SIMPLE ITERATIVE ORDINARY DIFFERENTIAL EQUATIONS

PIMPAK PHATARANAVIK

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หัวข้อวิทยานิพนธ์	สมการเชิงอนุพันธ์สามัญซ้ำเชิงเดียว
นักศึกษา	นางสาวพิมพ์ภัค ภัทรนาวิก
รหัสประจำตัว	40065300
ปริญญา	วิทยาศาสตรมหาบัณฑิต
สาขาวิชา	คณิตศาสตร์ประยุกต์
พ.ศ.	2542
อาจารย์ผู้ควบคุมวิทยานิพนธ์	รศ.ดร. ไมตรี โพธิ์สุข

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เนื้อหาของวิทยานิพนธ์เล่มนี้ กล่าวถึงการมีผลเฉลยหนึ่งเดียวของสมการเชิงอนุพันธ์สามัญที่อยู่ในรูป

$$y'(x) = y^m(x) \quad , \quad x \in [0,a]$$

กับเงื่อนไขค่าเริ่มต้น

$$y(0) = c$$

เมื่อ m เป็นจำนวนเต็มบวกที่มากกว่าหนึ่ง และ

$$y^2(x) = y(y(x))$$
$$y^3(x) = y(y^2(x)) = y(y(y(x)))$$
$$\vdots$$
$$y^m(x) = y(y^{m-1}(x))$$

พร้อมทั้งศึกษาถึงวิธีการหาผลเฉลยและผลเฉลยเชิงตัวเลขของสมการเชิงอนุพันธ์สามัญนี้

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Student	Miss Pimpak Phataranavik
Student ID.	40065300
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ABSTRACT

In this thesis we will discuss the uniqueness and existence of the solutions of the ordinary differential equations

$$y'(x) = y^m(x) \quad , \quad x \in [0,a]$$

with the initial condition

$$y(0) = c$$

where m is a positive integer greater than 1 and

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y^2(x)) = y(y(y(x))) \\ &\vdots \\ y^m(x) &= y(y^{m-1}(x)). \end{aligned}$$

And we shall give the method to obtain the analytic solutions and the numerical method to obtain the numerical solutions of the equations.

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Pimpak Phataranavik

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Introduction

The functional differential equations is one type of the differential equations. The iterative ordinary differential equations is one field of the functional differential equations. There are many mathematicians who gave their efforts in the study of this field. We may group the study of this field of the functional differential equations into three main types.

The first type of the functional differential equations is so called "***the differential equation with deviating argument***" which has the classical form

$$y'(x)=F(x,y(x)y(q(x))). \tag{1.1}$$

In [1], W.B. Fite proved the existence theorem for the solution of the ordinary differential equation

$$y^{(n)}(x)+\sum_{i=1}^{n}p_i(x)y^{(n-1)}(x)+r(x)y(q(x))=0. \tag{1.2}$$

In [2], S. Doss and S.K. Nasr found the conditions that make the functional equation

$$\frac{dy}{dx}=f(x,y(x),y(x+h)),h>0 \tag{1.3}$$

with the initial condition

$$y(x_0)=y_0 \tag{1.4}$$

has a unique solution.

In [3], W.R. Utz solved the equation

$$f'(x)=af(g(x)). \tag{1.5}$$

In [4], D.R. Anderson proved the existence theorem for a solution of the equation

$$f'(x)=F(x,f(g(x))). \tag{1.6}$$

The second type of the functional differential equations is so called "***iterative ordinary differential equation***" which has the classical form

$$y'(x)=G\left(x,y(x),y^2(x),\ldots,y^m(x)\right) \quad (1.7)$$

where

$$\begin{aligned} y^2(x)&=y(y(x)) \\ y^3(x)&=y\left(y^2(x)\right)=y(y(y(x))) \\ &\vdots \\ y^m(x)&=y\left(y^{m-1}(x)\right). \end{aligned} \quad (1.8)$$

For this differential equation, G. Barba [5] found the solution of the ordinary differential equation

$$f(x)f'(x)=f(f(x)). \quad (1.9)$$

In [6], [7] and [8], A. Pelczar proved the existent and unique solution of the equation

$$\frac{dy}{dx}=f(x,y(x),y(y(x))) \quad (1.10)$$

on the interval $[0,a]$ with the initial condition

$$y(0)=c. \quad (1.11)$$

And in [9], M. Podisuk proved the existent and unique solution of the ordinary differential equation

$$\frac{dy}{dx}=f\left(x,y(x),y^2(x),\ldots,y^m(x)\right) \quad (1.12)$$

on the interval $[0,a]$ with the initial condition

$$y(0)=c \quad (1.13)$$

where $y^k(x)$ denotes the k^{th} iteration of the (unknown) function $y(x)$ and satisfies (1.12) - (1.13). This work is the main interest of the author.

The third type of the functional differential equations is the mixed type of the first type and the second type. The following equation is one of this type:

$$y'(x)=H(x,y(x),y(g(x)),y(q(y(x)))). \quad (1.14)$$

Such equation have been considered by V.P. Skripnik [10] and [11].

In this thesis, we shall prove the existence and uniqueness theorems for the solution of the iterative differential equation

$$y'(x) = y^m(x) \tag{1.15}$$

which is so call ***" simple iterative ordinary differential equation "***,

on the interval $[0,a]$ with the initial condition

$$y(0) = c \tag{1.16}$$

where $y^k(x)$ denotes the k^{th} iteration of the (unknown) function $y(x)$ and satisfies (1.15) - (1.16).

We shall group the simple iterative ordinary differential equations into three types as follows :

The first type is ***the first order simple iterative ordinary differential equations***

$$y'(x) = y^m(x) \tag{1.17}$$

on the interval $[0,a]$ with the initial condition

$$y(0) = c \tag{1.18}$$

where a, c are positive real numbers,

m is a positive integer greater than 1 and

$y^k(x)$ denotes the k^{th} iteration of the (unknown) function $y(x)$ and satisfies (1.17) -(1.18).

The second type is ***the system of the first order simple iterative ordinary differential equations***

$$y'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix} = \begin{bmatrix} y_1{}^m(x) \\ y_2{}^m(x) \\ \vdots \\ y_n{}^m(x) \end{bmatrix} = y^m(x) \tag{1.19}$$

on the interval $[0,a]$ with the initial conditions

$$y(0)=\begin{bmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_n(0) \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}=c \tag{1.20}$$

where a, c_i are positive real numbers for $i=1,2,\ldots,n$,

m is a positive integer greater than 1 and

$$y^2(x)=y(y(x))=\begin{bmatrix} y_1(y_1(x)) \\ y_2(y_2(x)) \\ \vdots \\ y_n(y_n(x)) \end{bmatrix}$$

$$y^3(x)=y(y^2(x))=y(y(y(x)))=\begin{bmatrix} y_1(y_1(y_1(x))) \\ y_2(y_2(y_2(x))) \\ \vdots \\ y_n(y_n(y_n(x))) \end{bmatrix}$$

$$\vdots$$

$$y^m(x)=y(y^{m-1}(x))=\begin{bmatrix} y_1(y_1^{m-1}(x)) \\ y_2(y_2^{m-1}(x)) \\ \vdots \\ y_n(y_n^{m-1}(x)) \end{bmatrix}. \tag{1.21}$$

The third type is ***the higher order simple iterative ordinary differential equations***

$$y^{(n)}(x)=y^m(x) \tag{1.22}$$

on the interval $[0,a]$ with the initial conditions

$$y(0)=c_1, \quad y'(0)=c_2, \quad y''(0)=c_3, \ldots, \quad y^{(n-1)}(0)=c_n \tag{1.23}$$

such that we shall change to the system of the first order simple iterative ordinary differential equations

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_{n-1}'(x) \\ y_n'(x) \end{bmatrix}=\begin{bmatrix} y_2(x) \\ y_3(x) \\ \vdots \\ y_n(x) \\ y^m(x) \end{bmatrix}. \tag{1.24}$$

with the initial conditions

$$y(0)=\begin{bmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_n(0) \end{bmatrix}=\begin{bmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad (1.25)$$

where a, c_i are positive real numbers for $i = 1,2,\ldots,n$,

m is a positive integer greater than 1 and

$$\begin{aligned} y_1(x) &= y(x) \\ y_2(x) &= y_1'(x) = y'(x) \\ y_3(x) &= y_2'(x) = y''(x) \\ &\vdots \\ y_{n-1}(x) &= y_{n-2}'(x) = y^{(n-2)}(x) \\ y_n'(x) &= y^{(n)}(x) = y^m(x). \end{aligned} \qquad (1.26)$$

In Chapter 2, we shall determine the existence and uniqueness theorems for the solutions of the first order simple iterative ordinary differential equations, while the existence and uniqueness theorems for the solutions of the system of the first order simple iterative ordinary differential equations and the higher order simple iterative ordinary differential equations are determined in Chapter 3.

In Chapter 4, we shall find the solutions of some simple iterative ordinary differential equations for all three types.

In Chapter 5, we shall look for a numerical method for solving the simple iterative ordinary differential equations.

Chapter 2

First Order Simple Iterative Ordinary Differential Equations

In this chapter, we shall discuss the uniqueness and existence theorems for the solutions of the first order simple iterative ordinary differential equations

$$y'(x) = y^m(x) \tag{2.1}$$

on the interval $[0,a]$ with the initial condition

$$y(0) = c \tag{2.2}$$

where a, c are positive real numbers,

m is a positive integer greater than 1 and

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y^2(x)) = y(y(y(x))) \\ &\vdots \\ y^m(x) &= y(y^{m-1}(x)). \end{aligned} \tag{2.3}$$

By a solution of the initial value problem (2.1) - (2.2), we mean a function $y(x)$ in class $C^1[0,a]$ satisfying (2.1) and (2.2) and its value also in $[0,a]$. Thus, the initial value problem (2.1) - (2.2) is equivalent to the integral equation

$$y(x) = c + \int_0^x y^m(t)\,dt \qquad \text{for } x \in [0,a]. \tag{2.4}$$

2.1 Uniqueness of solution

Let $y(x)$ be a function in class C^1 map $[0,a]$ to $[0,a]$, that is

$$|y(x)| \le a \qquad \text{for all } x \in [0,a] \tag{2.5}$$

and let

$$|y(x) - y(\bar{x})| \le K|x - \bar{x}| \tag{2.6}$$

for all $x, \bar{x} \in [0,a]$ and for some positive constant K

and let $$S_m = K^{m-2} + K^{m-3} + \ldots + K^2 + K + 1 \tag{2.7}$$

$$T_m = K^{m-1} + K^{m-2} + \ldots + K^2 + K + 1 \tag{2.8}$$

then we have the following lemma.

Lemma 2.1.1. Let $u(x)$ and $v(x)$ be any functions in class C^1 map $[0,a]$ to $[0,a]$ which satisfy the conditions (2.5) and (2.6). Then we have

$$|u^m(x) - v^m(x)| \le K^{m-1}w(x) + K^{m-2}w(v(x)) + K^{m-3}w(v^2(x)) + \ldots + Kw(v^{m-2}(x)) + w(v^{m-1}(x))$$

where $w(x) = |u(x) - v(x)|$,

m is a positive integer greater than 1 and

$$\begin{matrix} u^2(x) = u(u(x)) & & v^2(x) = v(v(x)) \\ u^3(x) = u(u^2(x)) = u(u(u(x))) & , & v^3(x) = v(v^2(x)) = v(v(v(x))) \\ \vdots & & \vdots \\ u^m(x) = u(u^{m-1}(x)) & & v^m(x) = v(v^{m-1}(x)). \end{matrix}$$

Proof. Let $u(x)$ and $v(x)$ be any functions in class C^1 map $[0,a]$ to $[0,a]$ which satisfy the conditions (2.5) and (2.6) and let $w(x) = |u(x) - v(x)|$.

We shall prove the assertion by induction.

$$\begin{aligned} |u^2(x) - v^2(x)| &= |u(u(x)) - v(v(x))| \\ &= |u(u(x)) - u(v(x)) + u(v(x)) - v(v(x))| \\ &\le |u(u(x)) - u(v(x))| + |u(v(x)) - v(v(x))| \\ &\le K|u(x) - v(x)| + w(v(x)) \\ &\le Kw(x) + w(v(x)). \end{aligned}$$

Assume that

$$|u^n(x) - v^n(x)| \le K^{n-1}w(x) + K^{n-2}w(v(x)) + K^{n-3}w(v^2(x)) + \ldots + Kw(v^{n-2}(x)) + w(v^{n-1}(x)).$$

Let us consider

$$\begin{aligned} |u^{n+1}(x) - v^{n+1}(x)| &= |u(u^n(x)) - v(v^n(x))| \\ &= |u(u^n(x)) - u(v^n(x)) + u(v^n(x)) - v(v^n(x))| \\ &\le |u(u^n(x)) - u(v^n(x))| + |u(v^n(x)) - v(v^n(x))| \\ &\le K|u^n(x) - v^n(x)| + w(v^n(x)) \\ &\le K^n w(x) + K^{n-1}w(v(x)) + K^{n-2}w(v^2(x)) + \ldots \\ &\quad + K^2 w(v^{n-2}(x)) + Kw(v^{n-1}(x)) + w(v^n(x)). \end{aligned}$$

This completes the proof of Lemma 2.1.1. #

Remark 2.1.2. If $u(x)$ and $v(x)$ are functions which satisfy the conditions of Lemma 2.1.1. Then it can be seen that $w(v^{m-1}(x)) \le a$ where m is a positive integer greater than 1.

Theorem 2.1.3. If $aS_m < e^{-aK^{m-1}}$ and $y(x)$ satisfies the conditions of Lemma 2.1.1, then there exists at most one solution of the initial value problem (2.1) - (2.2).

Proof. Let $u(x)$ and $v(x)$ be two solutions of the initial value problem (2.1) - (2.2).

Hence $$u(x) = c + \int_0^x u^m(t)\,dt$$

and $$v(x) = c + \int_0^x v^m(t)\,dt \qquad \text{for } x \in [0,a].$$

Let $w(x) = |u(x) - v(x)|$ then we have

$$\begin{aligned} w(x) &= \left| \int_0^x u^m(t)dt - \int_0^x v^m(t)dt \right| \\ &\le \int_0^x \left| u^m(t) - v^m(t) \right| dt \\ &\le K^{m-1}\int_0^x w(t)dt + K^{m-2}\int_0^x w(v(t))dt + K^{m-3}\int_0^x w(v^2(t))dt + \ldots \\ &\quad + K\int_0^x w(v^{m-2}(t))dt + \int_0^x w(v^{m-1}(t))dt \qquad (2.9) \\ &\le K^{m-1}\int_0^x w(t)dt + K^{m-2}\int_0^x a\,dt + K^{m-3}\int_0^x a\,dt + \ldots + K\int_0^x a\,dt + \int_0^x a\,dt . \end{aligned}$$

$$\begin{aligned} \therefore \quad w(x) &\le K^{m-1}\int_0^x w(t)dt + a^2\left(K^{m-2} + K^{m-3} + \cdots + K + 1\right) \\ &\le K^{m-1}\int_0^x w(t)dt + a^2 S_m . \qquad (2.10) \end{aligned}$$

From (2.10) and well known theorems concerning the differential (integral) inequalities , Gronwall – Reid - Bellman inequality , it follows that

$$w(x) \le a^2 S_m e^{xK^{m-1}} \le a^2 S_m e^{aK^{m-1}} . \qquad (2.11)$$

From (2.9) and (2.11) it follows that

$$w(x) \le K^{m-1}\int_0^x w(t)dt + ae^{aK^{m-1}}(aS_m)^2$$
$$\le a\left(aS_m e^{aK^{m-1}}\right)^2.$$

By induction on n, it can be shown that

$$w(x) \le a\left(aS_m e^{aK^{m-1}}\right)^n \quad , \quad n = 1,2,3,\dots .$$

Since $aS_m < e^{-aK^{m-1}}$, so $aS_m e^{aK^{m-1}} < 1$ and $w(x)$ is upperly bounded by the sequence which tends to zero as $n \to \infty$. Since $w(x) \ge 0$, it must be equal to zero. This ends the proof of Theorem 2.1.3. #

Theorem 2.1.4. If $aT_m < 1$ and $y(x)$ satisfies the conditions of Theorem 2.1.3, then there exists at most one solution of the initial value problem (2.1) - (2.2).

Proof. Assume that $u(x)$ and $v(x)$ are two solutions of the initial value problem (2.1) - (2.2) and let

$$w(x) = |u(x) - v(x)|$$

and

$$P = \max_{x\in[0,a]} |u(x) - v(x)|.$$

It follows from the proof of Theorem 2.1.3 that

$$w(x) \le K^{m-1}\int_0^x w(t)dt + K^{m-2}\int_0^x w(v(t))dt + K^{m-3}\int_0^x w(v^2(t))dt + \dots$$
$$+ K\int_0^x w(v^{m-2}(t))dt + \int_0^x w(v^{m-1}(t))dt \tag{2.12}$$
$$\le K^{m-1}\int_0^x P\,dt + K^{m-2}\int_0^x P\,dt + K^{m-3}\int_0^x P\,dt + \dots + K\int_0^x P\,dt + \int_0^x P\,dt.$$

$$\therefore \quad w(x) \le aP\left(K^{m-1} + K^{m-2} + K^{m-3} + \dots + K + 1\right) = PaT_m. \tag{2.13}$$

From (2.12) and (2.13) it follows that

$$w(x) \le Pa^2T_m\left(K^{m-1} + K^{m-2} + K^{m-3} + \dots + K + 1\right) = P(aT_m)^2.$$

By induction on n, it can be shown that

$$w(x) \le P(aT_m)^n \quad , \quad n = 1,2,3,\dots .$$

Since $aT_m < 1$, P must be zero. This ends the proof of Theorem 2.1.4. #

2.2 Existence of solution

Let us assume that

$$c + a^2 \leq a \tag{2.14}$$

and $$a\left(K^{m-1} + K^{m-2} + \ldots + K + 1\right) < 1 \tag{2.15}$$

and let consider the following sequences :

$$y_{1,n+1}(x) = c + \int_0^x y_{1,n}^m(t)\,dt \tag{2.16.1}$$

$$y_{2,n+1}(x) = c + \int_0^x y_{2,n}^{m-1}\left(y_{2,n+1}(t)\right)dt \tag{2.16.2}$$

$$y_{3,n+1}(x) = c + \int_0^x y_{3,n}^{m-2}\left(y_{3,n+1}^2(t)\right)dt \tag{2.16.3}$$

$$\vdots$$

$$y_{m,n+1}(x) = c + \int_0^x y_{m,n}\left(y_{m,n+1}^{m-1}(t)\right)dt \tag{2.16.m}$$

where $n = 0,1,2,\ldots$ and

$y_{1,0}(x), y_{2,0}(x), \ldots, y_{n,0}(x)$ are fixed functions of the class C^1 map $[0,a]$ to $[0,a]$ such that

$$\left|y'_{1,0}(x)\right|, \left|y'_{2,0}(x)\right|, \ldots, \left|y'_{n,0}(x)\right| \leq a.$$

Hence, we have the following theorem.

Theorem 2.2.1. Let the assumptions of Theorem 2.1.3 , the conditions (2.14) and (2.15) hold. Then the sequences (2.16.1) - (2.16.m) converge uniformly to the (unique) solution $y = y(x)$ of the initial value problem (2.1) - (2.2).

