SIMPLE ITERATIVE PARTIAL DIFFERENTIAL EQUATIONS

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บทคัดย่อ

เนื้อหาของวิทยานิพนธ์เล่มนี้กล่าวถึงการมีผลเฉลยอยู่จริงและมีเพียงหนึ่งเดียวของสมการเชิงอนุพันธ์ย่อยซ้ำเชิงเดียวที่มีรูปแบบเป็น

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = u^m(x)$$

โดยที่ $x \in [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$ เมื่อ $a_1, a_2, \ldots, a_n$ เป็นจำนวนจริงบวก

และมีเงื่อนไขเริ่มต้นคือ

$$u(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = g_{1,i}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \quad i = 1, 2, \ldots n$$

$$u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) = g_{2,i,j}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \quad i \neq j, i, j = 1, 2, \ldots n$$

$$\vdots$$

$$\begin{aligned} u(x_1, 0, \ldots, 0) &= g_{n-1,2,3,\ldots,n}(x_1) \\ u(0, x_2, 0, \ldots, 0) &= g_{n-1,1,3,\ldots,n}(x_2) \\ &\vdots \\ u(0, \ldots, 0, x_n) &= g_{n-1,1,2,\ldots,n-1}(x_n) \\ u(0, \ldots, 0) &= c = [c_1 \ c_2 \ \ldots \ c_n]^T \end{aligned}$$

หรือจะกล่าวได้ว่าเงื่อนไขเริ่มต้นคือ

$$g(x)=\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

$$\begin{aligned}
= \quad & (-1)^2\left(g_{1,1}(x_2, x_3, \ldots, x_n) + \ldots + g_{1,n}(x_1, x_2, \ldots, x_{n-1})\right) \\
& + (-1)^3\left(g_{2,1,2}(x_3, x_4, \ldots, x_n) + \ldots + g_{2,n-1,n}(x_1, x_2, \ldots, x_{n-2})\right) \\
& \qquad \vdots \\
& + (-1)^n\left(g_{n-1,2,3,\ldots,n}(x_1) + \ldots + g_{n-1,1,2,\ldots,n-1}(x_n)\right) \\
& + (-1)^{n+1} c
\end{aligned}$$

เมื่อ m เป็นจำนวนเต็มบวกที่มีค่ามากกว่า 1,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}$$

และ

$$u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x), u_2(x), \ldots, u_n(x)) \\ u_2(u_1(x), u_2(x), \ldots, u_n(x)) \\ \vdots \\ u_n(u_1(x), u_2(x), \ldots, u_n(x)) \end{bmatrix}$$

$$\begin{aligned}
u^3(x) &= u\left(u^2(x)\right) \\
&\ \ \vdots \\
u^m(x) &= u\left(u^{m-1}(x)\right)
\end{aligned}$$

พร้อมทั้งศึกษาวิธีการหาผลเฉลยในรูปอนุกรมกำลังของหลายตัวแปรของสมการเชิงอนุพันธ์ย่อยรูปแบบดังกล่าว

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ABSTRACT

In this thesis we will study the existence and uniqueness of the solutions of the simple iterative partial differential equations,

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = u^m(x)$$

where $x \in [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$ and $a_1, a_2, \ldots, a_n$ are positive real numbers

with the initial conditions

$$u(x_1,\ x_2, \ldots,\ x_{i-1},\ 0,\ x_{i+1}, \ldots,\ x_n) = g_{1,i}(x_1,\ x_2, \ldots,\ x_{i-1},\ x_{i+1}, \ldots, x_n) \qquad i = 1, 2, \ldots n$$

$$u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$$
$$= g_{2,i,j}(x_1, \ldots,\ x_{i-1},\ x_{i+1}, \ldots, x_{j-1},\ x_{j+1}, \ldots,\ x_n)$$
$$i \neq j, i, j = 1, 2, \ldots n$$

$$\vdots$$

$$u(x_1, 0, \ldots, 0) = g_{n-1,2,3,\ldots,n}(x_1)$$
$$u(0,\ x_2, 0, \ldots,\ 0) = g_{n-1,1,3,\ldots,n}(x_2)$$

