)^2\right) \; &\leq \; 2\operatorname{tr}(A^2) + 2\operatorname{tr}(B^2) \end{aligned}$$

which prove (4.6) and

$$\begin{aligned} \operatorname{tr}\left(A^2 + AB + BA + B^2\right) \quad &\leq \quad 2\operatorname{tr}(A^2) + 2\operatorname{tr}(B^2), \\ \operatorname{tr}(A^2) + \operatorname{tr}(AB) + \operatorname{tr}(BA) + \operatorname{tr}(B^2) \quad &\leq \quad 2\operatorname{tr}(A^2) + 2\operatorname{tr}(B^2), \\ \operatorname{tr}(AB) \quad &\leq \quad \frac{1}{2}\operatorname{tr}(A^2) + \frac{1}{2}\operatorname{tr}(B^2) \end{aligned}$$

which proved (4.7). The case of equalities is same as Theorem 4.1.1. □

Remark 4.1.5.

Note that for any positive definite matrices A and B, AB is not necessarily positive definite (in fact, it is not necessarily Hermitian) but $\operatorname{tr}(AB)$ is always positive real since

$$\operatorname{tr}(AB) \;=\; \sum_{i=1}^{n} \lambda_i(AB) \;=\; \sum_{i=1}^{n} \lambda_i(AB^{\frac{1}{2}}B^{\frac{1}{2}}) \;=\; \sum_{i=1}^{n} \lambda_i(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \;>\; 0.$$

Corollary 4.1.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\frac{1}{2}\operatorname{tr}\left((A^{\frac{1}{2}} + B^{\frac{1}{2}})^2\right) \;\le\; \operatorname{tr}(A) + \operatorname{tr}(B) \;\le\; \sqrt{2}\operatorname{tr}\left((A^2 + B^2)^{\frac{1}{2}}\right), \tag{4.8}$$

$$2\operatorname{tr}(A^{\frac{1}{2}}B^{\frac{1}{2}}) \;\le\; \operatorname{tr}(A) + \operatorname{tr}(B) \tag{4.9}$$

with equalities hold if $A = B$. The left equality in (4.8) *holds if and only if $A = B$. If, in addition, AB is positive definite, then for $n > 1$*

$$2\left(\operatorname{tr}(AB)\right)^{\frac{1}{2}} \;<\; \operatorname{tr}(A) + \operatorname{tr}(B). \tag{4.10}$$

Proof. By setting $\alpha = \beta$ in Corollary 4.1.2, we get

$$\begin{aligned}
\frac{1}{\alpha+\alpha}\operatorname{tr}\left((\alpha A^{\frac{1}{2}} + \alpha B^{\frac{1}{2}})^2\right) &\le \alpha\operatorname{tr}(A) + \alpha\operatorname{tr}(B) \le (\alpha+\alpha)^{\frac{1}{2}}\operatorname{tr}\left((\alpha A^2 + \alpha B^2)^{\frac{1}{2}}\right),\\
\frac{\alpha^2}{2\alpha}\operatorname{tr}\left((A^{\frac{1}{2}} + B^{\frac{1}{2}})^2\right) &\le \alpha\left(\operatorname{tr}(A) + \operatorname{tr}(B)\right) \le (2\alpha)^{\frac{1}{2}}(\alpha)^{\frac{1}{2}}\operatorname{tr}\left((A^2 + B^2)^{\frac{1}{2}}\right),\\
\frac{1}{2}\operatorname{tr}\left((A^{\frac{1}{2}} + B^{\frac{1}{2}})^2\right) &\le \quad \operatorname{tr}(A) + \operatorname{tr}(B) \quad \le \sqrt{2}\operatorname{tr}\left((A^2 + B^2)^{\frac{1}{2}}\right).
\end{aligned}$$

The case of equality is same as Corollary 4.1.2. By replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in the inequality (4.7) in Corollary 4.1.4, we get (4.9). Since AB is positive definite, by applying Theorem2.6.4 to Corollary 3.1.2.5, we get

$$\begin{aligned}
\operatorname{tr}\left((AB)^{\frac{1}{2}}\right) &\le \operatorname{tr}\left(\frac{1}{2}(A+B)\right),\\
2\operatorname{tr}\left((AB)^{\frac{1}{2}}\right) &\le \operatorname{tr}(A) + \operatorname{tr}(B).
\end{aligned}$$

For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned}
2\operatorname{tr}\left((A^2)^{\frac{1}{2}}\right) &= 2\operatorname{tr}(A)\\
&= \operatorname{tr}(A) + \operatorname{tr}(A).
\end{aligned}$$

So the equality holds. Since for $n > 1$, $\operatorname{tr}\left((AB)^{\frac{1}{2}}\right) > \left(\operatorname{tr}(AB)\right)^{\frac{1}{2}}$, it follows that

$$2\left(\operatorname{tr}(AB)\right)^{\frac{1}{2}} \;<\; \operatorname{tr}(A) + \operatorname{tr}(B).$$

The inequality (4.10) can be alternatively derived from (4.9) as follows.
Since AB is positive definite, we have $AB = BA$ and

$$(A^{\frac{1}{2}}B^{\frac{1}{2}})^2 = A^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}} = A^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}} = AB.$$

Again, the positive definiteness of AB implies

$$(AB)^{\frac{1}{2}} = A^{\frac{1}{2}}B^{\frac{1}{2}}.$$

Hence, from (4.9) we have

$$2\,\mathrm{tr}\left((AB)^{\frac{1}{2}}\right) \quad < \quad \mathrm{tr}(A) + \mathrm{tr}(B).$$

The proof is completed. □

Remark 4.1.7.

1. If A and B are positive definite symmetric and AB is positive definite, then AB is also symmetric since AB is Hermitian and

   $$\overline{AB} \quad = \quad \overline{A}\,\overline{B} \quad = \quad AB,$$

   i.e., AB is real.

2. Corollary 4.1.6 shows an upper bound and some lower bounds of $\mathrm{tr}(A) + \mathrm{tr}(B)$.

3. What happen when $n = 1$ in this corollary ?
   Let $A = [a]$ and $B = [b]$, the inequality (4.9) becomes

   $$\begin{aligned} 2a^{\frac{1}{2}}b^{\frac{1}{2}} &\leq a + b, \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a+b) \end{aligned}$$

   with equality holds if and only if $a = b$.
   This is the AM–GM inequality for positive real numbers. Analogously, we can get this inequality from any inequality in Corollary 4.1.4 and Corollary 4.1.6.

4. Note that for positive definite matrices A and B, $A^{\frac{1}{2}}B^{\frac{1}{2}}$ is not necessarily positive definite (in fact, it is not necessarily Hermitian) but $\mathrm{tr}(A^{\frac{1}{2}}B^{\frac{1}{2}})$ is always positive real since

   $$\mathrm{tr}(A^{\frac{1}{2}}B^{\frac{1}{2}}) = \sum_{i=1}^{n} \lambda_i(A^{\frac{1}{2}}B^{\frac{1}{2}}) = \sum_{i=1}^{n} \lambda_i(A^{\frac{1}{2}}B^{\frac{1}{4}}B^{\frac{1}{4}}) = \sum_{i=1}^{n} \lambda_i(B^{\frac{1}{4}}A^{\frac{1}{2}}B^{\frac{1}{4}}) > 0.$$

5. $\left(\mathrm{tr}(AB)\right)^{\frac{1}{2}}$ is real and positive since $\mathrm{tr}(AB)$ is real and positive.

6. Note that the trace of any positive definite matrix is a positive real number. So every terms in previous theorems and corollaries are positive real number.

Observation 4.1.8.

1. Let $A, B \in \mathbb{M}_n$ be positive definite. Then for any real numbers r and s, $\operatorname{tr}(A^r B^s)$ is real and positive.
2. Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then for any nonnegative real numbers r and s, $\operatorname{tr}(A^r B^s)$ is real and nonnegative.

Proof. First assume that A and B are positive definite.
Then for any real numbers r and s, we obtain

$$\begin{aligned}\operatorname{tr}(A^r B^s) &= \operatorname{tr}(A^r B^{\frac{s}{2}} B^{\frac{s}{2}}) \\ &= \operatorname{tr}(B^{\frac{s}{2}} A^r B^{\frac{s}{2}}).\end{aligned}$$

Alternatively, if we consider its eigenvalues, we get

$$\begin{aligned}\operatorname{tr}(A^r B^s) &= \sum_{i=1}^{n} \lambda_i(A^r B^s) \\ &= \sum_{i=1}^{n} \lambda_i(A^r B^{\frac{s}{2}} B^{\frac{s}{2}}) \\ &= \sum_{i=1}^{n} \lambda_i(B^{\frac{s}{2}} A^r B^{\frac{s}{2}}).\end{aligned}$$

Since $B^{\frac{s}{2}} A^r B^{\frac{s}{2}}$ is positive definite, its trace or all its eigenvalues are positive. This implies that $\operatorname{tr}(A^r B^s)$ is real and positive.
Now assume that A and B are positive semidefinite.
Then for any nonnegative real numbers r, s, we analogously obtain

$$\begin{aligned}\operatorname{tr}(A^r B^s) &= \operatorname{tr}(B^{\frac{s}{2}} A^r B^{\frac{s}{2}}) \\ \text{or} \qquad \operatorname{tr}(A^r B^s) &= \sum_{i=1}^{n} \lambda_i(B^{\frac{s}{2}} A^r B^{\frac{s}{2}}).\end{aligned}$$

Since $B^{\frac{s}{2}} A^r B^{\frac{s}{2}}$ is positive semidefinite, its trace and all its eigenvalues are nonnegative. This implies that $\operatorname{tr}(A^r B^s)$ is real and nonnegative. □

Observation 4.1.9.

Let $A \in \mathbb{M}_n$ be Hermitian.

1. Then for any even positive integer k, $\operatorname{tr}(A^k) = 0$ if and only if $A = 0$.
2. If A is positive semidefinite, then for any positive real number p, $\operatorname{tr}(A^p) = 0$ if and only if $A = 0$.

Proof. For the second assertion, let A be positive semidefinite and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of A. Assume first that $\mathrm{tr}(A^p) = 0$. We get

$$\mathrm{tr}(A^p) \quad = \quad \sum_{i=1}^{n} \lambda_i^p.$$

Since all eigenvalues of A are nonnegative and p is positive real, we obtain that $\lambda_i = 0$ for all $i = 1, 2, \ldots, n$. From Observation 3.1.3.5, we can conclude that $A = 0$. The converse statement is obvious. The first assertion can be proved in analogous way. $\square$

4.2 Inequalities of Product of Trace Involving the Power of Linear Combination of Matrices

In this section, inequalities of product of trace involving the power of linear combination of matrices are established. Theorem 4.2.1 is the main result. The special cases of this theorem yield a sequence of corollaries.

Theorem 4.2.1. *Let $A, B, C, D \in \mathrm{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\mathrm{tr}\left((\alpha A + \beta B)^r\right) \mathrm{tr}\left((\alpha C + \beta D)^s\right) \quad \geq \quad \alpha\, \mathrm{tr}(A^r)\, \mathrm{tr}(C^s) + \beta\, \mathrm{tr}(B^r)\, \mathrm{tr}(D^s) \quad (4.11)$$

with equality holds if $A = B, C = D$.

Proof. Since A, B, C and D are positive definite and α and β are positive, so $(\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s$ and $\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)$ are also positive definite.
By applying Theorem 2.6.4 to Theorem 3.2.1.5 and then using properties of trace, we obtain

$$\begin{aligned}
\mathrm{tr}\left((\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s\right) &\geq \mathrm{tr}\left(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)\right), \\
\mathrm{tr}\left((\alpha A + \beta B)^r\right) \mathrm{tr}\left((\alpha C + \beta D)^s\right) &\geq \mathrm{tr}\left(\alpha(A^r \otimes C^s)\right) + \mathrm{tr}\left(\beta(B^r \otimes D^s)\right) \\
&= \alpha\, \mathrm{tr}(A^r \otimes C^s) + \beta\, \mathrm{tr}(B^r \otimes D^s) \\
&= \alpha\, \mathrm{tr}(A^r)\, \mathrm{tr}(C^s) + \beta\, \mathrm{tr}(B^r)\, \mathrm{tr}(D^s).
\end{aligned}$$

For the case of equality, assume that $A = B, C = D$.
We obtain from the linearity of trace that

$$\begin{aligned}
\mathrm{tr}\left((\alpha A + \beta A)^r\right) \mathrm{tr}\left((\alpha C + \beta C)^s\right) &= \mathrm{tr}\left((\alpha + \beta)^r A^r\right) \mathrm{tr}\left((\alpha + \beta)^s C^s\right) \\
&= (\alpha + \beta)^r (\alpha + \beta)^s\, \mathrm{tr}\left(A^r\right) \mathrm{tr}\left(C^s\right) \\
&= (\alpha + \beta)\, \mathrm{tr}\left(A^r\right) \mathrm{tr}\left(C^s\right) \\
&= \alpha\, \mathrm{tr}(A^r)\, \mathrm{tr}(C^s) + \beta\, \mathrm{tr}(A^r)\, \mathrm{tr}(C^s).
\end{aligned}$$

So the equality holds. The proof is completed. □

From the main result (Theorem 4.2.1), there are some special cases. There are 3 types of them: the equality between the matrices; the equality between the coefficients (scalars); and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C$, $B = D$;
2. the case when $A = D$, $B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

Combining all of them, there are 16 possible cases (which include the general case). The details for each case are shown in Table 4.1 (except for the general case).

Table 4.1: The Special Cases of Theorem 4.2.1

Conditions			Results	
A, B, C, D	$r. s$	α, β	Corollary	Inequality
-	-	$\alpha = \beta$	4.2.2	4.12
-	$r = s$	-	4.2.3	4.13,4.14
-	$r = s$	$\alpha = \beta$	4.2.5	4.17,4.18
$A = C, B = D$	-	-	4.2.4	4.15
$A = C. B = D$	-	$\alpha = \beta$	4.2.6	4.19
$A = C, B = D$	$r = s$	-	4.2.7	4.21,4.23
$A = C. B = D$	$r = s$	$\alpha = \beta$	4.2.8	4.25,4.27
$A = D, B = C$	-	-	4.2.4	4.16
$A = D, B = C$	-	$\alpha = \beta$	4.2.6	4.20
$A = D, B = C$	$r = s$	-	4.2.7	4.22,4.24
$A = D, B = C$	$r = s$	$\alpha = \beta$	4.2.8	4.26,4.28
$A = B = C = D$	-	-	Equality Case	
$A = B = C = D$	-	$\alpha = \beta$	Equality Case	
$A = B = C = D$	$r = s$	-	Equality Case	
$A = B = C = D$	$r = s$	$\alpha = \beta$	Equality Case	

We summarize these cases into 7 corollaries as follows (depend on the hypotheses).

Corollary 4.2.2. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\operatorname{tr}\big((A+B)^r\big)\operatorname{tr}\big((C+D)^s\big) \quad \geq \quad \operatorname{tr}(A^r)\operatorname{tr}(C^s) + \operatorname{tr}(B^r)\operatorname{tr}(D^s) \tag{4.12}$$

with equality holds if $A = B, C = D$.

Proof. By setting $\alpha = \beta$ in Theorem 4.2.1, we obtain

$$\begin{aligned}
\operatorname{tr}\big((\alpha A + \alpha B)^r\big)\operatorname{tr}\big((\alpha C + \alpha D)^s\big) &\geq \alpha\operatorname{tr}(A^r)\operatorname{tr}(C^s) + \alpha\operatorname{tr}(B^r)\operatorname{tr}(D^s)\,,\\
\operatorname{tr}\big(\alpha^r(A+B)^r\big)\operatorname{tr}\big(\alpha^s(C+D)^s\big) &\geq \alpha\big(\operatorname{tr}(A^r)\operatorname{tr}(C^s) + \operatorname{tr}(B^r)\operatorname{tr}(D^s)\big)\,,\\
\alpha^r\operatorname{tr}\big((A+B)^r\big)\alpha^s\operatorname{tr}\big((C+D)^s\big) &\geq \alpha\big(\operatorname{tr}(A^r)\operatorname{tr}(C^s) + \operatorname{tr}(B^r)\operatorname{tr}(D^s)\big)\,,\\
\alpha^r\alpha^s\operatorname{tr}\big((A+B)^r\big)\operatorname{tr}\big((C+D)^s\big) &\geq \alpha\big(\operatorname{tr}(A^r)\operatorname{tr}(C^s) + \operatorname{tr}(B^r)\operatorname{tr}(D^s)\big)\,,\\
\operatorname{tr}\big((A+B)^r\big)\operatorname{tr}\big((C+D)^s\big) &\geq \operatorname{tr}(A^r)\operatorname{tr}(C^s) + \operatorname{tr}(B^r)\operatorname{tr}(D^s).
\end{aligned}$$

The case of equality is same as Theorem 4.2.1. The proof is completed. ☐

Corollary 4.2.3. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\operatorname{tr}\big((\alpha A+\beta B)^{\frac{1}{2}}\big)\operatorname{tr}\big((\alpha C+\beta D)^{\frac{1}{2}}\big) \quad \geq \quad \alpha\operatorname{tr}(A^{\frac{1}{2}})\operatorname{tr}(C^{\frac{1}{2}})+\beta\operatorname{tr}(B^{\frac{1}{2}})\operatorname{tr}(D^{\frac{1}{2}}) \tag{4.13}$$

with equality holds if $A = B, C = D$.
Moreover, if $n > 1$, this inequality can be improved further to a following inequality:

$$\operatorname{tr}\big((\alpha A + \beta B)^{\frac{1}{2}}\big)\operatorname{tr}\big((\alpha C + \beta D)^{\frac{1}{2}}\big) \quad > \quad \alpha\big(\operatorname{tr}(A)\operatorname{tr}(C)\big)^{\frac{1}{2}} + \beta\big(\operatorname{tr}(B)\operatorname{tr}(D)\big)^{\frac{1}{2}}. \tag{4.14}$$

Proof. By setting $r = s$ in Theorem 4.2.1, we have $r = 1/2$ and then (4.13). The case of equality is same as Theorem 4.2.1. From Lemma 4.1.3, for $n > 1$, we get

$$\begin{aligned}
\operatorname{tr}\big((\alpha A + \beta B)^{\frac{1}{2}}\big)\operatorname{tr}\big((\alpha C + \beta D)^{\frac{1}{2}}\big) &> \alpha\big(\operatorname{tr}(A)\big)^{\frac{1}{2}}\big(\operatorname{tr}(C)\big)^{\frac{1}{2}} + \beta\big(\operatorname{tr}(B)\big)^{\frac{1}{2}}\big(\operatorname{tr}(D)\big)^{\frac{1}{2}}\\
&= \alpha\big(\operatorname{tr}(A)\operatorname{tr}(C)\big)^{\frac{1}{2}} + \beta\big(\operatorname{tr}(B)\operatorname{tr}(D)\big)^{\frac{1}{2}}.
\end{aligned}$$

The proof is completed. ☐

Corollary 4.2.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\operatorname{tr}\big((\alpha A + \beta B)^r\big)\operatorname{tr}\big((\alpha A + \beta B)^s\big) \quad \geq \quad \alpha\operatorname{tr}(A^r)\operatorname{tr}(A^s) + \beta\operatorname{tr}(B^r)\operatorname{tr}(B^s)\,, \tag{4.15}$$

$$\operatorname{tr}\big((\alpha A + \beta B)^r\big)\operatorname{tr}\big((\beta A + \alpha B)^s\big) \quad \geq \quad \alpha\operatorname{tr}(A^r)\operatorname{tr}(B^s) + \beta\operatorname{tr}(A^s)\operatorname{tr}(B^r) \tag{4.16}$$

with equalities hold if $A = B$. The inequality (4.15) *also becomes an equality when $n = 1$.*

Proof. By setting $A = C, B = D$ in Theorem 4.2.1, we get (4.15). By setting $A = D, B = C$ in Theorem 4.2.1, we get

$$\begin{aligned} \operatorname{tr}\left((\alpha A + \beta B)^r\right)\operatorname{tr}\left((\beta A + \alpha B)^s\right) &\geq \alpha\operatorname{tr}(A^r)\operatorname{tr}(B^s) + \beta\operatorname{tr}(B^r)\operatorname{tr}(A^s) \\ &= \alpha\operatorname{tr}(A^r)\operatorname{tr}(B^s) + \beta\operatorname{tr}(A^s)\operatorname{tr}(B^r). \end{aligned}$$

For the inequality (4.15) if $A = B$, we obtain

$$\begin{aligned} \operatorname{tr}\left((\alpha A + \beta A)^r\right)\operatorname{tr}\left((\alpha A + \beta A)^s\right) &= \operatorname{tr}\left((\alpha + \beta)^r A^r\right)\operatorname{tr}\left((\alpha + \beta)^s A^s\right) \\ &= (\alpha + \beta)^r\operatorname{tr}(A^r)(\alpha + \beta)^s\operatorname{tr}(A^s) \\ &= (\alpha + \beta)\operatorname{tr}(A^r)\operatorname{tr}(A^s) \\ &= \alpha\operatorname{tr}(A^r)\operatorname{tr}(A^s) + \beta\operatorname{tr}(A^r)\operatorname{tr}(A^s). \end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the inequality (4.15) also becomes an equality when $n = 1$. □

Corollary 4.2.5. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\operatorname{tr}\left((A + B)^{\frac{1}{2}}\right)\operatorname{tr}\left((C + D)^{\frac{1}{2}}\right) \geq \operatorname{tr}(A^{\frac{1}{2}})\operatorname{tr}(C^{\frac{1}{2}}) + \operatorname{tr}(B^{\frac{1}{2}})\operatorname{tr}(D^{\frac{1}{2}}). \tag{4.17}$$

with the equality holds if $A = B, C = D$.
Moreover, if $n > 1$, this inequality can be improved further to a following inequality:

$$\operatorname{tr}\left((A + B)^{\frac{1}{2}}\right)\operatorname{tr}\left((C + D)^{\frac{1}{2}}\right) > (\operatorname{tr}(A)\operatorname{tr}(C))^{\frac{1}{2}} + (\operatorname{tr}(B)\operatorname{tr}(D))^{\frac{1}{2}}. \tag{4.18}$$