Proof. We put

$$Y_{1,n} = \max_{x\in[0,a]} \left| y_{1,n}(x) - y_{1,n-1}(x) \right|$$

$$Y_{2,n} = \max_{x\in[0,a]} \left| y_{2,n}(x) - y_{2,n-1}(x) \right|$$

$$\vdots$$

$$Y_{m,n} = \max_{x\in[0,a]} \left| y_{m,n}(x) - y_{m,n-1}(x) \right| \quad \text{where } n = 1,2,3,\ldots .$$

It can be shown by induction n on that

$$
\begin{aligned}
Y_{1,n} &\le W_1^{n-1} Y_{1,1} \\
Y_{2,n} &\le W_2^{n-1} Y_{2,1} \\
&\vdots \\
Y_{m,n} &\le W_m^{n-1} Y_{m,1}
\end{aligned}
$$

where $W_1 = \dfrac{aU_1}{V_1},\ \ W_2 = \dfrac{aU_2}{V_2}, \ldots,\ \ W_n = \dfrac{aU_m}{V_m}$,

$$
\begin{aligned}
& U_1 = K^{m-1} + K^{m-2} + \ldots + K + 1 \ , \\
& U_2 = K^{m-2} + K^{m-3} + \ldots + K + 1 \ , \\
& U_3 = K^{m-3} + K^{m-4} + \ldots + K + 1 \ , \\
& \vdots \\
& U_m = 1 \ , \\
& V_1 = 1 \ , \\
& V_2 = 1 - aK^{m-1} \ , \\
& V_3 = 1 - aK^{m-1} - aK^{m-2} \ , \\
& \vdots \\
& V_m = 1 - aK^{m-1} - aK^{m-2} - aK^{m-3} - \ldots - aK \, .
\end{aligned}
$$

Since $aT_m < 1$, so $W_j < 1$ and W_j^{n-1} tends to zero as $n \to \infty$ for $j = 1,2,\ldots,m$. Hence $Y_{j,n}$ tends to zero as $n \to \infty$ for $j = 1,2,\ldots,m$. This means that for $i = 1,2,\ldots,m$, if $\{y_{i,n_j}(z)\}$ is a subsequence of $\{y_{i,n}(z)\}$ tending uniformly to some $\bar{y}_i(z)$ then $\bar{y}_i(z)$ is a solution of the initial value problem (2.1) - (2.2). Since the family $\{y_{i,n}\}$ is the Arzela-Ascoli family, thus for every subsequence $\{y_{i,n_j}\}$ of $\{y_{i,n}\}$ there exists a subsequence $\{y_{i,m_j}\}$ uniformly convergent and the limit needs to be a solution of the initial value problem (2.1) - (2.2). Thus , the sequence $\{y_{i,n}\}$ tend uniformly to the (unique) solution $y = y(x)$ of the initial value problem (2.1) - (2.2). This ends the proof of the Theorem 2.2.1. #

Remark 2.2.2. In (2.2) , if $c = 0$, then the solutions of the initial value problem (2.1) - (2.2) are identically zero.

2.3 Fixed-point method

In this section we shall use the classical method for proving the existence and uniqueness theorem. We shall now apply fix-pointed techniques which are somewhat and elegant.

Theorem 2.3.1. The initial value problem (2.1) - (2.2) has a unique solution defined on the interval $[0,a]$ if

i) $y(x)$ is continuous on $[0,a]$,

ii) $|y(x)| \leq a$ for all $x \in [0,a]$,

iii) $|y(x) - y(\bar{x})| \leq K|x - \bar{x}|$ for all $x, \bar{x} \in [0,a]$ and for some positive constant K and

iv) $aT_m < 1$.

Proof. Let S denote the family of all continuous functions on the interval $[0,a]$. We define the norm of any function y in $[0,a]$ as

$$\|y\|_S = \sup_{x\in[0,a]} |y(x)|.$$

It can be seen that, with this norm, S is a Banach space.

Define the set $S(\rho)$ as

$$S(\rho) = \{y \in S / \|y\|_S \leq \rho\}$$

and the operator T by

$$(Ty)(x) = c + \int_0^x y^m(t)dt \quad \text{for } y \in S(\rho) \text{ and } x \in [0,a]$$

where $\rho = c + a^2$.

Then, we have

$$|(Ty)(x)| \leq c + \int_0^x |y^m(t)|dt$$

and hence, $$\|Ty\|_S \leq c + a^2 = \rho .$$

Thus, $T : S(\rho) \to S(\rho)$.

Further, for $u,v \in S(\rho)$, we have

$$
\begin{aligned}
\left|(Tu)(x)-(Tv)(x)\right| &\le \int_0^x \left|u^m(t)-v^m(t)\right| dt \\
&\le K^{m-1}\int_0^x \left|u(t)-v(t)\right| dt + K^{m-2}\int_0^x \left|u(v(t))-v(v(t))\right| dt + \\
&\quad K^{m-3}\int_0^x \left|u\left(v^2(t)\right)-v\left(v^2(t)\right)\right| dt + \ldots + K\int_0^x \left|u\left(v^{m-2}(t)\right)-v\left(v^{m-2}(t)\right)\right| dt \\
&\quad + \int_0^x \left|u\left(v^{m-1}(t)\right)-v\left(v^{m-1}(t)\right)\right| dt.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \left\|Tu-Tv\right\|_S &\le aK^{m-1}\left\|u-v\right\|_S + aK^{m-2}\left\|u-v\right\|_S + aK^{m-3}\left\|u-v\right\|_S + \ldots \\
&\quad + aK\left\|u-v\right\|_S + a\left\|u-v\right\|_S \\
&\le a\left\|u-v\right\|_S\left(K^{m-1}+K^{m-2}+K^{m-3}+\ldots+K+1\right) \\
&= aT_m\left\|u-v\right\|_S.
\end{aligned}
$$

Since $aT_m < 1$, so T is a contraction on $S(\rho)$. By the Banach Contraction Mapping Principle, we have a unique fixed point. This means that $u \equiv v$, which ends the proof of Theorem 2.3.1. #

Chapter 3

System of The First Order Simple Iterative Ordinary Differential Equations and Higher Order Simple Iterative Ordinary Differential Equations

In this chapter, we shall discuss the uniqueness and existence theorems for the solutions of the system of the first order simple iterative ordinary differential equations and higher order simple iterative ordinary differential equations.

3.1 System of the first order simple iterative ordinary differential equations

We shall consider the equations

$$y'(x)=\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}=\begin{bmatrix} y_1^m(x) \\ y_2^m(x) \\ \vdots \\ y_n^m(x) \end{bmatrix}=y^m(x) \qquad (3.1)$$

on the interval $[0,a]$ with the initial conditions

$$y(0)=\begin{bmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_n(0) \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}=c \qquad (3.2)$$

where a, c_i are positive real numbers for $i=1,2,\cdots,n$,

m is a positive integer greater than 1 and

$$y^2(x)=y(y(x))=\begin{bmatrix} y_1(y_1(x)) \\ y_2(y_2(x)) \\ \vdots \\ y_n(y_n(x)) \end{bmatrix}$$

$$y^3(x) = y(y^2(x)) = y(y(y(x))) = \begin{bmatrix} y_1(y_1(y_1(x))) \\ y_2(y_2(y_2(x))) \\ \vdots \\ y_n(y_n(y_n(x))) \end{bmatrix}$$

$$\vdots$$

$$y^m(x) = y(y^{m-1}(x)) = \begin{bmatrix} y_1(y_1^{m-1}(x)) \\ y_2(y_2^{m-1}(x)) \\ \vdots \\ y_n(y_n^{m-1}(x)) \end{bmatrix} \tag{3.3}$$

and $y_i : [0,a] \to [0,a]$ for $i = 1,2,\cdots,n$.

By a solution of the initial value problem (3.1) - (3.2), we mean a function $y(x)$ which is continuous differentiable function and satisfying (3.1) and (3.2) in the interval $[0,a]$. Thus, the initial value problem (3.1) - (3.2) is equivalent to the integral equation

$$y(x) = c + \int_0^x y^m(t)\, dt \quad \text{for } x \in [0,a] \tag{3.4}$$

or

$$y_i(x) = c_i + \int_0^x y_i^m(t)\, dt \ , \ i = 1,2,\cdots,n. \tag{3.5}$$

3.1.1 Uniqueness of solution

Let $\|.\|$ be the absolute value norm, let $y(x)$ be a continuous differentiable function on the interval $[0,a]$ and let

$$|y_i(x)| \le a \ , \ i = 1,2,\ldots,n \quad \text{for all } x \in [0,a] \tag{3.6}$$

which implies that

$$\|y(x)\| \le na = N \tag{3.7}$$

and let

$$|y_i(x) - y_i(\bar{x})| \le K_i |x - \bar{x}| \tag{3.8}$$

for all $x, \bar{x} \in [0,a]$ and for some positive constant K_i , $i = 1,2,\ldots,n$.

If we let $K = \sum_{i=1}^{n} K_i$, then we have

$$|y_i(x) - y_i(\bar{x})| \le K|x - \bar{x}| \quad \text{for } i = 1,2,\cdots,n \tag{3.9}$$

Now , let

$$S_m = K^{m-2} + K^{m-3} + \ldots + K^2 + K + 1 \tag{3.10}$$

$$T_m = K^{m-1} + K^{m-2} + \ldots + K^2 + K + 1 \tag{3.11}$$

then we have the following theorems.

Theorem 3.1.1.1. If $aS_m < e^{-aK^{m-1}}$ and $y(x)$ satisfies the conditions (3.6) – (3.9), then there exists at most one solution of the initial value problem (3.1) - (3.2).

Proof. Let $u(x)$ and $v(x)$ be two solutions of the initial value problem (3.1) - (3.2).

Then $$u(x) = c + \int_0^x u^m(t)\,dt \qquad \text{for all } x \in [0,a]$$

or $$u_i(x) = c_i + \int_0^x u_i^m(t)\,dt \;, \qquad i = 1,2,\ldots,n$$

and $$v(x) = c + \int_0^x v^m(t)\,dt \qquad \text{for all } x \in [0,a]$$

or $$v_i(x) = c_i + \int_0^x v_i^m(t)\,dt \;, \qquad i = 1,2,\ldots,n.$$

Let
$$\begin{aligned} w(x) &= \| u(x) - v(x) \| \\ &= |u_1(x) - v_1(x)| + |u_2(x) - v_2(x)| + \ldots + |u_n(x) - v_n(x)| \\ &= \sum_{i=1}^{n} |u_i(x) - v_i(x)|. \end{aligned}$$

Thus, we have

$$\begin{aligned} w(x) &= \sum_{i=1}^{n} \left| \int_0^x u_i^m(t)dt - \int_0^x v_i^m(t)dt \right| \\ &\le \sum_{i=1}^{n} \int_0^x \left| u_i^m(t) - v_i^m(t) \right| dt \\ &\le K^{m-1} \sum_{i=1}^{n} \int_0^x |u_i(t) - v_i(t)|dt + K^{m-2} \sum_{i=1}^{n} \int_0^x |u_i(v_i(t)) - v_i(v_i(t))|dt + \\ &\quad K^{m-3} \sum_{i=1}^{n} \int_0^x |u_i(v_i^2(t)) - v_i(v_i^2(t))|dt + \ldots + K \sum_{i=1}^{n} \int_0^x |u_i(v_i^{m-2}(t)) - v_i(v_i^{m-2}(t))|dt \\ &\quad + \sum_{i=1}^{n} \int_0^x | u_i(v_i^{m-1}(t)) - v_i(v_i^{m-1}(t))| \,dt \end{aligned}$$

$$\le K^{m-1}\int_0^x \|u(t)-v(t)\|dt + K^{m-2}\int_0^x \|u(v(t))-v(v(t))\|dt +$$

$$K^{m-3}\int_0^x \left\|u\left(v^2(t)\right)-v\left(v^2(t)\right)\right\|dt + \ldots + K\int_0^x \left\|u\left(v^{m-2}(t)\right)-v\left(v^{m-2}(t)\right)\right\|dt$$

$$+\int_0^x \left\|u\left(v^{m-1}(t)\right)-v\left(v^{m-1}(t)\right)\right\|dt \qquad (3.12)$$

$$\le K^{m-1}\int_0^x w(t)dt + K^{m-2}\int_0^x N\,dt + K^{m-3}\int_0^x N\,dt + \ldots + K\int_0^x N\,dt + \int_0^x N\,dt .$$

$$\therefore\ w(x) \le K^{m-1}\int_0^x w(t)dt + aN\left(K^{m-2}+K^{m-3}+\cdots+K+1\right)$$

$$\le K^{m-1}\int_0^x w(t)dt + aNS_m . \qquad (3.13)$$

From (3.13) and well known theorems concerning the differential (integral) inequalities , Gronwall – Reid – Bellman inequality , it follows that

$$w(x) \le aNS_m e^{xK^{m-1}} \le aNS_m e^{aK^{m-1}} . \qquad (3.14)$$

From (3.12) and (3.14) it follows that

$$w(x) \le K^{m-1}\int_0^x w(t)dt + Ne^{aK^{m-1}}\left(aS_m\right)^2$$

$$\le N\left(aS_m e^{aK^{m-1}}\right)^2 .$$

By induction on n , it can be shown that

$$w(x) \le N\left(aS_m e^{aK^{m-1}}\right)^n \quad , \quad n = 1,2,3,\ldots .$$

Since $aS_m < e^{-aK^{m-1}}$, so $aS_m e^{aK^{m-1}} < 1$ and $w(x)$ is upperly bounded by the sequence with tends to zero as $n \to \infty$. Since $w(x) \ge 0$, it must be equal to zero. This ends the proof of Theorem 3.1.1.1. #

Theorem 3.1.1.2. If $aT_m < 1$ and $y(x)$ satisfies the condition of Theorem 3.1.1.1, then there exists at most one solution of the initial value problem (3.1) - (3.2).

Proof. Assume that $u(x)$ and $v(x)$ are two solutions of the initial value problem (3.1) - (3.2) and let $w(x) = \|u(x)-v(x)\|$

and $P = \max_{x\in[0,a]} \|u(x)-v(x)\|$.

It follows from the proof of Theorem 3.1.1.1 that

$$w(x) \le K^{m-1}\int_0^x \|u(t)-v(t)\|dt + K^{m-2}\int_0^x \|u(v(t))-v(v(t))\|dt$$

$$+K^{m-3}\int_0^x \|u(v^2(t))-v(v^2(t))\|dt + \ldots + K\int_0^x \|u(v^{m-2}(t))-v(v^{m-2}(t))\|dt$$

$$+\int_0^x \|u(v^{m-1}(t))-v(v^{m-1}(t))\|dt \qquad (3.15)$$

$$\le K^{m-1}\int_0^x P\,dt + K^{m-2}\int_0^x P\,dt + K^{m-3}\int_0^x P\,dt + \ldots + K\int_0^x P\,dt + \int_0^x P\,dt.$$

$$\therefore \; w(x) \le aP\left(K^{m-1}+K^{m-2}+K^{m-3}+\cdots+K+1\right) = PaT_m. \qquad (3.16)$$

From (3.15) and (3.16) it follows that

$$w(x) \le Pa^2T_m\left(K^{m-1}+K^{m-2}+K^{m-3}+\cdots+K+1\right) = P(aT_m)^2.$$

By induction on n, it can be shown that

$$w(x) \le P(aT_m)^n \quad , \quad n = 1,2,3,\ldots .$$

Since $aT_m < 1$, P must be zero, which ends the proof of Theorem 3.1.1.2. #

3.1.2 Existence of solution

Let us assume that

$$c_i + a^2 \le a \quad , \quad i = 1,2,\cdots,n \qquad (3.17)$$

and $$a\left(K^{m-1}+K^{m-2}+\ldots+K+1\right) < 1 \qquad (3.18)$$

and let consider the following sequences :

$$y_{1,n+1}(x) = c + \int_0^x y_{1,n}^m(t)dt \qquad (3.19.1)$$

$$y_{2,n+1}(x) = c + \int_0^x y_{2,n}^{m-1}\left(y_{2,n+1}(t)\right)dt \qquad (3.19.2)$$

$$y_{3,n+1}(x) = c + \int_0^x y_{3,n}^{m-2}\left(y_{3,n+1}^2(t)\right)dt \qquad (3.19.3)$$

$$\vdots \qquad\qquad \vdots$$

$$y_{m,n+1}(x) = c + \int_0^x y_{m,n}\left(y_{m,n+1}^{m-1}(t)\right)dt \qquad (3.19.m)$$

where $n = 0,1,2,\ldots$ and

$y_{1,0}(x), y_{2,0}(x),\dots, y_{n,0}(x)$ are fixed functions of the class C^1 map $[0,a]$ to $[0,a]^n$ such that

$$\|y'_{1,0}(x)\|, \|y'_{2,0}(x)\|,\dots, \|y'_{n,0}(x)\| \le N.$$

Hence, we have the following theorem.

Theorem 3.1.2.1. Let the assumptions of Theorem 3.1.1.1 , the conditions (3.17) and (3.18) hold. Then the sequences (3.19.1) - (3.19.m) converge uniformly to the (unique) solution $y = y(x)$ of the initial value problem (3.1) - (3.2).

Proof. We put

$$\begin{aligned} Y_{1,n} &= \max_{x\in[0,a]} \|y_{1,n}(x) - y_{1,n-1}(x)\| \\ Y_{2,n} &= \max_{x\in[0,a]} \|y_{2,n}(x) - y_{2,n-1}(x)\| \\ &\vdots \\ Y_{m,n} &= \max_{x\in[0,a]} \|y_{m,n}(x) - y_{m,n-1}(x)\| \quad \text{where } n = 1,2,3,\dots . \end{aligned}$$

It can be shown by induction on n that

$$\begin{aligned} Y_{1,n} &\le W_1^{n-1} Y_{1,1} \\ Y_{2,n} &\le W_2^{n-1} Y_{2,1} \\ &\vdots \\ Y_{m,n} &\le W_m^{n-1} Y_{m,1} \end{aligned}$$

where $W_1 = \dfrac{aU_1}{V_1}$, $W_2 = \dfrac{aU_2}{V_2}, \dots,$ $W_n = \dfrac{aU_m}{V_m}$,

$U_1 = K^{m-1} + K^{m-2} + \dots + K + 1$,

$U_2 = K^{m-2} + K^{m-3} + \dots + K + 1$,

$U_3 = K^{m-3} + K^{m-4} + \dots + K + 1$,

$\vdots$

$U_m = 1$,

$V_1 = 1$,

$V_2 = 1 - aK^{m-1}$,

$V_3 = 1 - aK^{m-1} - aK^{m-2}$,

$\vdots$

$V_m = 1 - aK^{m-1} - aK^{m-2} - aK^{m-3} - \dots - aK$.

Since $aT_m < 1$, so $W_j < 1$ and W_j^{n-1} tends to zero as $n \to \infty$ for $j = 1,2,\ldots,m$. Hence $Y_{j,n}$ tends to zero as $n \to \infty$ for $j = 1,2,\ldots,m$. This means that for $i = 1,2,\ldots,m$, if $\{y_{i,n_j}(z)\}$ is a subsequence of $\{y_{i,n}(z)\}$ tending uniformly to some $\bar{y}_i(z)$ then $\bar{y}_i(z)$ is a solution of the initial value problem (3.1), (3.2). Since the family $\{y_{i,n}\}$ is the Arzela-Ascoli family, thus for every subsequence $\{y_{i,n_j}\}$ of $\{y_{i,n}\}$ there exists a subsequence $\{y_{i,m_j}\}$ uniformly convergent and the limit needs to be a solution of the initial value problem (3.1) - (3.2). Thus, the sequence $\{y_{i,n}\}$ tend uniformly to the (unique) solution $y = y(x)$ of the initial value problem (3.1) - (3.2). This ends the proof of the Theorem 3.1.2.1. #

3.2 Higher order simple iterative ordinary differential equations

We shall consider the equations

$$y^{(n)}(x) = y^m(x) \tag{3.20}$$

on the interval $[0,a]$ with the initial conditions

$$y(0) = c_1 \, , \, y'(0) = c_2 \, , \, y''(0) = c_3 \, ,\ldots, \, y^{(n-1)}(0) = c_n \tag{3.21}$$

where a, c_i are positive real numbers for $i = 1,2,\ldots,n$,

m is a positive integer greater than 1 and

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y^2(x)) = y(y(y(x))) \\ &\vdots \\ y^m(x) &= y(y^{m-1}(x)). \end{aligned} \tag{3.22}$$

If we let

$$\begin{aligned} y_1(x) &= y(x) \\ y_2(x) &= y_1'(x) \\ y_3(x) &= y_2'(x) \\ &\vdots \\ y_n(x) &= y_{n-1}'(x) \end{aligned} \tag{3.23}$$

then we have the following system of the first order simple iterative ordinary differential equations

$$
\begin{aligned}
y_1'(x) &= y_2(x) \quad , \quad y_1(0) = c_1 \\
y_2'(x) &= y_3(x) \quad , \quad y_2(0) = c_2 \\
&\vdots \\
y_{n-1}'(x) &= y_n(x) \quad , \quad y_{n-1}(0) = c_{n-1} \\
y_n'(x) &= y^m(x) \quad , \quad y_n(0) = c_n \ .
\end{aligned} \tag{3.24}
$$

Thus, the Theorem 3.1.1.1, Theorem 3.1.1.2 and Theorem 3.1.2.1 can be applied to the system (3.24).