$$\vdots$$

$$u(0, \ldots, 0, x_n) = g_{n-1,1,2,\ldots,n-1}(x_n)$$
$$u(0, \ldots, 0) = c = [c_1 \ \ c_2 \ \ldots \ c_n]^T$$

or the compatibility initial condition

$$g(x)=\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

$$\begin{aligned} = \quad & (-1)^2\left(g_{1,1}(x_2, x_3, \ldots, x_n) + \ldots + g_{1,n}(x_1, x_2, \ldots, x_{n-1})\right) \\ & + (-1)^3\left(g_{2,1,2}(x_3, x_4, \ldots, x_n) + \ldots + g_{2,n-1,n}(x_1, x_2, \ldots, x_{n-2})\right) \\ & \qquad \vdots \\ & + (-1)^n\left(g_{n-1,2,3,\ldots,n}(x_1) + \ldots + g_{n-1,1,2,\ldots,n-1}(x_n)\right) \\ & + (-1)^{n+1} c \end{aligned}$$

where m is a positive integer greater than 1,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}$$

and $$u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x), u_2(x), \ldots, u_n(x)) \\ u_2(u_1(x), u_2(x), \ldots, u_n(x)) \\ \vdots \\ u_n(u_1(x), u_2(x), \ldots, u_n(x)) \end{bmatrix}$$

$$\begin{aligned} u^3(x) &= u\left(u^2(x)\right) \\ &\vdots \\ u^m(x) &= u\left(u^{m-1}(x)\right) \end{aligned}$$

including finding the power series solutions to approximate the solutions of these equations.

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Jaipong Wongsawasdi

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CHAPTER 1

INTRODUCTION

Iterative Differential Equations is one type of the functional differential equations. The study of the functional differential equations of this type was started more than 90 years ago. There are many mathematicians around the world who gave their efforts in the study of this field. We may classify the study of the iterative differential equations into two main types.

1. Iterative Ordinary Differential Equations.

Iterative ordinary differential equation in the form

$$y'(x) = G\left(x, y(x), y^2(x), \ldots, y^m(x)\right) \tag{1.1}$$

where

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y(y(x))) = y\left(y^2(x)\right) \\ &\vdots \\ y^m(x) &= y\left(y^{m-1}(x)\right) \end{aligned} \tag{1.2}$$

are been used extensively as a reference in many studies.

Many studies of this type of differential equation have used this equation as a basis of its research. One such study is the work of G. Barba [1], which was presented in 1930. The aim of this work is to find the solution of the ordinary differential equation of the form

$$f(x)f'(x) = f(f(x)). \tag{1.3}$$

In another study, presented in 1968-1971, A.Pelczar [2-4] presented the proof of the uniqueness theorem of the solution of the problem in following form

$$y'(x) = f(x, y(x), y(y(x))) \tag{1.4}$$

with the initial condition

$$y(0) = c. \tag{1.5}$$

In 1992, M.Podisuk [5] studied the existence and uniqueness for the solution of the iterative ordinary differential equations of the form

$$y'(x) = f(x, y(x), y^2(x), \ldots, y^m(x)), \quad x \in [0, a] \tag{1.6}$$

with the initial condition

$$y(0) = c, \quad c \in [0, a]. \tag{1.7}$$

In a recent study presented in 1999, P.Pataranavik [6] discussed the existence and uniqueness for the solution of the simple iterative ordinary differential equations of the form

$$y'(x) = y^m(x), \quad x \in [0, a] \tag{1.8}$$

with the initial condition

$$y(0) = c, \quad c \in [0, a] \tag{1.9}$$

where a is positive real number

c is positive real number in interval $[0, a]$

m is positive integer greater than 1, and

$$
\begin{aligned}
y^2(x) &= y(y(x)) \\
y^3(x) &= y(y(y(x))) = y(y^2(x)) \\
y^4(x) &= y(y^3(x)) \\
&\vdots \\
y^m(x) &= y(y^{m-1}(x)).
\end{aligned}
$$