Proof. By setting $r = s$ in Corollary 4.2.2, we get $r = 1/2$ and then (4.17). The case of equality is same as Corollary 4.2.2. From Lemma 4.1.3, for $n > 1$, we get

$$\begin{aligned} \operatorname{tr}\left((A + B)^{\frac{1}{2}}\right)\operatorname{tr}\left((C + D)^{\frac{1}{2}}\right) &> \left(\operatorname{tr}(A)\right)^{\frac{1}{2}}\left(\operatorname{tr}(C)\right)^{\frac{1}{2}} + \left(\operatorname{tr}(B)\right)^{\frac{1}{2}}\left(\operatorname{tr}(D)\right)^{\frac{1}{2}} \\ &= (\operatorname{tr}(A)\operatorname{tr}(C))^{\frac{1}{2}} + (\operatorname{tr}(B)\operatorname{tr}(D))^{\frac{1}{2}}. \end{aligned}$$

The proof is completed. □

Corollary 4.2.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\operatorname{tr}\left((A + B)^r\right)\operatorname{tr}\left((A + B)^s\right) \geq \operatorname{tr}(A^r)\operatorname{tr}(A^s) + \operatorname{tr}(B^r)\operatorname{tr}(B^s), \tag{4.19}$$

$$\operatorname{tr}\left((A + B)^r\right)\operatorname{tr}\left((A + B)^s\right) \geq \operatorname{tr}(A^r)\operatorname{tr}(B^s) + \operatorname{tr}(A^s)\operatorname{tr}(B^r) \tag{4.20}$$

with equalities hold if $A = B$. The inequality (4.19) *also becomes an equality when $n = 1$.*

Proof. By setting $\alpha = \beta$ in (4.15) in Corollary 4.2.4, we obtain

$$\begin{aligned}
\operatorname{tr}\left((\alpha A+\alpha B)^r\right)\operatorname{tr}\left((\alpha A+\alpha B)^s\right) &\geq \alpha\operatorname{tr}(A^r)\operatorname{tr}(A^s)+\alpha\operatorname{tr}(B^r)\operatorname{tr}(B^s),\\
\operatorname{tr}\left(\alpha^r(A+B)^r\right)\operatorname{tr}\left(\alpha^s(A+B)^s\right) &\geq \alpha\operatorname{tr}(A^r)\operatorname{tr}(A^s)+\alpha\operatorname{tr}(B^r)\operatorname{tr}(B^s),\\
\alpha^r\alpha^s\operatorname{tr}\left((A+B)^r\right)\operatorname{tr}\left((A+B)^s\right) &\geq \alpha\left(\operatorname{tr}(A^r)\operatorname{tr}(A^s)+\operatorname{tr}(B^r)\operatorname{tr}(B^s)\right),\\
\alpha\operatorname{tr}\left((A+B)^r\right)\operatorname{tr}\left((A+B)^s\right) &\geq \alpha\left(\operatorname{tr}(A^r)\operatorname{tr}(A^s)+\operatorname{tr}(B^r)\operatorname{tr}(B^s)\right),\\
\operatorname{tr}\left((A+B)^r\right)\operatorname{tr}\left((A+B)^s\right) &\geq \operatorname{tr}(A^r)\operatorname{tr}(A^s)+\operatorname{tr}(B^r)\operatorname{tr}(B^s).
\end{aligned}$$

Similarly, by setting $\alpha = \beta$ in (4.16) in Corollary 4.2.4, we get (4.20).
The case of equalities is same as Corollary 4.2.4. □

Corollary 4.2.7. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\begin{aligned}
\left(\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}}\right)\right)^2 &\geq \alpha\left(\operatorname{tr}(A^{\frac{1}{2}})\right)^2+\alpha\left(\operatorname{tr}(B^{\frac{1}{2}})\right)^2, &(4.21)\\
\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}}\right)\operatorname{tr}\left((\beta A+\alpha B)^{\frac{1}{2}}\right) &\geq (\alpha+\beta)\operatorname{tr}(A^{\frac{1}{2}})\operatorname{tr}(B^{\frac{1}{2}}) &(4.22)
\end{aligned}$$

with the equalities hold if $A = B$. The inequality (4.21) *also becomes an equality when $n = 1$. Moreover, if $n > 1$, these inequalities can be improved further to the following inequalities:*

$$\begin{aligned}
\left(\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}}\right)\right)^2 &> \alpha\operatorname{tr}(A)+\beta\operatorname{tr}(B), &(4.23)\\
\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}}\right)\operatorname{tr}\left((\beta A+\alpha B)^{\frac{1}{2}}\right) &> (\alpha+\beta)\left(\operatorname{tr}(A)\operatorname{tr}(B)\right)^{\frac{1}{2}}. &(4.24)
\end{aligned}$$

Proof. By setting $r = s$ in (4.15) in Corollary 4.2.4, we get $r = 1/2$ and then (4.21). Obviously, the equality holds when $n = 1$. By setting $r = s$ in (4.16) in Corollary 4.2.4, we get $r = 1/2$ and

$$\begin{aligned}
\left(\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}}\right)\right)^2 &\geq \alpha\operatorname{tr}(A^{\frac{1}{2}})\operatorname{tr}(B^{\frac{1}{2}})+\beta\operatorname{tr}(B^{\frac{1}{2}})\operatorname{tr}(A^{\frac{1}{2}})\\
&= (\alpha+\beta)\operatorname{tr}(A^{\frac{1}{2}})\operatorname{tr}(B^{\frac{1}{2}}).
\end{aligned}$$

The case of equalities is same as Corollary 4.2.4.
By using Lemma 4.1.3 in (4.21) and (4.22), we obtain that for $n > 1$,

$$\begin{aligned}
\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}}\right)\operatorname{tr}\left((\beta A+\alpha B)^{\frac{1}{2}}\right) &> \alpha\left(\left(\operatorname{tr}(A)\right)^{\frac{1}{2}}\right)^2+\beta\left(\left(\operatorname{tr}(B)\right)^{\frac{1}{2}}\right)^2\\
&= \alpha\operatorname{tr}(A)+\beta\operatorname{tr}(B),\\
\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}}\right)\operatorname{tr}\left((\beta A+\alpha B)^{\frac{1}{2}}\right) &> (\alpha+\beta)\left(\operatorname{tr}(A)\right)^{\frac{1}{2}}\left(\operatorname{tr}(B)\right)^{\frac{1}{2}}\\
&= (\alpha+\beta)\left(\operatorname{tr}(A)\operatorname{tr}(B)\right)^{\frac{1}{2}}.
\end{aligned}$$

The proof is completed. □

Corollary 4.2.8. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\left(\operatorname{tr}\left((A+B)^{\frac{1}{2}}\right)\right)^2 \geq \left(\operatorname{tr}(A^{\frac{1}{2}})\right)^2 + \left(\operatorname{tr}(B^{\frac{1}{2}})\right)^2, \tag{4.25}$$

$$\left(\operatorname{tr}\left((A+B)^{\frac{1}{2}}\right)\right)^2 \geq 2\operatorname{tr}(A^{\frac{1}{2}})\operatorname{tr}(B^{\frac{1}{2}}) \tag{4.26}$$

with the equalities hold if $A = B$. The inequality (4.25) *also becomes an equality when $n = 1$. Moreover, if $n > 1$, these inequalities can be improved further to the following inequalities*:

$$\left(\operatorname{tr}\left((A+B)^{\frac{1}{2}}\right)\right)^2 > \operatorname{tr}(A) + \operatorname{tr}(B). \tag{4.27}$$

$$\left(\operatorname{tr}\left((A+B)^{\frac{1}{2}}\right)\right)^2 > 2\left(\operatorname{tr}(A)\operatorname{tr}(B)\right)^{\frac{1}{2}}. \tag{4.28}$$

Proof. Set $\alpha = \beta$ in Corollary 4.2.7. □

Remark 4.2.9.

For the inequality (4.26) in Corollary 4.2.8, when $n = 1$, let $A = [a]$ and $B = [b]$, we obtain

$$\begin{aligned} \left((a+b)^{\frac{1}{2}}\right)^2 &\geq 2a^{\frac{1}{2}}b^{\frac{1}{2}}, \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a+b). \end{aligned}$$

This is the AM–GM inequality for real numbers.

4.3 Inequalities of Trace Involving the Hadamard Product of Power of Linear Combination of Matrices

In this section, we present inequalities of trace involving the Hadamard product of power of linear combination of matrices. The main result is Theorem 4.3.1. This theorem leads to a sequence of corollaries as its special cases.

Theorem 4.3.1. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\operatorname{tr}\left((\alpha A+\beta B)^r \circ (\alpha C+\beta D)^s\right) \quad \geq \quad \alpha \operatorname{tr}(A^r \circ C^s) + \beta \operatorname{tr}(B^r \circ D^s) \tag{4.29}$$

with equality holds if $A = B, C = D$.

Proof. Since A, B, C and D are positive definite and α and β are positive, by Schur product theorem (Theorem 2.5.5), we have that $(\alpha A+\beta B)^r \circ (\alpha C+\beta B)^s$ and $\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)$ are also positive definite.
Then by applying Theorem 2.6.4 to Theorem 3.3.1.3 and using the linearity of trace, we obtain

$$\begin{aligned}
\operatorname{tr}\left((\alpha A+\beta B)^r \circ (\alpha C+\beta B)^s\right) \quad &\geq \quad \operatorname{tr}\left(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)\right) \\
&= \quad \alpha \operatorname{tr}(A^r \circ C^s) + \beta \operatorname{tr}(B^r \circ D^s).
\end{aligned}$$

For the case of equality, assume that $A = B, C = D$.
We obtain from the linearity of trace that

$$\begin{aligned}
\operatorname{tr}\left((\alpha A+\beta A)^r \circ (\alpha C+\beta C)^s\right) \quad &= \quad \operatorname{tr}\left((\alpha+\beta)^r A^r \circ (\alpha+\beta)^s C^s\right) \\
&= \quad (\alpha+\beta)^r(\alpha+\beta)^s \operatorname{tr}(A^r \circ C^s) \\
&= \quad (\alpha+\beta) \operatorname{tr}(A^r \circ C^s) \\
&= \quad \alpha \operatorname{tr}(A^r \circ C^s) + \beta \operatorname{tr}(A^r \circ C^s).
\end{aligned}$$

So the equality holds. The proof is completed. □

There are 3 types of special cases of the main result (Theorem 4.3.1): the equality between the matrices; the equality between the coefficients (scalars); and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C$, $B = D$;
2. the case when $A = D$, $B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

Combining all of them, there are 16 possible cases (which include the general case). The details for each case are shown in Table 4.2 (except for the general case).

Table 4.2: The Special Cases of Theorem 4.3.1

Conditions			Results	
A, B, C, D	r, s	α, β	Corollary	Inequality
-	-	$\alpha = \beta$	4.3.2	4.30
-	$r = s$	-	4.3.3	4.31
-	$r = s$	$\alpha = \beta$	4.3.5	4.34
$A = C, B = D$	-	-	4.3.4	4.32
$A = C, B = D$	-	$\alpha = \beta$	4.3.6	4.35
$A = C, B = D$	$r = s$	-	4.3.7	4.37
$A = C, B = D$	$r = s$	$\alpha = \beta$	4.3.8	4.39
$A = D, B = C$	-	-	4.3.4	4.33
$A = D, B = C$	-	$\alpha = \beta$	4.3.6	4.35
$A = D, B = C$	$r = s$	-	4.3.7	4.38
$A = D, B = C$	$r = s$	$\alpha = \beta$	4.3.8	4.40
$A = B = C = D$	-	-	Equality Case	
$A = B = C = D$	-	$\alpha = \beta$	Equality Case	
$A = B = C = D$	$r = s$	-	Equality Case	
$A = B = C = D$	$r = s$	$\alpha = \beta$	Equality Case	

We summarize these cases into 7 corollaries as follows (depend on the hypotheses).

Corollary 4.3.2. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\operatorname{tr}\left((A+B)^r \circ (C+D)^s\right) \quad \geq \quad \operatorname{tr}(A^r \circ C^s) + \operatorname{tr}(B^r \circ D^s) \tag{4.30}$$

with equality holds if $A = B, C = D$.

Proof. By setting $\alpha = \beta$ in Theorem 4.3.1, we obtain

$$\begin{aligned}
\operatorname{tr}\left((\alpha A+\alpha B)^r \circ (\alpha C+\alpha D)^s\right) &\geq \alpha \operatorname{tr}(A^r \circ C^s) + \alpha(B^r \circ D^s)\,, \\
\operatorname{tr}\left(\alpha^r (A+B)^r \circ \alpha^s (C+D)^s\right) &\geq \alpha \operatorname{tr}(A^r \circ C^s) + \alpha(B^r \circ D^s)\,, \\
\operatorname{tr}\left(\alpha^r \alpha^s ((A+B)^r \circ (C+D)^s)\right) &\geq \alpha \operatorname{tr}(A^r \circ C^s) + \alpha(B^r \circ D^s)\,, \\
\alpha \operatorname{tr}\left((A+B)^r \circ (C+D)^s\right) &\geq \alpha\left(\operatorname{tr}(A^r \circ C^s) + (B^r \circ D^s)\right), \\
\operatorname{tr}\left((A+B)^r \circ (C+D)^s\right) &\geq \operatorname{tr}(A^r \circ C^s) + \operatorname{tr}(B^r \circ D^s).
\end{aligned}$$

The case of equality is same as Theorem 4.3.1. The proof is completed. ☐

Corollary 4.3.3. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\operatorname{tr}\left((\alpha A+\beta B)^{\frac{1}{2}} \circ (\alpha C+\beta D)^{\frac{1}{2}}\right) \quad \geq \quad \alpha \operatorname{tr}(A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + \beta \operatorname{tr}(B^{\frac{1}{2}} \circ D^{\frac{1}{2}}) \tag{4.31}$$

with equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Theorem 4.3.1, we get $r = 1/2$ and then (4.31). The case of equality is same as Theorem 4.3.1. ☐

Corollary 4.3.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\operatorname{tr}\left((\alpha A+\beta B)^r \circ (\alpha A+\beta B)^s\right) \;\geq\; \alpha \operatorname{tr}(A^r \circ A^s) + \beta \operatorname{tr}(B^r \circ B^s)\,, \tag{4.32}$$

$$\operatorname{tr}\left((\alpha A+\beta B)^r \circ (\beta A+\alpha B)^s\right) \;\geq\; \alpha \operatorname{tr}(A^r \circ B^s) + \beta \operatorname{tr}(A^s \circ B^r) \tag{4.33}$$

with equalities hold if $A = B$. The inequality (4.32) *also becomes an equality if $n = 1$.*

Proof. By setting $A = C, B = D$ in Theorem 4.3.1, we get (4.32). By setting $A = D, B = C$ in Theorem 4.3.1, we obtain from the commutativity of the Hadamard product that

$$\begin{aligned}
\operatorname{tr}\left((\alpha A+\beta B)^r \circ (\beta A+\alpha B)^s\right) &\geq \alpha \operatorname{tr}(A^r \circ B^s) + \beta \operatorname{tr}(B^r \circ A^s) \\
&= \alpha \operatorname{tr}(A^r \circ B^s) + \beta \operatorname{tr}(A^s \circ B^r).
\end{aligned}$$

For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned}
\operatorname{tr}\left((\alpha A+\beta A)^r \circ (\alpha A+\beta A)^s\right) &= \operatorname{tr}\left((\alpha+\beta)^r A^r \circ (\alpha+\beta)^s A^s\right) \\
&= \operatorname{tr}\left((\alpha+\beta)^r(\alpha+\beta)^s(A^r \circ A^s)\right) \\
&= \operatorname{tr}\left((\alpha+\beta)(A^r \circ A^s)\right) \\
&= (\alpha+\beta)\operatorname{tr}(A^r \circ A^s) \\
&= \alpha \operatorname{tr}(A^r \circ A^s) + \beta \operatorname{tr}(A^r \circ A^s).
\end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the inequality (4.32) also becomes an equality when $n = 1$. □

Corollary 4.3.5. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\operatorname{tr}\left((A+B)^{\frac{1}{2}} \circ (C+D)^{\frac{1}{2}}\right) \geq \operatorname{tr}(A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + \operatorname{tr}(B^{\frac{1}{2}} \circ D^{\frac{1}{2}}) \tag{4.34}$$

with equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Corollary 4.3.2, we get $r = 1/2$ and then (4.34). The case of equality is same as Corollary 4.3.2. □

Corollary 4.3.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\operatorname{tr}\left((A+B)^r \circ (A+B)^s\right) \geq \operatorname{tr}(A^r \circ A^s) + \operatorname{tr}(B^r \circ B^s), \tag{4.35}$$

$$\operatorname{tr}\left((A+B)^r \circ (A+B)^s\right) \geq \operatorname{tr}(A^r \circ B^s) + \operatorname{tr}(A^s \circ B^r) \tag{4.36}$$

with equalities hold if $A = B$. The inequality (4.35) *also becomes an equality if $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 4.3.4. □

Corollary 4.3.7. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$. Then*

$$\operatorname{tr}\left(\left((\alpha A+\beta B)^{\frac{1}{2}}\right)^{(2)}\right) \geq \alpha \operatorname{tr}\left((A^{\frac{1}{2}})^{(2)}\right) + \beta \operatorname{tr}\left((B^{\frac{1}{2}})^{(2)}\right), \tag{4.37}$$

$$\operatorname{tr}\left(\left((\alpha A+\beta B)^{\frac{1}{2}}\right)^{(2)}\right) \geq (\alpha+\beta)\operatorname{tr}(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) \tag{4.38}$$

with equalities hold if $A = B$. The inequalities (4.37) *also becomes an equality when $n = 1$.*

Proof. Set $r = s$ in Corollary 4.3.4. □

Corollary 4.3.8. *Let* $A, B \in \mathbb{M}_n$ *be positive definite symmetric and let* $\alpha, \beta > 0$. *Then*

$$\operatorname{tr}\left(((A+B)^{\frac{1}{2}})^{(2)}\right) \geq \operatorname{tr}\left((A^{\frac{1}{2}})^{(2)}\right) + \operatorname{tr}\left((B^{\frac{1}{2}})^{(2)}\right), \tag{4.39}$$

$$\operatorname{tr}\left(((A+B)^{\frac{1}{2}})^{(2)}\right) \geq 2\operatorname{tr}(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) \tag{4.40}$$

with equalities hold if $A = B$. *The inequality* (4.39) *also becomes an equality when* $n = 1$.

Proof. Set $\alpha = \beta$ in Corollary 4.3.7. □

Remark 4.3.9.

For the inequality (4.40) in Corollary 4.3.8, when $n = 1$, let $A = [a]$ and $B = [b]$, we obtain

$$\begin{aligned} ((a+b)^{\frac{1}{2}})^2 &\geq 2(a^{\frac{1}{2}}b^{\frac{1}{2}}), \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a+b). \end{aligned}$$

This is the AM–GM inequality for real numbers.

4.4 Numerical Examples

In order to support our main results, some numerical examples for the inequalities of trace are given here. The tool we use is the mathematical program which is called Maple 9.5.

Example 4.4.1.

Consider the inequality of trace involving the 2nd power of linear combination of the given matrices

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

in the case when $\alpha = 4,\ \beta = 3$.

We see that A and B are positive definite symmetric and α and β satisfy the hypothesis of Theorem 4.1.1. We get

$$\begin{aligned} A^2 &= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \\ B^2 &= \begin{bmatrix} 8 & 14 \\ 14 & 29 \end{bmatrix}, \\ (\alpha A + \beta B)^2 &= \begin{bmatrix} 296 & 370 \\ 370 & 629 \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{\alpha+\beta}\operatorname{tr}\left((\alpha A + \beta B)^2\right) &\approx 131.571, \\ \alpha \operatorname{tr}(A^2) + \beta \operatorname{tr}(B^2) &= 151. \end{aligned}$$

Therefore,

$$\frac{1}{\alpha+\beta}\operatorname{tr}\left((\alpha A + \beta B)^2\right) \leq \alpha \operatorname{tr}(A^2) + \beta \operatorname{tr}(B^2).$$

This example supports the result of Theorem 4.1.1. □

Example 4.4.2.

Consider the inequality of product of trace involving the power of linear combination of the given matrices

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha = 2,\ \beta = 1,\ r = 1/3,\ s = 2/3$.

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 4.2.1. We get

$$\begin{aligned}
(\alpha A + \beta B)^r &\approx \begin{bmatrix} 2.208 & 0.453 \\ 0.453 & 1.574 \end{bmatrix}, \ (\alpha C + \beta D)^s \approx \begin{bmatrix} 2.886 & -0.691 \\ -0.691 & 4.614 \end{bmatrix}, \\
A^r &\approx \begin{bmatrix} 1.520 & 0.614 \\ 0.614 & 0.906 \end{bmatrix}, \ C^s \approx \begin{bmatrix} 1.587 & 0 \\ 0 & 1.587 \end{bmatrix}, \\
B^r &\approx \begin{bmatrix} 1.198 & -0.292 \\ -0.292 & 0.906 \end{bmatrix}, \ D^s \approx \begin{bmatrix} 0.787 & -0.970 \\ -0.970 & 3.214 \end{bmatrix}.
\end{aligned}$$

It follows that,

$$\begin{aligned}
\operatorname{tr}\left((\alpha A + \beta B)^r\right) &\approx 3.782, \\
\operatorname{tr}\left((\alpha C + \beta D)^s\right) &\approx 7.500, \\
\operatorname{tr}(A^r) &\approx 2.426, \\
\operatorname{tr}(B^r) &\approx 2.104, \\
\operatorname{tr}(C^s) &\approx 3.174, \\
\operatorname{tr}(D^s) &\approx 4.001.
\end{aligned}$$

Hence,

$$\begin{aligned}
\operatorname{tr}\left((\alpha A + \beta B)^r\right) \operatorname{tr}\left((\alpha C + \beta D)^s\right) &\approx 28.365, \\
\alpha \operatorname{tr}(A^r) \operatorname{tr}(C^s) + \beta \operatorname{tr}(B^r) \operatorname{tr}(D^s) &\approx 23.818.
\end{aligned}$$

Thus,

$$\operatorname{tr}\left((\alpha A + \beta B)^r\right) \operatorname{tr}\left((\alpha C + \beta D)^s\right) \geq \alpha \operatorname{tr}(A^r) \operatorname{tr}(C^s) + \beta \operatorname{tr}(B^r) \operatorname{tr}(D^s).$$

This example supports the result of Theorem 4.2.1. □

Example 4.4.3.