Chapter 4

Some Examples of The Simple Iterative Ordinary Differential Equations

In this chapter, we shall find the solutions of some equations of the simple iterative ordinary differential equations.

Example 4.1. Find the solution on the interval $\left[0,\frac{1}{2}\right]$ of the equation

$$y'(x)=y^2(x)=y(y(x)) \qquad (4.1)$$

with the initial condition

$$y(0)=0.25. \qquad (4.2)$$

Let $y_0(x)=0.25$ then by the equation (2.16.1), we get

$$y_1(x) = 0.25+\int_0^x y_0(y_0(t))dt = 0.25+(0.25)x$$

$$y_2(x) = 0.25+\int_0^x y_1(y_1(t))dt = 0.25+(0.3125)x+(0.03125)x^2$$

$$\begin{aligned} y_3(x) &= 0.25+\int_0^x y_2(y_2(t))dt \\ &= 0.25+(3.30078\times10^{-1})x+(5.12695\times10^{-2})x^2+(4.435221\times10^{-3})x^3 \\ &\quad +(1.52587\times10^{-4})x^4+(6.10351\times10^{-6})x^5 \end{aligned}$$

$$\begin{aligned} y_4(x) &= 0.25+\int_0^x y_3(y_3(t))dt \\ &= 0.25+(3.36086\times10^{-1})x+(5.92627\times10^{-2})x^2+(8.369\times10^{-3})x^3 \\ &\quad +(1.03805\times10^{-3})x^4+(1.65254\times10^{-4})x^5+(5.13064\times10^{-5})x^6 \\ &\quad +(2.27891\times10^{-5})x^7+(1.11981\times10^{-5})x^8+(5.64832\times10^{-6})x^9 \\ &\quad +(2.88698\times10^{-6})x^{10}+(1.49034\times10^{-6})x^{11}+(6.62681\times10^{-7})x^{12} \\ &\quad +(1.75158\times10^{-7})x^{13}+(6.74152\times10^{-8})x^{14}+(3.35821\times10^{-8})x^{15} \\ &\quad +(1.77152\times10^{-8})x^{16}+(8.02786\times10^{-9})x^{17}+(2.12115\times10^{-9})x^{18} \\ &\quad +(8.02153\times10^{-10})x^{19}+(3.93583\times10^{-10})x^{20}+(2.04969\times10^{-10})x^{21} \\ &\quad +(8.59010\times10^{-11})x^{22}+(1.12052\times10^{-11})x^{23}+(8.48685\times10^{-13})x^{24} \\ &\quad +(2.81777\times10^{-14})x^{25}+(1.01015\times10^{-15})x^{26}. \end{aligned}$$

We may use $y_4(x)$ as the polynomial approximation to the solution of the initial value problem (4.1) – (4.2). #

Example 4.2. Find the solution on the interval $\left[0,\frac{1}{2}\right]$ of the equation

$$y'(x) = y^3(x) = y(y(y(x))) \tag{4.3}$$

with the initial condition

$$y(0) = 0.2\ . \tag{4.4}$$

Let $y_0(x) = 0.2$ then by the equation (2.16.1), we get

$$\begin{aligned}
y_1(x) &= 0.2 + \int_0^x y_0(y_0(y_0(t)))dt \\
&= 0.2 + (0.2)x \\
y_2(x) &= 0.2 + \int_0^x y_1(y_1(y_1(t)))dt \\
&= 0.2 + (0.248)x + (0.004)x^2 \\
y_3(x) &= 0.2 + \int_0^x y_2(y_2(y_2(t)))dt \\
&= 0.2 + (2.62190\times10^{-1})x + (7.73754\times10^{-3})x^2 + (1.08809\times10^{-4})x^3 \\
&\quad + (6.50056\times10^{-7})x^4 + (5.22482\times10^{-9})x^5 + (1.84497\times10^{-11})x^6 \\
&\quad + (1.27008\times10^{-13})x^7 + (5.07904\times10^{-16})x^8 + (1.82044\times10^{-18})x^9\ .
\end{aligned}$$

We may use $y_3(x)$ as the polynomial approximation to the solution of the initial value problem (4.3) – (4.4). #

Example 4.3. Find the solution on the interval $\left[0,\frac{1}{2}\right]$ of the equation

$$y'(x) = y^4(x) = y(y(y(y(x)))) \tag{4.5}$$

with the initial condition

$$y(0) = 0.25\ . \tag{4.6}$$

Let $y_0(x) = 0.25$ then by the equation (2.16.1), we get

$$y_1(x) = 0.25 + \int_0^x y_0(y_0(y_0(y_0(t))))dt$$
$$= 0.25 + (0.25)x$$
$$y_2(x) = 0.25 + \int_0^x y_1(y_1(y_1(y_1(t))))dt$$
$$= 0.25 + (3.32031\times10^{-1})x + (1.95312\times10^{-3})x^2$$
$$y_3(x) = 0.25 + \int_0^x y_2(y_2(y_2(y_2(t))))dt$$
$$= 0.25 + (3.70059\times10^{-1})x + (6.14468\times10^{-3})x^2 + (3.56125\times10^{-5})x^3$$
$$+ (1.12804\times10^{-7})x^4 + (4.09421\times10^{-10})x^5 + (1.02371\times10^{-12})x^6$$
$$+ (3.33276\times10^{-15})x^7 + (8.54887\times10^{-18})x^8 + (2.206\times10^{-20})x^9$$
$$+ (5.17225\times10^{-23})x^{10} + (1.36131\times10^{-25})x^{11} + (3.08724\times10^{-28})x^{12}$$
$$+ (6.39176\times10^{-31})x^{13} + (1.13262\times10^{-33})x^{14} + (1.77301\times10^{-36})x^{15}.$$

We may use $y_3(x)$ as the polynomial approximation to the solution of the initial value problem (4.5) – (4.6). #

Example 4.4. Find the solution on the interval $\left[0,\frac{1}{2}\right]$ of the equation

$$y_1'(x) = y_1(y_1(x)) \quad , \quad y(0) = 0.25 \tag{4.7}$$
$$y_2'(x) = y_2(y_2(x)) \quad , \quad y(0) = 0.2. \tag{4.8}$$

Let $y_{1,0}(x) = 0.25$ and $y_{2,0}(x) = 0.2$.

Then by the equation (3.19.1), we get

$$y_{1,1}(x) = 0.25 + (0.25)x$$
$$y_{2,1}(x) = 0.2 + (0.2)x$$

$$y_{1,2}(x) = 0.25 + (0.3125)x + (0.03125)x^2$$
$$y_{2,2}(x) = 0.2 + (0.24)x + (0.02)x^2$$

$$y_{1,3}(x) = 0.25 + (3.30078\times10^{-1})x + (5.12695\times10^{-2})x^2 + (4.43522\times10^{-3})x^3$$
$$+ (1.52587\times10^{-4})x^4 + (6.10351\times10^{-6})x^5$$
$$y_{2,3}(x) = 0.2 + (2.488\times10^{-1})x + (2.976\times10^{-2})x^2 + (2.03733\times10^{-3})x^3$$
$$+ (4.8\times10^{-5})x^4 + (1.6\times10^{-6})x^5$$

$$
\begin{aligned}
y_{1,4}(x) = {} & 0.25 + \left(3.36086\times10^{-1}\right)x + \left(5.92627\times10^{-2}\right)x^2 + \left(8.369\times10^{-3}\right)x^3 \\
& + \left(1.03805\times10^{-3}\right)x^4 + \left(1.65254\times10^{-4}\right)x^5 + \left(5.13064\times10^{-5}\right)x^6 \\
& + \left(2.27891\times10^{-5}\right)x^7 + \left(1.11981\times10^{-5}\right)x^8 + \left(5.64832\times10^{-6}\right)x^9 \\
& + \left(2.88698\times10^{-6}\right)x^{10} + \left(1.49034\times10^{-6}\right)x^{11} + \left(6.62681\times10^{-7}\right)x^{12} \\
& + \left(1.75158\times10^{-7}\right)x^{13} + \left(6.74152\times10^{-8}\right)x^{14} + \left(3.35821\times10^{-8}\right)x^{15} \\
& + \left(1.77152\times10^{-8}\right)x^{16} + \left(8.02786\times10^{-9}\right)x^{17} + \left(2.12115\times10^{-9}\right)x^{18} \\
& + \left(8.02153\times10^{-10}\right)x^{19} + \left(3.93583\times10^{-10}\right)x^{20} + \left(2.04969\times10^{-10}\right)x^{21} \\
& + \left(8.59010\times10^{-11}\right)x^{22} + \left(1.12052\times10^{-11}\right)x^{23} + \left(8.48685\times10^{-13}\right)x^{24} \\
& + \left(2.81777\times10^{-14}\right)x^{25} + \left(1.01015\times10^{-15}\right)x^{26} \\
y_{2,4}(x) = {} & 0.2 + \left(2.51051\times10^{-1}\right)x + \left(3.25722\times10^{-2}\right)x^2 + \left(3.29223\times10^{-3}\right)x^3 \\
& + \left(2.76651\times10^{-4}\right)x^4 + \left(2.52334\times10^{-5}\right)x^5 + \left(4.72305\times10^{-6}\right)x^6 \\
& + \left(1.42397\times10^{-6}\right)x^7 + \left(5.03046\times10^{-7}\right)x^8 + \left(1.84082\times10^{-7}\right)x^9 \\
& + \left(6.83323\times10^{-8}\right)x^{10} + \left(2.56258\times10^{-8}\right)x^{11} + \left(8.3797\times10^{-9}\right)x^{12} \\
& + \left(1.38547\times10^{-9}\right)x^{13} + \left(3.31057\times10^{-10}\right)x^{14} + \left(1.14845\times10^{-10}\right)x^{15} \\
& + \left(4.41662\times10^{-11}\right)x^{16} + \left(1.49258\times10^{-11}\right)x^{17} + \left(2.68385\times10^{-12}\right)x^{18} \\
& + \left(6.8457\times10^{-13}\right)x^{19} + \left(2.40016\times10^{-13}\right)x^{20} + \left(9.16573\times10^{-14}\right)x^{21} \\
& + \left(2.93793\times10^{-14}\right)x^{22} + \left(2.93889\times10^{-15}\right)x^{23} + \left(1.73655\times10^{-16}\right)x^{24} \\
& + \left(3.99939\times10^{-18}\right)x^{25} + \left(1.18396\times10^{-19}\right)x^{26}.
\end{aligned}
$$

We may use $y_{1,4}(x)$ and $y_{2,4}(x)$ as the polynomial approximation to the solution of the initial value problem (4.7) – (4.8). #

Example 4.5. Find the solution on the interval $\left[0,\frac{1}{2}\right]$ of the equation

$$y''(x) = y(y(x)) \tag{4.9}$$

with the initial conditions

$$y(0) = 0.25 \quad , \quad y'(0) = 0.25 \; . \tag{4.10}$$

Let $y_1(x) = y(x)$ and

$y_2(x) = y_1'(x)$.

Thus, we have the following system of the first order simple iterative ordinary differential equations

$$y_1'(x) = y_2(x)$$
$$y_2'(x) = y_1(y_1(x))$$

with the initial conditions

$$y_1(0) = 0.25 \quad , \quad y_2(0) = 0.25 \ .$$

Let $y_{1,0}(x) = 0.25$ and

$y_{2,0}(x) = 0.25$.

Then by the equation (3.19.1) we get

$$\begin{aligned} y_{1,1}(x) &= 0.25 + \int_0^x y_{2,0}(t)\,dt \\ &= 0.25 + (0.25)x \\ y_{2,1}(x) &= 0.25 + \int_0^x y_{1,0}(y_{1,0}(t))\,dt \\ &= 0.25 + (0.25)x \end{aligned}$$

$$\begin{aligned} y_{1,2}(x) &= 0.25 + \int_0^x y_{2,1}(t)dt \\ &= 0.25 + (0.25)x + (0.125)x^2 \\ y_{2,2}(x) &= 0.25 + \int_0^x y_{1,1}(y_{1,1}(t))dt \\ &= 0.25 + (0.3125)x + (0.03125)x^2 \end{aligned}$$

$$\begin{aligned} y_{1,3}(x) &= 0.25 + \int_0^x y_{2,2}(t)dt \\ &= 0.25 + (0.25)x + (0.15625)x^2 + (1.04166\times 10^{-2})x^3 \\ y_{2,3}(x) &= 0.25 + \int_0^x y_{1,2}(y_{1,2}(t))dt \\ &= 0.25 + (3.20312\times 10^{-1})x + (3.90625\times 10^{-2})x^2 + (1.5625\times 10^{-2})x^3 \\ &= +(1.95312\times 10^{-3})x^4 + (3.90625\times 10^{-4})x^5 \end{aligned}$$

$$
\begin{aligned}
y_{1,4}(x) &= 0.25+\int_0^x y_{2,3}(t)dt \\
&= 0.25+(0.25)x+\left(1.60156\times10^{-1}\right)x^2+\left(1.30208\times10^{-2}\right)x^3 \\
&= \left(3.90625\times10^{-3}\right)x^4+\left(3.90625\times10^{-4}\right)x^5+\left(6.51041\times10^{-5}\right)x^6 \\
y_{2,4}(x) &= 0.25+\int_0^x y_{1,3}\left(y_{1,3}(t)\right)dt \\
&= 0.25+\left(1.05794\times10^{-2}\right)x+\left(1.07065\times10^{-2}\right)x^2+\left(8.18358\times10^{-3}\right)x^3 \\
&= +\left(3.82242\times10^{-3}\right)x^4+\left(1.25025\times10^{-3}\right)x^5+\left(2.55806\times10^{-4}\right)x^6 \\
&= +\left(9.79356\times10^{-5}\right)x^7+\left(4.30497\times10^{-5}\right)x^8+\left(1.27952\times10^{-5}\right)x^9 \\
&= +\left(6.96453\times10^{-7}\right)x^{10}.
\end{aligned}
$$

We may use $y_{1,4}(x)$ as the polynomial approximation to the solution of the initial value problem (4.9) – (4.10). #

Chapter 5

A Survey of Numerical Method for Solving The Simple Iterative Ordinary Differential Equations

In this chapter, we shall look for a numerical method for solving the simple iterative ordinary differential equations

$$y'(x) = y^m(x) \tag{5.1}$$

on the interval $[0,a]$ with the initial condition

$$y(0) = c. \tag{5.2}$$

5.1 First Method

We shall begin by partitioning the interval $[0,a]$ into n equally subintervals, that is $P \ : \ x_0 = 0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = a$.

Choose any $k \le n$ and a polynomial $P_{k,0}(x)$ which will approximate $y(x)$ at the points $x_0, x_1, x_2, \ldots, x_k$ so that

$$P_{k,0}(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k$$

where $P_{k,0}(x_0) = c$.

Find the approximated values of $y(x)$ at the points $x_1, x_2, \ldots, x_k$ by the integral equation

$$y(x_i) = c + \int_{x_0}^{x_i} P_{k,0}^m(t)\,dt \quad , i = 1,2,\ldots,k. \tag{5.3}$$

Then use these values to find the polynomial $P_{k,1}(x)$ by Newton's Divided-Difference Interpolation. Let $P_{k,1}(x)$ is of the form

$$P_{k,1}(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_k x^k.$$

Let $E = |b_0 - a_0| + |b_1 - a_1| + \ldots + |b_k - a_k|$.

If $E < \varepsilon$ where ε is any small positive real number that we have chosen ahead of time or the number of iterations reach M where M is any large positive integer that we have chosen ahead of time then we shall use the values

$$T_1 = c + \int_{x_0}^{x_1} P_{k,1}^m(t)dt \tag{5.4}$$

as the approximated value for $y(x_1)$. If not, we shall let $a_i = b_i$, $i = 0,1,2\ldots,k$ and repeat the algorithm until it is satisfied.

Again choose any $P_{k,0}(x)$ which will approximate $y(x)$ at the points $x_1, x_2, \ldots, x_k, x_{k+1}$ so that

$$P_{k,0}(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k$$

where $$P_{k,0}(x_1) = T_1 .$$

Find the approximated values of $y(x)$ at the points $x_2, x_3, \ldots, x_k, x_{k+1}$ by the integral equation

$$y(x_i) = T_1 + \int_{x_1}^{x_i} P_{k,0}^m(t)dt \ , i = 2,3,\ldots,k+1 \ . \tag{5.5}$$

Then again use these values to find the polynomial $P_{k,1}(x)$ by Newton's Divided-Difference Interpolation. Do the same method as before then we get T_2 to approximated value for $y(x_2)$. Continue the algorithm until we have had the other points which are less than k points and then we use Newton's Divided-Difference Interpolation to find $P_{k-1,1}(x)$, $P_{k-2,1}(x)$, ..., $P_{1,1}(x)$. Thus, we shall have $T_1, T_2, \ldots, T_n$ as the approximated values of $y(x)$ at the points $x_1, x_2, \ldots, x_n$ respectively.

5.2 Second Method

This method is the same idea as the first method but instead of using only one value each step we shall use all k values , that is we shall use the formula (5.4) to find $T_1, T_2, \ldots, T_k$ as the approximated values of $y(x)$ at the points $x_1, x_2, \ldots, x_k$ by the formula

$$T_i = c + \int_{x_0}^{x_i} P_{k,1}^m(t)dt \ , i = 1,2,\ldots,k . \tag{5.6}$$

Then use T_k as the initial value of $y(x)$ at x_k, that is we shall use the formula (5.5) to find the approximated values of $y(x)$ at the points $x_{k+1}, x_{k+2}, \ldots, x_{2k-1}, x_{2k}$ by the integral equation

$$y(x_i) = T_k + \int_{x_k}^{x_i} P_{k,0}^m(t)\,dt \quad , i = k+1, k+2, \ldots, 2k. \tag{5.7}$$

Continue the process as the first method.

5.3 Third Method

We shall begin by partitioning the interval $[0,a]$ into n equally subintervals, that is $\quad P \ : \ x_0 = 0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = a$.

Choose any $k \le n$ and a polynomial $P_{k,0}(x)$ which will approximate $y(x)$ at the points $x_0, x_1, x_2, \ldots, x_k$ so that

$$P_{k,0}(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k$$

where $\quad P_{k,0}(x_0) = c$.

Find the approximated values of $y(x)$ at the points $x_1, x_2, \ldots, x_k$ by the fourth Runge- Kutta formula for $i = 0,1,2\ldots,k-1$,

$$y(x_{i+1}) = y(x_i) + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{5.8}$$

where

$$\begin{aligned} h &= \frac{x_n - x_0}{n} \\ k_1 &= P_{k,0}^m(x_i) \\ k_2 &= k_3 = P_{k,0}^m\left(x_i + \frac{h}{2}\right) \\ k_4 &= P_{k,0}^m(x_i + h). \end{aligned} \tag{5.9}$$

Then use these values to find the polynomial $P_{k,1}(x)$ by Newton's Divided-Difference Interpolation. Let $P_{k,1}(x)$ is of the form

$$P_{k,1}(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_k x^k.$$

Let $E = |\,b_0 - a_0\,| + |\,b_1 - a_1\,| + \ldots + |\,b_k - a_k\,|$.

If $E < \varepsilon$ where ε is any small positive real number that we have chosen ahead of time or the number of iterations reach M where M is any large positive integer that we have chosen ahead of time then we shall use the values

$$T_1 = y(x_0) + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{5.10}$$

where

$$\begin{aligned} h &= \frac{x_n - x_0}{n} \\ k_1 &= P_{k,1}^m(x_0) \\ k_2 = k_3 &= P_{k,1}^m\left(x_0 + \frac{h}{2}\right) \\ k_4 &= P_{k,1}^m(x_0 + h). \end{aligned} \tag{5.11}$$

as the approximated value for $y(x_1)$. If not, we shall let $a_i = b_i$, $i = 0,1,2\ldots,k$ and repeat the algorithm until it is satisfied.