2. Iterative Partial Differential Equations

In 1992, M.Podisuk studied the existence and uniqueness for the solution of iterative partial differential equation of the form

$$
\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = f\left(x, u(x), u^2(x), \ldots, u^m(x)\right) \tag{1.10}
$$

with the initial conditions

$$
u(x_1,\ x_2,\ldots,\ x_{i-1},\ 0,\ x_{i+1},\ldots,\ x_n) = g_{1,i}(x_1,\ x_2,\ldots,\ x_{i-1},\ x_{i+1},\ldots,x_n) \quad i=1,2,\ldots,n
$$

$$
\begin{aligned}
u(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_{j-1},0,x_{j+1},\ldots,x_n) \\
= g_{2,i,j}(x_1,\ldots,\ x_{i-1},\ x_{i+1},\ldots,x_{j-1},\ x_{j+1},\ldots,\ x_n) \\
i \neq j,\ i, j=1,2,\ldots,n
\end{aligned}
$$

$$
\vdots
$$

$$
\begin{aligned}
u(x_1, 0,\ldots,0) &= g_{n-1,2,3,\ldots,n}(x_1) \\
u(0,\ x_2, 0,\ldots,\ 0) &= g_{n-1,1,3,\ldots,n}(x_2) \\
&\vdots \\
u(0,\ldots,0, x_n) &= g_{n-1,1,2,\ldots,n-1}(x_n) \\
u(0,\ldots,0) &= c = [c_1 \quad c_2 \ldots c_n]^T
\end{aligned} \tag{1.11}
$$

or the compatibility initial condition

$$
g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}
$$

$$
\begin{aligned}
= \; & (-1)^2\left(g_{1,1}(x_2,x_3,\dots,x_n)+\dots+g_{1,n}(x_1,x_2,\dots,x_{n-1})\right) \\
& +(-1)^3\left(g_{2,1,2}(x_3,x_4,\dots,x_n)+\dots+g_{2,n-1,n}(x_1,x_2,\dots,x_{n-2})\right) \\
& \qquad\qquad\vdots \\
& +(-1)^n\left(g_{n-1,2,3,\dots,n}(x_1)+\dots+g_{n-1,1,2,\dots,n-1}(x_n)\right) \\
& +(-1)^{n+1}c
\end{aligned}
\tag{1.12}
$$

where m is positive integer greater than 1 and

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}
$$

and

$$
f(x) = \begin{bmatrix} f_1\left(x, u(x), u^2(x), \dots, u^m(x)\right) \\ f_2\left(x, u(x), u^2(x), \dots, u^m(x)\right) \\ \vdots \\ f_n\left(x, u(x), u^2(x), \dots, u^m(x)\right) \end{bmatrix}
$$

$$
u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x), u_2(x), \dots, u_n(x)) \\ u_2(u_1(x), u_2(x), \dots, u_n(x)) \\ \vdots \\ u_n(u_1(x), u_2(x), \dots, u_n(x)) \end{bmatrix}
$$

$$
\begin{aligned}
u^3(x) &= u\left(u^2(x)\right) \\
&\;\;\vdots \\
u^m(x) &= u\left(u^{m-1}(x)\right)
\end{aligned}
$$

and

$$
Z = [0, a_1]\times[0, a_2]\times\dots\times[0, a_n]
$$

and

$$
\begin{aligned}
& u_i : Z \to R, \quad f_i : Z \times R^{mn} \to R, \quad i = 1, 2, \dots, n \\
& u : Z \to R^n, \quad f : Z \times R^{mn} \to R^n
\end{aligned}
$$

when f and g are continuous, then the problem (1.10) – (1.12) is equivalent to the problem of finding the continuous solution of the integral equation

$$u(x) = g(x) + \int_0^x f\left(t,\ u(t), u^2(t), \ldots, u^m(t)\right)dt\,. \tag{1.13}$$

The research presented in this thesis is influenced by the study in 1992 by M. Podisuk. The focus of this thesis is equation 1.14 which is derived from equation 1.10 by Podisuk. In this thesis, we will prove the existence and uniqueness theorems of the simple iterative partial differential equations in the form