Consider the inequality of trace involving the Hadamard product of power of linear combination of the given matrices

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha = 2,\ \beta = 1,\ r = 2/3,\ s = 1/3$.

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 4.2.1. We get

$$\begin{aligned} (\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s &\approx \begin{bmatrix} 8.583 & -0.309 \\ -0.309 & 5.742 \end{bmatrix}, \\ A^r \circ C^s &\approx \begin{bmatrix} 3.387 & 0 \\ 0 & 1.509 \end{bmatrix}, \\ B^r \circ D^s &\approx \begin{bmatrix} 1.219 & 0.238 \\ 0.238 & 1.587 \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{tr}\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) &\approx 14.325, \\ \operatorname{tr}(A^r \circ C^s) &\approx 4.896, \\ \operatorname{tr}(B^r \circ D^s) &\approx 2.806, \\ \alpha \operatorname{tr}(A^r \circ C^s) + \beta \operatorname{tr}(B^r \circ D^s) &\approx 12.598. \end{aligned}$$

Thus,

$$\operatorname{tr}\left((\alpha A + \beta B)^r\right) \operatorname{tr}\left((\alpha C + \beta D)^s\right) \geq \alpha \operatorname{tr}(A^r) \operatorname{tr}(C^s) + \beta \operatorname{tr}(B^r) \operatorname{tr}(D^s).$$

This example supports the result of Theorem 4.3.1. ☐

Example 4.4.4.

Consider the inequality of trace involving the Hadamard product of of power of linear combination of the given matrices

$$A = \begin{bmatrix} \sqrt{3} & 1/2 & -1 \\ 1/2 & \sqrt{7}-1 & 0.851 \\ -1 & 0.851 & 2\sqrt{2} \end{bmatrix}, B = \begin{bmatrix} \sqrt{7} & 17/4 & 1 \\ 17/4 & 6\sqrt{6}-4 & 3.851 \\ 1 & 3.851 & 3\sqrt{3} \end{bmatrix},$$

$$C = \begin{bmatrix} \pi & 2 & 0 \\ 2 & 5.219 & \sqrt[4]{29} \\ 0 & \sqrt[4]{29} & 8 \end{bmatrix}, D = \begin{bmatrix} e & -2 & -0.1 \\ -2 & 4.567 & \sqrt[3]{17} \\ -0.1 & \sqrt[3]{17} & 8 \end{bmatrix}$$

in the case when $\alpha = 1,\ \beta = 2,\ r = 1/4,\ s = 3/4$.

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 4.3.1. We get

$$(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s \approx \begin{bmatrix} 17.275 & -0.298 & 0.008 \\ -0.298 & 34.082 & 1.742 \\ 0.008 & 1.742 & 45.444 \end{bmatrix},$$

$$A^r \circ C^s \approx \begin{bmatrix} 2.481 & 0.142 & 0.010 \\ 0.142 & 3.602 & 0.171 \\ 0.010 & 0.171 & 5.859 \end{bmatrix},$$

$$B^r \circ D^s \approx \begin{bmatrix} 2.200 & -0.390 & 0.00008 \\ -0.390 & 5.021 & 0.287 \\ 0.00008 & 0.287 & 6.814 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \operatorname{tr}\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) &\approx 96.801, \\ \operatorname{tr}(A^r \circ C^s) &\approx 11.942, \\ \operatorname{tr}(B^r \circ D^s) &\approx 14.035, \\ \alpha \operatorname{tr}(A^r \circ C^s) + \beta \operatorname{tr}(B^r \circ D^s) &\approx 91.965. \end{aligned}$$

Thus,

$$\operatorname{tr}\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) \geq \alpha \operatorname{tr}(A^r \circ C^s) + \beta \operatorname{tr}(B^r \circ D^s).$$

This example supports the result of Theorem 4.3.1. □

CHAPTER 5

INEQUALITIES OF DETERMINANT

The aim of this chapter is to present inequalities of determinant for positive definite symmetric matrices. These inequalities are shown in Section 5.1–Section 5.3. We also give some numerical examples to guarantee our results in Section 5.4.

In order to obtain these inequalities, we use properties of the determinant in Proposition 2.1.4 and Theorem 2.6.4 which is stated that for any positive definite matrices $A.B \in \mathbb{M}_n$,

$$A \geq B \quad \Longrightarrow \quad \det(A) \geq \det(B).$$

Note that the determinants of positive definite symmetric matrices are always positive real numbers. This assures that we can compare them.

5.1 Inequalities of Determinant Involving the 2nd Power of Linear Combination of Two Matrices

In this section, we present inequalities of determinants involving the 2nd power of linear combination of two positive definite symmetric matrices. First we prove Theorem 5.1.1 which is the main result. Then we state some corollaries which are followed from this theorem.

Theorem 5.1.1. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\frac{1}{(\alpha+\beta)^n}\big(\det(\alpha A + \beta B)\big)^2 \quad \leq \quad \det\left(\alpha A^2 + \beta B^2\right) \tag{5.1}$$

with the equality holds if $A = B$.

Proof. Since A and B are positive definite and α, β are positive, so $\frac{1}{\alpha+\beta}(\alpha A + \beta B)^2$ and $\alpha A^2 + \beta B^2$ are also positive definite. By applying Theorem 2.6.4 to Theorem 3.1.1.1 and using properties of determinant, we obtain

$$\begin{aligned}\det(\alpha A^2+\beta B^2) &\geq \det\left(\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2\right)\\ &= \left(\frac{1}{\alpha+\beta}\right)^n \det\left((\alpha A+\beta B)^2\right)\\ &= \frac{1}{(\alpha+\beta)^n}\big(\det(\alpha A+\beta B)\big)^2.\end{aligned}$$

For the case of equality, assume that $A = B$.
We obtain from the properties of determinant that

$$\begin{aligned}\frac{1}{(\alpha+\beta)^n}\big(\det(\alpha A+\beta A)\big)^2 &= \frac{1}{(\alpha+\beta)^n}\Big(\det\big((\alpha+\beta)A\big)\Big)^2\\ &= \frac{(\alpha+\beta)^{2n}}{(\alpha+\beta)^n}\det\left(A^2\right)\\ &= (\alpha+\beta)^n\det(A^2)\\ &= \det(\alpha A^2+\beta A^2).\end{aligned}$$

So the equality holds. The proof is completed. □

Lemma 5.1.2.

Let $A \in \mathbb{M}_n$.

1. If A are positive definite, then for any real number r,

$$\det(A^r) = \big(\det(A)\big)^r. \tag{5.2}$$

2. If A are positive semidefinite, then for any nonnegative real number s,

$$\det(A^s) = \big(\det(A)\big)^s. \tag{5.3}$$

Proof. Assume that A is positive definite. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of A. Since A is Hermitian, there exists a unitary matrix U such that

$$A = U\,diag[\lambda_1, \lambda_2, \dots, \lambda_n]\,U^*.$$

Hence,

$$A^r = U\,diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\,U^*.$$

It follows that

$$\begin{aligned}
\det(A^r) &= \det\left(U\, diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\, U^*\right) \\
&= \det(U)\, \det\left(diag[\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r]\right)\, \det(U^*) \\
&= \det(U) \prod_{i=1}^{n} \lambda_i^r\, \det(U^*) \\
&= \det(U^r) \left(\prod_{i=1}^{n} \lambda_i\right)^r \det\left((U^*)^r\right) \\
&= \left(\det(U)\right)^r \left(\det(A)\right)^r \left(\det(U^*)\right)^r \\
&= \left(\det(U)\, \det(A)\, \det(U^*)\right)^r \\
&= \left(\det(A)\right)^r.
\end{aligned}$$

Now assume that A is positive semidefinite.
Analogously, for any nonnegative real number s, we get

$$\det(A^s) = \left(\det(A)\right)^s.$$

The proof is completed. □

Corollary 5.1.3. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\frac{1}{(\alpha+\beta)^n}\left(\det(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})\right)^2 \le \det(\alpha A+\beta B) \le (\alpha+\beta)^{\frac{n}{2}}\left(\det(\alpha A^2+\beta B^2)\right)^{\frac{1}{2}} \quad (5.4)$$

with the equality holds if $A = B$.

Proof. By replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in Corollary 5.1.1, we get

$$\frac{1}{(\alpha+\beta)^n}\left(\det(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})\right)^2 \le \det(\alpha A+\beta B) \quad (5.5)$$

with equality holds if $A = B$.
Then by using Löwner–Heinz inequality (Theorem 2.8.2) to Theorem 3.1.1.1, we get

$$\begin{aligned}
\left(\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2\right)^{\frac{1}{2}} &\le (\alpha A+\beta B)^{\frac{1}{2}}, \\
\frac{1}{(\alpha+\beta)^{\frac{1}{2}}}(\alpha A+\beta B) &\le (\alpha A+\beta B)^{\frac{1}{2}}, \\
\alpha A+\beta B &\le (\alpha+\beta)^{\frac{1}{2}}(\alpha A+\beta B)^{\frac{1}{2}}.
\end{aligned}$$

Since $\alpha A+\beta B$ and $(\alpha+\beta)^{\frac{1}{2}}(\alpha A+\beta B)^{\frac{1}{2}}$ are positive definite, it follows from Theorem 2.6.4 that

$$\det(\alpha A+\beta B) \le (\alpha+\beta)^{\frac{n}{2}} \det\left((\alpha A^2+\beta B^2)^{\frac{1}{2}}\right). \quad (5.6)$$

Combining (5.5) and (5.6), we get (5.4).

For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned}(\alpha+\beta)^{\frac{n}{2}}\det\left((\alpha A^2+\beta A^2)^{\frac{1}{2}}\right) &= (\alpha+\beta)^{\frac{n}{2}}\det\left((\alpha+\beta)^{\frac{1}{2}}A)\right) \\ &= (\alpha+\beta)^{\frac{n}{2}}(\alpha+\beta)^{\frac{n}{2}}\det(A)) \\ &= \det(\alpha A+\beta A).\end{aligned}$$

So the equality holds.

This corollary can be alternatively proved as follows.

By applying Theorem 2.6.4 to Corollary 3.1.1.4, we obtain

$$\begin{aligned}\det\left(\frac{1}{\alpha+\beta}(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\right) &\le \det(\alpha A+\beta B) \le \det\left((\alpha+\beta)^{\frac{1}{2}}(\alpha A^2+\beta B^2)^{\frac{1}{2}}\right), \\ \frac{1}{(\alpha+\beta)^n}\det\left(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\right) &\le \det(\alpha A+\beta B) \le (\alpha+\beta)^{\frac{n}{2}}\det\left((\alpha A^2+\beta B^2)^{\frac{1}{2}}\right), \\ \frac{1}{(\alpha+\beta)^n}\left(\det(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})\right)^2 &\le \det(\alpha A+\beta B) \le (\alpha+\beta)^{\frac{n}{2}}\left(\det(\alpha A^2+\beta B^2)\right)^{\frac{1}{2}}.\end{aligned}$$

The proof is completed. □

Corollary 5.1.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\left(\det(A+B)\right)^2 \le 2^n\det(A^2+B^2) \tag{5.7}$$

with equalities hold if $A = B$.

Proof. By setting $\alpha = \beta$ in Theorem 5.1.1, we obtain

$$\begin{aligned}\frac{1}{(2\alpha)^n}\left(\det(\alpha A+\alpha B)\right)^2 &\le \det(\alpha A^2+\alpha B^2), \\ \frac{\alpha^{2n}}{2^n\alpha^n}\left(\det(A+B)\right)^2 &\le \alpha^n\det(A^2+B^2), \\ \left(\det(A+B)\right)^2 &\le 2^n\det(A^2+B^2).\end{aligned}$$

The case of equality is same as Theorem 5.1.1. □

Corollary 5.1.5. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. If AB is positive definite, then*

$$2^n\det(A)\det(B) \le \det(A^2+B^2) \tag{5.8}$$

with equality holds if $A = B$.

Proof. Since AB are positive definite, from Observation 3.1.11, we have $AB = BA$. It follows from the inequality (3.5) in Corollary 3.1.2.3 that

$$2AB \le A^2+B^2. \tag{5.9}$$

Since $2AB$ and A^2+B^2 are positive definite, by applying Theorem 2.6.4 to the inequality (5.9), we get

$$\begin{aligned} \det(A^2+B^2) &\geq \det(2AB) \\ &= 2^n \det(A)\det(B). \end{aligned}$$

The case of equality is same as Corollary 3.2. □

Corollary 5.1.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\frac{1}{2^n}\left(\det(A^{\frac{1}{2}}+B^{\frac{1}{2}})\right)^2 \leq \det(A+B) \leq 2^{\frac{n}{2}}\Big(\det(A^2+B^2)\Big)^{\frac{1}{2}}. \tag{5.10}$$

If, in addition, AB is positive definite, then

$$2^n\left(\det(A)\det(B)\right)^{\frac{1}{2}} \leq \det(A+B). \tag{5.11}$$

All equalities hold if $A=B$.

Proof. By setting $\alpha=\beta$ in Corollary 5.1.3, we obtain

$$\begin{aligned} \frac{1}{(2\alpha)^n}\left(\det(\alpha A^{\frac{1}{2}}+\alpha B^{\frac{1}{2}})\right)^2 &\leq \det(\alpha A+\alpha B) \leq (2\alpha)^{\frac{n}{2}}\Big(\det(\alpha A^2+\alpha B^2)\Big)^{\frac{1}{2}}, \\ \frac{1}{2^n\alpha^n}\Big(\det\left(\alpha(A^{\frac{1}{2}}+B^{\frac{1}{2}})\right)\Big)^2 &\leq \alpha^n\det(A+B) \leq 2^{\frac{n}{2}}\alpha^{\frac{n}{2}}\Big(\det\left(\alpha(A^2+B^2)\right)\Big)^{\frac{1}{2}}, \\ \frac{\alpha^{2n}}{2^n\alpha^n}\Big(\det(A^{\frac{1}{2}}+B^{\frac{1}{2}}\Big)^2 &\leq \alpha^n\det(A+B) \leq 2^{\frac{n}{2}}\alpha^n\left(\det(A^2+B^2)\right)^{\frac{1}{2}}, \\ \frac{1}{2^n}\left(\det(A^{\frac{1}{2}}+B^{\frac{1}{2}})\right)^2 &\leq \det(A+B) \leq 2^{\frac{n}{2}}\left(\det(A^2+B^2)\right)^{\frac{1}{2}}. \end{aligned}$$

The equality case is same as Corollary 5.1.3.
By replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in the inequality (5.7) in Corollary 5.1.4, we obtain

$$\begin{aligned} \det(A+B) &\geq \det(2A^{\frac{1}{2}}B^{\frac{1}{2}}) \\ &= 2^n\det(A^{\frac{1}{2}})\det(B^{\frac{1}{2}}) \\ &= 2^n\left(\det(A)\right)^{\frac{1}{2}}\left(\det(B)\right)^{\frac{1}{2}} \\ &= 2^n\left(\det(A)\det(B)\right)^{\frac{1}{2}}. \end{aligned}$$

The case of quality is same as Corollary 5.1.5. □

Remark 5.1.7.

1. When $n = 1$, let $A = [a]$ and $B = [b]$, the inequality (5.11) becomes

$$\begin{aligned} 2(ab)^{\frac{1}{2}} &\leq a + b, \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a + b) \end{aligned}$$

   with equality holds if and only if $a = b$. This is the AM–GM inequality for positive real numbers. Analogously, we can get this inequality from any inequality in Corollary 5.1.4, Corollary 5.1.5 and Corollary 5.1.6.

2. Corollary 5.1.6 shows an upper bound and some lower bounds of $\det(A + B)$.

3. Remember that the determinant of any positive definite matrix is always positive real numbers.

4. Note that for positive definite matrices A and B, AB is not necessarily positive definite (in fact, it is not necessarily Hermitian) but $\det(AB)$ is always real and positive since

$$\det(AB) = \det(A)\det(B) > 0$$

   or alternatively, if we consider its eigenvalues, we have

$$\det(AB) = \prod_{i=1}^{n} \lambda_i(AB) = \prod_{i=1}^{n} \lambda_i(AB^{\frac{1}{2}}B^{\frac{1}{2}}) = \prod_{i=1}^{n} \lambda_i(B^{\frac{1}{2}}AB^{\frac{1}{2}}) > 0.$$

   A generalization of this statement is Observation 5.1.8.

Observation 5.1.8.

Let $A, B \in \mathbb{M}_n$.

1. If A and B are positive definite (not necessarily symmetric, i.e., not necessarily real), then for any real numbers r and s, $\det(A^r B^s)$ is real and positive.

2. If A and B are positive semidefinite (not necessarily symmetric, i.e., not necessarily real), then for any nonnegative real numbers r and s, $\det(A^r B^s)$ is real and nonnegative.

3. If A and B are nonsingular Hermitian (not necessarily real), then for any even integers p and q, $\det(A^p B^q)$ is real and positive.

4. If A and B are Hermitian (not necessarily real), then for any even nonnegative integers p and q, $\det(A^p B^q)$ is real and nonnegative.

Proof. Assume that A and B are positive definite. From Lemma 5.1.2, we have

$$\begin{aligned} \det(A^r B^s) &= \det(A^r)\,\det(B^s) \\ &= \left(\det(A)\right)^r \left(\det(B)\right)^s \\ &> 0. \end{aligned}$$

Alternatively, if we consider its eigenvalues, we get

$$\begin{aligned} \det(A^r B^s) &= \prod_{i=1}^{n} \lambda_i(A^r B^s) \\ &= \prod_{i=1}^{n} \lambda_i(A^r B^{\frac{s}{2}} B^{\frac{s}{2}}) \\ &= \prod_{i=1}^{n} \lambda_i(B^{\frac{s}{2}} A^r B^{\frac{s}{2}}) \\ &> 0. \end{aligned}$$

Now assume that A and B are positive semidefinite.
Analogously, for any nonnegative real number r and s, $\det(A^r B^s)$ is real and nonnegative. Notice that in this case A^r and B^s exist if r and s are nonnegative. The cases of nonsingular Hermitian and Hermitian are followed like that. □

5.2 Inequalities of Determinant Involving the Kronecker Product of Power of Linear Combination of Matrices

In this section, we show inequalities of determinant involving the Kronecker product of power of linear combination of positive definite symmetric matrices. Theorem 5.2.1 is the main result. Others inequalities are followed from this theorem as its special cases.

Theorem 5.2.1. *Let* $A, B, C, D \in \mathbb{M}_n$ *be positive definite symmetric and let* $\alpha, \beta, r, s > 0$ *be scalars such that* $r + s = 1$. *Then*

$$\left(\det(\alpha A+\beta B)\right)^{nr}\left(\det(\alpha C+\beta D)\right)^{ns} \quad \geq \quad \det\left(\alpha(A^r\otimes C^s)+\beta(B^r\otimes D^s)\right) \tag{5.12}$$

with equality holds if $A = B, C = D$.

Proof. Since A and B are positive definite and α and β are positive, so $(\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s$ and $\alpha(A^r\otimes C^s)+\beta(B^r\otimes D^s)$ are also positive definite. Then by applying Theorem 2.6.4 to Theorem 3.2.1.5 and using properties of determinant and Lemma 5.1.2, we obtain

$$\begin{aligned} \det\left(\alpha(A\otimes B)^r+\beta(A\otimes B)^s\right) &\leq \det\left((\alpha A+\beta B)^r\otimes(\alpha C+\beta D)^s\right) \\ &= \left(\det(\alpha A+\beta B)^r\right)^n\left(\det(\alpha C+\beta D)^s\right)^n \\ &= \left(\det(\alpha A+\beta B)\right)^{nr}\left(\det(\alpha C+\beta D)\right)^{ns}. \end{aligned}$$

For the case of equality, assume that $A = B, C = D$.
We obtain from properties of determinant and Lemma 5.1.2 that

$$\begin{aligned} &\left(\det(\alpha A+\beta A)\right)^{nr}\left(\det(\alpha C+\beta C)\right)^{ns} \\ &\quad = \Big(\det\left((\alpha+\beta)A\right)\Big)^{nr}\Big(\det\left((\alpha+\beta)C\right)\Big)^{ns} \\ &\quad = \Big(\det\Big(\left((\alpha+\beta)A\right)^r\Big)\Big)^n\Big(\det\Big(\left((\alpha+\beta)C\right)^s\Big)\Big)^n \\ &\quad = \det\Big(\left((\alpha+\beta)A\right)^r\otimes\left((\alpha+\beta)C\right)^s\Big) \\ &\quad = \det\left((\alpha+\beta)^rA^r\otimes(\alpha+\beta)^sC^s\right) \\ &\quad = \det\left((\alpha+\beta)^r(\alpha+\beta)^s(A^r\otimes C^s)\right) \\ &\quad = \det\left((\alpha+\beta)(A^r\otimes C^s)\right) \\ &\quad = \det\left(\alpha(A^r\otimes C^s)+\beta(A^r\otimes C^s)\right). \end{aligned}$$

So the equality holds. The proof is completed. □

Now, let us consider special cases of the main result (Theorem 5.2.1). There are 3 types of them: the equality between the matrices, the equality between the coefficients (scalars) and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C,\ B = D$;
2. the case when $A = D,\ B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

When we combine all of them, we have 16 possible cases (which include the general case). The details for each case are given in Table 5.1 (except for the general case).

Table 5.1: The Special Cases of Theorem 5.2.1

Conditions			Results	
A, B, C, D	r, s	α, β	Corollary	Inequality
-	-	$\alpha = \beta$	5.2.2	5.13
-	$r = s$	-	5.2.3	5.14
-	$r = s$	$\alpha = \beta$	5.2.5	5.17
$A = C. B = D$	-	-	5.2.4	5.15
$A = C, B = D$	-	$\alpha = \beta$	5.2.6	5.18
$A = C. B = D$	$r = s$	-	5.2.7	5.20
$A = C, B = D$	$r = s$	$\alpha = \beta$	5.2.8	5.22
$A = D, B = C$	-	-	5.2.4	5.16
$A = D, B = C$	-	$\alpha = \beta$	5.2.6	5.19
$A = D, B = C$	$r = s$	-	5.2.7	5.21
$A = D. B = C$	$r = s$	$\alpha = \beta$	5.2.8	5.23
$A = B = C = D$	-	-	Equality Case	
$A = B = C = D$	-	$\alpha = \beta$	Equality Case	
$A = B = C = D$	$r = s$	-	Equality Case	
$A = B = C = D$	$r = s$	$\alpha = \beta$	Equality Case	

These cases yield following corollaries:

Corollary 5.2.2. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\big(\det(A+B)\big)^{nr}\big(\det(C+D)\big)^{ns} \geq \det\big((A\otimes C)^r + (B\otimes D)^s\big) \qquad (5.13)$$

with equality holds if $A = B, C = D$.