Again choose any $P_{k,0}(x)$ which will approximate $y(x)$ at the points $x_1, x_2, \ldots, x_k, x_{k+1}$ so that

$$P_{k,0}(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k$$

where
$$P_{k,0}(x_1) = T_1.$$

Find the approximated values of $y(x)$ at the points $x_2, x_3, \ldots, x_k, x_{k+1}$ by the fourth Runge–Kutta formula for $i = 1,2,\ldots,k$,

$$y(x_{i+1}) = y(x_i) + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{5.12}$$

where

$$\begin{aligned} h &= \frac{x_n - x_0}{n} \\ k_1 &= P_{k,0}^m(x_i) \\ k_2 = k_3 &= P_{k,0}^m\left(x_i + \frac{h}{2}\right) \\ k_4 &= P_{k,0}^m(x_i + h) \end{aligned} \tag{5.13}$$

Then again use these values to find the polynomial $P_{k,1}(x)$ by Newton's Divided-Difference Interpolation. Do the same method as before then we get T_2 to approximated value for $y(x_2)$. Continue the algorithm until we have had the other points which are less than k points and then we use Newton's Divided-Difference

Interpolation to find $P_{k-1,1}(x)$, $P_{k-2,1}(x)$, ..., $P_{1,1}(x)$. Thus , we shall have $T_1, T_2, \ldots, T_n$ as the approximated values of $y(x)$ at the points $x_1, x_2, \ldots, x_n$ respectively.

5.4 Fourth Method

This method is the same idea as the third method but instead of using only one value each step we shall use all k values , that is we shall use the formula (5.10) and (5.11) to find $T_1, T_2, \ldots, T_k$ as the approximated values of $y(x)$ at $x_1, x_2, \ldots, x_k$ by the fourth Runge-Kutta formula for $i = 0,1, 2, \ldots, k-1$,

$$y(x_{i+1}) = y(x_i) + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{5.13}$$

where

$$\begin{aligned} h &= \frac{x_n - x_0}{n} \\ k_1 &= P_{k,1}^m(x_i) \\ k_2 = k_3 &= P_{k,1}^m\left(x_i + \frac{h}{2}\right) \\ k_4 &= P_{k,1}^m(x_i + h). \end{aligned} \tag{5.14}$$

Then use T_k as the initial value of $y(x)$ at x_k, that is we shall use the formula (5.12) and (5.13) to find the approximated values of $y(x)$ at the points $x_{k+1}, x_{k+2}, \ldots, x_{2k-1}, x_{2k}$ by the fourth Runge-Kutta formula for $i = k, k+1, \ldots, 2k-1$,

$$y(x_{i+1}) = y(x_i) + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{5.16}$$

where

$$\begin{aligned} h &= \frac{x_n - x_0}{n} \\ k_1 &= P_{k,0}^m(x_i) \\ k_2 = k_3 &= P_{k,0}^m\left(x_i + \frac{h}{2}\right) \\ k_4 &= P_{k,0}^m(x_i + h). \end{aligned} \tag{5.17}$$

Continue the process as the third method.

5.5 Examples

Example 5.5.1. We shall use the method mentioned above to solve the problem

$$y'(x) = y(y(x)) \tag{5.18}$$

$$y(0) = 0.25 \quad , \quad 0 \le x \le \frac{1}{2}. \tag{5.19}$$

First, divide the interval $\left[0, \frac{1}{2}\right]$ into 8 equally subintervals. So we have nine points, they are 0, 0.0625, 0.125, 0.1875,. 0.25, 0.3125, 0.375, 0.4375 and 0.5.

We shall use $k = 1$, $k = 2$ and $k = 3$, that is we use

$$P_{1,i}(x) = a_0 + a_1 x \tag{5.20}$$

$$P_{2,i}(x) = a_0 + a_1 x + a_2 x^2 \tag{5.21}$$

$$P_{3,i}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{5.22}$$

and $\varepsilon = 10^{-10}$ and $M = 200$.

The results of the first method are in the Table 5.1, Table 5.2 and Table 5.3 respectively. The results of the second method are in the Table 5.4 and Table 5.5 for (5.21) and (5.22) since the result of (5.20) will be the same as the first method. The results of the third method are in the Table 5.6, Table 5.7 and Table 5.8 respectively. The results of the fourth method are in the Table 5.9 and Table 5.10 for (5.21) and (5.22) since the result of (5.20) will be the same as the third method.

Table 5.1 The result, Example 5.5.1, of the first method ($k=1$)

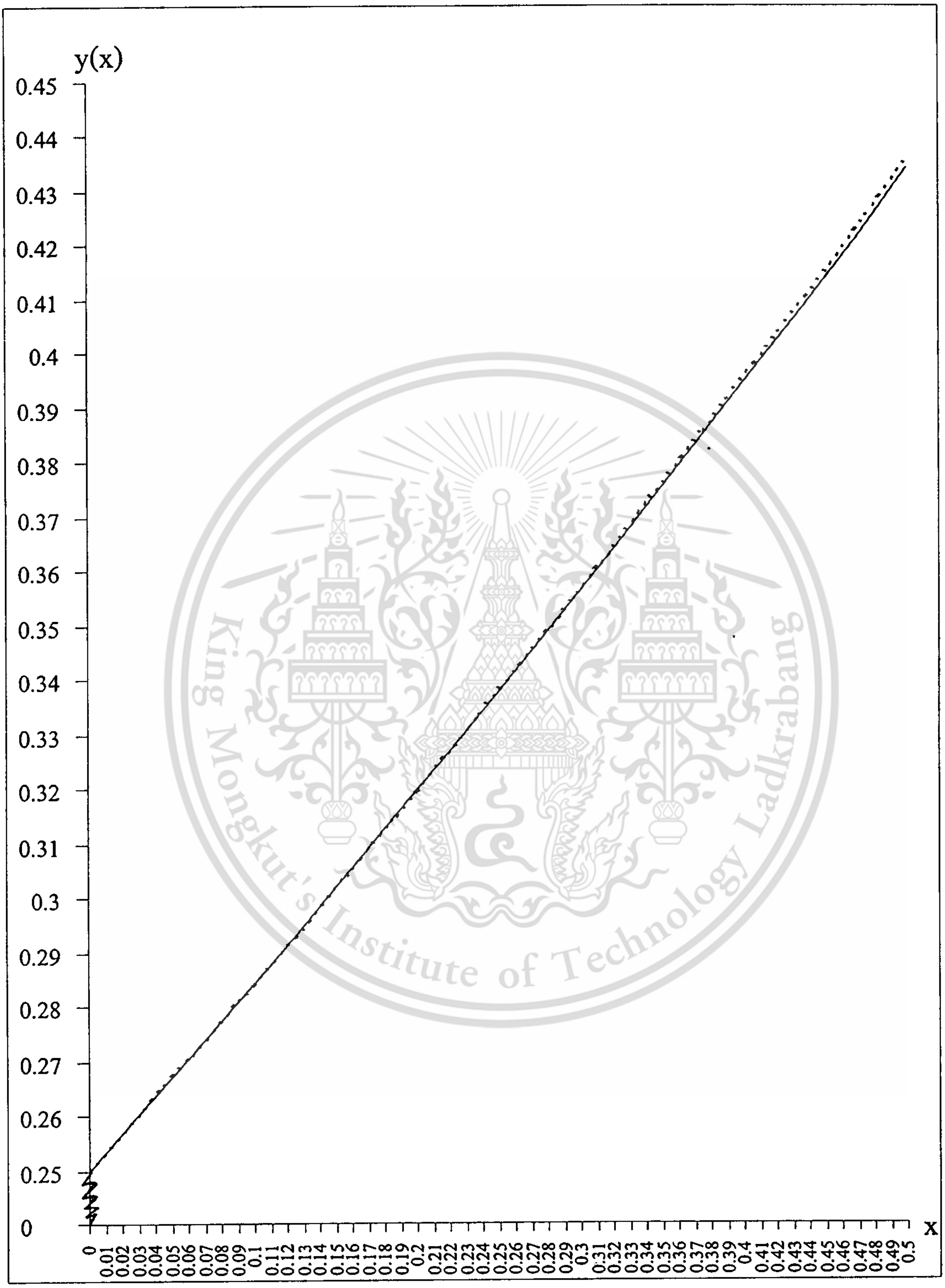
x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2711310131	0.2712389291	0.0001079160
0.1250	0.2928419612	0.2929533341	0.0001113729
0.1875	0.3151347295	0.3151560698	0.0000213403
0.2500	0.3380123167	0.3378604148	0.0001519020
0.3125	0.3614787595	0.3610800974	0.0003986621
0.3750	0.3855390758	0.3848293263	0.0007097494
0.4375	0.4101992247	0.4091228263	0.0010763984
0.5000	0.4354660785	0.4339758805	0.0014901979

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999994127.

Table 5.2 The result, Example 5.5.1, of the first method ($k=2$)

x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714170935	0.2712389291	0.0001781643
0.1250	0.2933467486	0.2929533341	0.0003934145
0.1875	0.3158039238	0.3151560698	0.0006478540
0.2500	0.3388048448	0.3378604148	0.0009444301
0.3125	0.3623669873	0.3610800974	0.0012868899
0.3750	0.3865090818	0.3848293263	0.0016797555
0.4375	0.4112511363	0.4091228263	0.0021283100
0.5000	0.4366102267	0.4339758805	0.0026343462

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998172.



---------- approximated value , ——— $y_4(x)$ (Example 4.1)

Figure 5.1 Graph of Table 5.1

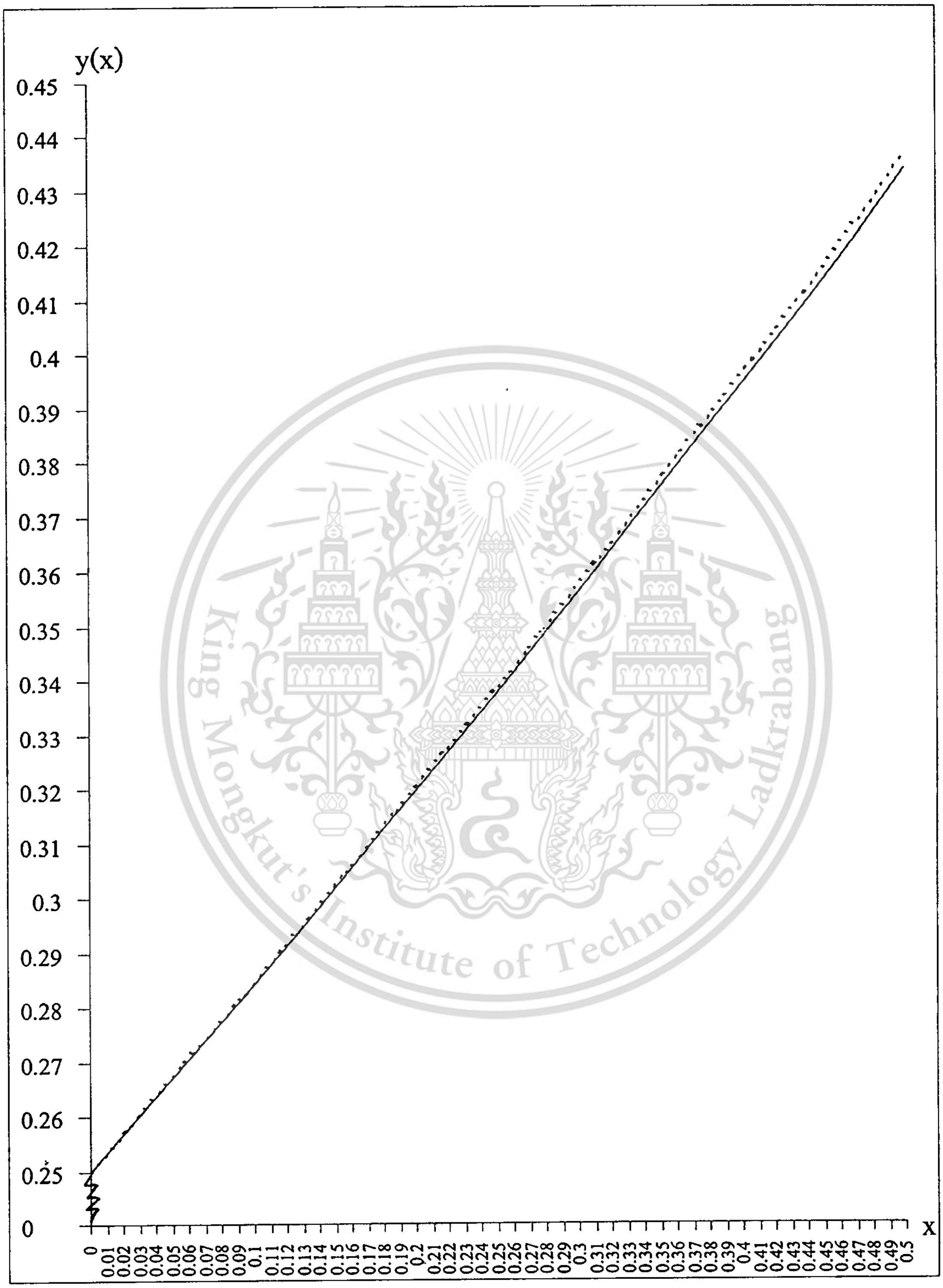


Figure 5.2 Graph of Table 5.2

Table 5.3 The result, Example 5.5.1, of the first method ($k=3$)

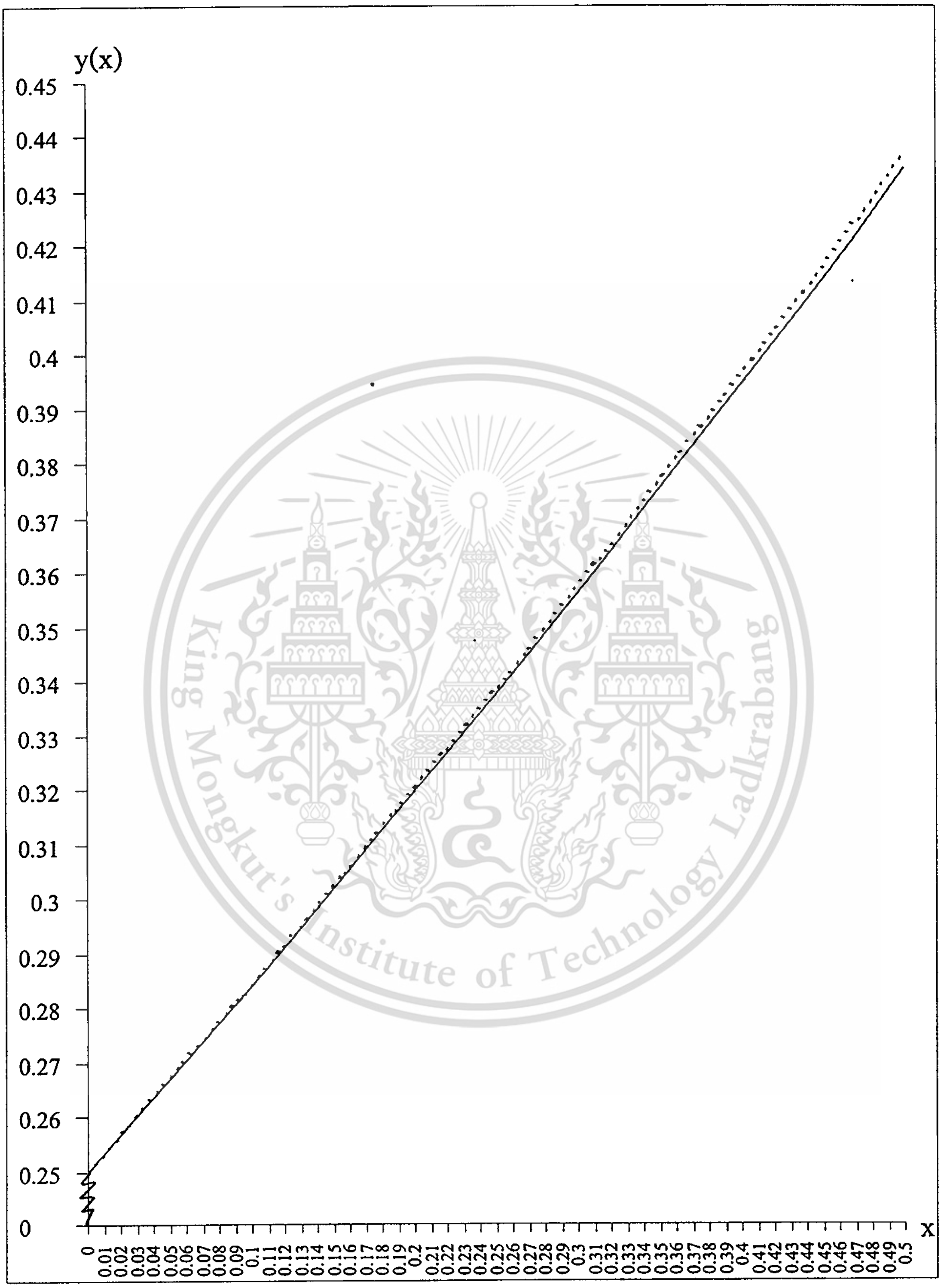
x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714253748	0.2712389291	0.0001864457
0.1250	0.2933597522	0.2929533341	0.0004064181
0.1875	0.3158196505	0.3151560698	0.0006635807
0.2500	0.3388224061	0.3378604148	0.0009619913
0.3125	0.3623862065	0.3610800974	0.0013061091
0.3750	0.3865301310	0.3848293263	0.0017008047
0.4375	0.4112740883	0.4091228263	0.0021512620
0.5000	0.4366351935	0.4339758805	0.0026593130

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998235.