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = u^m(x) \tag{1.14}$$

with the initial conditions

$$\begin{aligned}
u(x_1,\ x_2, \ldots,\ x_{i-1},\ 0,\ x_{i+1}, \ldots,\ x_n) &= g_{1,i}(x_1,\ x_2, \ldots,\ x_{i-1},\ x_{i+1},\ x_n) \qquad i = 1, 2, \ldots, n \\
u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) & \\
&= g_{2,i,j}(x_1, \ldots,\ x_{i-1},\ x_{i+1}, \ldots, x_{j-1},\ x_{j+1}, \ldots,\ x_n) \\
& \qquad i \neq j, i, j = 1, 2, \ldots, n \\
&\ \vdots \\
u(x_1, 0, \ldots, 0) &= g_{n-1,2,3,\ldots,n}(x_1) \\
u(0,\ x_2, 0, \ldots,\ 0) &= g_{n-1,1,3,\ldots,n}(x_2) \\
&\ \vdots \\
u(0, \ldots, 0, x_n) &= g_{n-1,1,2,\ldots,n-1}(x_n) \\
u(0, \ldots, 0) &= c = [c_1 \quad c_2 \ldots c_n]^T
\end{aligned} \tag{1.15}$$

or the compatibility initial condition

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

$$\begin{aligned} = \quad & (-1)^2\left(g_{1,1}(x_2, x_3, \ldots, x_n) + \ldots + g_{1,n}(x_1, x_2, \ldots, x_{n-1})\right) \\ & + (-1)^3\left(g_{2,1,2}(x_3, x_4, \ldots, x_n) + \ldots + g_{2,n-1,n}(x_1, x_2, \ldots, x_{n-2})\right) \\ & \qquad \vdots \\ & + (-1)^n\left(g_{n-1,2,3,\ldots,n}(x_1) + \ldots + g_{n-1,1,2,\ldots,n-1}(x_n)\right) \\ & + (-1)^{n+1} c \end{aligned} \tag{1.16}$$

where m is positive integer greater than 1 and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}$$

and
$$u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x), u_2(x), \ldots, u_n(x)) \\ u_2(u_1(x), u_2(x), \ldots, u_n(x)) \\ \vdots \\ u_n(u_1(x), u_2(x), \ldots, u_n(x)) \end{bmatrix}$$

$$\begin{aligned} u^3(x) &= u\left(u^2(x)\right) \\ &\vdots \\ u^m(x) &= u\left(u^{m-1}(x)\right). \end{aligned}$$

Examples will be given of solutions in the form of power series of several variables for simple iterative partial differential equations.

In chapter 2, we will define the existence and uniqueness theorems of iterative partial differential equations.

Chapter 3 will present a discussion on the existence and uniqueness theorems for the solutions of the simple iterative partial differential equations.

We will find the approximating solutions for the simple iterative partial differential equations of order 2 and 3 degree 2 and 3 in chapter 4.

In closing, a conclusion and suggestions will be offered in chapter 5.

CHAPTER 2

LITERATURE REVIEWS

In this chapter, we will state the existence and uniqueness theorems of the iterative partial differential equations. We will study and apply the steps for proving the existence and uniqueness theorems by using the previous studies for the simple iterative partial differential equations.

Consider the iterative partial differential equation of the form

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = f\left(x, u(x), u^2(x), \ldots, u^m(x)\right) \tag{2.1}$$

with the initial conditions

$$\begin{aligned}
&u(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = g_{1,i}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \quad i = 1, 2, \ldots, n \\
&u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) \\
&\qquad = g_{2,i,j}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \\
&\qquad\qquad i \neq j, i, j = 1, 2, \ldots, n \\
&\qquad \vdots \\
&u(x_1, 0, \ldots, 0) = g_{n-1,2,3,\ldots,n}(x_1) \\
&u(0, x_2, 0, \ldots, 0) = g_{n-1,1,3,\ldots,n}(x_2) \\
&\qquad \vdots \\
&u(0, \ldots, 0, x_n) = g_{n-1,1,2,\ldots,n-1}(x_n) \\
&u(0, \ldots, 0) = c = [c_1 \;\; c_2 \ldots c_n]^T
\end{aligned} \tag{2.2}$$