Proof. By setting $\alpha = \beta$ in Theorem 5.2.1, we obtain

$$\begin{aligned}
\big(\det(\alpha A+\alpha B)\big)^{nr}\big(\det(\alpha C+\alpha D)\big)^{ns} &\geq \det\big(\alpha(A\otimes C)^r + \alpha(B\otimes D)^s\big),\\
\alpha^{n^2 r}\big(\det(A+B)\big)^{nr}\alpha^{n^2 s}\big(\det(C+D)\big)^{ns} &\geq \alpha^{n^2}\det\big((A\otimes C)^r + (B\otimes D)^s\big),\\
\alpha^{n^2}\big(\det(A+B)\big)^{nr}\big(\det(C+D)\big)^{ns} &\geq \alpha^{n^2}\det\big((A\otimes C)^r + (B\otimes D)^s\big),\\
\big(\det(A+B)\big)^{nr}\big(\det(C+D)\big)^{ns} &\geq \det\big((A\otimes C)^r + (B\otimes D)^s\big).
\end{aligned}$$

The case of equality is same as Theorem 5.2.1. The proof is completed. ☐

Corollary 5.2.3. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\big(\det(\alpha A+\beta B)\,\det(\alpha C+\beta D)\big)^{\frac{n}{2}} \geq \det\big(\alpha(A\otimes C)^{\frac{1}{2}} + \beta(B\otimes D)^{\frac{1}{2}}\big) \qquad (5.14)$$

with equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Theorem 5.2.1, we get $r = 1/2$ and then

$$\begin{aligned}
\det\big(\alpha(A\otimes C)^{\frac{1}{2}} + \beta(B\otimes D)^{\frac{1}{2}}\big) &\leq \big(\det(\alpha A+\beta B)\big)^{\frac{n}{2}}\big(\det(\alpha C+\beta D)\big)^{\frac{n}{2}}\\
&= \big(\det(\alpha A+\beta B)\,\big(\det(\alpha C+\beta D)\big)^{\frac{n}{2}}.
\end{aligned}$$

For the case of equality, assume that $A = B, C = D$. We obtain from properties of determinant and Lemma 5.1.2 that

$$\begin{aligned}
\det\big(\alpha(A\otimes B)^{\frac{1}{2}} + \beta(A\otimes B)^{\frac{1}{2}}\big) &= \det\big((\alpha+\beta)(A\otimes B)^{\frac{1}{2}}\big)\\
&= (\alpha+\beta)^{n^2}\det\big((A\otimes B)^{\frac{1}{2}}\big)\\
&= (\alpha+\beta)^{\frac{n^2}{2}}(\alpha+\beta)^{\frac{n^2}{2}}\big(\det(A)\big)^{\frac{n}{2}}\big(\det(A)\big)^{\frac{n}{2}}\\
&= \big((\alpha+\beta)^{n}\det(A)\big)^{\frac{n}{2}}\big((\alpha+\beta)^{n}\det(B)\big)^{\frac{n}{2}}\\
&= \big(\det(\alpha A+\beta B)\big)^{\frac{n}{2}}\big(\det(\alpha A+\beta B)\big)^{\frac{n}{2}}\\
&= \big(\det(\alpha A+\beta B)\det(\alpha A+\beta B)\big)^{\frac{n}{2}}.
\end{aligned}$$

So the equality holds. The proof is completed. ☐

Corollary 5.2.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\left(\det(\alpha A + \beta B)\right)^n \geq \det\left(\alpha(A^r \otimes A^s) + \beta(B^r \otimes B^s)\right), \quad (5.15)$$

$$\left(\det(\alpha A + \beta B)\right)^{nr}\left(\det(\beta A + \alpha B)\right)^{ns} \geq \det\left(\alpha(A^r \otimes B^s) + \beta(B^r \otimes A^s)\right) \quad (5.16)$$

with equalities hold if $A = B$. The inequalities (5.15) *also becomes an equality when $n = 1$.*

Proof. By setting $A = C, B = D$ in Theorem 5.2.1, we get (5.15). By setting $A = D, B = C$ in Theorem 5.2.1, we get (5.16). For the case of equality, assume that $A = B$. We obtain from properties of determinant and Lemma 5.1.2 that

$$\begin{aligned}
\det\left(\alpha(A^r \otimes A^s) + \beta(A^r \otimes A^s)\right) &= \det\left((\alpha + \beta)(A^r \otimes A^s)\right) \\
&= (\alpha + \beta)^{n^2} \det(A^r \otimes A^s) \\
&= (\alpha + \beta)^{n^2}\left(\det(A^r)\right)^n\left(\det(A^s)\right)^n \\
&= (\alpha + \beta)^{n^2}\left(\det(A)\right)^{nr}\left(\det(A)\right)^{ns} \\
&= (\alpha + \beta)^{n^2}\left(\det(A)\right)^n \\
&= \left(\det(\alpha A + \beta A)\right)^n.
\end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$. Obviously, the inequality (5.15) also becomes an equality when $n = 1$. □

Corollary 5.2.5. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\left(\det(A + B)\ \det(C + D)\right)^{\frac{n}{2}} \geq \det\left((A \otimes C)^{\frac{1}{2}} + (B \otimes D)^{\frac{1}{2}}\right) \quad (5.17)$$

with equality holds if $A = B, C = D$.

Proof. Set $r = s$ in Corollary 5.2.2 or set $\alpha = \beta$ in Corollary 5.2.3. □

Corollary 5.2.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\begin{aligned}
\left(\det(A + B)\right)^n &\geq \det\left((A^r \otimes A^s) + (B^r \otimes B^s)\right), \quad &(5.18) \\
\left(\det(A + B)\right)^n &\geq \det\left((A^r \otimes B^s) + (B^r \otimes A^s)\right) \quad &(5.19)
\end{aligned}$$

with equalities hold if $A = B$. The inequalities (5.18) *also becomes an equality when $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 5.2.4. $\square$

Corollary 5.2.7. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\begin{aligned} \big(\det(\alpha A + \beta B)\big)^n &\geq \det\big(\alpha(A^{\frac{1}{2}})^{\otimes 2} + \beta(B^{\frac{1}{2}})^{\otimes 2}\big), && (5.20)\\ \big(\det(\alpha A + \beta B)\det(\beta A + \alpha B)\big)^{\frac{n}{2}} &\geq \det\big(\alpha(A \otimes B)^{\frac{1}{2}} + \beta(B \otimes A)^{\frac{1}{2}}\big) && (5.21) \end{aligned}$$

with equality holds if $A = B$. The inequalities (5.20) *also becomes an equality when $n = 1$.*

Proof. By setting $r = s$ in (5.15) in Corollary 5.2.4, we get $r = 1/2$ and

$$\begin{aligned} \big(\det(\alpha A + \beta B)\big)^n &\geq \det\big(\alpha(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \otimes B^{\frac{1}{2}})\big), \\ &= \det\big(\alpha(A^{\frac{1}{2}})^{\otimes 2} + \beta(B^{\frac{1}{2}})^{\otimes 2})\big). \end{aligned}$$

By setting $r = s$ in (5.16) in Corollary 5.2.4, we get $r = 1/2$ and

$$\begin{aligned} \big(\det(\alpha A + \beta B)\ \det(\beta A + \alpha B)\big)^{n/2} &\geq \det\big(\alpha(A^{\frac{1}{2}} \otimes B^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \otimes A^{\frac{1}{2}})\big), \\ &= \det\big(\alpha(A \otimes B)^{\frac{1}{2}} + \beta(B \otimes A)^{\frac{1}{2}}\big). \end{aligned}$$

The case of equalities is same as Corollary 5.2.4. $\square$

Corollary 5.2.8. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\begin{aligned} \big(\det(A + B)\big)^n &\geq \det\big((A^{\frac{1}{2}})^{\otimes 2} + (B^{\frac{1}{2}})^{\otimes 2}\big), && (5.22)\\ \big(\det(A + B)\big)^n &\geq \det\big((A \otimes B)^{\frac{1}{2}} + (B \otimes A)^{\frac{1}{2}}\big) && (5.23) \end{aligned}$$

with equality holds if $A = B$. The inequalities (5.22) *also becomes an equality when $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 5.2.7. $\square$

Remark 5.2.9.

1. For the inequality (5.23) in Corollary 5.2.8, when $n = 1$, let $A = [a]$ and $B = [b]$, we obtain

   $$\begin{aligned} a + b &\geq (ab)^{\frac{1}{2}} + (ba)^{\frac{1}{2}}, \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a + b). \end{aligned}$$

   This is the AM–GM inequality for real numbers.

2. For the inequality (5.23) in Corollary 5.2.8, if $A \otimes B = B \otimes A$, we obtain

$$\begin{aligned} \left(\det(A+B)\right)^n &\geq \det\left(2(A \otimes B)^{\frac{1}{2}}\right) \\ &= 2^{n^2} \det\left(A \otimes B\right)^{\frac{1}{2}} \\ &= 2^{n^2}\left(\det(A^{\frac{1}{2}})\right)^n \left(\det(B^{\frac{1}{2}})\right)^n, \\ \det(A+B) &\geq 2^n \det(A^{\frac{1}{2}}) \det(B^{\frac{1}{2}}). \end{aligned}$$

Then by replacing A with A^2 and B with B^2, we get

$$2^n \det(A) \det(B) \leq \det(A^2+B^2).$$

This is the inequality (5.8) in Corollary 5.1.5.

Observation 5.2.10.

Let $A \in \mathbb{M}_n$. Then for any positive integer k,

$$\det(A^{\otimes k}) = \left(\det(A)\right)^{nk}. \tag{5.24}$$

Proof. Since $\det(A \otimes A) = \left(\det(A)\right)^n \left(\det(A)\right)^n$, we get

$$\det(A^{\otimes 2}) = \left(\det(A)\right)^n \left(\det(A)\right)^n.$$

It follows inductively that, for any positive integer k,

$$\begin{aligned} \det(A^{\otimes k}) &= \det(\underbrace{A \otimes A \otimes \cdots \otimes A}_{k}) \\ &= \left(\det(A)\right)^n \left(\det(A)\right)^n \cdots \left(\det(A)\right)^n \\ &= \left(\left(\det(A)\right)^n\right)^k \\ &= \left(\det(A)\right)^{nk}. \end{aligned}$$

The proof is completed. □

5.3 Inequalities of Determinant Involving the Hadamard Product of Power of Linear Combination of Matrices

In this section, inequalities of determinant involving the Hadamard product of power of linear combination of matrices are presented. Theorem 5.3.1 is the main result. Special cases of this theorem are also considered.

Theorem 5.3.1. *Let* $A, B, C, D \in \mathbb{M}_n$ *be positive definite symmetric and let* $\alpha, \beta, r, s > 0$ *be scalars such that* $r + s = 1$. *Then*

$$\det\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) \quad \geq \quad \det\left(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)\right) \qquad (5.25)$$

with equality holds if $A = B, C = D$.

Proof. Since $A, B. C, D$ are positive definite and α, β are positive, by Schur product theorem (Theorem 2.8.2), we have $(\alpha A+\beta B)^r \circ (\alpha C+\beta B)^s$ and $\alpha(A^r \circ C^s)+\beta(B^r \circ D^s)$ are also positive definite.
Then by applying Theorem 2.6.4 to Theorem 3.3.1.3, we get (5.25).
For the case of equality, assume that $A = B, C = D$.
We obtain from the properties of the Hadamard product that

$$\begin{aligned}
\det\left((\alpha A + \beta A)^r \circ (\alpha C + \beta C)^s\right) &= \det\left((\alpha + \beta)^r A^r \circ (\alpha + \beta)^s C^s\right) \\
&= \det\left((\alpha + \beta)^r (\alpha + \beta)^s (A^r \circ C^s)\right) \\
&= \det\left((\alpha + \beta)(A^r \circ C^s)\right) \\
&= \det\left(\alpha(A^r \circ C^s) + \beta(A^s \circ C^s)\right).
\end{aligned}$$

So the equality holds. The proof is completed. □

From the main result (Theorem 5.3.1), there are many special cases. There are 3 types of them: the equality between the matrices; the equality between the coefficients (scalars); and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C,\ B = D$;
2. the case when $A = D,\ B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

Combining all of them, there are 16 possible cases (which include the general case). The details for each case are shown in Table 5.2 (except for the general case).

Table 5.2: The Special Cases of Theorem 5.3.1

Conditions			Results	
A, B, C, D	r, s	α, β	Corollary	Inequality
-	-	$\alpha = \beta$	5.3.2	5.26
-	$r = s$	-	5.3.3	5.27
-	$r = s$	$\alpha = \beta$	5.3.5	5.30
$A = C, B = D$	-	-	5.3.4	5.28
$A = C. B = D$	-	$\alpha = \beta$	5.3.6	5.29
$A = C, B = D$	$r = s$	-	5.3.7	5.31
$A = C, B = D$	$r = s$	$\alpha = \beta$	5.3.8	5.35
$A = D, B = C$	-	-	5.3.4	5.29
$A = D, B = C$	-	$\alpha = \beta$	5.3.6	5.32
$A = D, B = C$	$r = s$	-	5.3.7	5.34
$A = D, B = C$	$r = s$	$\alpha = \beta$	5.3.8	5.36
$A = B = C = D$	-	-	Equality Case	
$A = B = C = D$	-	$\alpha = \beta$	Equality Case	
$A = B = C = D$	$r = s$	-	Equality Case	
$A = B = C = D$	$r = s$	$\alpha = \beta$	Equality Case	

We summarize these cases into 7 corollaries as follows (depend on the hypotheses).

Corollary 5.3.2. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\det\left((A+B)^r \circ (C+D)^s\right) \quad \geq \quad \det\left((A^r \circ C^s) + (B^r \circ D^s)\right) \tag{5.26}$$

with equality holds if $A = B, C = D$.

Proof. By setting $\alpha = \beta$ in Theorem 5.3.1, we obtain

$$\begin{aligned}
\det\left((\alpha A + \alpha B)^r \circ (\alpha C + \alpha D)^s\right) &\geq \det\left(\alpha(A^r \circ C^s) + \alpha(B^s \circ D^s)\right), \\
\det\left(\alpha^r (A+B)^r \circ \alpha^s (C+D)^s\right) &\geq \det\left(\alpha(A^r \circ C^s) + \alpha(B^s \circ D^s)\right), \\
\det\left(\alpha^r \alpha^s ((A+B)^r \circ (C+D)^s)\right) &\geq \det\left(\alpha((A^r \circ C^s) + (B^s \circ D^s))\right), \\
\alpha^n \det\left((A+B)^r \circ (C+D)^s\right) &\geq \alpha^n \det\left((A^r \circ C^s) + (B^s \circ D^s)\right), \\
\det\left((A+B)^r \circ (C+D)^s\right) &\geq \det\left((A^r \circ C^s) + (B^r \circ D^s)\right).
\end{aligned}$$

The case of equality is same as Theorem 5.3.1. The proof is completed. □

Corollary 5.3.3. *Let* $A, B, C, D \in \mathbb{M}_n$ *be positive definite symmetric and let* $\alpha, \beta > 0$ *be scalars. Then*

$$\det\left((\alpha A+\beta B)^{\frac{1}{2}} \circ (\alpha C+\beta D)^{\frac{1}{2}}\right) \quad \geq \quad \det\left(\alpha(A^{\frac{1}{2}} \circ C^{\frac{1}{2}})+\beta(B^{\frac{1}{2}} \circ D^{\frac{1}{2}})\right) \quad (5.27)$$

with equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Theorem 5.3.1, we get $r = 1/2$ and then (5.27).
For the case of equality, assume that $A = B, C = D$.
We obtain from the properties of the Hadamard product that

$$\begin{aligned}
\det\left((\alpha A+\beta A)^{\frac{1}{2}} \circ (\alpha C+\beta C)^{\frac{1}{2}}\right) &= \det\left((\alpha+\beta)^{\frac{1}{2}}A^{\frac{1}{2}} \circ (\alpha+\beta)^{\frac{1}{2}}C^{\frac{1}{2}}\right) \\
&= \det\left((\alpha+\beta)^{\frac{1}{2}}(\alpha+\beta)^{\frac{1}{2}}(A^{\frac{1}{2}} \circ C^{\frac{1}{2}})\right) \\
&= \det\left((\alpha+\beta)(A^{\frac{1}{2}} \circ C^{\frac{1}{2}})\right) \\
&= \det\left(\alpha(A^{\frac{1}{2}} \circ C^{\frac{1}{2}})+\beta(A^{\frac{1}{2}} \circ C^{\frac{1}{2}})\right).
\end{aligned}$$

So the equality holds. The proof is completed. □

Corollary 5.3.4. *Let* $A, B \in \mathbb{M}_n$ *be positive definite symmetric and let* $\alpha, \beta, r, s > 0$ *be scalars such that* $r + s = 1$. *Then*

$$\det\left((\alpha A+\beta B)^r \circ (\alpha A+\beta B)^s\right) \;\geq\; \det\left(\alpha(A^r \circ A^s)+\beta(B^r \circ B^s)\right), \quad (5.28)$$

$$\det\left((\alpha A+\beta B)^r \circ (\beta A+\alpha B)^s\right) \;\geq\; \det\left(\alpha(A^r \circ B^s)+\beta(A^s \circ B^r)\right) \quad (5.29)$$

with equalities hold if $A = B, C = D$. *The equality* (5.28) *also becomes an equality if* $n = 1$.

Proof. By setting $A = C$, $B = D$ in Theorem 5.3.1, we get (5.28).
By setting $A = D$, $B = C$ in Theorem 5.3.1, we get (5.29).
For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned}
\det\left((\alpha A+\beta A)^r \circ (\alpha A+\beta A)^s\right) &= \det\left((\alpha+\beta)^r A^r \circ (\alpha+\beta)^s A^s\right) \\
&= \det\left((\alpha+\beta)^r(\alpha+\beta)^s(A^r \circ A^s)\right) \\
&= \det\left((\alpha+\beta)(A^r \circ A^s)\right) \\
&= \det\left(\alpha(A^r \circ A^s)+\beta(A^r \circ A^s)\right).
\end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the equality (5.28) also becomes an equality when $n = 1$. □

Corollary 5.3.5. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\det\left((A+B)^{\frac{1}{2}} \circ (C+D)^{\frac{1}{2}}\right) \geq \det\left((A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + (B^{\frac{1}{2}} \circ D^{\frac{1}{2}})\right) \tag{5.30}$$

with equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Corollary 5.3.2, we get $r = 1/2$ and then (5.30). The case of equality is same as Corollary 5.3.2. □

Corollary 5.3.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\det\left((A+B)^r \circ (A+B)^s\right) \geq \det\left((A^r \circ A^s) + (B^r \circ B^s)\right), \tag{5.31}$$

$$\det\left((A+B)^r \circ (A+B)^s\right) \geq \det\left((A^r \circ B^s) + (A^s \circ B^r)\right) \tag{5.32}$$

with equalities hold if $A = B$. The inequality (5.31) *also becomes an equality if $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 5.3.4. □

Corollary 5.3.7. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\det\left(\left((\alpha A + \beta B)^{\frac{1}{2}}\right)^{(2)}\right) \geq \det\left(\alpha(A^{\frac{1}{2}})^{(2)} + \beta(B^{\frac{1}{2}})^{(2)}\right), \tag{5.33}$$

$$\det\left((\alpha A + \beta B)^{\frac{1}{2}} \circ (\beta A + \alpha B)^{\frac{1}{2}}\right) \geq \det\left(\alpha(A^{\frac{1}{2}})^{(2)} + \beta(B^{\frac{1}{2}})^{(2)}\right) \tag{5.34}$$

with equalities hold if $A = B$. The inequality (5.33) *also becomes an equality if $n = 1$.*

Proof. By setting $r = s$ in (5.28) and (5.29) in Corollary 5.3.4, we get (5.33) and (5.34), respectively. For the inequality (5.33), if $A = B$, we obtain

$$\begin{aligned}
\det\left(\left((\alpha A + \beta A)^{\frac{1}{2}}\right)^{(2)}\right) &= \det\left(\left((\alpha+\beta)^{\frac{1}{2}} A^{\frac{1}{2}}\right)^{(2)}\right) \\
&= \det\left(\left((\alpha+\beta)^{\frac{1}{2}}\right)^2 (A^{\frac{1}{2}})^{(2)}\right) \\
&= \det\left((\alpha+\beta)(A^{\frac{1}{2}})^{(2)}\right) \\
&= \det\left(\alpha(A^{\frac{1}{2}})^{(2)} + \beta(A^{\frac{1}{2}})^{(2)}\right).
\end{aligned}$$

So the equality holds. Similarly, the others one hold if $A = B$.
Obviously, the inequality (5.33) becomes an equality when $n = 1$. □

Corollary 5.3.8. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\det\left(\left((A+B)^{\frac{1}{2}}\right)^{(2)}\right) \geq \det\left((A^{\frac{1}{2}})^{(2)} + (B^{\frac{1}{2}})^{(2)}\right), \tag{5.35}$$

$$\det(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) \leq \frac{1}{2^n} \det\left(\left((A+B)^{\frac{1}{2}}\right)^{(2)}\right), \tag{5.36}$$

with equalities hold if $A = B$. The inequality (5.35) *also becomes an equality if $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 5.3.7. Obviously, the inequality (5.35) becomes an equality when $n = 1$. □

Remark 5.3.9.

For the inequality (5.36) in Corollary 5.3.8, when $n = 1$, let $A = [a]$ and $B = [b]$, we obtain

$$\begin{aligned} (a^{\frac{1}{2}}b^{\frac{1}{2}}) &\leq \frac{1}{2}((a+b)^{\frac{1}{2}})^2, \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a+b). \end{aligned}$$

This is the AM–GM inequality for real numbers.

Observation 5.3.10.