Table 5.4 The result, Example 5.5.1, of the second method ($k=2$)

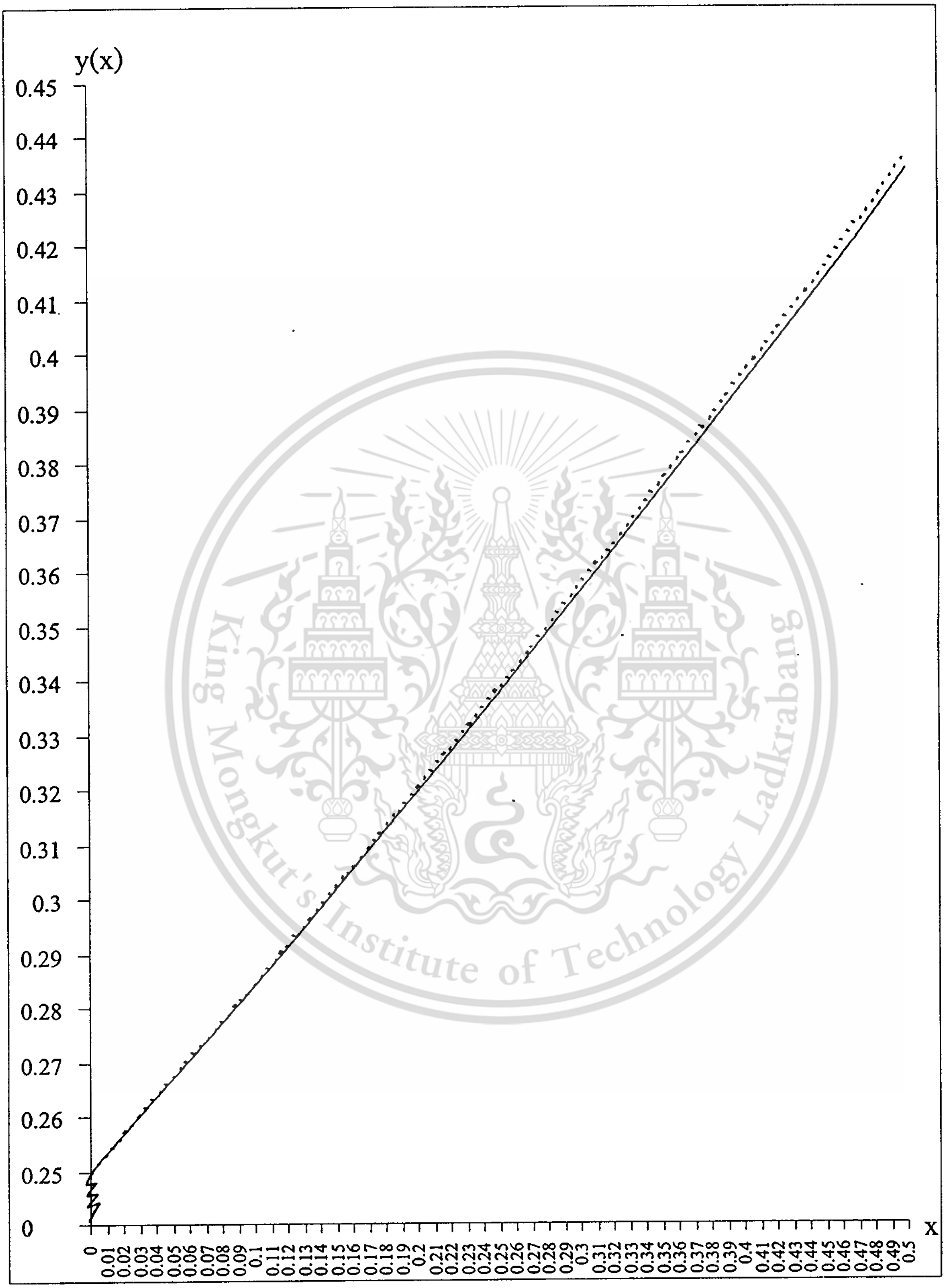
x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714170935	0.2712389291	0.0001781643
0.1250	0.2933404828	0.2929533341	0.0003871488
0.1875	0.3157969825	0.3151560698	0.0006409128
0.2500	0.3387953235	0.3378604148	0.0009349087
0.3125	0.3623565340	0.3610800974	0.0012764366
0.3750	0.3864976897	0.3848293263	0.0016683633
0.4375	0.4112387143	0.4091228263	0.0021158880
0.5000	0.4366008210	0.4339758805	0.0026249404

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998111



---------- approximated value , ———— $y_4(x)$ (Example 4.1)

Figure 5.3 Graph of Table 5.3



---------- approximated value , ———— $y_4(x)$ (Example 4.1)

Figure 5.4 Graph of Table 5.4

Table 5.5 The result, Example 5.5.1, of the second method ($k = 3$)

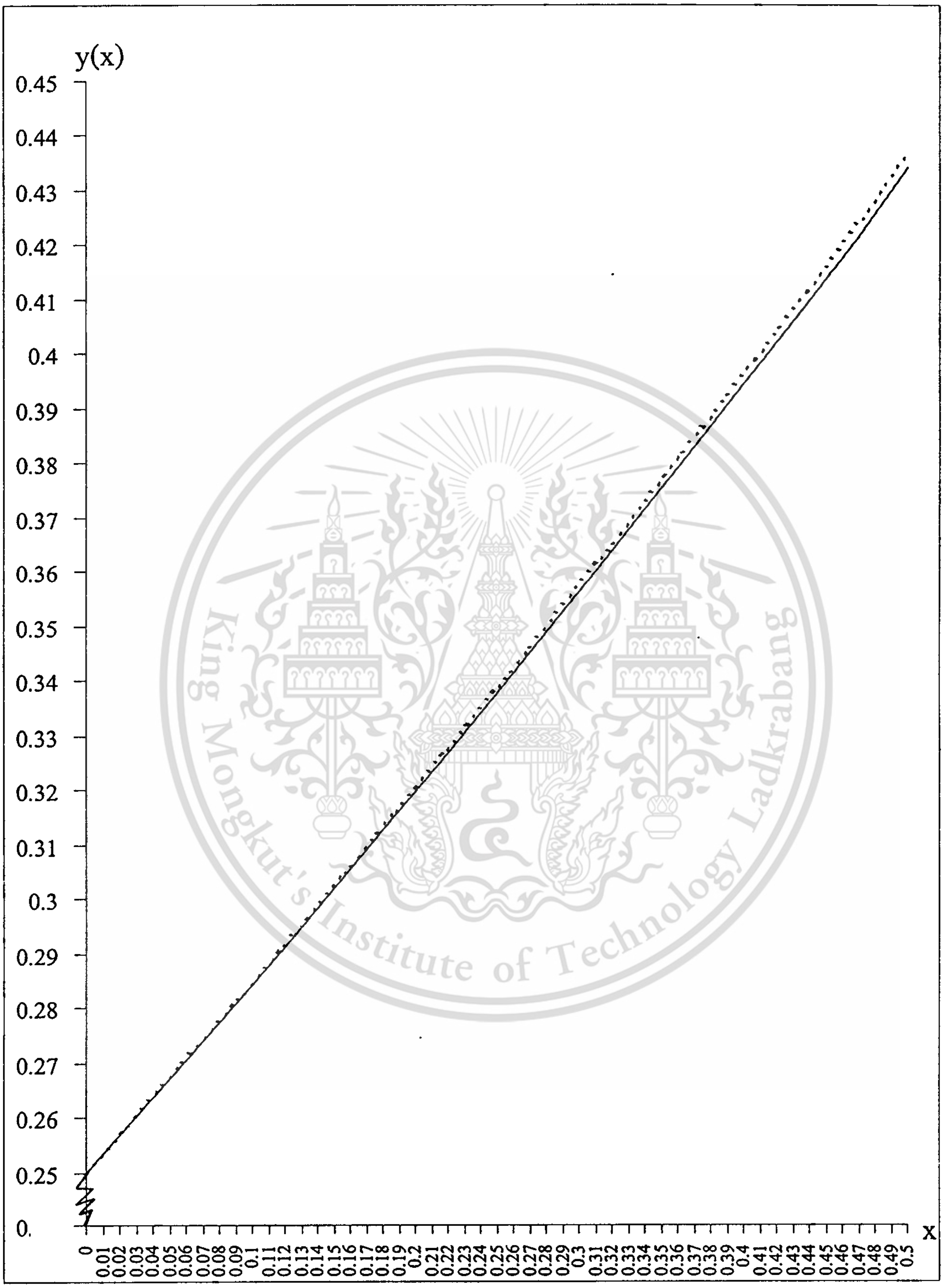
x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714253748	0.2712389291	0.0001864457
0.1250	0.2933595802	0.2929533341	0.0004062461
0.1875	0.3158191722	0.3151560698	0.0006631025
0.2500	0.3388218788	0.3378604148	0.0009614641
0.3125	0.3623856305	0.3610800974	0.0013055332
0.3750	0.3865295006	0.3848293263	0.0017001742
0.4375	0.4112734009	0.4091228263	0.0021505745
0.5000	0.4366385373	0.4339758805	0.0026626568

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998209.

Table 5.6 The result, Example 5.5.1, of the third method ($k = 1$)

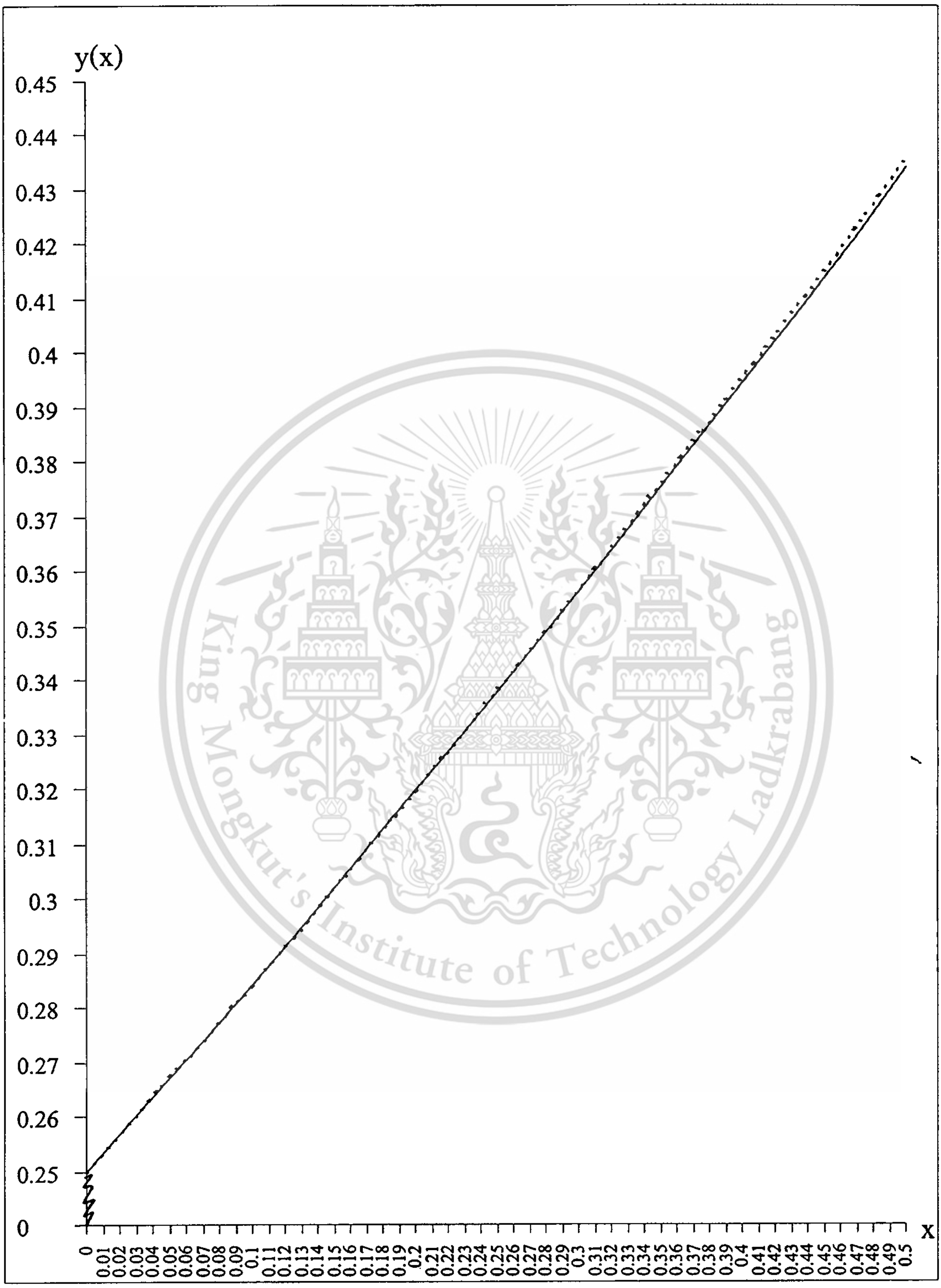
x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2711310131	0.2712389291	0.0001079160
0.1250	0.2928419612	0.2929533341	0.0001113729
0.1875	0.3151347295	0.3151560698	0.0000213403
0.2500	0.3380123167	0.3378604148	0.0001519020
0.3125	0.3614787595	0.3610800974	0.0003986621
0.3750	0.3855390758	0.3848293263	0.0007097494
0.4375	0.4101992247	0.4091228263	0.0010763984
0.5000	0.4354660785	0.4339758805	0.0014901979

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999994127.



---------- approximated value , ———— $y_4(x)$ (Example 4.1)

Figure 5.5 Graph of Table 5.5



---------- approximated value , ———— $y_4(x)$ (Example 4.1)

Figure 5.6 Graph of Table 5.6

Table 5.7 The result, Example 5.5.1, of the third method ($k = 2$)

x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714170935	0.2712389291	0.0001781643
0.1250	0.2933467486	0.2929533341	0.0003934145
0.1875	0.3158039238	0.3151560698	0.0006478541
0.2500	0.3388048448	0.3378604148	0.0009444301
0.3125	0.3623669873	0.3610800974	0.0012868899
0.3750	0.3865090819	0.3848293263	0.0016797555
0.4375	0.4112511363	0.4091228263	0.0021283100
0.5000	0.4366102268	0.4339758805	0.0026343462

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998172.

Table 5.8 The result of, Example 5.5.1, the third method ($k = 3$)

x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714253748	0.2712389291	0.0001864457
0.1250	0.2933597522	0.2929533341	0.0004064182
0.1875	0.3158196505	0.3151560698	0.0006635808
0.2500	0.3388224061	0.3378604148	0.0009619914
0.3125	0.3623862066	0.3610800974	0.0013061092
0.3750	0.3865301311	0.3848293263	0.0017008048
0.4375	0.4112740884	0.4091228263	0.0021512621
0.5000	0.4366351937	0.4339758805	0.0026593131

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998235.

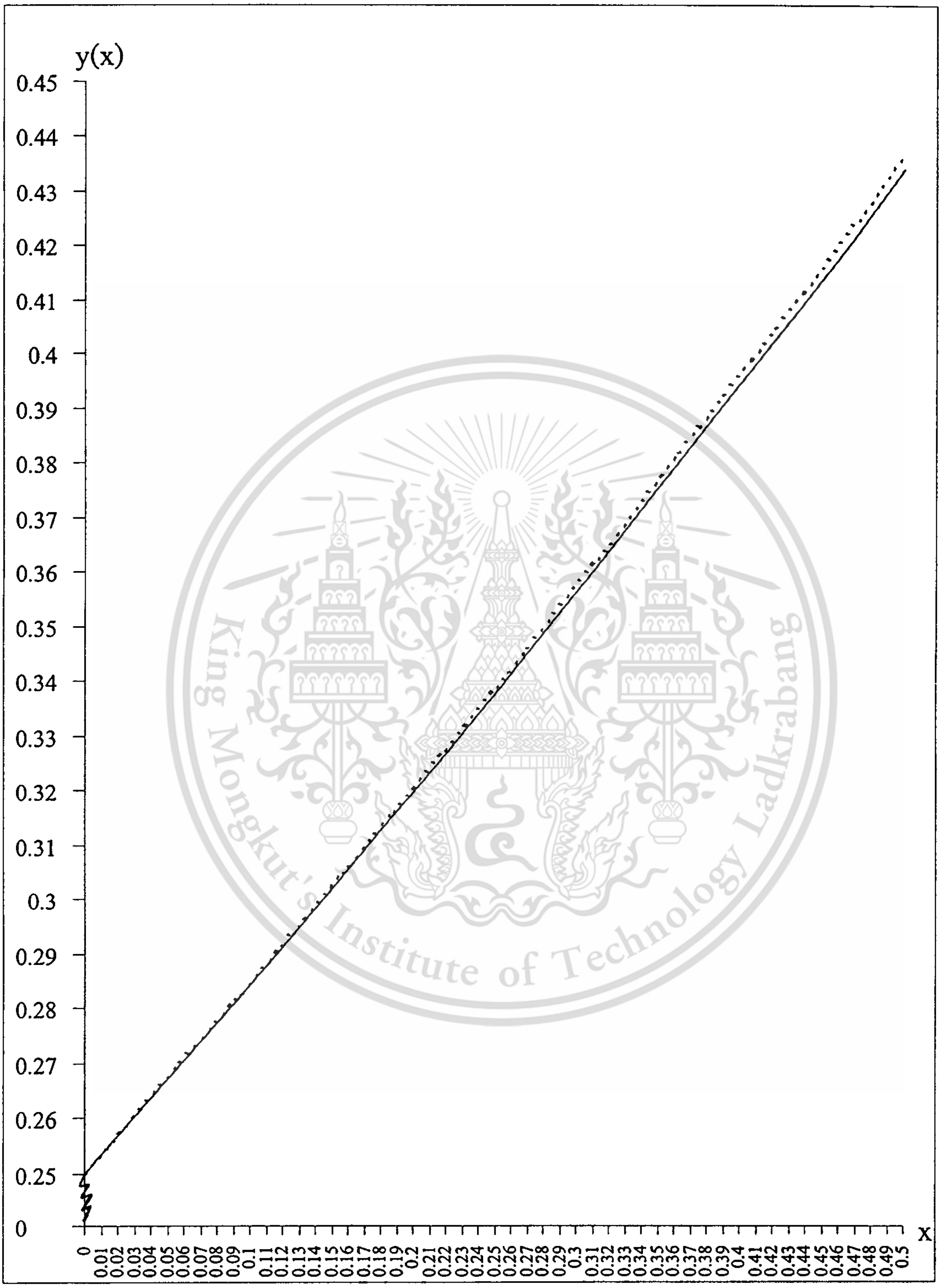


Figure 5.7 Graph of Table 5.7

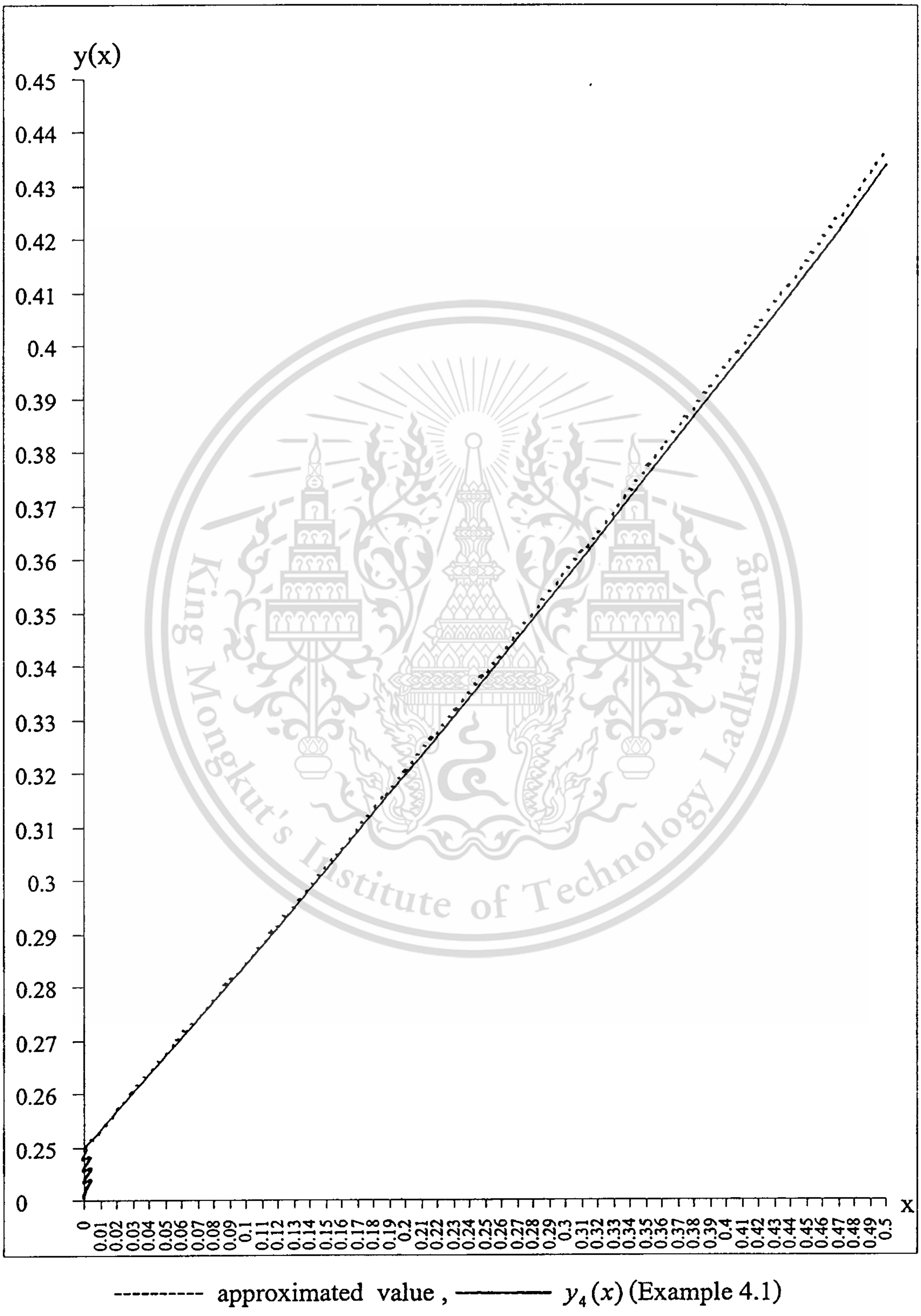


Figure 5.8 Graph of Table 5.8

Table 5.9 The result of, Example 5.5.1, the fourth method ($k=2$)