or the compatibility initial condition

$$g(x)=\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

$$\begin{aligned} =\ & (-1)^2\left(g_{1,1}(x_2,x_3,\ldots,x_n)+\ldots+g_{1,n}(x_1,x_2,\ldots,x_{n-1})\right) \\ & +(-1)^3\left(g_{2,1,2}(x_3,x_4,\ldots,x_n)+\ldots+g_{2,n-1,n}(x_1,x_2,\ldots,x_{n-2})\right) \\ & \qquad\vdots \\ & +(-1)^n\left(g_{n-1,2,3,\ldots,n}(x_1)+\ldots+g_{n-1,1,2,\ldots,n-1}(x_n)\right) \\ & +(-1)^{n+1}c \end{aligned} \tag{2.3}$$

where m is positive integer greater than 1,

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u(x)=\begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}$$

and
$$f(x)=\begin{bmatrix} f_1\left(x,u(x),u^2(x),\ldots,u^m(x)\right) \\ f_2\left(x,u(x),u^2(x),\ldots,u^m(x)\right) \\ \vdots \\ f_n\left(x,u(x),u^2(x),\ldots,u^m(x)\right) \end{bmatrix}$$

$$u^2(x)=u(u(x))=\begin{bmatrix} u_1(u_1(x),u_2(x),\ldots,u_n(x)) \\ u_2(u_1(x),u_2(x),\ldots,u_n(x)) \\ \vdots \\ u_n(u_1(x),u_2(x),\ldots,u_n(x)) \end{bmatrix}$$

$$\begin{aligned} u^3(x) &= u\left(u^2(x)\right) \\ &\ \vdots \\ u^m(x) &= u\left(u^{m-1}(x)\right) \end{aligned}$$

and
$$Z=[0,a_1]\times[0,a_2]\times\ldots\times[0,a_n]$$

and
$$u_i : Z \to R \ , \ f_i : Z \times R^{mn} \to R \quad , \quad i = 1, 2, \ldots, n$$
$$u : Z \to R^n, \ f : Z \times R^{mn} \to R^n$$

when f and g are continuous, then the problem (2.1) – (2.2) is equivalent to the problem of the continuous solution of the integral equation

$$u(x) = g(x) + \int_0^x f\left(t, \ u(t), u^2(t), \ldots, u^m(t)\right) dt \,. \tag{2.4}$$

2.1 Uniqueness

Let $f\left(x, u(x), u^2(x), \ldots, u^m(x)\right)$ be defined and continuous in the set $Z \times R^{mn}$, say D, and g be defined and continuous in Z and let

$$\left\| f\left(x, z_1, z_2, \ldots, z_m\right) \right\| \leq K \tag{2.5}$$

$$\left\| f\left(x, z_1, z_2, \ldots, z_m\right) - f\left(x, \overline{z_1}, \overline{z_2}, \ldots, \overline{z_m}\right) \right\|$$
$$\leq M_1 \left\| z_1 - \overline{z}_1 \right\| + M_2 \left\| z_2 - \overline{z}_2 \right\| + \ldots + M_m \left\| z_m - \overline{z}_m \right\| \tag{2.6}$$

$$\left\| g(x) \right\| \leq L \leq K \tag{2.7}$$

for all $(x, z_1, z_2, \ldots, z_m)$, $\left(x, \overline{z_1}, \overline{z_2}, \ldots, \overline{z_m}\right)$ in D and for x in Z and $K, L, M_1, M_2, \ldots, M_m$ are in R^+. The norm $\|.\|$ is the Euclidean norm.

We are looking for the solution $u(x)$ of the problem (2.1) – (2.2) or (2.4)

where $u(x)$ belongs to Z for all x in Z (2.8)

$\left\| u(x) - u(y) \right\| \leq N \left\| x - y \right\|$ for all x and y in Z and N in R^+. (2.9)

Theorem 2.1 If $Be^{dC} \leq 1$ and f and g satisfy the above conditions then there exist at most one solution to the problem (2.1) – (2.2).

Proof reference [5].