Let $A \in \mathbb{M}_n$ be positive definite. Then for any positive integer k,

$$\det(A^{(k)}) \geq (\det(A))^k \tag{5.37}$$

with equality holds if $n = 1$ or $k = 1$.

Proof. Hadamard's inequality (Theorem 2.3.5) and Oppenheim's inequality (Theorem 2.5.7) imply

$$\det(A \circ A) \geq \det(A)\det(A).$$

It follows inductively that, for any positive integer k,

$$\begin{aligned} \det(A^{(k)}) &= \det(\underbrace{A \circ A \circ \cdots \circ A}_{k}) \\ &\geq \det(A)\,\det(A)\cdots\det(A) \\ &= (\det(A))^k. \end{aligned}$$

Obviously, the equality holds when $n = 1$ or $k = 1$. □

5.4 Numerical Examples

In this section, we give some numerical examples to support our main results. The tool we use is the mathematical program which is called Maple 9.5.

Example 5.4.1.

Consider the inequality of determinant involving the 2nd power of linear combination of the given matrices

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

in the case when $\alpha = 1, \ \beta = 2$.

We see that A and B are positive definite symmetric and α and β satisfy the hypothesis of Theorem 5.1.1. We get

$$\begin{aligned} \alpha A + \beta B &= \begin{bmatrix} 6 & 5 \\ 5 & 2 \end{bmatrix}, \\ \alpha A^2 + \beta B^2 &= \begin{bmatrix} 21 & 32 \\ 32 & 63 \end{bmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{(\alpha+\beta)^n}\big(\det(\alpha A + \beta B)\big)^2 &\approx 254.444, \\ \det\big(\alpha(A^2) + \beta(B^2)\big) &= 299. \end{aligned}$$

Therefore,

$$\frac{1}{(\alpha+\beta)^n}\big(\det(\alpha A + \beta B)\big)^2 \leq \det\big(\alpha(A^2) + \beta(B^2)\big).$$

This example supports the result of Theorem 4.1.1. □

Example 5.4.2.

Consider the inequality of determinant involving the Kronecker product of power of linear combination of the given matrices

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha = 1,\ \beta = 2,\ r = 2/3,\ s = 1/3$.

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 5.2.1. We get

$$\alpha A + \beta B = \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}, \alpha C + \beta D = \begin{bmatrix} 4 & -4 \\ -4 & 14 \end{bmatrix},$$

$$\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s) \approx \begin{bmatrix} 5.825 & -1.156 & 0.893 & 0.466 \\ -1.156 & 8.713 & 0.466 & -0.275 \\ 0.893 & 0.466 & 2.963 & -0.688 \\ 0.466 & -0.275 & -0.688 & 4.683 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \det(\alpha A + \beta B) &= 35, \\ \det(\alpha C + \beta D) &= 40, \\ \left(\det(\alpha A + \beta B)\right)^{nr} \left(\det(\alpha C + \beta D)\right)^{ns} &\approx 1339.052, \\ \det\left(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)\right) &\approx 609.600. \end{aligned}$$

Thus,

$$\left(\det(\alpha A + \beta B)\right)^{nr} \left(\det(\alpha C + \beta D)\right)^{ns} \geq \det\left(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)\right).$$

This example supports the result of Theorem 5.2.1. □

Example 5.4.3.

Consider the inequality of determinant involving the Hadamard product of power of linear combination of the given matrices

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha = 2,\ \beta = 1,\ r = 1/3,\ s = 2/3.$

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 5.2.1. We get

$$(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s \approx \begin{bmatrix} 6.372 & -0.313 \\ -0.313 & 7.262 \end{bmatrix},$$

$$\alpha(A^r \circ C^s) + \beta(B^r \circ D^s) \approx \begin{bmatrix} 5.767 & 0.238 \\ 0.238 & 5.788 \end{bmatrix}.$$

Hence,

$$\det\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) \approx 46.175,$$
$$\det\left(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)\right) \approx 33.323.$$

Thus,

$$\det\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) \geq \det\left(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)\right).$$

This examples supports the result of Theorem 5.3.1. □

Example 5.4.4.

Consider the inequality of determinant involving the Hadamard product of power of linear combination of the given matrices

$$A = \begin{bmatrix} \sqrt{3} & 1/2 & -1 \\ 1/2 & \sqrt{7}-1 & 0.851 \\ -1 & 0.851 & 2\sqrt{2} \end{bmatrix}, \; B = \begin{bmatrix} \sqrt{7} & 17/4 & 1 \\ 17/4 & 6\sqrt{6}-4 & 3.851 \\ 1 & 3.851 & 3\sqrt{3} \end{bmatrix},$$

$$C = \begin{bmatrix} \pi & 2 & 0 \\ 2 & 5.219 & \sqrt[4]{29} \\ 0 & \sqrt[4]{29} & 8 \end{bmatrix}, \; D = \begin{bmatrix} e & -2 & -0.1 \\ -2 & 4.567 & \sqrt[3]{17} \\ -0.1 & \sqrt[3]{17} & 8 \end{bmatrix}$$

in the case when $\alpha = 4, \; \beta = 3, \; r = 3/4, \; s = 1/4$.

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 5.3.1. We get

$$(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s \approx \begin{bmatrix} 14.442 & 0.233 & 0.010 \\ 0.233 & 35.337 & 1.248 \\ 0.010 & 1.248 & 30.944 \end{bmatrix},$$

$$\alpha(A^r \circ C^s) + \beta(B^r \circ D^s) \approx \begin{bmatrix} 13.868 & -1.113 & 0.108 \\ -1.113 & 31.479 & 1.306 \\ 0.108 & 1.306 & 30.442 \end{bmatrix}.$$

Hence,

$$\det\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) \approx 25697.513,$$

$$\det\left(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)\right) \approx 21312.460.$$

Thus,

$$\det\left((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s\right) \geq \det\left(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)\right).$$

This examples supports the result of Theorem 5.3.1. □

CHAPTER 6

INEQUALITIES OF EIGENVALUE

In this chapter, inequalities of eigenvalue for positive definite symmetric matrices are presented. These inequalities are shown in Section 6.1–Section 6.3. Some numerical examples in Section 6.4 support our results.

To prove these inequalities, we use properties of eigenvalue in Theorem 2.1.8 and Corollary 2.1.10 and Theorem 2.6.4 which is stated that for any positive definite matrices $A, B \in \mathbb{M}_n$,

$$A \geq B \quad \Longrightarrow \quad \lambda_i(A) \geq \lambda_i(B)$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of A and B are arranged in the same order.

Since eigenvalues of positive definite symmetric matrices are always real (in fact, they are positive), so we can compare them.

6.1 Inequalities of Eigenvalue Involving the 2nd Power of Linear Combination of Two Matrices

In this section, we prove inequalities of eigenvalue involving the 2nd power of linear combination of two matrices. The main result is Theorem 6.1.1. The consequent results and special cases are also considered.

Theorem 6.1.1. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\frac{1}{\alpha+\beta}\left(\lambda_i(\alpha A + \beta B)\right)^2 \quad \leq \quad \lambda_i(\alpha A^2 + \beta B^2) \tag{6.1}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B$.

Proof. Since A and B are positive definite and α and β are positive, so $\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2$ and $\alpha A^2+\beta B^2$ are also positive definite. By applying Theorem 2.6.4 to Theorem 4.1.1 and using properties of eigenvalue, we obtain

$$\begin{aligned}\lambda_i(\alpha A^2+\beta B^2) &\geq \lambda_i\left(\frac{1}{\alpha+\beta}(\alpha A+\beta B)^2\right)\\ &= \frac{1}{\alpha+\beta}\lambda_i\left((\alpha A+\beta B)^2\right)\\ &= \frac{1}{\alpha+\beta}\left(\lambda_i(\alpha A+\beta B)\right)^2\end{aligned}$$

for all $i = 1,2,\ldots,n$ if the respective eigenvalues of them are arranged in the same order.

For the case of equality, assume that $A = B$.

We obtain from the properties of eigenvalue that

$$\begin{aligned}\frac{1}{\alpha+\beta}\left(\lambda_i(\alpha A+\beta A)\right)^2 &= \frac{1}{\alpha+\beta}\left(\lambda_i((\alpha+\beta)A)\right)^2\\ &= \frac{(\alpha+\beta)^2}{\alpha+\beta}\lambda_i(A^2)\\ &= (\alpha+\beta)\lambda_i(A^2)\\ &= \lambda_i(\alpha A^2+\beta A^2).\end{aligned}$$

So the equality holds. The proof is completed. □

Lemma 6.1.2. *Let* $A \in \mathrm{M}_n$.

1. *If A are positive definite, then for any real number* r,

$$\lambda_i(A^r) = \left(\lambda_i(A)\right)^r \tag{6.2}$$

for all $i = 1,2,\ldots,n$ *if the respective eigenvalues of them are arranged in the same order.*

2. *If A are positive semidefinite, then for any nonnegative real number* s,

$$\lambda_i(A^s) = \left(\lambda_i(A)\right)^s \tag{6.3}$$

for all $i = 1,2,\ldots,n$ *if the respective eigenvalues of them are arranged in the same order.*

Proof. Assume that A is positive definite. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be eigenvalues of A. Since A is Hermitian, there exists a unitary matrix U such that

$$A = U\,diag[\lambda_1,\lambda_2,\ldots,\lambda_n]\,U^*.$$

Hence,

$$A^r = U\, diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^*.$$

For each $i = 1, 2, \dots, n$, let μ_i be eigenvalues of A^r with associated eigenvectors x_i. It follows that

$$\begin{aligned} U\, diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^* x_i &= \mu_i x_i\,, \\ (U\, diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^* - \mu_i I) x_i &= \mathbf{0}. \end{aligned}$$

Hence, $U\, diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^* - \mu_i I$ is nonsingular and consequently,

$$\begin{aligned} \det\left(U\, diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^* - \mu_i I\right) &= 0\,, \\ \det(U^*) \det\left(U\, diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^* - \mu_i I\right) \det(U) &= 0\,, \\ \det\left(U^* U\, diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^* U - \mu_i U^* U\right) &= 0\,, \\ \det\left(diag[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r] - \mu_i I\right) &= 0\,, \\ (\lambda_1^r - \mu_i)(\lambda_2^r - \mu_i) \cdots (\lambda_n^r - \mu_i) &= 0. \end{aligned}$$

That is for each i, $\mu_i = \lambda_j^r$ for some $j = 1, 2, \dots, n$.
So if the eigenvalues of A and B are arranged in the same order, we get

$$\lambda_i(A^r) = (\lambda_i(A))^r$$

for all $i = 1, 2, \dots, n$.
Now assume that A is positive semidefinite.
Analogously, for any nonnegative real number s, we get

$$\lambda_i(A^s) = (\lambda_i(A))^s$$

for all $i = 1, 2, \dots, n$ if the respective eigenvalues of them are arranged in the same order.
The proof is completed. □

Corollary 6.1.3. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\frac{1}{\alpha+\beta}\left(\lambda_i(\alpha A^{\frac{1}{2}} + \beta B^{\frac{1}{2}})\right)^2 \leq \lambda_i(\alpha A + \beta B) \leq (\alpha+\beta)^{\frac{1}{2}}\left(\lambda_i(\alpha A^2 + \beta B^2)\right)^{\frac{1}{2}} \quad (6.4)$$

for all $i = 1, 2, \dots, n$ if the respective eigenvalues of them are arranged in the same order. The equalities hold if $A = B$.

Proof. By replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in Theorem 6.1.1, we get

$$\frac{1}{\alpha+\beta}\left(\lambda_i(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})\right)^2 \quad\leq\quad \lambda_i(\alpha A+\beta B) \tag{6.5}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order.

For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned}
\frac{1}{\alpha+\beta}\left(\lambda_i(\alpha A^{\frac{1}{2}}+\beta A^{\frac{1}{2}})\right)^2 &= \frac{1}{\alpha+\beta}\Big(\lambda_i\big((\alpha+\beta)A^{\frac{1}{2}}\big)\Big)^2\\
&= \frac{1}{\alpha+\beta}\lambda_i\big((\alpha+\beta)^2 A\big)\\
&= (\alpha+\beta)\lambda_i(A)\\
&= \lambda_i(\alpha A+\beta B).
\end{aligned}$$

So the equality holds.

Then by taking square root in (6.1), we get

$$\lambda_i(\alpha A+\beta B) \quad\leq\quad (\alpha+\beta)^{\frac{1}{2}}\Big(\lambda_i(\alpha A^2+\beta B^2)\Big)^{\frac{1}{2}} \tag{6.6}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order.

Combining (6.5) and (6.6), we get (6.4).

For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned}
(\alpha+\beta)^{\frac{1}{2}}\Big(\lambda_i(\alpha A^2+\beta A^2)\Big)^{\frac{1}{2}} &= (\alpha+\beta)^{\frac{1}{2}}\Big(\lambda_i\big((\alpha+\beta)A^2\big)\Big)^{\frac{1}{2}}\\
&= (\alpha+\beta)^{\frac{1}{2}}\lambda_i\big((\alpha+\beta)^{\frac{1}{2}}A\big)\\
&= (\alpha+\beta)\lambda_i(A)\\
&= \lambda_i(\alpha A+\beta A).
\end{aligned}$$

So the equality holds.

This corollary can be alternatively proved as follows.

By applying Theorem 2.6.4 to Corollary 3.1.1.4, we obtain

$$\begin{aligned}
\lambda_i\Big(\frac{1}{\alpha+\beta}(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\Big) &\leq \lambda_i(\alpha A+\beta B) \leq \lambda_i\Big((\alpha+\beta)^{\frac{1}{2}}(\alpha A^2+\beta B^2)^{\frac{1}{2}}\Big),\\
\frac{1}{\alpha+\beta}\lambda_i\big((\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})^2\big) &\leq \lambda_i(\alpha A+\beta B) \leq (\alpha+\beta)^{\frac{1}{2}}\lambda_i\Big((\alpha A^2+\beta B^2)^{\frac{1}{2}}\Big),\\
\frac{1}{\alpha+\beta}\left(\lambda_i(\alpha A^{\frac{1}{2}}+\beta B^{\frac{1}{2}})\right)^2 &\leq \lambda_i(\alpha A+\beta B) \leq (\alpha+\beta)^{\frac{1}{2}}\Big(\lambda_i(\alpha A^2+\beta B^2)\Big)^{\frac{1}{2}}
\end{aligned}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. □

Corollary 6.1.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\left(\lambda_i(A+B)\right)^2 \quad \leq \quad 2\lambda_i(A^2+B^2) \tag{6.7}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B$.

Proof. By setting $\alpha = \beta$ in Theorem 6.1.1, we obtain

$$\begin{aligned} \frac{1}{2\alpha}\left(\lambda_i(\alpha A+\alpha B)\right)^2 &\leq \lambda_i(\alpha A^2+\alpha B^2), \\ \frac{\alpha^2}{2\alpha}\left(\lambda_i(A+B)\right)^2 &\leq \alpha\lambda_i(A^2+B^2), \\ \left(\lambda_i(A+B)\right)^2 &\leq 2\lambda_i(A^2+B^2). \end{aligned}$$

The case of equality is same as Theorem 6.1.1. □

Corollary 6.1.5. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. If AB is positive definite, then*

$$2\lambda_i(AB) \quad \leq \quad \lambda_i(A^2+B^2) \tag{6.8}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B$.

Proof. Since AB are positive definite, from Observation 3.1.11, we have $AB = BA$. It follows from the inequality (3.5) in Corollary 3.1.2.3 that

$$2AB \quad \leq \quad A^2+B^2.$$

Since $2AB$ and A^2+B^2 are positive definite, by applying Theorem 2.6.4 to this inequality, we get

$$\begin{aligned} \lambda_i(A^2+B^2) &\geq \lambda_i(2AB) \\ &= 2\lambda_i(AB) \end{aligned}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order.

It is easy to see that the equality holds when $A = B$. □

Corollary 6.1.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\frac{1}{2}(\lambda_i(A^{\frac{1}{2}} + B^{\frac{1}{2}}))^2 \;\leq\; \lambda_i(A+B) \;\leq\; \sqrt{2}\Big(\lambda_i(A^2+B^2)\Big)^{\frac{1}{2}} \tag{6.9}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order.

If, in addition, AB is positive definite, then

$$2(\lambda_i(AB))^{\frac{1}{2}} \quad\leq\quad \lambda_i(A+B) \tag{6.10}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order.

All equalities hold if $A = B$.

Proof. By setting $\alpha = \beta$ in Theorem 6.1.3, we get

$$\begin{aligned}
\frac{1}{2\alpha}(\lambda_i(\alpha A^{\frac{1}{2}} + \alpha B^{\frac{1}{2}}))^2 \;&\leq\; \lambda_i(\alpha A + \alpha B) \;\leq\; (2\alpha)^{\frac{1}{2}}\Big(\lambda_i(\alpha A^2 + \alpha B^2)\Big)^{\frac{1}{2}},\\
\frac{1}{2\alpha}\Big(\lambda_i(\alpha(A^{\frac{1}{2}} + B^{\frac{1}{2}}))\Big)^2 \;&\leq\; \alpha\lambda_i(A+B) \;\leq\; \sqrt{2}\alpha^{\frac{1}{2}}\Big(\lambda_i(\alpha(A^2+B^2))\Big)^{\frac{1}{2}},\\
\frac{\alpha^2}{2\alpha}\Big(\lambda_i(A^{\frac{1}{2}} + B^{\frac{1}{2}}\Big)^2 \;&\leq\; \alpha\lambda_i(A+B) \;\leq\; \sqrt{2}\alpha(\lambda_i(A^2+B^2))^{\frac{1}{2}},\\
\frac{1}{2}(\lambda_i(A^{\frac{1}{2}} + B^{\frac{1}{2}}))^2 \;&\leq\; \lambda_i(A+B) \;\leq\; \sqrt{2}(\lambda_i(A^2+B^2))^{\frac{1}{2}}.
\end{aligned}$$

The equality case is same as Corollary 6.1.3.

By replacing A with $A^{\frac{1}{2}}$ and B with $B^{\frac{1}{2}}$ in the inequality (6.8) in Corollary 6.1.5, we obtain

$$\begin{aligned}
\lambda_i(A+B) \quad&\geq\quad \lambda_i(2A^{\frac{1}{2}}B^{\frac{1}{2}})\\
&=\quad 2\lambda_i((AB)^{\frac{1}{2}})\\
&=\quad 2(\lambda_i(AB))^{\frac{1}{2}}.
\end{aligned}$$

The case of quality is same as Corollary 6.1.5. □

Remark 6.1.7.

1. When $n = 1$, let $A = [a]$ and $B = [b]$, the inequality (6.10) becomes

$$\begin{aligned}
2(ab)^{\frac{1}{2}} \quad&\leq\quad a+b,\\
(ab)^{\frac{1}{2}} \quad&\leq\quad \frac{1}{2}(a+b)
\end{aligned}$$

with equality holds if and only if $a = b$. This is the AM–GM inequality for positive real numbers. Analogously, we can also get this inequality from any inequality in Corollary 6.1.4, Corollary 6.1.5 and Corollary 6.1.6.

2. Corollary 6.1.6 shows an upper bound and some lower bounds of $\lambda_i(A + B)$.

3. Remember that the eigenvalues of any positive definite matrices are always positive real numbers.

4. Note that for positive definite matrices A and B, AB is not necessarily positive definite (in fact, it is not necessarily Hermitian) but $\lambda(AB)$ is always real and positive since

$$\lambda(AB) = \lambda(AB^{\frac{1}{2}}B^{\frac{1}{2}}) = \lambda(B^{\frac{1}{2}}AB^{\frac{1}{2}}) > 0.$$

A generalization of this statement is Observation 6.1.8.

Observation 6.1.8.

Let $A, B \in \mathbb{M}_n$.

1. If A and B are positive definite (not necessarily symmetric, i.e., not necessarily real), then for any real numbers r and s, $\lambda(A^rB^s)$ is real and positive.

2. If A and B are positive semidefinite (not necessarily symmetric, i.e., not necessarily real), then for any nonnegative real numbers r and s, $\lambda(A^rB^s)$ is real and nonnegative.

3. If A and B are nonsingular Hermitian (not necessarily real), then for any even integers p and q, $\lambda(A^pB^q)$ is real and positive.

4. If A and B are Hermitian (not necessarily real), then for any even nonnegative integers p and q, $\lambda(A^pB^q)$ is real and nonnegative.

Proof. Assume that A and B are positive definite. We obtain

$$\begin{aligned}\lambda(A^rB^s) &= \lambda(A^rB^{\frac{s}{2}}B^{\frac{s}{2}}) \\ &= \lambda(B^{\frac{s}{2}}A^rB^{\frac{s}{2}}) \\ &> 0.\end{aligned}$$

Now assume that A and B are positive semidefinite.
Analogously, for any nonnegative real number r and s, $\lambda(A^rB^s)$ is real and nonnegative.
Notice that in this case A^r and B^s exist if r and s are nonnegative.
The cases of nonsingular Hermitian and Hermitian can be proved in analogous ways. □

6.2 Inequalities of Eigenvalue Involving the Kronecker Product of Power of Linear Combination of Matrices

In this section, we present inequalities of eigenvalue involving the Kronecker product of power of linear combination of matrices. First we establish Theorem 6.2.1 which is the main result. Then we state a sequence of corollaries which is come from special cases of this theorem.

Theorem 6.2.1. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$(\lambda_i(\alpha A + \beta B))^r \, (\lambda_j(\alpha C + \beta D))^s \quad \geq \quad \lambda_k(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)) \tag{6.11}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
The equality holds if $A = B, C = D$.