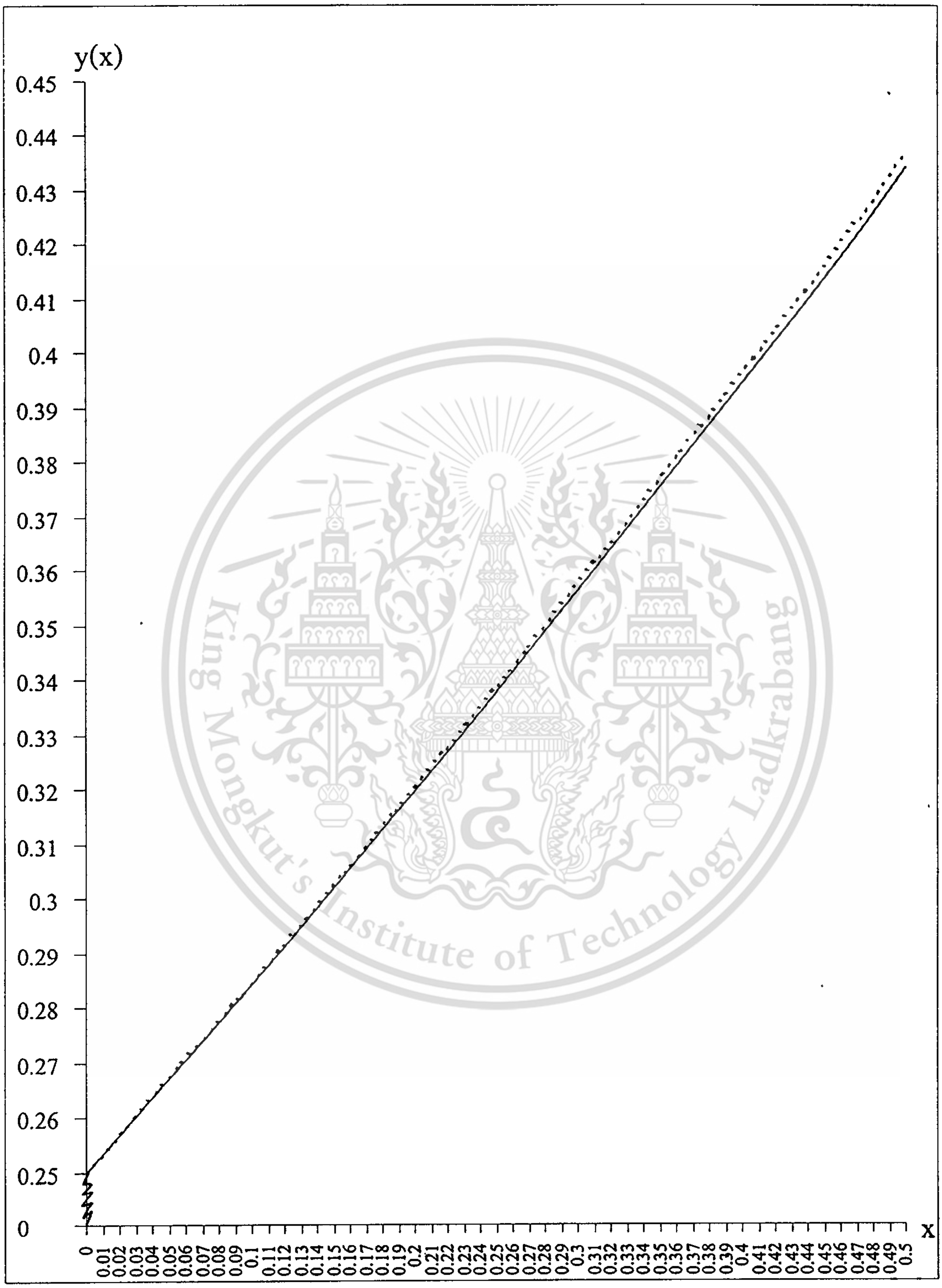
x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714170935	0.2712389291	0.0001781643
0.1250	0.2933404828	0.2929533341	0.0003871488
0.1875	0.3157969825	0.3151560698	0.0006409128
0.2500	0.3387953235	0.3378604148	0.0009349087
0.3125	0.3623565340	0.3610800974	0.0012764366
0.3750	0.3864976897	0.3848293263	0.0016683633
0.4375	0.4112387144	0.4091228263	0.0021158881
0.5000	0.4366008210	0.4339758805	0.0026249405

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998111.

Table 5.10 The result of, Example 5.5.1, the fourth method ($k=3$)

x	approximated value	$y_4(x)$ (Example 4.1)	absolute error
0.0000	0.2500000000	0.2500000000	0.0000000000
0.0625	0.2714253748	0.2712389291	0.0001864457
0.1250	0.2933595802	0.2929533341	0.0004062462
0.1875	0.3158191723	0.3151560698	0.0006631025
0.2500	0.3388218789	0.3378604148	0.0009614641
0.3125	0.3623856306	0.3610800974	0.0013055332
0.3750	0.3865295007	0.3848293263	0.0017001743
0.4375	0.4112734010	0.4091228263	0.0021505747
0.5000	0.4366385374	0.4339758805	0.0026626569

The Correlation Coefficient between approximated value and $y_4(x)$ is 0.999998209.



---------- approximated value , ———— $y_4(x)$ (Example 4.1)

Figure 5.9 Graph of Table 5.9

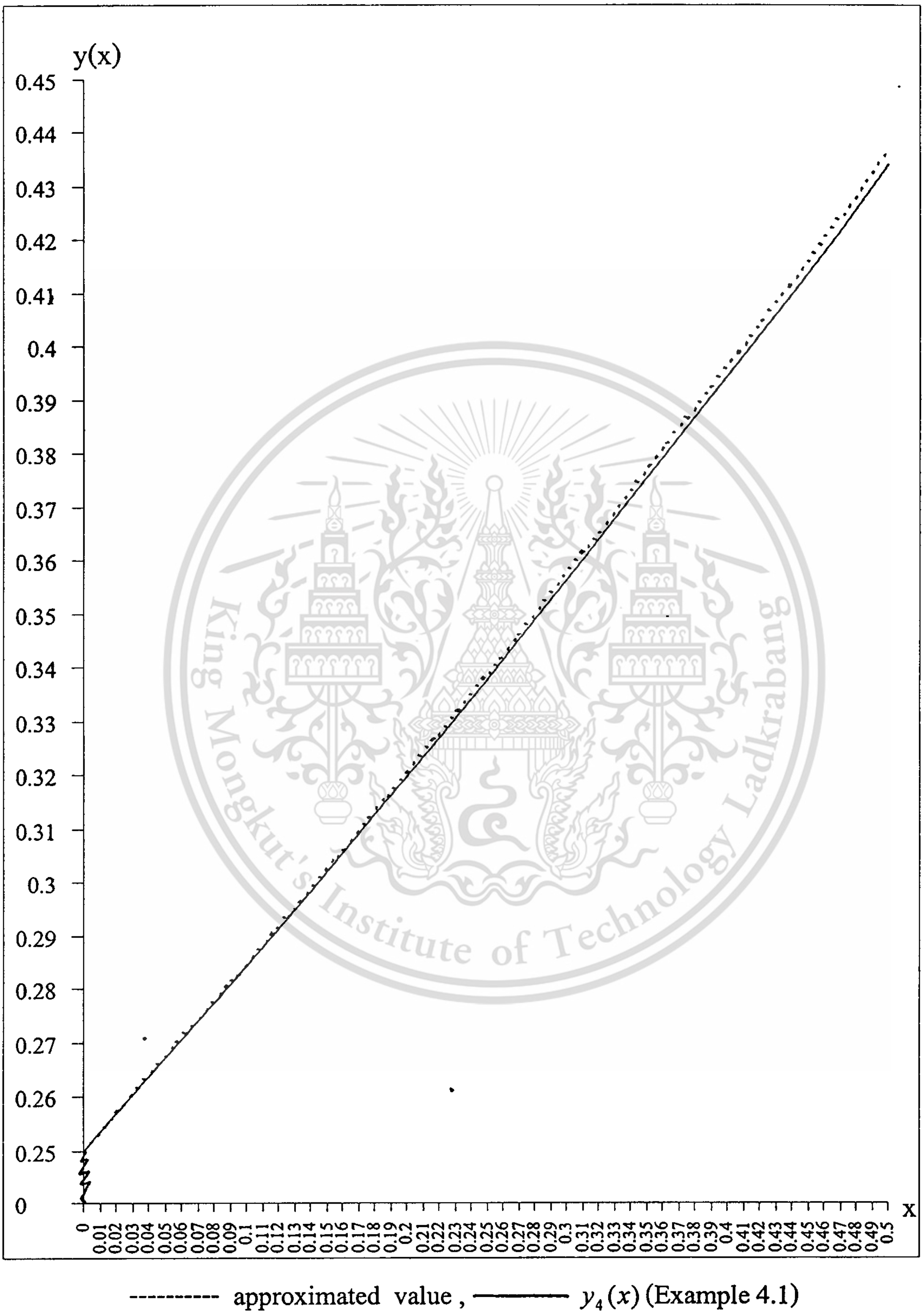


Figure 5.10 Graph of Table 5.10

Example 5.5.2. We shall use the method mentioned above to solve the problem

$$y'(x) = y(y(y(x))) \tag{5.23}$$

$$y(0) = 0.2 \quad , \quad 0 \le x \le \frac{1}{2}. \tag{5.24}$$

First, divide the interval $\left[0, \frac{1}{2}\right]$ into 8 equally subintervals. So we have nine points, they are 0, 0.0625, 0.125, 0.1875, 0.25, 0.3125, 0.375, 0.4375 and 0.5.

We shall use $k = 1$, $k = 2$ and $k = 3$, that is we use

$$P_{1,i}(x) = a_0 + a_1 x \tag{5.25}$$

$$P_{2,i}(x) = a_0 + a_1 x + a_2 x^2 \tag{5.26}$$

$$P_{3,i}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{5.27}$$

and $\varepsilon = 10^{-10}$ and $M = 200$.

The results of the first method are in the Table 5.11, Table 5.12 and Table 5.13 respectively. The results of the second method are in the Table 5.14 and Table 5.15 for (5.26) and (5.27) since the result of (5.25) will be the same as the first method. The results of the third method are in the Table 5.16, Table 5.17 and Table 5.18 respectively. The results of the fourth method are in the Table 5.19 and Table 5.20 for (5.26) and (5.27) since the result of (5.25) will be the same as the third method.

Table 5.11 The result , Example 5.5.2 , of the first method ($k=1$)

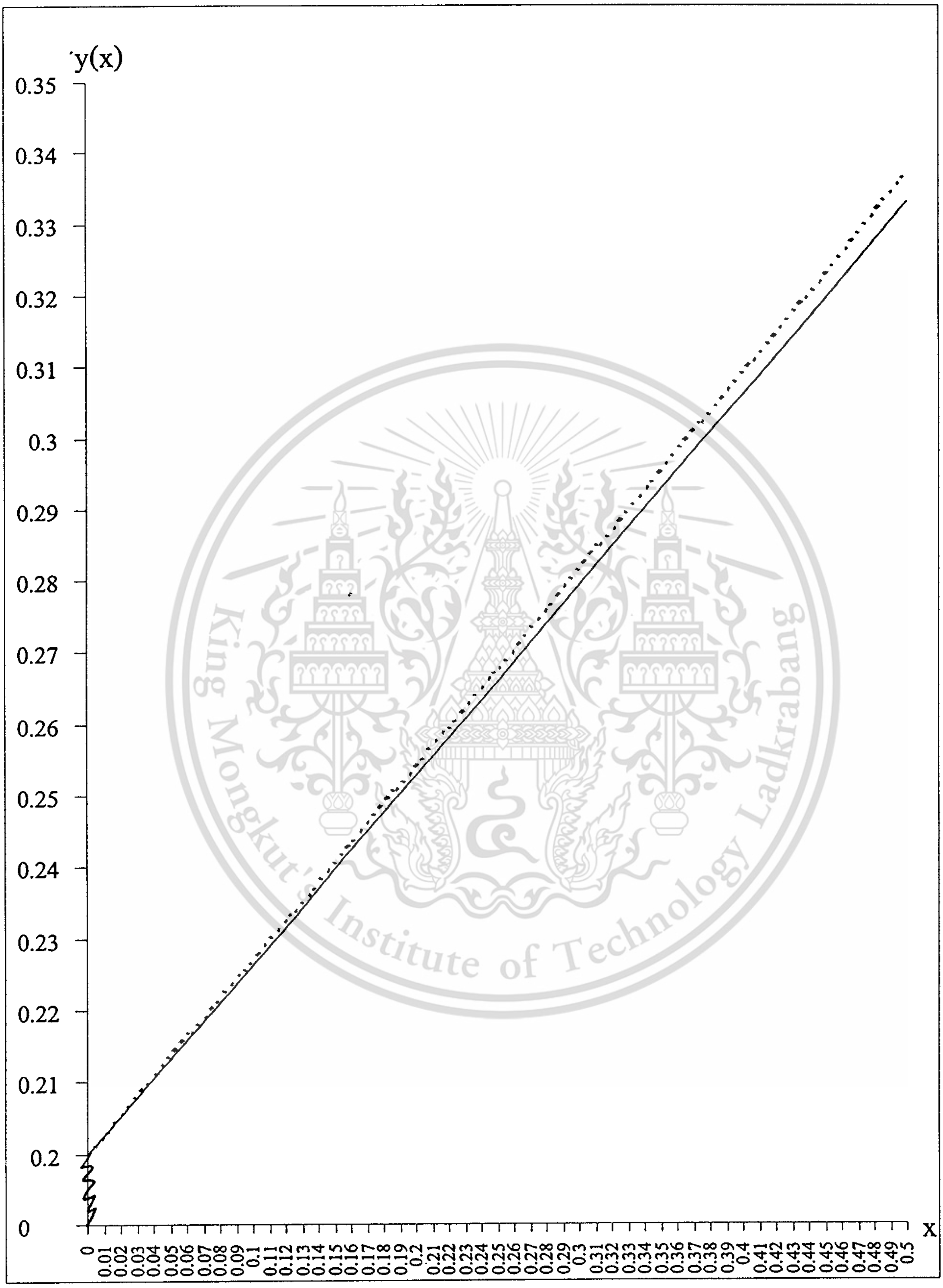
x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168016057	0.2164171263	0.0003844794
0.1250	0.2337036355	0.2328948617	0.0008087738
0.1875	0.2506992593	0.2494333659	0.0012658934
0.2500	0.2677826091	0.2660327989	0.0017498101
0.3125	0.2849485847	0.2826933209	0.0022552637
0.3750	0.3021927099	0.2994150924	0.0027776174
0.4375	0.3195110219	0.3161982741	0.0033127478
0.5000	0.3368999860	0.3330430269	0.0038569591

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999044.

Table 5.12 The result , Example 5.5.2 , of the first method ($k=2$)

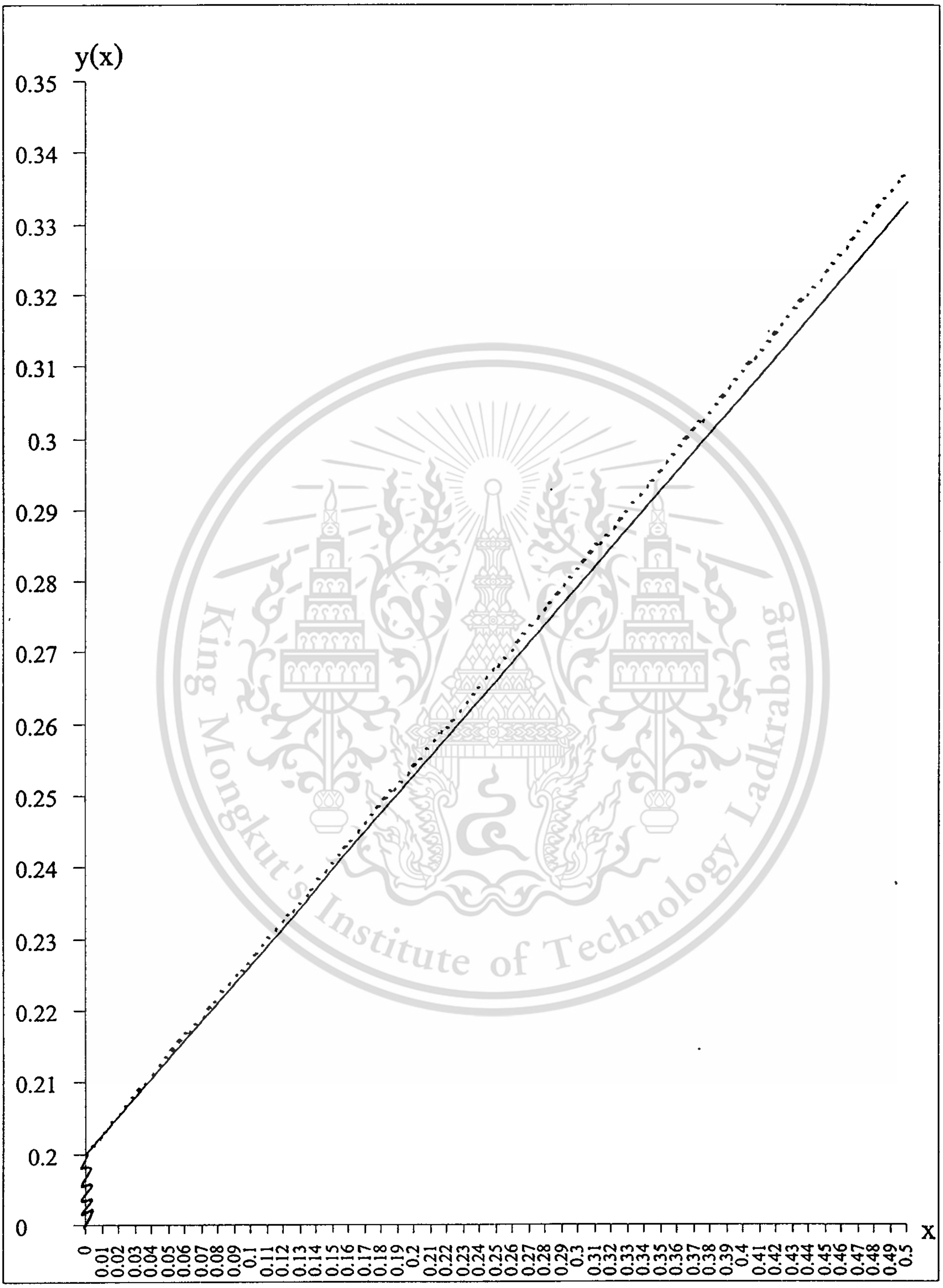
x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168551503	0.2164171263	0.0004380239
0.1250	0.2337896719	0.2328948617	0.0008948102
0.1875	0.2508039707	0.2494333659	0.0013706048
0.2500	0.2678985054	0.2660327989	0.0018657065
0.3125	0.2850737765	0.2826933209	0.0023804555
0.3750	0.3023303185	0.2994150924	0.0029152261
0.4375	0.3196686946	0.3161982741	0.0034704204
0.5000	0.3370690169	0.3330430269	0.0040259900

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999395.



---------- approximated value , ——— $y_3(x)$ (Example 4.2)

Figure 5.11 Graph of Table 5.11



---------- approximated value , ———— $y_3(x)$ (Example 4.2)

Figure 5.12 Graph of Table 5.12

Table 5.13 The result , Example 5.5.2 , of the first method ($k = 3$)

x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168553921	0.2164171263	0.0004382657
0.1250	0.2337900062	0.2328948617	0.0008951444
0.1875	0.2508043439	0.2494333659	0.0013709780
0.2500	0.2678989119	0.2660327989	0.0018661130
0.3125	0.2850742214	0.2826933209	0.0023809005
0.3750	0.3023307880	0.2994150924	0.0029156956
0.4375	0.3196691999	0.3161982741	0.0034709258
0.5000	0.3370695587	0.3330430269	0.0040265318

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999396.