2.2 Existence

Let us suppose that

$$\|g(x)\| + aK \le a \qquad \text{for all } x \text{ in } Z \text{ and} \qquad (2.10)$$

$$a\left[M_1 + (N+1)M_2 + (N^2+N+1)M_3 + (N^{m-1} + N^{m-2} + \ldots + N + 1)M_m\right] < 1 \qquad (2.11)$$

and let consider the following sequences

$$u_{1,k+1}(x) = g(x) + \int_0^x f\left(t, u_{1,k}(t), u_{1,k}^2(t), \ldots, u_{1,k}^m(t)\right)dt \qquad (2.12.1)$$

$$u_{2,k+1}(x) = g(x) + \int_0^x f\left(t, u_{2,k}(t), u_{2,k}^2(t), \ldots, u_{2,k}^{m-1}(t), u_{2,k}^{m-1}(u_{2,k+1}(t))\right)dt \qquad (2.12.2)$$

$$u_{3,k+1}(x) = g(x) + \int_0^x f\left(t, u_{3,k}(t), u_{3,k}^2(t), \ldots, u_{3,k}^{m-2}(t), u_{3,k}^{m-2}(u_{3,k+1}(t)), u_{3,k}^{m-1}(u_{3,k+1}(t))\right)dt \qquad (2.12.3)$$

$$\vdots$$

$$u_{m+1,k+1}(x) = g(x) + \int_0^x f\left(t, u_{m+1,k+1}(t), u_{m+1,k}(u_{m+1,k+1}(t)), u_{m+1,k}^2(u_{m+1,k+1}(t)) u_{m+1,k}^{m-1}(u_{m+1,k+1}(t))\right)dt \qquad (2.12.m+1)$$

where $k = 0, 1, 2, \ldots$ and $u_{1,0}(x), u_{2,0}(x), \ldots, u_{m+1,0}(x)$ are fixed functions of the class C^1 map Z' to Z such that $\left\|\dfrac{\partial^n u_{i,0}(x)}{\partial x_1 \partial x_2 \ldots \partial x_n}\right\| \le K$, $i = 1, 2, \ldots, m+1$.

Hence we have the following theorem.

Theorem 2.2 Let the condition of theorem 2.1 holds and the conditions (2.10) and (2.11) be satisfied then the sequences (2.12.1), (2.12.2),…, (2.12.m+1) converge uniformly to the (unique) solution $u = u(x)$ of the problem (2.1) – (2.2).

Proof reference [5].

CHAPTER 3

EXISTENCE AND UNIQUENESS THEOREMS

In this chapter, we will discuss the existence and uniqueness theorems for the solutions of the simple iterative partial differential equations. In order to gain a better understanding we will begin from studying the problems of two variables, three variables and then finally we will study the problems of n variables.

3.1 Existence and Uniqueness Theorems for the Second Order.

In this section, we will show the steps and conditions in proving of the existence and uniqueness theorems for the solutions of the second order simple iterative partial differential equations degree m

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = u^m(x,y) \tag{3.1}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= g_{1,2}(x) \\ u(0,y) &= g_{1,1}(y) \\ u(0,0) &= c = [\,c_1 \quad c_2\,]^T \end{aligned} \tag{3.2}$$

or the compatibility initial condition

$$g(x) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix} = (-1)^2\left(g_{1,1}(y) + g_{1,2}(x)\right) + (-1)^3 c \tag{3.3}$$

where m is a positive integer greater than 1 and,

$$u(x,y) = \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix}$$

and
$$u^2(x,y) = u(u(x,y)) = \begin{bmatrix} u_1(u_1(x,y), u_2(x,y)) \\ u_2(u_1(x,y), u_2(x,y)) \end{bmatrix}$$

$$\begin{aligned} u^3(x,y) &= u(u^2(x,y)) \\ &\vdots \\ u^m(x,y) &= u(u^{m-1}(x,y)) \end{aligned}$$

and
$$Z = [0, a_1] \times [0, a_2]$$

$$\begin{aligned} &u_1 : Z \to R, \quad u_2 : Z \to R \\ &u : Z \to R^2 \end{aligned}$$

and
$$Y = [0, b_1] \times [0, b_2] \subset Z \text{ where } b_1 \le a_1 \text{ and } b_2 \le a_2$$

$$\begin{aligned} &g_1 : Y \to R, \quad g_2 : Y \to R \\ &g : Y \to R^2. \end{aligned}$$

Remark 3.1.1 We will use the following graph to consider the boundary of the function $u(x, y)$.