Proof. Since A and B are positive definite and α and β are positive, so $(\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s$ and $\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)$ are also positive definite.
Then by applying Theorem 2.6.4 to Theorem 3.2.1.5 and using properties of eigenvalue and Lemma 6.1.2, we obtain

$$\begin{aligned}
\lambda_k(\alpha(A \otimes B)^r + \beta(A \otimes B)^s) &\leq \lambda_k((\alpha A + \beta B)^r \otimes (\alpha C + \beta D)^s) \\
&= \lambda_i((\alpha A + \beta B)^r) \, \lambda_j((\alpha C + \beta D)^s) \\
&= (\lambda_i(\alpha A + \beta B))^r \, (\lambda_j(\alpha C + \beta D))^s
\end{aligned}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \cdots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
For the case of equality, assume that $A = B, C = D$.
We obtain from properties of eigenvalue and Lemma 6.1.2 that

$$\begin{aligned}
(\lambda_i(\alpha A + \beta A))^r (\lambda_j(\alpha C + \beta C))^s &= \Big(\lambda_i((\alpha + \beta)A)\Big)^r \Big(\lambda_j((\alpha + \beta)C)\Big)^s \\
&= \lambda_i\Big(((\alpha + \beta)A)^r\Big) \, \lambda_j\Big(((\alpha + \beta)C)^s\Big) \\
&= \lambda_k\Big(((\alpha + \beta)A)^r \otimes ((\alpha + \beta)C)^s\Big) \\
&= \lambda_k((\alpha + \beta)^r A^r \otimes (\alpha + \beta)^s C^s) \\
&= \lambda_k((\alpha + \beta)^r (\alpha + \beta)^s (A^r \otimes C^s)) \\
&= \lambda_k(\alpha(A^r \otimes C^s) + \beta(A^r \otimes C^s)).
\end{aligned}$$

So the equality holds. The proof is completed. □

Remark 6.2.2.

Note that for the inequality (6.11) in Theorem 6.2.1, on the left hand side there are n^2 possible products of the eigenvalues. What about for the right hand side? Since the size of $\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)$ is n^2, there are n^2 eigenvalues (counting multiplicities). So each side has the same number of real numbers to compare.

From the main result (Theorem 6.2.1), there are many special cases. There are 3 types of them: the equality between the matrices; the equality between the coefficients (scalars); and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C$, $B = D$;
2. the case when $A = D$, $B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

Combining all of them, there are 16 possible cases (which include the general case). The details for each case are shown in Table 6.1 (except for the general case).

Table 6.1: The Special Cases of Theorem 6.2.1

Conditions			Results	
A, B, C, D	$r.s$	α, β	Corollary	Inequality
-	-	$\alpha = \beta$	6.2.3	6.12
-	$r = s$	-	6.2.4	6.13
-	$r = s$	$\alpha = \beta$	6.2.6	6.16
$A = C.B = D$	-	-	6.2.5	6.14
$A = C, B = D$	-	$\alpha = \beta$	6.2.7	6.17
$A = C, B = D$	$r = s$	-	6.2.8	6.19
$A = C, B = D$	$r = s$	$\alpha = \beta$	6.2.9	6.21
$A = D, B = C$	-	-	6.2.5	6.15
$A = D, B = C$	-	$\alpha = \beta$	6.2.7	6.18
$A = D, B = C$	$r = s$	-	6.2.8	6.20
$A = D, B = C$	$r = s$	$\alpha = \beta$	6.2.9	6.22
$A = B = C = D$	-	-	Equality Case	
$A = B = C = D$	-	$\alpha = \beta$	Equality Case	
$A = B = C = D$	$r = s$	-	Equality Case	
$A = B = C = D$	$r = s$	$\alpha = \beta$	Equality Case	

We summarize these cases into 7 corollaries as follows (depend on the hypotheses).

Corollary 6.2.3. *Let $A, B, C, D \in \mathrm{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$(\lambda_i(A+B))^r \, (\lambda_j(C+D))^s \quad \geq \quad \lambda_k((A^r \otimes C^s) + (B^r \otimes D^s)) \tag{6.12}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
The equality holds if $A = B, C = D$.

Proof. By setting $\alpha = \beta$ in Theorem 6.2.1, we obtain

$$\begin{aligned}
(\lambda_i(\alpha A + \alpha B))^r \, (\lambda_j(\alpha C + \alpha D))^s &\geq \lambda_k(\alpha(A^r \otimes C^s) + \alpha(B^r \otimes D^s)), \\
\alpha^r (\lambda_i(A+B))^r \, \alpha^s (\lambda_j(C+D))^s &\geq \alpha\lambda_k((A^r \otimes C^s) + (B^r \otimes D^s)), \\
\alpha(\lambda_i(A+B))^r \, (\lambda_j(C+D))^s &\geq \alpha\lambda_k((A^r \otimes C^s) + (B^r \otimes D^s)), \\
(\lambda_i(A+B))^r \, (\lambda_j(C+D))^s &\geq \lambda_k((A^r \otimes C^s) + (B^r \otimes D^s))
\end{aligned}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
The case of equality is same as Theorem 6.2.1. The proof is completed. □

Corollary 6.2.4. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\left(\lambda_i(\alpha A + \beta B)\, \lambda_j(\alpha C + \beta D)\right)^{\frac{1}{2}} \quad \geq \quad \lambda_k\left(\alpha(A \otimes C)^{\frac{1}{2}} + \beta(B \otimes D)^{\frac{1}{2}}\right) \qquad (6.13)$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
The equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Theorem 6.2.1, we get $r = 1/2$ and then

$$\begin{aligned}\lambda_k\left(\alpha(A \otimes C)^{\frac{1}{2}} + \beta(B \otimes D)^{\frac{1}{2}}\right) \quad &\leq \quad (\lambda_i(\alpha A + \beta B))^{\frac{1}{2}} \left(\lambda_j(\alpha C + \beta D)\right)^{\frac{1}{2}} \\ &= \quad (\lambda_i(\alpha A + \beta B) \left(\lambda_j(\alpha C + \beta D)\right)^{\frac{1}{2}}\end{aligned}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
For the case of equality, assume that $A = B, C = D$. We obtain

$$\begin{aligned}\lambda_k\left(\alpha(A \otimes B)^{\frac{1}{2}} + \beta(A \otimes B)^{\frac{1}{2}}\right) \quad &= \quad \lambda_k\left((\alpha + \beta)(A \otimes B)^{\frac{1}{2}}\right) \\ &= \quad (\alpha + \beta)\lambda_k\left((A \otimes B)^{\frac{1}{2}}\right) \\ &= \quad (\alpha + \beta)^{\frac{1}{2}}(\alpha + \beta)^{\frac{1}{2}}(\lambda_i(A))^{\frac{1}{2}}(\lambda_j(B))^{\frac{1}{2}} \\ &= \quad ((\alpha + \beta)\lambda_i(A))^{\frac{1}{2}}((\alpha + \beta)^n\lambda_j(B))^{\frac{1}{2}} \\ &= \quad (\lambda_i(\alpha A + \beta B))^{\frac{1}{2}}(\lambda_j(\alpha A + \beta B))^{\frac{1}{2}} \\ &= \quad (\lambda_i(\alpha A + \beta B)\lambda_j(\alpha A + \beta B))^{\frac{1}{2}}.\end{aligned}$$

So the equality holds. The proof is completed. □

Corollary 6.2.5. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\begin{aligned}(\lambda_i(\alpha A + \beta B))^r(\lambda_j(\alpha A + \beta B))^s \quad &\geq \quad \lambda_k(\alpha(A^r \otimes A^s) + \beta(B^r \otimes B^s)), \quad (6.14) \\ (\lambda_i(\alpha A + \beta B))^r(\lambda_j(\beta A + \alpha B))^s \quad &\geq \quad \lambda_k(\alpha(A^r \otimes B^s) + \beta(B^r \otimes A^s)) \quad (6.15)\end{aligned}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
The equalities hold if $A = B$. The inequality (6.14) *also becomes an equality if $n = 1$.*

Proof. By setting $A = C, B = D$ in Theorem 6.2.1, we get (6.14). By setting $A = D, B = C$ in Theorem 6.2.1, we get (6.15).
For the case of equality, assume that $A = B$. We obtain

$$\begin{aligned}
\lambda_k(\alpha(A^r \otimes A^s) + \beta(A^r \otimes A^s)) &= \lambda_k((\alpha+\beta)(A^r \otimes A^s)) \\
&= (\alpha+\beta)\lambda_k(A^r \otimes A^s) \\
&= (\alpha+\beta)\lambda_k(A^r)\lambda_i(A^s) \\
&= (\alpha+\beta)(\lambda_i(A))^r(\lambda_j(A))^s \\
&= (\alpha+\beta)\lambda_k(A) \\
&= \lambda_k(\alpha A + \beta A).
\end{aligned}$$

So the equality holds. Similarly, the other one holds when $A = B$.
Obviously, the inequality (6.14) also becomes an equality when $n = 1$. □

Corollary 6.2.6. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$(\lambda_i(A+B)\,\lambda_j(C+D))^{\frac{1}{2}} \quad \geq \quad \lambda_k((A\otimes C)^{\frac{1}{2}} + (B\otimes D)^{\frac{1}{2}}) \tag{6.16}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
The equality holds if $A = B, C = D$.

Proof. Set $r = s$ in Corollary 6.2.3 or set $\alpha = \beta$ in Corollary 6.2.4. □

Corollary 6.2.7. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$(\lambda_i(A+B))^r(\lambda_j(A+B))^s \;\geq\; \lambda_k((A^r \otimes A^s) + (B^r \otimes B^s))\,, \tag{6.17}$$

$$(\lambda_i(A+B))^r(\lambda_j(A+B))^s \;\geq\; \lambda_k((A^r \otimes B^s) + (B^r \otimes A^s)) \tag{6.18}$$

for all $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n^2$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
The equalities hold if $A = B$. The inequality (6.17) *also becomes an equality if $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 6.2.5. □

Corollary 6.2.8. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$(\lambda_i(\alpha A + \beta B)\lambda_j(\alpha A + \beta B))^{\frac{1}{2}} \;\geq\; \lambda_k(\alpha(A^{\frac{1}{2}})^{\otimes 2} + \beta(B^{\frac{1}{2}})^{\otimes 2})\,, \tag{6.19}$$

$$(\lambda_i(\alpha A + \beta B)\,\lambda_j(\beta A + \alpha B))^{\frac{1}{2}} \;\geq\; \lambda_k(\alpha(A\otimes B)^{\frac{1}{2}} + \beta(B\otimes A)^{\frac{1}{2}}) \tag{6.20}$$

for all $i, j = 1, 2, \dots, n$ *and* $k = 1, 2, \dots, n^2$ *if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.*
The equalities hold if $A = B$. *The inequality* (6.19) *also becomes an equality if* $n = 1$.

Proof. By setting $r = s$ in (6.14) in Corollary 6.2.5, we get $r = 1/2$ and

$$\begin{aligned}(\lambda_i(\alpha A + \beta B)\,\lambda_j(\alpha A + \beta B))^{\frac{1}{2}} &\geq \lambda_k(\alpha(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \otimes B^{\frac{1}{2}}))\,, \\ &= \lambda_k(\alpha(A^{\frac{1}{2}})^{\otimes 2} + \beta(B^{\frac{1}{2}})^{\otimes 2})).\end{aligned}$$

By setting $r = s$ in (6.15) in Corollary 6.2.5, we get $r = 1/2$ and

$$\begin{aligned}(\lambda_i(\alpha A + \beta B)\,\lambda_j(\beta A + \alpha B))^{\frac{1}{2}} &\geq \lambda_k(\alpha(A^{\frac{1}{2}} \otimes B^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \otimes A^{\frac{1}{2}}))\,, \\ &= \lambda_k(\alpha(A \otimes B)^{\frac{1}{2}} + \beta(B \otimes A)^{\frac{1}{2}}).\end{aligned}$$

The case of equalities is same as Corollary 6.2.5.
Obviously, the inequality (6.19) becomes an equality when $n = 1$. □

Corollary 6.2.9. *Let* $A, B \in \mathbb{M}_n$ *be positive definite symmetric. Then*

$$(\lambda_i(A + B)\,\lambda_j(A + B))^{\frac{1}{2}} \geq \lambda_k((A^{\otimes 2})^{\frac{1}{2}} + \beta(B^{\otimes 2})^{\frac{1}{2}})\,, \tag{6.21}$$

$$(\lambda_i(A + B)\,\lambda_j(A + B))^{\frac{1}{2}} \geq \lambda_k((A \otimes B)^{\frac{1}{2}} + (B \otimes A)^{\frac{1}{2}}) \tag{6.22}$$

for all $i, j = 1, 2, \dots, n$ *and* $k = 1, 2, \dots, n^2$ *if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.*
The equalities hold if $A = B$. *The inequality* (6.21) *also becomes an equality if* $n = 1$.

Proof. Set $\alpha = \beta$ in Corollary 6.2.8. □

Remark 6.2.10.

For the inequality (6.22) in Corollary 6.2.9, when $n = 1$, let $A = [a]$ and $B = [b]$, we obtain

$$\begin{aligned} a + b &\geq (ab)^{\frac{1}{2}} + (ba)^{\frac{1}{2}}\,, \\ (ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a + b).\end{aligned}$$

This is the AM–GM inequality for real numbers.

Observation 6.2.11.

Let $A \in \mathbb{M}_n$. For any positive integer k, if $\lambda \in \sigma(A)$, then $\lambda^k \in \sigma(A^{\otimes k})$.

Proof. From Theorem 2.4.6, we have $\lambda\lambda \in \sigma(A \otimes A)$. That is $\lambda^2 \in \sigma(A^{\otimes 2})$.
It follows inductively that $\lambda^k \in \sigma(A^{\otimes k})$ for any positive integer k. □

6.3 Inequalities of Eigenvalue Involving the Hadamard Product of Power of Linear Combination of Matrices

We present inequalities of eigenvalue involving the Hadamard product of power of linear combination of matrices in this section. The next theorem which we establish is the main result. The special cases of this theorem are considered in a sequence of corollaries.

Theorem 6.3.1. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\lambda_i\big((\alpha A+\beta B)^r \circ (\alpha C+\beta D)^s\big) \quad \geq \quad \lambda_i\big(\alpha(A^r \circ C^r)+\beta(B^r \circ D^s)\big) \qquad (6.23)$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B, C = D$.

Proof. Since A, B, C and D are positive definite and α and β are positive, by Schur product theorem (Theorem 2.5.5), we have that $(\alpha A+\beta B)^r \circ (\alpha C+\beta B)^s$ and $\alpha(A^r \circ C^s)+\beta(B^r \circ D^s)$ are also positive definite.
Then by applying Theorem 2.6.4 to Theorem 3.3.1.3, we get (6.23).
For the case of equality, assume that $A = B, C = D$. We obtain

$$\begin{aligned}
\lambda_i\big((\alpha A+\beta A)^r \circ (\alpha C+\beta C)^s\big) &= \lambda_i\big((\alpha+\beta)^r A^r \circ (\alpha+\beta)^s C^s\big) \\
&= \lambda_i\big((\alpha+\beta)^r (\alpha+\beta)^s (A^r \circ C^s)\big) \\
&= \lambda_i\big((\alpha+\beta)(A^r \circ C^s)\big) \\
&= \lambda_i\big(\alpha(A^r \circ C^s)+\beta(A^s \circ C^s)\big).
\end{aligned}$$

So the equality holds. The proof is completed. ☐

There are 3 types of special cases of the main result (Theorem 6.3.1): the equality between the matrices; the equality between the coefficients (scalars); and the equality between the powers of matrices. We divide them into 4 cases as follows:

1. the case when $A = C$, $B = D$;
2. the case when $A = D$, $B = C$;
3. the case when $r = s$; and
4. the case when $\alpha = \beta$.

Combining all of them, there are 16 possible cases (which include the general case). Except for the general case, the details for each case are given here.

Table 6.2: The Special Cases of Theorem 6.3.1

Conditions			Results	
A,B,C,D	r,s	α,β	Corollary	Inequality
-	-	$\alpha=\beta$	6.3.2	6.24
-	$r=s$	-	6.3.3	6.25
-	$r=s$	$\alpha=\beta$	6.3.5	6.28
$A=C,B=D$	-	-	6.3.4	6.26
$A=C,B=D$	-	$\alpha=\beta$	6.3.6	6.29
$A=C,B=D$	$r=s$	-	6.3.7	6.31
$A=C,B=D$	$r=s$	$\alpha=\beta$	6.3.8	6.33
$A=D,B=C$	-	-	6.3.4	6.27
$A=D,B=C$	-	$\alpha=\beta$	6.3.6	6.30
$A=D,B=C$	$r=s$	-	6.3.7	6.32
$A=D,B=C$	$r=s$	$\alpha=\beta$	6.3.8	6.34
$A=B=C=D$	-	-	Equality Case	
$A=B=C=D$	-	$\alpha=\beta$	Equality Case	
$A=B=C=D$	$r=s$	-	Equality Case	
$A=B=C=D$	$r=s$	$\alpha=\beta$	Equality Case	

We summarize these cases into 7 corollaries as follows (depend on the hypotheses).

Corollary 6.3.2. *Let $A,B,C,D \in \mathbb{M}_n$ be positive definite symmetric and let $r,s>0$ be scalars such that $r+s=1$. Then*

$$\lambda_i\big((A+B)^r \circ (C+D)^s\big) \quad \geq \quad \lambda_i\big((A^r \circ C^s) + (B^r \circ D^s)\big) \tag{6.24}$$

for all $i = 1,2,\dots,n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A=B, C=D$.

Proof. By setting $\alpha = \beta$ in Theorem 6.3.1, we obtain

$$\begin{aligned}
\lambda_i\big((\alpha A + \alpha B)^r \circ (\alpha C + \alpha D)^s\big) &\geq \lambda_i\big(\alpha(A^r \circ C^s) + \alpha(B^s \circ D^s)\big), \\
\lambda_i\big(\alpha^r (A + B)^r \circ \alpha^s (C + D)^s\big) &\geq \lambda_i\big(\alpha(A^r \circ C^s) + \alpha(B^s \circ D^s)\big), \\
\lambda_i\Big(\alpha^r \alpha^s \big((A + B)^r \circ (C + D)^s\big)\Big) &\geq \lambda_i\Big(\alpha\big((A^r \circ C^s) + (B^s \circ D^s)\big)\Big). \\
\alpha\lambda_i\big((A + B)^r \circ (C + D)^s\big) &\geq \alpha\lambda_i\big((A^r \circ C^s) + (B^s \circ D^s)\big), \\
\lambda_i\big((A + B)^r \circ (C + D)^s\big) &\geq \lambda_i\big((A^r \circ C^s) + (B^r \circ D^s)\big).
\end{aligned}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order.

The case of equality is same as Theorem 6.3.1. The proof is completed. □

Corollary 6.3.3. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\lambda_i\big((\alpha A + \beta B)^{\frac{1}{2}} \circ (\alpha C + \beta D)^{\frac{1}{2}}\big) \geq \lambda_i\big(\alpha(A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \circ D^{\frac{1}{2}})\big) \qquad (6.25)$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Theorem 6.3.1, we have $r = 1/2$ and then (6.25).
For the case of equality, assume that $A = B, C = D$. We obtain

$$\begin{aligned}
\lambda_i\big((\alpha A + \beta A)^{\frac{1}{2}} \circ (\alpha C + \beta C)^{\frac{1}{2}}\big) &= \lambda_i\big((\alpha + \beta)^{\frac{1}{2}} A^{\frac{1}{2}} \circ (\alpha + \beta)^{\frac{1}{2}} C^{\frac{1}{2}}\big) \\
&= \lambda_i\big((\alpha + \beta)^{\frac{1}{2}} (\alpha + \beta)^{\frac{1}{2}} (A^{\frac{1}{2}} \circ C^{\frac{1}{2}})\big) \\
&= \lambda_i\big((\alpha + \beta)(A^{\frac{1}{2}} \circ C^{\frac{1}{2}})\big) \\
&= \lambda_i\big(\alpha(A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + \beta(A^{\frac{1}{2}} \circ C^{\frac{1}{2}})\big)
\end{aligned}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. So the equality holds. □

Corollary 6.3.4. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta, r. s > 0$ be scalars such that $r + s = 1$. Then*

$$\begin{aligned}
\lambda_i\big((\alpha A + \beta B)^r \circ (\alpha A + \beta B)^s\big) &\geq \lambda_i\big(\alpha(A^r \circ A^s) + \beta(B^r \circ B^s)\big), \qquad (6.26) \\
\lambda_i\big((\alpha A + \beta B)^r \circ (\beta A + \alpha B)^s\big) &\geq \lambda_i\big((A^r \circ B^s) + (A^s \circ B^r)\big) \qquad (6.27)
\end{aligned}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B$. The inequality (6.26) *also becomes an equality if $n = 1$.*

Proof. By setting $A = C, B = D$ in Theorem 6.3.1, we get (6.26). By setting $A = D,\ B = C$ in Theorem 6.3.1, we obtain from the commutativity of the Hadamard product that

$$\begin{aligned}\lambda_i\big((\alpha A+\beta B)^r \circ (\beta A+\alpha B)^s\big) &\geq \lambda_i\big((A^r \circ B^s)+(B^r \circ A^s)\big)\\ &= \lambda_i\big((A^r \circ B^s)+(A^s \circ B^r)\big)\end{aligned}$$

for all $i = 1.2,\dots,n$ if the respective eigenvalues of them are arranged in the same order.
The case of equalities is same as Theorem 6.3.1.
Obviously, the inequality (6.26) also becomes an equality when $n = 1$ □

Corollary 6.3.5. *Let $A, B, C, D \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\lambda_i\big((A+B)^{\frac{1}{2}} \circ (C+D)^{\frac{1}{2}}\big) \geq \lambda_i\big((A^{\frac{1}{2}} \circ C^{\frac{1}{2}})+(B^{\frac{1}{2}} \circ D^{\frac{1}{2}})\big) \qquad (6.28)$$

for all $i = 1,2,\dots,n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B, C = D$.