Table 5.14 The result , Example 5.5.2 , of the second method ($k = 2$)

x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168551503	0.2164171263	0.0004380239
0.1250	0.2337895003	0.2328948617	0.0008946386
0.1875	0.2508037815	0.2494333659	0.0013704156
0.2500	0.2678982899	0.2660327989	0.0018654910
0.3125	0.2850735422	0.2826933209	0.0023802212
0.3750	0.3023300512	0.2994150924	0.0029149588
0.4375	0.3196684068	0.3161982741	0.0034701327
0.5000	0.3370890548	0.3330430269	0.0040460279

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999305.

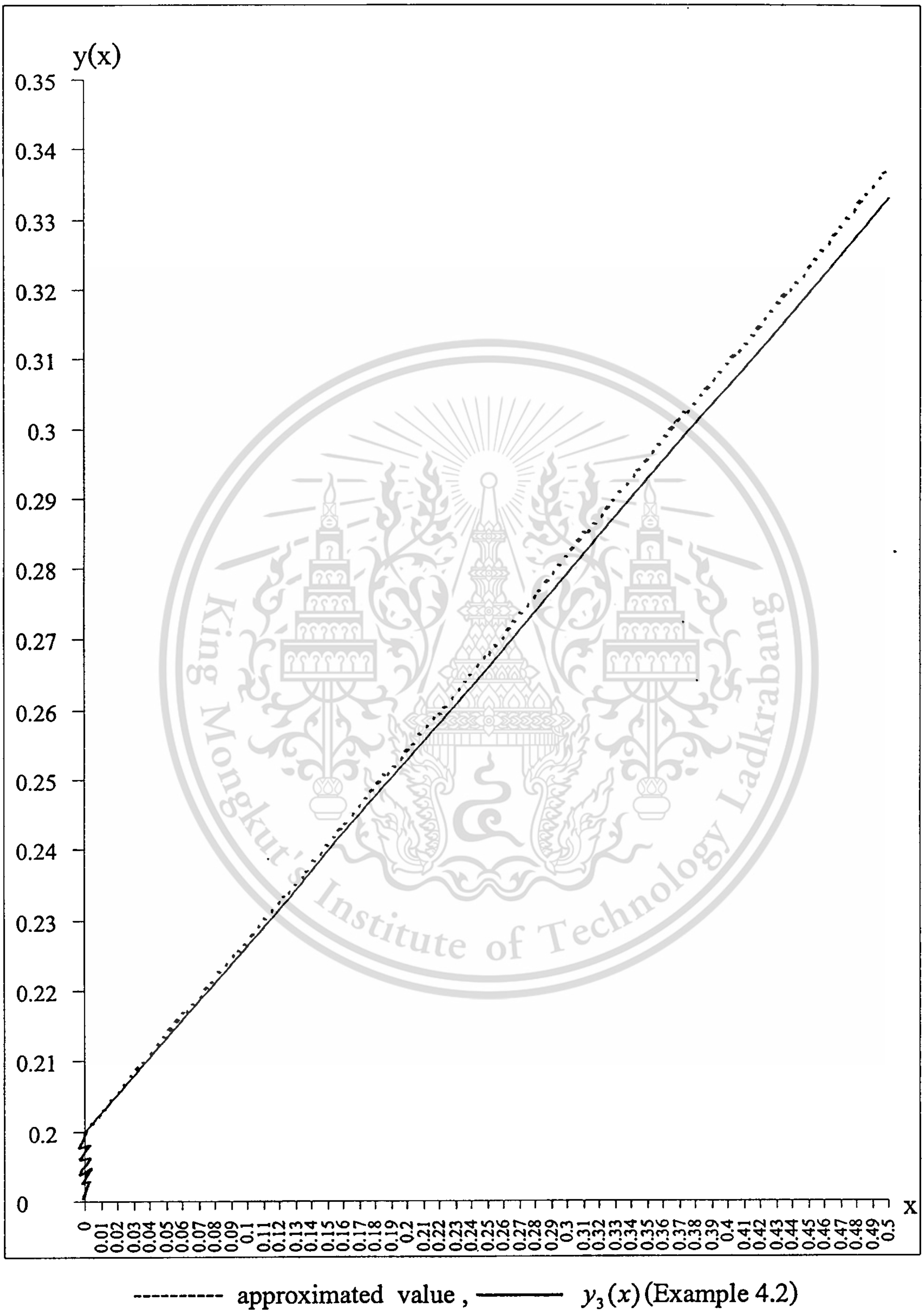
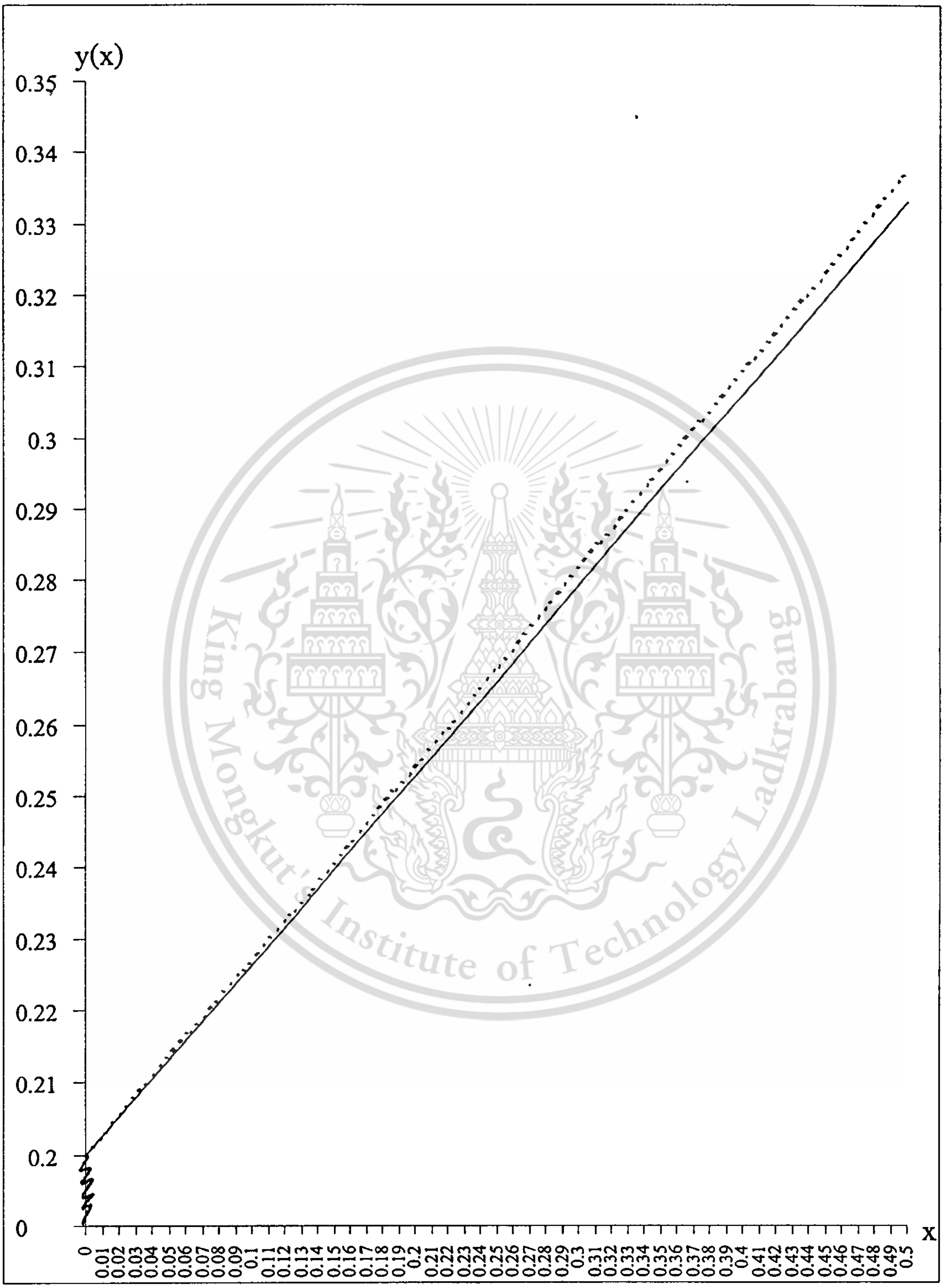


Figure 5.13 Graph of Table 5.13



---------- approximated value , ———— $y_3(x)$ (Example 4.2)

Figure 5.14 Graph of Table 5.14

Table 5.15 The result , Example 5.5.2 , of the second method ($k = 3$)

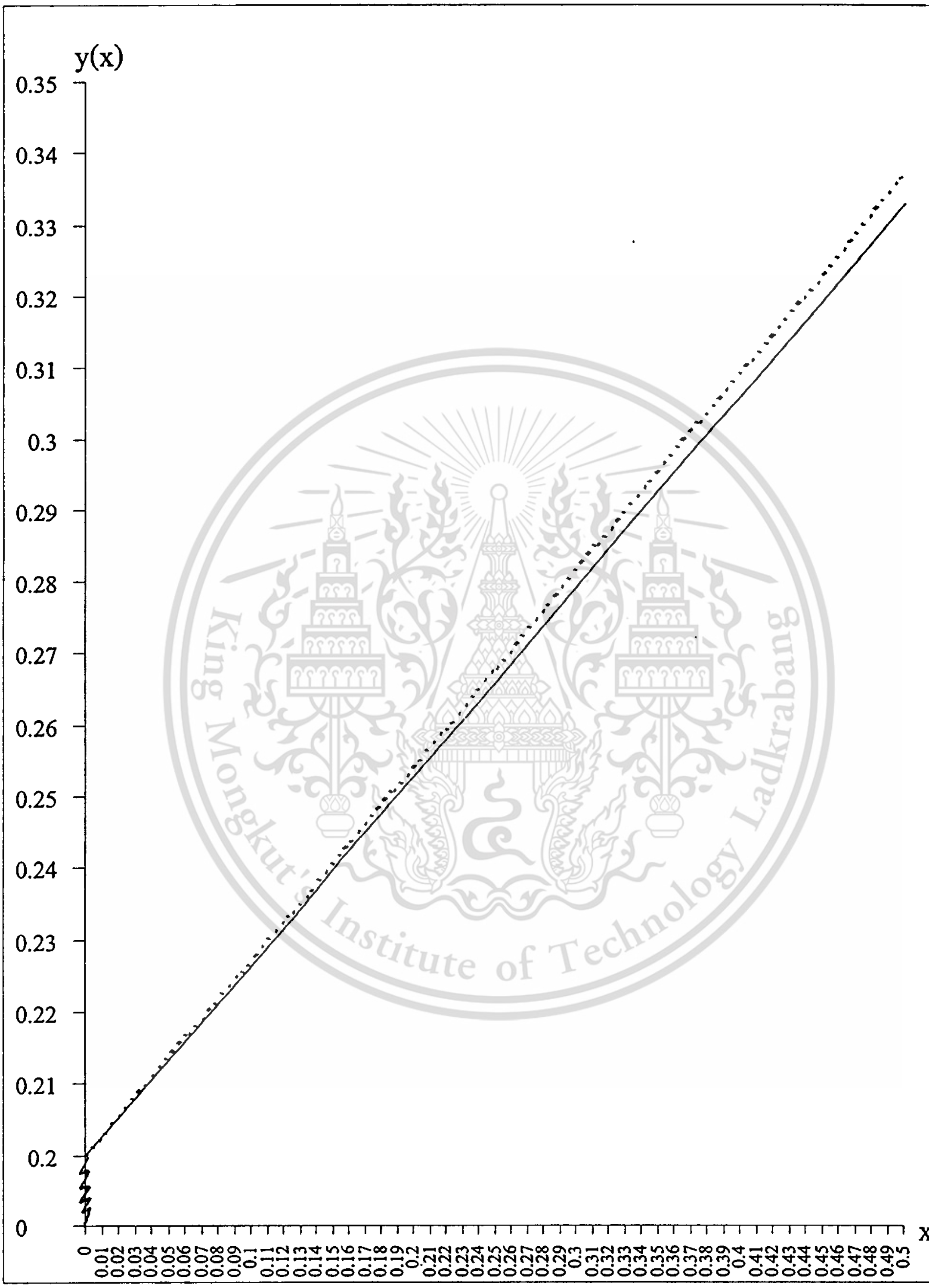
x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168553921	0.2164171263	0.0004382657
0.1250	0.2337900056	0.2328948617	0.0008951439
0.1875	0.2508043427	0.2494333659	0.0013709767
0.2500	0.2678989105	0.2660327989	0.0018661116
0.3125	0.2850742198	0.2826933209	0.0023808989
0.3750	0.3023307864	0.2994150924	0.0029156940
0.4375	0.3196691982	0.3161982741	0.0034709241
0.5000	0.3370899032	0.3330430269	0.0040468763

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999307.

Table 5.16 The result , Example 5.5.2 , of the third method ($k = 1$)

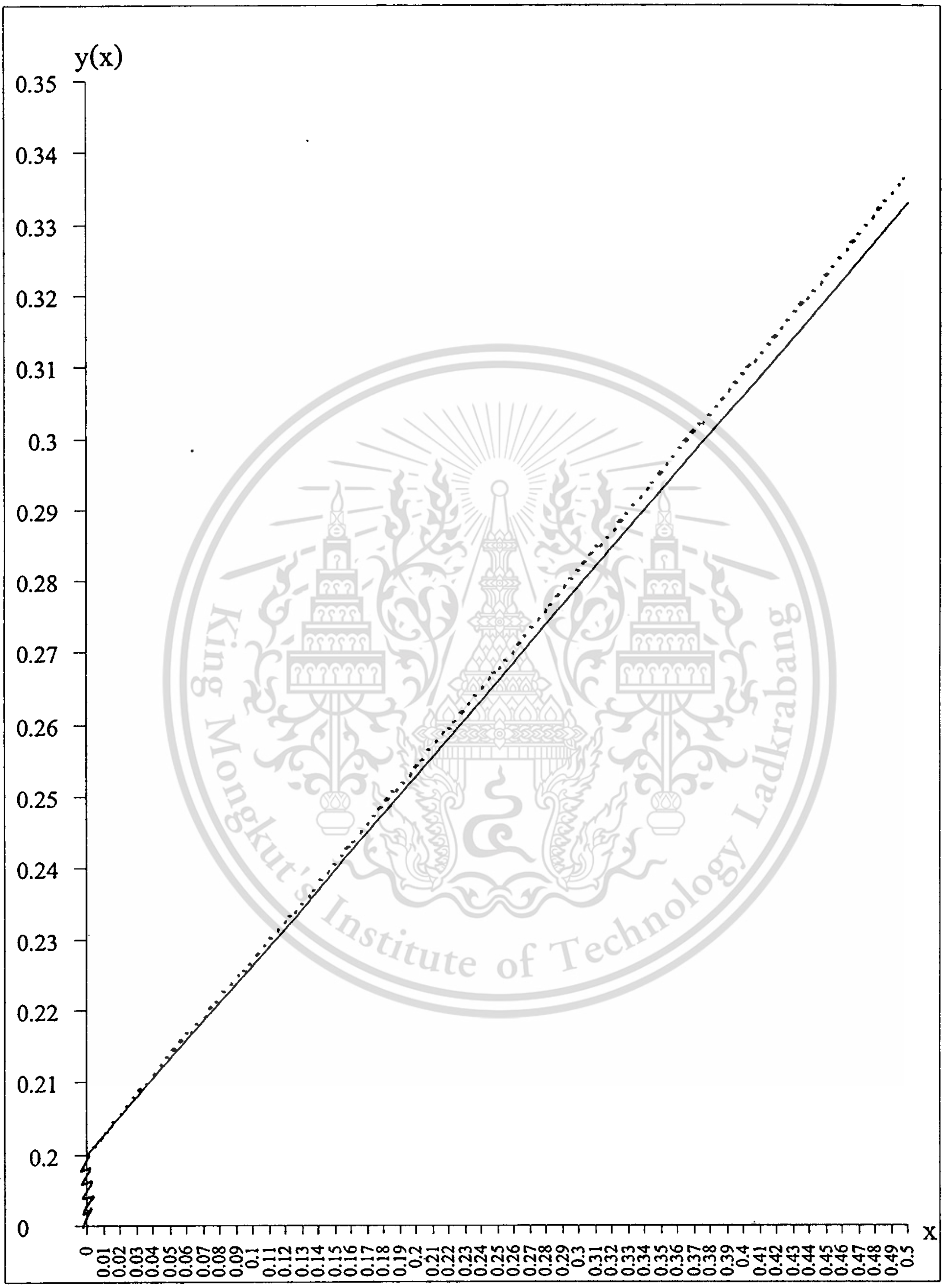
x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168016057	0.2164171263	0.0003844794
0.1250	0.2337036355	0.2328948617	0.0008087738
0.1875	0.2506992593	0.2494333659	0.0012658934
0.2500	0.2677826091	0.2660327989	0.0017498101
0.3125	0.2849485847	0.2826933209	0.0022552637
0.3750	0.3021927099	0.2994150924	0.0027776174
0.4375	0.3195110219	0.3161982741	0.0033127478
0.5000	0.3368999860	0.3330430269	0.0038569591

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999044.



---------- approximated value , ——— $y_3(x)$ (Example 4.2)

Figure 5.15 Graph of Table 5.15



---------- approximated value , ——— $y_3(x)$ (Example 4.2)

Figure 5.16 Graph of Table 5.16

Table 5.17 The result , Example 5.5.2 , of the third method ($k = 2$)

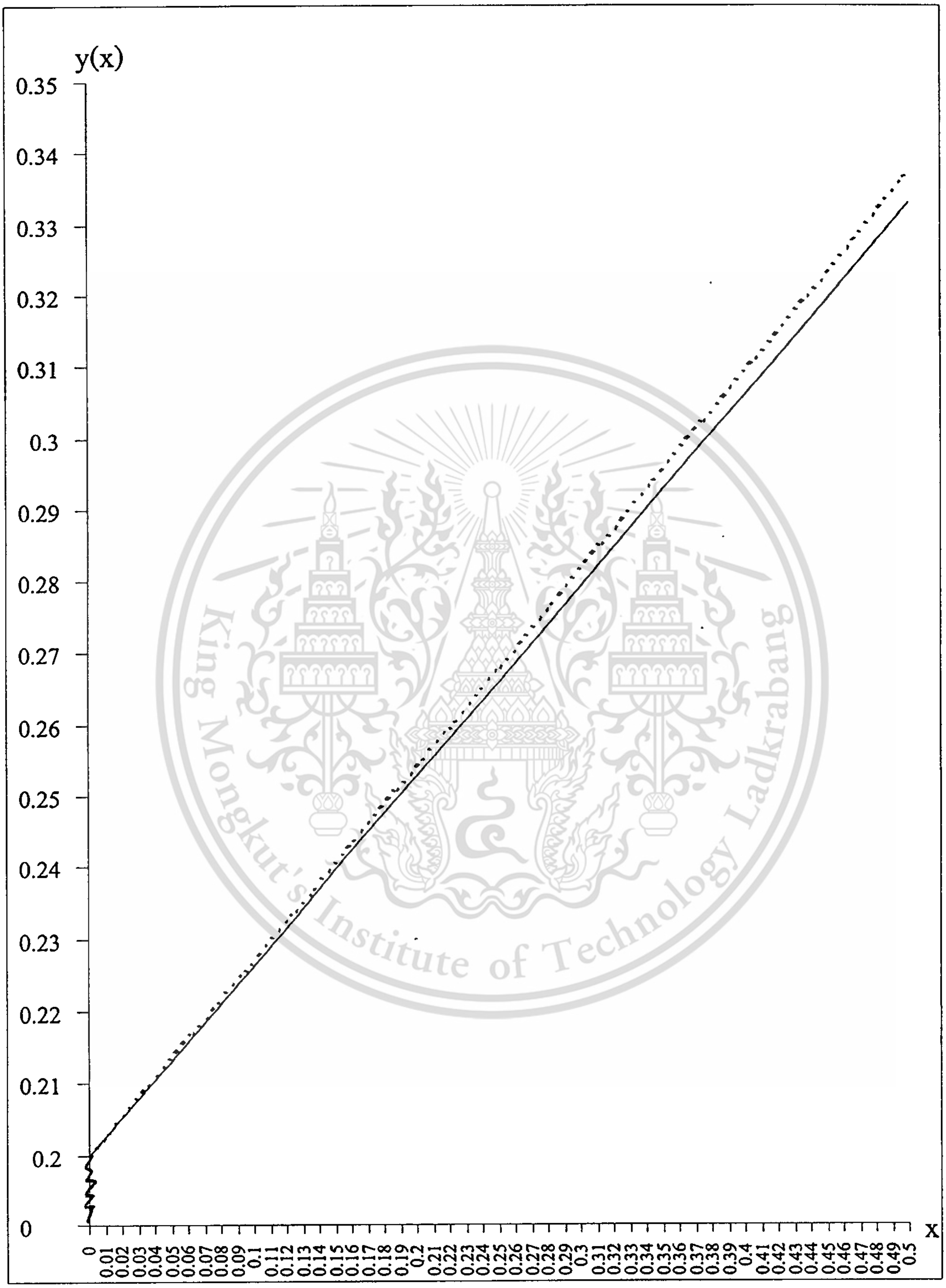
x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168551503	0.2164171263	0.0004380239
0.1250	0.2337896719	0.2328948617	0.0008948102
0.1875	0.2508039707	0.2494333659	0.0013706048
0.2500	0.2678985054	0.2660327989	0.0018657065
0.3125	0.2850737765	0.2826933209	0.0023804555
0.3750	0.3023303185	0.2994150924	0.0029152261
0.4375	0.3196686946	0.3161982741	0.0034704204
0.5000	0.3370690169	0.3330430269	0.0040259900

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999395.

Table 5.18 The result , Example 5.5.2 , of the third method ($k = 3$)

x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168553921	0.2164171263	0.0004382657
0.1250	0.2337900062	0.2328948617	0.0008951444
0.1875	0.2508043439	0.2494333659	0.0013709780
0.2500	0.2678989119	0.2660327989	0.0018661130
0.3125	0.2850742214	0.2826933209	0.0023809005
0.3750	0.3023307880	0.2994150924	0.0029156956
0.4375	0.3196691999	0.3161982741	0.0034709258
0.5000	0.3370695587	0.3330430269	0.0040265318

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999396.



---------- approximated value , ——— $y_3(x)$ (Example 4.2)

Figure 5.17 Graph of Table 5.17

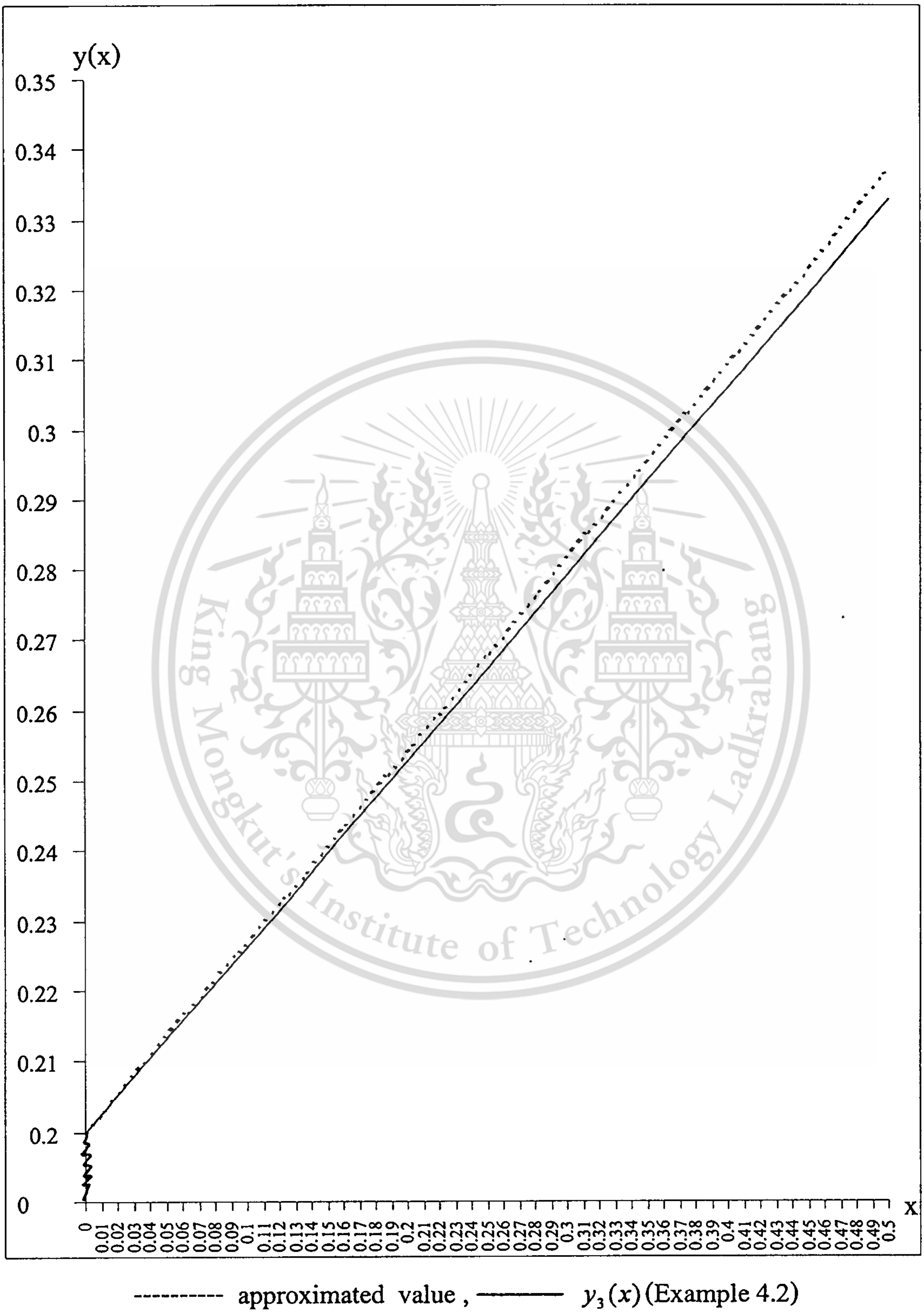


Figure 5.18 Graph of Table 5.18

Table 5.19 The result, Example 5.5.2, of the fourth method ($k = 2$)

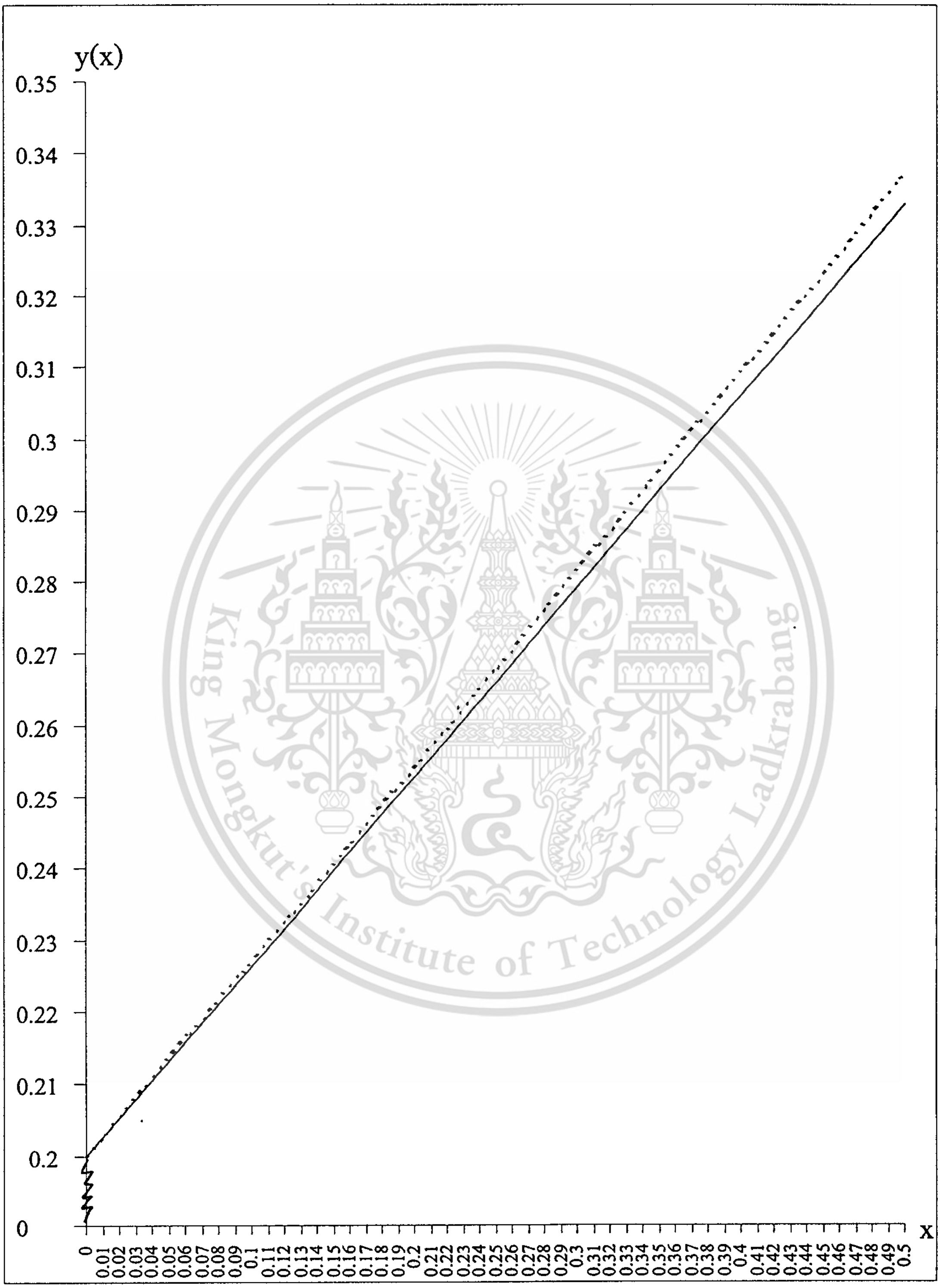
x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168551503	0.2164171263	0.0004380239
0.1250	0.2337895003	0.2328948617	0.0008946386
0.1875	0.2508037815	0.2494333659	0.0013704156
0.2500	0.2678982899	0.2660327989	0.0018654910
0.3125	0.2850735422	0.2826933209	0.0023802212
0.3750	0.3023300512	0.2994150924	0.0029149588
0.4375	0.3196684068	0.3161982741	0.0034701327
0.5000	0.3370890548	0.3330430269	0.0040460279

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999305.

Table 5.20 The result, Example 5.5.2, of the fourth method ($k = 3$)

x	approximated value	$y_3(x)$ (Example 4.2)	absolute error
0.0000	0.2000000000	0.2000000000	0.0000000000
0.0625	0.2168553921	0.2164171263	0.0004382657
0.1250	0.2337900056	0.2328948617	0.0008951439
0.1875	0.2508043427	0.2494333659	0.0013709767
0.2500	0.2678989105	0.2660327989	0.0018661116
0.3125	0.2850742198	0.2826933209	0.0023808989
0.3750	0.3023307864	0.2994150924	0.0029156940
0.4375	0.3196691982	0.3161982741	0.0034709241
0.5000	0.3370899032	0.3330430269	0.0040468763

The Correlation Coefficient between approximated value and $y_3(x)$ is 0.999999307.



---------- approximated value , ——— $y_3(x)$ (Example 4.2)

Figure 5.19 Graph of Table 5.19

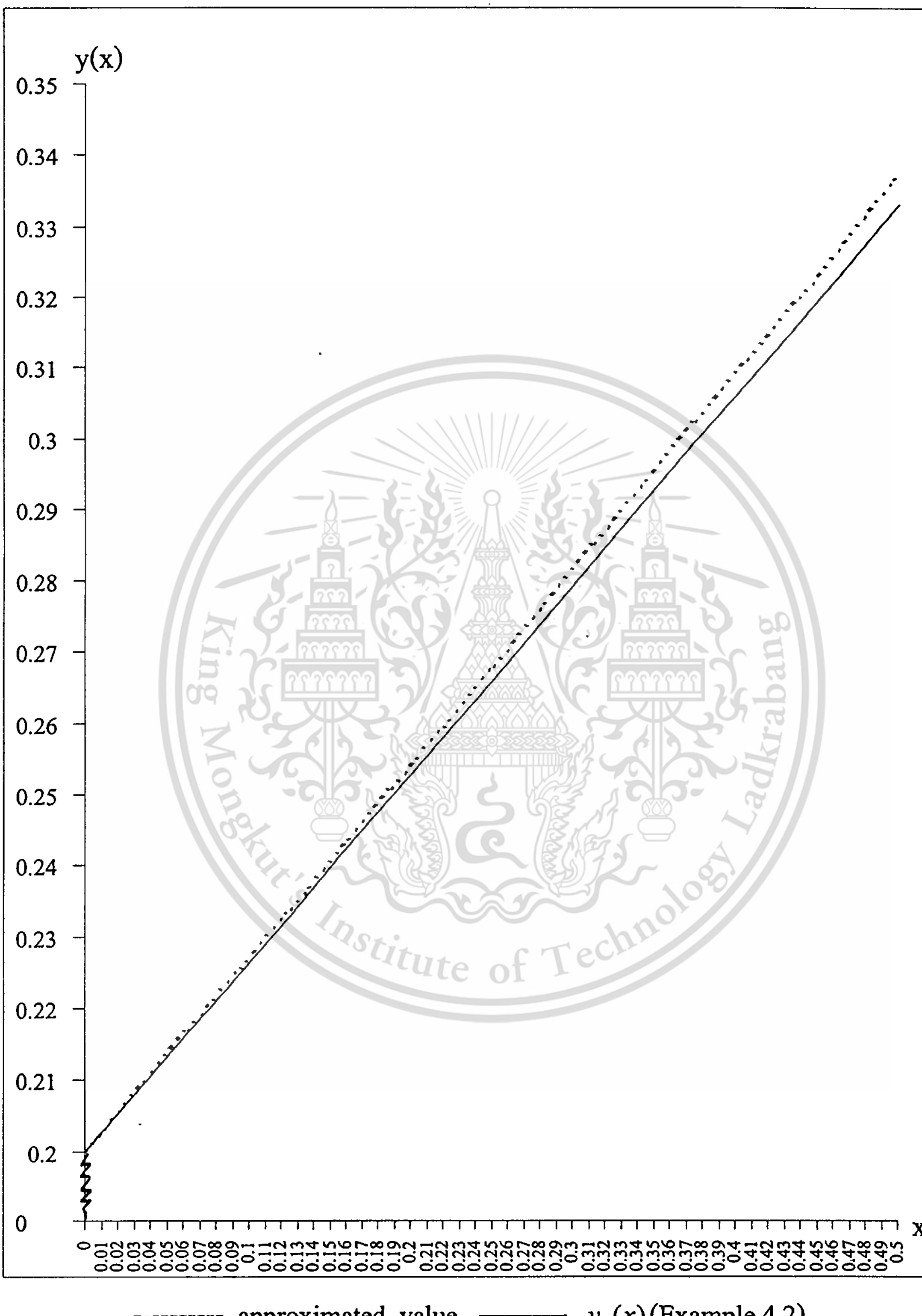


Figure 5.20 Graph of Table 5.20

5.6 Conclusion

Table 5.21 The Correlation Coefficient between approximated value and $y_4(x)$ in Example 5.5.1

	Method			
	First	Second	Third	Fourth
$k=1$	0.999994127	0.999994127	0.999994127	0.999994127
$k=2$	0.999998172	0.999998111	0.999998172	0.999998111
$k=3$	0.999998235	0.999998209	0.999998235	0.999998209

Table 5.22 The Correlation Coefficient between approximated value and $y_3(x)$ in Example 5.5.2

	Method			
	First	Second	Third	Fourth
$k=1$	0.999999044	0.999999044	0.999999044	0.999999044
$k=2$	0.999999395	0.999999305	0.999999395	0.999999305
$k=3$	0.999999396	0.999999307	0.999999396	0.999999307

From the Table 5.21 and Table 5.22, if we consider that the Correlation Coefficients have five significant figures, we will find that the Correlation Coefficients of the four methods are the same values (0.99999). This means that we can choose each one of the four methods for solving the problems. Anyway we should prefer to use the second method. Because of each step for the first method and the third method, we can have the approximate value of $y(x)$ in one point. While in each step of the second method, we can have the approximate value of $y(x)$ in k points. Also in the fourth method, we can have the approximate value of $y(x)$ in k points. But it is difficult to do. So we suggest that the second method is the best. If we consider that the

Correlation Coefficients have nine significant figures, the first method or the third method are suitable . The reason is the both of two methods have the maximal Correlation Coefficient. However we should prefer to use the first method more than the third method. Because of using the step of the first method is easier than the third method.

REFERENCES

[1] W.B.Fite. "Properties of The Solutions of Certain Functional Differential Equations ." Trans. Amer. Math. Soc., vol. 22, 1921. pp. 311-319.

[2] S. Doss and S.K. Nasr. "On the Functional Equation $\frac{dy}{dx} = f(x, y(x), y(x+h))$, $h > 0$." Amer. J. Math., vol 75, 1953. pp. 713-716.

[3] W.R.Utz. "The Equation $f(x) = a(f(g(x)))$." Bull. Amer. Math. Soc., vol 71, 1965. pp. 138.

[4] D.R.Anderson. "An Existence Theorem for a Solution of $f'(x) = F(x, f(g(x)))$." Siam Rev., vol. 8, 1966. pp. 359-362.

[5] G.Barba. "Sopra l'equazione funzionale $f(x)f'(x) = f(f(x))$ collegata ad un problema geometrico." Atti Accad. Lincei, Rendiconti, vol. 11, 1930. pp. 655-658.

[6] A.Pelczar. "On Some Iterative-Differential Equations I." Zeszyty Naukowe UJ., Prace Mat., vol. 12, 1968. pp. 53-56.

[7] A.Pelczar. "On Some Iterative-Differential Equations II .", Zeszyty Naukowe UJ., Prace Mat., vol. 13, 1969. pp.49-51.

[8] A.Pelczar. "On Some Iterative-Differential Equations III." , Zeszyty Naukowe UJ., Prace Mat., vol. 15, 1971. pp.125-130.

[9] Maitree Podisuk. " Iterative Differential Equations." Ph.D.Thesis of Jagiellonian University. Cracow. Poland. 1992.

[10] V.P.Skripnik. "Systems with Transformed Argument, Boundary Value Problems and the Cauchy Problem." , Mat. Sb. (N.S.)., vol. 62, 1963. pp.385-396.

[11] V.P.Skripnik. "System with Transformed Argument Depended on Its Solutions." Mat. Sb. (N.S.)., vol. 68, 1965. pp.274-281.

[12] Coddington E.A.. "An Introduction to Ordinary Differential Equations." Englewood Cliffs, New Tersey. Prentice-Hall. 1961.

[13] F.Brauer and J.A.Nohel. "Qualitative Theory of Ordinary Differential Equations." W.A.Benjamin, Inc. 1969.

[14] S.D.Conte and de Boor. "Elementary Numerical Analysis." 2nd Edition. McGraw-Hill Book Company. 1972.

[15] R.R. Goldberg. "Methods of Real Analysis." 2nd Edition. John Wiley and Sons, Inc. 1976.

[16] H.L.Royden. "Real Analysis." 2nd Edition. The Macmillan Company. 1968.

AUTHOR BIOGRAPHY

Author	:	Pimpak Phataranavik
Date of Birth	:	August 2, 1973
Bachelor Degree	:	B.Sc. in Mathematics
Institution	:	Faculty of Science
		Chulalongkong University, Bangkok, Thailand
Year of Graduation	:	1994
Position	:	Instructor of Mathematics, Huachew Chalermprakiet University, Samutprakarn, Thailand