Proof. By setting $r = s$ in Corollary 6.3.2, we get (6.28). The case of equality is same as Corollary 6.3.2. □

Corollary 6.3.6. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $r, s > 0$ be scalars such that $r + s = 1$. Then*

$$\lambda_i\big((A+B)^r \circ (A+B)^s\big) \geq \lambda_i\big((A^r)^{(2)}+(B^r)^{(2)}\big), \qquad (6.29)$$

$$\lambda_i\big((A+B)^r \circ (A+B)^s\big) \geq \lambda_i\big((A^r \circ B^r)+(A^s \circ B^r)\big). \qquad (6.30)$$

for all $i = 1,2,\dots,n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B, C = D$. The inequality (6.29) *also becomes an equality if $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 6.3.4. □

Corollary 6.3.7. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric and let $\alpha, \beta > 0$ be scalars. Then*

$$\lambda_i\Big(\big((\alpha A+\beta B)^{\frac{1}{2}}\big)^{(2)}\Big) \geq \lambda_i\big(\alpha(A^{\frac{1}{2}})^{(2)}+\beta(B^{\frac{1}{2}})^{(2)}\big), \qquad (6.31)$$

$$\lambda_i\big((\alpha A+\beta B)^{\frac{1}{2}} \circ (\beta A+\alpha B)^{\frac{1}{2}}\big) \geq \lambda_i\big(\alpha(A^{\frac{1}{2}})^{(2)}+\beta(B^{\frac{1}{2}})^{(2)}\big) \qquad (6.32)$$

for all $i = 1,2,\dots,n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B$. The inequality (6.31) *also becomes an equality when $n = 1$.*

Proof. By setting $r = s$ in the inequalities (6.26) and (6.27) in Corollary 6.3.4, we get the inequalities (6.31) and (6.32), respectively. For the inequality (6.31), if $A = B$, we obtain

$$\begin{aligned}
\lambda_i\big((\alpha A+\beta A)^{\frac{1}{2}} \circ (\alpha A+\beta A)^{\frac{1}{2}}\big) &= \lambda_i\Big(\big((\alpha+\beta)^{\frac{1}{2}}A^{\frac{1}{2}}\big)^{(2)}\Big) \\
&= \lambda_i\Big(\big((\alpha+\beta)^{\frac{1}{2}}\big)^{2}(A^{\frac{1}{2}})^{(2)}\Big) \\
&= \lambda_i\big((\alpha+\beta)(A^{\frac{1}{2}})^{(2)}\big) \\
&= \lambda_i\big(\alpha(A^{\frac{1}{2}})^{(2)} + \beta(A^{\frac{1}{2}})^{(2)}\big).
\end{aligned}$$

So the equality holds. Similarly, the others one hold if $A = B$. □

Corollary 6.3.8. *Let $A, B \in \mathbb{M}_n$ be positive definite symmetric. Then*

$$\lambda_i\Big(\big((A+B)^{\frac{1}{2}}\big)^{(2)}\Big) \geq \lambda_i\big((A^{\frac{1}{2}})^{(2)} + (B^{\frac{1}{2}})^{(2)}\big), \tag{6.33}$$

$$\lambda_i(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}) \leq \frac{1}{2}\lambda_i\Big(\big((A+B)^{\frac{1}{2}}\big)^{(2)}\Big) \tag{6.34}$$

for all $i = 1, 2, \ldots, n$ if the respective eigenvalues of them are arranged in the same order. The equality holds if $A = B$. The inequality (6.33) *also becomes an equality when $n = 1$.*

Proof. Set $\alpha = \beta$ in Corollary 6.3.7. □

Remark 6.3.9.

For the inequality (6.34) in Corollary 6.3.8, when $n = 1$, let $A = [a]$ and $B = [b]$, we obtain

$$\begin{aligned}
a^{\frac{1}{2}}b^{\frac{1}{2}} &\leq \frac{1}{2}\big((a+b)^{\frac{1}{2}}\big)^{2}, \\
(ab)^{\frac{1}{2}} &\leq \frac{1}{2}(a+b).
\end{aligned}$$

This is the AM–GM inequality for real numbers.

6.4 Numerical Examples

In order to support our main results in the previous sections, we give some numerical examples in this section.

Example 6.4.1.

Consider the inequality of determinant involving the 2nd power of linear combination of the given matrices

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

in the case when $\alpha = 4, \; \beta = 3$.

We see that A and B are positive definite symmetric and α and β satisfy the hypothesis of Theorem 5.1.1. We get

$$\begin{aligned} \alpha A + \beta B &= \begin{bmatrix} 14 & 10 \\ 10 & 23 \end{bmatrix}, \\ \alpha A^2 + \beta B^2 &= \begin{bmatrix} 44 & 58 \\ 58 & 107 \end{bmatrix} \end{aligned}$$

When we solve the characteristic equations we get

$$\begin{aligned} \lambda_1(\alpha A + \beta B) &\approx 7.534\,, \\ \lambda_2(\alpha A + \beta B) &\approx 29.466\,, \\ \lambda_1(\alpha A^2 + \beta B^2) &\approx 9.498\,, \\ \lambda_2(\alpha A^2 + \beta B^2) &\approx 141.504. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{\alpha+\beta}\left(\lambda_1(\alpha A + \beta B)\right)^2 &\approx 8.109\,, \\ \frac{1}{\alpha+\beta}\left(\lambda_2(\alpha A + \beta B)\right)^2 &\approx 124.035. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{\alpha+\beta}\left(\lambda_1(\alpha A + \beta B)\right)^2 &\leq \lambda_1(\alpha(A^2) + \beta(B^2))\,, \\ \frac{1}{\alpha+\beta}\left(\lambda_2(\alpha A + \beta B)\right)^2 &\leq \lambda_2(\alpha(A^2) + \beta(B^2)). \end{aligned}$$

This example supports the result of Theorem 6.1.1. □

Example 6.4.2.

Consider the inequality of eigenvalue involving the Kronecker product of power of linear combination of the given matrices

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix},\ B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ \text{and } D = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha = 1,\ \beta = 2,\ r = 1/3,\ s = 2/3$.

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 5.2.1. We get

$$\begin{aligned} \alpha A + \beta B &= \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}, \alpha C + \beta D = \begin{bmatrix} 4 & -4 \\ -4 & 14 \end{bmatrix}, \\ \alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s) &\approx \begin{bmatrix} 4.298 & -2.324 & 0.514 & 0.476 \\ -2.324 & 10.112 & 0.566 & -0.902 \\ 0.514 & 0.566 & 2.864 & -1.758 \\ 0.566 & -0.902 & -1.758 & 7.262 \end{bmatrix}. \end{aligned}$$

When we solve the characteristic equations we get

$$\begin{aligned} \lambda_1(\alpha A + \beta B) &\approx 3.807, \\ \lambda_2(\alpha A + \beta B) &\approx 9.193, \\ \lambda_1(\alpha C + \beta D) &\approx 2.597, \\ \lambda_2(\alpha C + \beta D) &\approx 15.403, \\ \lambda_1(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)) &\approx 1.933, \\ \lambda_2(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)) &\approx 3.798, \\ \lambda_3(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)) &\approx 7.531, \\ \lambda_4(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s)) &\approx 11.274. \end{aligned}$$

Hence,

$$\begin{aligned} (\lambda_1(\alpha A + \beta B))^r (\lambda_1(\alpha C + \beta D))^s &\approx 2.499, \\ (\lambda_2(\alpha A + \beta B))^r (\lambda_1(\alpha C + \beta D))^s &\approx 3.957, \\ (\lambda_1(\alpha A + \beta B))^r (\lambda_2(\alpha C + \beta D))^s &\approx 9.664, \\ (\lambda_2(\alpha A + \beta B))^r (\lambda_2(\alpha C + \beta D))^s &\approx 12.938. \end{aligned}$$

Thus,

$$(\lambda_i(\alpha A + \beta B))^r (\lambda_j(\alpha C + \beta D))^s \geq \lambda_k(\alpha(A^r \otimes C^s) + \beta(B^r \otimes D^s))$$

for all $i, j = 1, 2$ and $k = 1, 2, 3, 4$ if the product of eigenvalues in the left hand side are arranged in the same order as the right hand side.
This example supports the result of Theorem 6.2.1. $\square$

Example 6.4.3.

Consider the inequality of determinant involving the Hadamard product of power of linear combination of the given matrices

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$$

in the case when $\alpha = 1,\ \beta = 2,\ r = 2/3,\ s = 1/3.$

We see that A, B, C and D are positive definite symmetric and α, β, r and s satisfy the hypothesis of Theorem 5.2.1. We get

$$(\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s \approx \begin{bmatrix} 6.460 & -0.126 \\ -0.126 & 5.933 \end{bmatrix},$$

$$\alpha(A^r \circ C^s) + \beta(B^r \circ D^s) \approx \begin{bmatrix} 5.825 & 0.466 \\ 0.466 & 4.683 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \lambda_1((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s) &\approx 5.904, \\ \lambda_2((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s) &\approx 6.489, \\ \lambda_1(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)) &\approx 4.516, \\ \lambda_2(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)) &\approx 5.987. \end{aligned}$$

Thus, for all $i = 1, 2$,

$$\lambda_i((\alpha A + \beta B)^r \circ (\alpha C + \beta D)^s) \geq \lambda_i(\alpha(A^r \circ C^s) + \beta(B^r \circ D^s)).$$

This example supports the result of Theorem 6.3.1. $\square$

CHAPTER 7

CONCLUSIONS

7.1 Discussions

1. Compare the special case of Theorem 3.1.1.1 when $\alpha+\beta = 1$ to the previous work, we see that it is the work by Ando [14]. When $n = 1$ this theorem becomes the well–known AM–GM inequality for positive real numbers. Hence, this theorem is generalizations of some previous works.

2. In Theorem 3.2.1.5 and Theorem 3.3.1.3, any matrix is necessarily positive definite since the proof of each theorem requires the nonsingularity of matrices (positive semidefinite or Hermitian matrices is not necessarily nonsingular).

3. The cases of equality in Theorem 3.2.1.5 and Theorem 3.3.1.3 follow from the equality cases of the jointly concave maps $\Phi(A, B) = A^r \otimes B^s$ and $\Phi(A, B) = A^r \circ B^s$, respectively.

4. For all inequalities in the Löwner partial order in Chapter 3, if the equalities do not hold, the sign "$\leq$ " is still "$\leq$" not "$<$" since $A \leq B$ does not mean that $A < B$ or $A = B$, similarly for "$\geq$ ".

5. For the inequalities of trace, determinant and eigenvalue, the proofs of them require the positive definiteness of matrices. So any matrix is necessarily positive definite.

6. There are some special cases of the main theorems for inequalities of trace, determinant and eigenvalue in Chapter 4–Chapter 6 which are reduced to the AM–GM inequality for real number. Hence, these theorems generalizes this well–known fact.

At the end, we observe that there are analogous notions between inequalities in the Löwner partial order for positive definite symmetric matrices and inequalities for positive real numbers (this is a total order) as well as the analogous notions between Hermitian matrices and real numbers. We knew that the zero matrix plays a role like the real number 0 and the identity matrix plays a role like the real number 1. Actually, a Hermitian matrix plays a role like a real number, a nonsingular Hermitian matrix plays a role like a nonzero real number, a positive semidefinite matrix (sometimes called nonnegative definite matrix) plays a role like a nonnegative real number and a positive definite matrix plays a role like a positive real number.

Hermitian Matrix is a generalization of a real number since the dimension of a number is 1 but the dimension of the matrix is $n \times n$ and the Hermitian Matrix has some properties similar to properties of the real number. So, as we see, for example the AM–GM inequality, if any matrix inequalities holds, it still holds for the case of real numbers. But there are some inequalities for real number such that they are not hold for Hermitian matrices. The important reasons are

1). the dimension of a number is 1 but the dimension of a matrix is $n \times n$,

2). there are two Hermitian matrices which are not comparable (the Löwner partial order is just a partial order),

3). the matrix multiplication is not commutative.

7.2 Summary of the Main Results

In this research, we use the concepts of Schur's complement and maps on matrix spaces to prove new inequalities in the Löwner partial order between the three linear combination forms which are involving the squares, the Kronecker product and the Hadamard product of positive definite symmetric matrices. These inequalities lead to inequalities of trace, inequalities of determinant and inequalities of eigenvalue of the same linear combination forms. Our results show the relationship between matrices in the form of inequalities and suggest the analogous notions between inequalities in the Löwner partial order for positive definite symmetric matrices and inequalities for positive real numbers.

7.3 Suggestions for Further Works

The ways to extend this research are followings:

1. For the case of equality in any inequality, you can improve these inequalities by finding more sufficient conditions for the equalities hold or, if possible, finding necessary and sufficient conditions for the equalities hold (this means that, in this case, we know that equality holds if and only if what).

2. Some inequalities of trace, determinant and eigenvalue which are special cases of theorems may be improved if we know more properties of trace, determinant and eigenvalue or if we add some assumptions. For example, we knew that $\operatorname{tr}(A^{\frac{1}{k}}) > (\operatorname{tr}(A))^{\frac{1}{k}}$ holds for any positive integers k and $n > 1$. Does it hold for any positive real number (or real number)?. If we know, we may improve the inequalities which are involved the traces of positive real (or real) power of matrices (i.e., $\operatorname{tr}(A^r)$ in this research).

3. In this research, any inequality holds for positive definite symmetric matrices. Does it still hold for positive semidefinite symmetric matrices ? But notice that positive semidefinite symmetric matrices is not necessarily nonsingular and the proofs of Theorem 3.2.1.5 and Theorem 3.3.1.3 require the nonsingularity of the matrices. So, maybe we need to add some assumptions.

4. In this research, we consider only real matrices. What happen when we extend the study to complex matrices? Note that, in this case, we consider positive definite Hermitian matrix which is not necessarily symmetric.

5. In this research, the power of matrices r, s are restricted to $r, s > 0$ and $r + s = 1$. We can generalize it by removing the condition $r + s = 1$ or extending the study to the case when

   - $r > 0, s < 0$,
   - $r < 0, s < 0$,
   - $r < 0, s < 0$ or

   in the general case: r, s are real numbers. An another way to extend this research is studying the cases when $r + s \neq 1$. So in this case r, s can be greater than 1 (in this research r, s must be less than 1 because of the positivity of r, s).

6. We can extend the study to finding the inequalities between $\alpha A^p + \beta B^p$ and $(\alpha A + \beta B)^p$ in the followings cases:

   - p is a positive integer,
   - p is a negative integer,
   - p is a real number and $p > 1$,
   - p is a real number and $0 < p < 1$,
   - p is a real number and $-1 < p < 0$,
   - p is a real number and $p < -1$.

   Then, if possible, find the inequalities between $\sum_{i=1}^{n} \alpha_i A_i^p$ and $(\sum_{i=1}^{n} \alpha_i A_i)^p$.

7. In the proofs of Theorem 3.2.1.5 and Theorem 3.3.1.3, the concept of maps on matrix spaces play an important role. We hope that we can get more inequalities in matrix theory by using these concepts.

8. Since matrices are linear operators, we may extend this study to bounded linear operators on (real or complex) separable Hilbert space or another spaces.

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APPENDICES

Appendix A

SOME RELATED APPLICATIONS

In Control theory, to analyze the stability of dynamical systems, we have to solve the differential equation

$$\frac{d}{dt}x(t) \quad = \quad Ax(t). \tag{A.1}$$

It is shown that this is equivalent to find a positive definite matrix X which satisfies the linear matrix inequality

$$A^{\mathsf{T}}X + XA \quad < \quad 0. \tag{A.2}$$

Indeed, we can pick $Q = Q^{\mathsf{T}} > 0$ and then solve the following matrix equation which is called the Lyapunov equation

$$A^{\mathsf{T}}X + XA \quad = \quad -Q \tag{A.3}$$

where Q is a positive definite symmetric matrix. The equation has a positive definite symmetric solution X if and only if matrix A is stable, i.e., all of its eigenvalues have negative real parts. Given a stable matrix A and a matrix Q of the same size, by taking trace, a necessary condition for X to be a solution to the equation is

$$\operatorname{tr}(XA) \quad = \quad -\frac{1}{2}\operatorname{tr}(Q). \tag{A.4}$$

This is one of the reasons leading to the study of the trace of the product of two matrices, one of which is assumed to be positive semidefinite.

There are many study of trace bounds on the solutions of Lyapunov equation. For example, it is shown by Wang [18] that for real symmetric matrix A and positive semidefinite symmetric matrix B of the same size

$$\lambda_{min}(A)\operatorname{tr}(B) \quad \leq \quad \operatorname{tr}(AB) \quad \leq \quad \lambda_{max}(A)\operatorname{tr}(B) \tag{A.5}$$

where $\lambda_{min}(A)$ and $\lambda_{max}(A)$ are the smallest and largest eigenvalues of A, respectively.

Lasserre [19] showed that for Hermitian matrices A and B, assuming that all eigenvalues are arranged in decreasing order $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and

$\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$, then

$$\sum_{i=1}^{n} \lambda_{n-i+1}(A)\,\lambda_i(B) \quad \leq \quad \mathrm{tr}(AB) \quad \leq \quad \sum_{i=1}^{n} \lambda_i(A)\lambda_i(B). \tag{A.6}$$

In another paper, Zhang [20] presents a family of eigenvalue inequalities for the product of a positive semidefinite matrix.

Some results of this research are inequalities of the trace of product of positive definite symmetric matrices and inequalities of product of eigenvalue of positive definite symmetric matrices. These inequalities may apply to find the solutions of the Lyapunov equation.

For more details about linear matrix inequality in control and system theory and the Lyapunov equation, see [1].

Appendix B

FAMOUS INEQUALITIES IN THE LÖWNER PARTIAL ORDER

Theorem B.1 (Löwner-Heinz Inequality [13, 17]). *If* $A \geq B \geq 0$ *and* $0 \leq r \leq 1$. *Then*

$$A^r \quad \geq \quad B^r. \tag{B.1}$$

Theorem B.2 (Furata's Inequalities [8, 21]). *If* $A \geq B \geq 0$, *then*

$$(B^r A^p B^r)^{1/q} \quad \geq \quad B^{(p+2r)/q} \tag{B.2}$$

and

$$A^{(p+2r)/q} \quad \geq \quad (A^r B^p A^r)^{1/q} \tag{B.3}$$

for $r > 0$, $p \geq 0$, $q \geq 1$ *with* $(1+2r)q \geq p+2r$.

The next two corollaries are the special cases of Furata's Inequalities.

Corollary B.3 ([8]). *If* $A \geq B \geq 0$, *then*

$$(B^r A^p B^r)^{1/p} \quad \geq \quad B^{(p+2r)/p} \tag{B.4}$$

and

$$A^{(p+2r)/p} \quad \geq \quad (A^r B^p A^r)^{1/p} \tag{B.5}$$

for all $r \geq 0$ *and* $p \geq 1$.

Corollary B.4 ([8]). *If* $A \geq B \geq 0$, *then*

$$(BA^2B)^{1/2} \quad \geq \quad B^2 \tag{B.6}$$

and

$$A^2 \quad \geq \quad (AB^2A)^{1/2}. \tag{B.7}$$

Theorem B.5 (Young's Inequality [8]). *Let $p, q > 1$ be real numbers such that $1/p + 1/q = 1$. If $A, B \geq 0$ is a commuting pair, however, $AB \geq 0$ and via a simultaneous unitary diagonalization, then*

$$AB \quad \leq \quad \frac{1}{p}A^p + \frac{1}{q}B^q. \tag{B.8}$$

Theorem B.6 (Ando-Hiai Inequality [8, 21]). *Let A, B be positive definite matrices. If for $0 \leq \alpha \leq 1$*

$$A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \leq \quad I,$$

then

$$A^{-\frac{r}{2}}(A^{-\frac{r}{2}}BA^{-\frac{r}{2}})^{\alpha}A^{\frac{r}{2}} \quad \leq \quad I$$

holds for all $r \geq 1$.

Theorem B.7 (Hölder's Inequalities [14, 8]). *Let A, B be positive definite matrices. Then*

$$\sum_{i=1}^{k} A_i \circ B_i \quad \leq \quad \left(\sum_{i=1}^{k} A_i^p\right)^{1/p} \circ \left(\sum_{i=1}^{k} B_i^q\right)^{1/q} \tag{B.9}$$

for $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\sum_{i=1}^{k} A_i \circ B_i \quad \geq \quad \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-1/p} \circ \left(\sum_{i=1}^{k} B_i^{-q}\right)^{-1/q} \tag{B.10}$$

for $p \geq 1 \geq q > 0$ and $\frac{1}{q} - \frac{1}{p} = 1$.

Theorem B.8 ([5, 8]). *If $A, B \in \mathbb{M}_n$ are positive definite. Then*

$$A^{-1} \circ B^{-1} \quad \geq \quad (A \circ B)^{-1}, \tag{B.11}$$

$$A^{-1} \circ A^{-1} \quad \geq \quad (A \circ A)^{-1}, \tag{B.12}$$

$$A^{-1} \circ A \quad \geq \quad I \quad \geq \quad (A^{-1} \circ A)^{-1}. \tag{B.13}$$

Lemma B.9 ([8]). *Let Φ be a unital positive linear map from $\mathbb{M}_m$ to $\mathbb{M}_n$. Then*

$$\Phi(A^2) \;\geq\; \Phi(A)^2 \quad (A \geq 0),$$

$$\Phi(A^{-1}) \;\geq\; \Phi(A)^{-1} \quad (A > 0).$$

Theorem B.10 ([8, 9]). *Let Φ be a unital positive linear map from $\mathbb{M}_m$ to $\mathbb{M}_n$ and f an operator monotone function on $[0,\infty)$. Then for every $A \geq \mathbf{0}$,*

$$f(\Phi(A)) \quad \geq \quad \Phi(f(A)). \tag{B.14}$$

Theorem B.11 ([8, 9]). *Let Φ be a unital positive linear map from $\mathbb{M}_m$ to $\mathbb{M}_n$ and g an operator convex function on $[0.\infty)$. Then for every $A \geq \mathbf{0}$,*

$$g(\Phi(A)) \quad \leq \quad \Phi(g(A)). \tag{B.15}$$

Corollary B.12 ([8]). *Let Φ be a unital positive linear map from $\mathbb{M}_m$ to $\mathbb{M}_n$. Then*

$$\begin{aligned} \Phi(A^r) &\leq \Phi(A)^r && , A \geq \mathbf{0},\ 0 < r \leq 1; \\ \Phi(A^r) &\geq \Phi(A)^r && , A > \mathbf{0},\ -1 \leq r \leq 0 \, or\, 1 \leq r \leq 2; \\ \Phi(\log A) &\leq \log(\Phi(A)) && , A > \mathbf{0}. \end{aligned}$$

Corollary B.13 ([8]).

$$\begin{aligned} A^r \circ B^r &\leq (A \circ B)^r && , A, B \geq \mathbf{0},\ 0 < r \leq 1; \\ A^r \circ B^r &\geq (A \circ B)^r && , A, B > \mathbf{0},\ 1 \leq r \leq 0 \, or\, 1 \leq r \leq 2; \\ (\log A + \log B) \circ I &\leq \log(A \circ B) && , A, B > \mathbf{0}. \end{aligned}$$

Appendix C

ANALYTIC INEQUALITIES

Theorem C.1 (A.M–G.M.–H.M. Inequality [22]). *Let* $a_1, a_2, \ldots, a_n$ *be positive real numbers. Then*

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \quad \leq \quad \sqrt[n]{a_1 a_2 \cdots a_n} \quad \leq \quad \frac{a_1 + a_2 + \cdots + a_n}{n}. \tag{C.1}$$

That is H.M. $\leq$ G.M. $\leq$ A.M.

Theorem C.2 (Weight A.M–G.M. Inequality [22]). *Let* $a_1, a_2, \ldots, a_n$ *and* $v_1 . v_2, \ldots, v_n$ *be positive real numbers such that* $v_1 + v_2 + \cdots + v_n = 1$. *Then*

$$v_1 a_1 + v_2 a_2 + \cdots + v_n a_n \quad \leq \quad a_1^{v_1} a_n^{v_n} \cdots a_n^{v_n}. \tag{C.2}$$

Theorem C.3 (Cauchy–Schwarz Inequality [22]). *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \quad \leq \quad \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{C.3}$$

Theorem C.4 (Triangle Inequality [22]). *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be real numbers. Then*

$$\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \quad \leq \quad \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}. \tag{C.4}$$

Theorem C.5 (Jensen Inequality [22]). *Let* $a_1, a_2, \ldots, a_n, r, s$ *and* $r < s$ *and* $n \geq 2$. *Then*

$$\left(\sum_{i=1}^{n} a_k^s\right)^{\frac{1}{s}} \quad < \quad \left(\sum_{i=1}^{n} a_k^r\right)^{\frac{1}{r}}. \tag{C.5}$$

Theorem C.6 (Power Mean Inequality [22]). *Let* $a_1, a_2, \ldots, a_n, r, s$ *and* $r < s$. *Then*

$$\left(\frac{a_1^r + a_1^r + \cdots + a_n^r}{n}\right)^{\frac{1}{r}} \quad \leq \quad \left(\frac{a_1^s + a_1^s + \cdots + a_n^s}{n}\right)^{\frac{1}{s}}. \tag{C.6}$$

Theorem C.7 (Young Inequality [22]). *Let a, b, p, q be positive real number such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \quad \leq \quad \frac{a^p}{p} + \frac{b^q}{q}. \tag{C.7}$$

Theorem C.8 (Hölder Inequality [22]). *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be positive real numbers and let p, q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \quad \leq \quad (x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}}(y_1^p + y_2^p + \cdots + y_n^p)^{\frac{1}{q}}. \tag{C.8}$$

Theorem C.9 (Minkowski Inequality [22]). *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive real numbers and let $p > 1$ be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \quad \leq \quad \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}. \tag{C.9}$$

Theorem C.10 (Bernoulli Inequality [22]). *For any real number $a \geq -1$ and positive integer $n \geq 2$. Then*

$$(a + 1)^n \quad \leq \quad 1 + na. \tag{C.10}$$

Appendix D

ABSTRACT CONES

Let $\mathbb{R}_+$ be the set of non-negative reals. An *abstract cone* is analogous to a real vector space, except that we take the non-negative reals as scalars. Since the non-negative reals do not form a field, we have to replace the vector space law $v + (-v) = 0$ by a *cancellation law* $v + u = w + u \Rightarrow v = w$. We also require strictness, which means, no non-zero element has a negative.

Definition D.1. (*Abstract cone*) *An abstract cone is a set C, together with two operations $+ : C \times C \to C$ and $\cdot : \mathbb{R}_+ \times C \to C$ and a distinguished element $0 \in C$, satisfying the following laws for all $v, w, u \in C$ and $\lambda, \mu \in \mathbb{R}_+$:*

Additive laws:

$$\begin{aligned} 0 + v &= v \\ v + (w + u) &= (v + w) + u \\ v + w &= w + v \\ v + u = w + u &\Rightarrow v = w \quad \textbf{(cancellation)} \\ v + w = 0 &\Rightarrow v = w = 0 \quad \textbf{(strictness)} \end{aligned}$$

Multiplicative laws:

$$\begin{aligned} 1v &= v \\ (\lambda\mu)v &= \lambda(\mu v) \\ (\lambda + \mu)v &= \lambda v + \mu v \\ \lambda(v + w) &= \lambda v + \lambda w, \end{aligned}$$

Example D.2. $\mathbb{R}_+$ *is an abstract cone. The set*

$$\mathbb{R}_+^n = \{(x_1, \ldots, x_n) | x_1, \ldots, x_n \in \mathbb{R}_+\}$$

is an abstract cone, with the coordinatewise operations. More generally, if $C_1, \ldots, C_n$ are abstract cones, then so is $C_1 \times \ldots \times C_n$. The set of all complex Hermitian positive $n \times n$-matrices,

$$\wp_n = \{A \in \mathbb{M}(\mathbb{C}) | A = A^*, \forall v \in \mathbb{C}^n . v^* A v \geq 0\}$$

is an abstract cone. Also, for any signature $\sigma = n_1, \ldots, n_s$, *the set of positive matrix tuples* $\wp_\sigma := P_{n_1} \times \cdots \times P_{n_s}$ *is an abstract cone.*

Definition D.3. (***Cone order***) *Let* C *be an abstract cone. The* ***cone order*** *is defined by setting* $v \sqsubseteq w$ *if there exists* $u \in C$ *such that* $v + u = w$. *Note that the cone order is a partial order. If* $v \sqsubseteq w$, *then the element* u *such that* $v + u = w$ *is necessarily unique, and thus we may also write* $u = w - v$.

You can see more concepts of abstract cones in [23].

Appendix E

MORE ON THE FUNCTIONAL CALCULUS

E.1 Function of One Variable

Consider a square matrix A. It is natural to define A^2 as the matrix AA and A^3 as the matrix AAA. In continuation of this idea we set $A^k = A^{k-1}A$ for $k \geq 2$ and

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_k A^k$$

for a polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$. We have thus learned to take a polynomial of a square matrix. Notice that $p(A)$ and A commute. If in particular A is a real symmetric (and thus Hermitian) matrix of order n, we may apply the spectral theorem and write A on the form

$$A = QDQ^{-1}$$

where Q is an orthogonal matrix, that is $Q^{-1} = Q^{\mathrm{T}}$, and D is a diagonal matrix with the eigenvalues of A counting multiplicity as diagonal elements. Setting

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

we calculate

$$A^2 = QDQ^{-1}QDQ^{-1} = QD^2Q^{-1} = Q \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} Q^{-1}$$

and more generally

$$A^k = QD^kQ^{-}1 = Q \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} Q^{-1}$$

and

$$p(A) = Q_p(D)Q^{-1} = Q \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{bmatrix} Q^{-1}$$

for any polynomial. We may use this construction to define the functional calculus for any function f defined on the spectrum of A simply by setting

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} Q^{-1}$$

If f is a polynomial, then this definition of $f(A)$ coincides with the elementary calculation given above. There is a certain ambiguity in the diagonalisation of A because the diagonal elements in D can be permutated corresponding to permutations of the columns in Q. However, the definition of $f(A)$ is unaffected by this ambiguity. It becomes easier if we consider the spectral representation of A given by

$$A = \sum_{i=1}^{p} \lambda_i P_i$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of A (not counting multiplicity) and $P_1, \ldots, P_p$ are orthogonal projections with the identity matrix as sum. This representation is unique and

$$f(A) = \sum_{i=1}^{p} f(\lambda_i) P_i$$

The functional calculus can be extended to self-adjoint operators acting on an infinite-dimensional Hilbert space, but we will consider the theory only for matrices in order to avoid unnecessary complications.

E.2 Function of Several Variables

Let us consider the function $f(t,s) = ts$ of two variables and two Hermitian matrices A and B of orders n and m. We would like to define the matrix $f(A, B)$. What would be a good definition? Korányi [27] proposed that the definition should be the Kronecker product

$$f(A,B) \quad := \quad A \otimes B$$

of A and B. If

$$A \quad = \quad \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$$

then the Kronecker product is given by the block matrix

$$A \otimes B \quad = \quad \begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ldots & a_{nn}B \end{bmatrix}$$

which is of order nm. If f can be written as a product $f(t,s) = f_1(t)f_2(s)$ of two functions each depending of only one variable, then we set

$$f(A,B) = f_1(A) \otimes f_2(B).$$

The definition can then be extended by linearity in f (we want the mapping $f \to f(A,B)$ to be linear) and continuity. We may also extend the definition to functions of more than two variables and obtain:

Definition E.2.1. *Let $f : I_1 \times \cdots \times I_k \to \mathbb{R}$ be a real function (of any kind) defined on a product of real intervals, and let $x = (x_1, \ldots, x_k)$ be a k-tuple of Hermitian matrices such that the eigenvalues of x_i are contained in I_i for $i = 1, \ldots, k$. We say that such a k-tuple is in the domain of f. If*

$$x_i = \sum_{t_i=1}^{p_i} \lambda_{t_i i} P_{t_i i} \quad ; \quad i = 1, \ldots, k$$

is the spectral resolution of x_i, we define

$$f(x) := \sum_{t_1=1}^{p_1} \cdots \sum_{t_k=1}^{p_k} f(\lambda_{t_1}, \ldots, \lambda_{t_k}) P_{t_1 1} \otimes \cdots \otimes P_{t_k k}$$

as the function f applied to the k-tuple $x = (x_1, \ldots, x_k)$.

If the k-tuple $x = (x_1, \ldots, x_k)$ is of order $(n_1, \ldots, n_k)$, then $f(x)$ is a Hermitian matrix of order $n_1 \cdot \ldots \cdot n_k$. We shall henceforth consider only real and symmetric matrices, although all statements remain valid for Hermitian matrices and in many cases also for bounded self-adjoint operators on a Hilbert space. It is not obvious to extend the concept of monotonicity from functions of one variable to functions of several variables. This is so because there is no natural order structure on tuples of matrices. It is completely trivial to define the notion of matrix convexity for functions of several variables, simply because the definition of matrix convexity only involves the order structure for matrices.

Definition E.2.2. *A function $f : I_1 \times \cdots \times I_k \to \mathbb{R}$ defined on a product of real intervals is said to be matrix convex of order $(n_1, \ldots, n_k)$, if the matrix inequality*

$$f(\epsilon x + (1-\epsilon)y) \leq \epsilon f(x) + (1-\epsilon) f(y)$$

holds for any $\epsilon \in [0,1]$ and all k-tuples of matrices $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ of order $(n_1, \ldots, n_k)$ in the domain of f.

The definition is meaningful since also $\epsilon x + (1 - \epsilon)y$ is contained in the domain of f. We say that f is operator convex, if f is matrix convex of every order $(n_1, \ldots, n_k)$.

See more advanced concepts in Hansen [26].

Appendix F

HISTORY AND APPLICATIONS OF THE KRONECKER PRODUCT

F.1 History of the Kronecker Product

The following information is interpreted from the paper "On the History of the Kronecker Product" by Henderson [24]. You can see it also in Schäcke [25].

Apparently, the first documented work on Kronecker products was written by Johann Georg Zehfuss between 1858 and 1868. It was he, who established the determinant result

$$\det(A \otimes B) = \left(\det(A)\right)^{m}\left(\det(B)\right)^{n}, \tag{F.1}$$

where A and B are square matrices of dimension n and m, respectively.

Zehfuss was acknowledged by Muir (1881) and his followers, who called the determinant $\det(A \otimes B)$ the Zehfuss determinant of A and B.

However, in the 1880's, Kronecker gave a series of lectures in Berlin, where he introduced the result (F.1) to his students. One of these students, Hensel, acknowledged in some of his papers that Kronecker presented (F.1) in his lectures.

Later, in the 1890's, Hurwitz and Stéphanos developed the same determinant equality and other results involving Kronecker products such as:

$$\begin{aligned} I_m \otimes I_n &= I_{mn}, \\ (A \otimes B)(C \otimes D) &= (AC) \otimes (BD), \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}, \\ (A \otimes B)^{\mathrm{T}} &= A^{\mathrm{T}} \otimes B^{\mathrm{T}}. \end{aligned}$$

Hurwitz used the symbol × to denote the operation. Furthermore, Stéphanos derives the result that the eigenvalues of $A \otimes B$ are the products of all eigenvalues of A with all eigenvalues of B.

There were other writers such as Rados in the late 1800's who also discovered property (F.1) independently. Rados even thought that he wrote the original paper on property (F.1) and claims it for himself in his paper published in 1900, questioning Hensel's contributing it to Kronecker.

Despite Rados' claim, the determinant result (F.1) continued to be associated with Kronecker. Later on, in the 1930's, even the definition of the matrix operation $A \otimes B$ was associated with Kronecker's name.

Therefore today, we know the Kronecker product as "Kronecker" product and not as "Zehfuss", "Hurwitz", "Stéphanos", or "Rados" product.

F.2 Applications of the Kronecker Product

The following facts are come from Schäcke [25].

The Kronecker product can be used to present linear equations in which the unknowns are matrices. Examples for such equations are:

$$AX = B, \tag{F.2}$$

$$AX + XB = C, \tag{F.3}$$

$$AXB = C, \tag{F.4}$$

$$AX + YB = C. \tag{F.5}$$

These equations are equivalent to the following systems of equations:

$$(I \otimes A)vecX = vecB \quad \text{corresponds to (F.2),} \tag{F.6}$$

$$[(I \otimes A) + (B^{\mathrm{T}} \otimes I)]vecX = vecC \quad \text{corresponds to (F.3),} \tag{F.7}$$

$$(B^{\mathrm{T}} \otimes A)vecX = vecC \quad \text{corresponds to (F.4),} \tag{F.8}$$

$$(I \otimes A)vecX + (B^{\mathrm{T}} \otimes I)vecY = vecC \quad \text{corresponds to (F.5).} \tag{F.9}$$

Note that with the notation of the Kronecker sum, equation (F.7) can be written as

$$(A \oplus B^{\mathrm{T}})vecX = vecC.$$

The above properties of the Kronecker product have some very nice applications. Equation (F.3) is known to numerical linear algebraists as the **Sylvester equation**. For given $A \in \mathbb{M}_m$, $B \in \mathbb{M}_n$, $C \in \mathbb{M}_{m,n}$, one wants to find all $X \in \mathbb{M}_{m,n}$ which satisfy the equation. This system of linear equations plays a central role in control theory, Poisson equation solving, or invariant subspace computation to name just a few applications. In the case of all matrices being square and of the same dimension, equation (F.3) appears frequently in system theory.

The question is often, whether there is a solution to this equation or not. In other words one wants to know if the Kronecker sum $A \oplus B^{\mathrm{T}}$ is nonsingular. From our knowledge about eigenvalues of the Kronecker sum, we can immediately conclude that this matrix is nonsingular if and only if the spectrum of A has no eigenvalue in common with the negative spectrum of B :

$$\sigma(A) \cap (-\sigma(B)) \quad = \quad \varnothing$$

An important special case of the Sylvester equation is the **Lyapunov equation**:

$$XA + A^{*}X \quad = \quad H,$$

where $A, H \in \mathbb{M}_n$ are given and H is Hermitian. This special type of matrix equation arises in the study of matrix stability. A solution of this equation can be found by transforming it into the equivalent system of equations:

$$[(A^{\mathrm{T}} \otimes I) + (I \otimes A^{*})]vec(X) \quad = \quad vec(H),$$

which is equivalent to

$$[A^{*} \oplus A^{\mathrm{T}}]vec(X) \quad = \quad vec(H).$$

It has a unique solution X if and only if A^{*} and $-A^{\mathrm{T}}$ have no eigenvalues in common. For example, consider the computation of the Nesterov-Todd search direction. The following equation needs to be solved:

$$\frac{1}{2}(D_V V + V D_V) = \mu I - V^2,$$

where V is a real symmetric positive definite matrix and the right hand side is real and symmetric, therefore Hermitian. Now, we can conclude that this equation has a unique symmetric solution since V is positive definite, and therefore V and $-V^{\mathrm{T}}$ have no eigenvalues in common.

Another application of the Kronecker product is the commutativity equation. Given a matrix $A \in \mathbb{M}_n$; we want to know all matrices $X \in \mathbb{M}_n$ that commute with A, i.e. $\{X : AX = XA\}$. This can be rewritten as $AX - XA = 0$, and hence as

$$[(I \otimes A) - (A^{\mathrm{T}} \otimes I)]vec(X) = 0.$$

Now we have transformed the commutativity problem into a null space problem which can be solved easily.

Graham mentions another interesting application of the Kronecker product. Given $A \in \mathbb{M}_n$ and $\mu \in \mathbb{K}$, we want to know when the equation

$$AX - XA = \mu K \tag{F.10}$$

has a nontrivial solution. By transforming the equation into

$$[(I \otimes A) - (A^{\mathrm{T}} \otimes I)]vec(X) = \mu vec(X).$$

which is equivalent to

$$[A \oplus (-A^{\mathrm{T}})]vec(X) = \mu vec(X),$$

we find that μ has to be an eigenvalue of $[A \oplus (-A^{\mathrm{T}})]$, and that all X satisfying (F.10) are eigenvectors of $[A \oplus (-A^{\mathrm{T}})]$ (after applying vec to X). From our results on the eigenvalues and eigenvectors of the Kronecker sum, we know that those X are therefore Kronecker products of eigenvectors of A^{T} with the eigenvectors of A.

This also ties in with our result on the commutativity equation. For $\mu = 0$; we get that 0 has to be an eigenvalue of $[A \oplus (-A^{\mathrm{T}})]$ in order for a nontrivial commutating X to exist.

There are many other applications of the Kronecker product in e.g. signal processing, image processing, quantum computing and semidefinite programming.

Appendix G

FAMOUS DETERMINANT INEQUALITIES

Corollary G.1 (Hadamard's Inequality [5]). *For any matrix* $B = [b_{ij}] \in \mathbb{M}_n$,

$$|\det(B)| \quad \leq \quad \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |b_{ij}|^2 \right)^{1/2} \tag{G.1}$$

and

$$|\det(B)| \quad \leq \quad \prod_{j=1}^{n} \left(\sum_{i=1}^{n} |b_{ij}|^2 \right)^{1/2} . \tag{G.2}$$

Furthermore, when B is nonsingular, then equality holds if and only if the rows (respectively, columns) of B are orthoganal.

Theorem G.2 (Fisher's Inequality [5]). *Suppose that*

$$P \quad = \quad \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

is positive definite matrix that is partitioned so that A and C are square and nonempty. Then

$$\det(P) \quad \leq \quad \det(A)\,\det(B). \tag{G.3}$$

Theorem G.3 (Ostrowski-Taussky's Inequality [5]). *If* $A \in \mathbb{M}_n$ *is such that* $H(A) := \frac{1}{2}(A + A^*)$ *is positive definite, then*

$$\det\left(H(A)\right) \quad \leq \quad |\det(A)|. \tag{G.4}$$

Equality holds if and only if A is Hermitian.

Theorem G.4 (Minkowski's Inequality [5]). *If* $A, B \in \mathbb{M}_n$ *are positive definite, then*

$$[\det(A + B)]^{1/n} \quad \geq \quad \det(A)^{1/n} + \det(B)^{1/n}. \tag{G.5}$$

Appendix H

EIGENVALUE INEQUALITIES

In this appendix we arrange the eigenvalues of matrices in the decreasing order.

Theorem H.1 ([9]). *Let A, B be positive semidefinite with $\lambda_1(A) \leq 1$, then for $0 \leq s \leq 1$*

$$\lambda_1(A^s B^s) \quad \leq \quad 1. \tag{H.1}$$

Theorem H.2 ([9]). *Let A, B be positive semidefinite with $\lambda_1(A) \leq 1$, then for $0 \leq s \leq 1$*

$$\lambda_1(A^s B^s) \quad \leq \quad (\lambda_1(AB))^s. \tag{H.2}$$

Theorem H.3 ([9]). *Let A, B be positive semidefinite with $\lambda_1(A) \leq 1$, then for $t \geq 1$*

$$(\lambda_1(AB))^t \quad \leq \quad \lambda_1(A^t B^t). \tag{H.3}$$

Theorem H.4 ([9]). *Let A, B be positive semidefinite. Then*

1. $[\lambda_1(A^{1/t} B^{1/t})]^t$ *is a monotonically decreasing function of t on* $(0, \infty)$.
2. $[\lambda_1(A^t B^t)]^{1/t}$ *is a monotonically increasing function of t on* $(0, \infty)$.

Appendix I

THE WELL-KNOWN TRACE INEQUALITIES

Theorem I.1 (Golden-Thompson Inequality [9]). *Let A, B be Hermitian matrices. Then*

$$\operatorname{tr}(e^{A+B}) \quad \leq \quad \operatorname{tr}(e^{A}e^{B}). \tag{I.1}$$

Theorem I.2 (Lieb-Thirring Inequality [9]). *Let $A, B \in \mathbb{M}_n$ be positive semidefinite and let m, n be positive integer such that $m \geq k$. Then*

$$\begin{aligned} \operatorname{tr}\left((A^m B^m)^k\right) \quad &\leq \quad \operatorname{tr}\left((A^m B^m)^k\right) && \text{(I.2)} \\ \text{the special case} \quad \operatorname{tr}\left((AB)^m\right) \quad &\leq \quad \operatorname{tr}(A^m B^m). && \text{(I.3)} \end{aligned}$$

Theorem I.3 ([9]). *Let A, B be Hermitian matrices. Then for every positive integer m*

$$\begin{aligned} |\operatorname{tr}\left((AB)^{2m}\right)| \quad &\leq \quad \operatorname{tr}(A^{2m}B^{2m}), && \text{(I.4)} \\ \operatorname{tr}\left((A^m B^m)^2\right) \quad &\leq \quad \operatorname{tr}(A^{2m}B^{2m}), && \text{(I.5)} \\ \operatorname{tr}\left((AB)^{4m}\right) \quad &\leq \quad \operatorname{tr}\left((A^{2m}B^{2m})^2\right). && \text{(I.6)} \end{aligned}$$

Theorem I.4 ([9]). *Let A, B be positive semidefinite matrices and let s, t be positive real numbers with $t \geq 1$. Then*

$$\operatorname{tr}\left((B^{1/2}AB^{1/2})^{st}\right) \quad \leq \quad \operatorname{tr}\left((B^{t/2}A^{t}B^{t/2})^{s}\right). \tag{I.7}$$

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