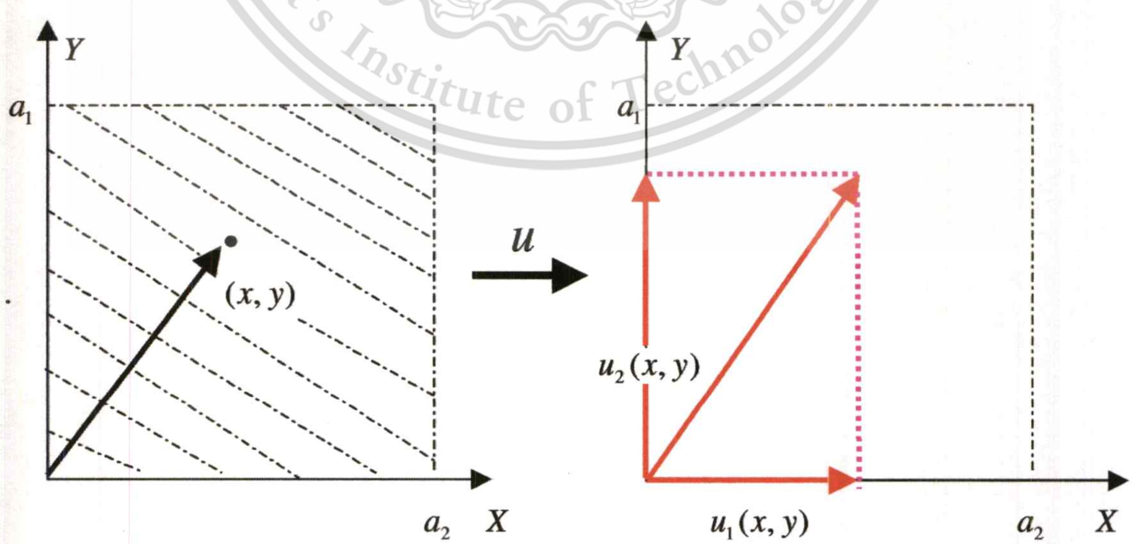

Figure 3.1 Graph of the boundary for the vector function $u(x, y)$.

When g and u_x , u_y u_{xy} are continuous on Z, then the problem (3.1) – (3.2) is equivalent to the problem of finding the continuous solution of the integral equation

$$u(x,y) = g(x,y) + \int_0^y \int_0^x u^m(s,t)dsdt. \tag{3.4}$$

Remark 3.2.2 We consider the boundary of the integral equation as follow:

Let $a = a_1 . a_2$ and $K = \sqrt{a_1^2 + a_2^2}$.

We will consider a vector function $u(x,y) = [u_1(x,y) \;\; u_2(x,y)]^T$ which u_1, u_2 satisfy the integral equations:

$$u_1(x,y) = g_1(x,y) + \int_0^y \int_0^x u_1^m(s,t)dsdt \quad \Rightarrow \quad b_1 + aa_1 \le a_1$$

$$u_2(x,y) = g_2(x,y) + \int_0^y \int_0^x u_2^m(s,t)dsdt \quad \Rightarrow \quad b_2 + aa_2 \le a_2 .$$

Thus, we obtain $a_1, a_2 < 1$ and also $a < 1$.

Next, we will apply the fixed – point technique to prove the existence and uniqueness theorems for the second order simple iterative partial differential equations. By using the integral equation in (3.4) and considering $g(x,y)$ into 2 cases which are $g(x,y) = c$ and $g(x,y) \neq c$ respectively:

Case 1 : $g(x,y) = c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Choose the Banach space $B = C[Z]$ equipped with norm $\| u \| = \max_{(x,y)\in Z} \| u(x,y) \|$.

Set $\qquad S(\rho) = \left\{ u \in B : \| u \| \le \rho , \left\| u(x) - u(\bar{x}) \right\| \le N \left\| x - \bar{x} \right\| \right\}$

where $\qquad \rho = L + aK$ and $\| c \| \le L \le K$

and $\qquad T_m = N^{m-1} + N^{m-2} + N^{m-3} + \ldots + N + 1$ where $N \in R^+$.

Define $\quad T : S(\rho) \to S(\rho)$ by

$$(Tu)(x) = c + \int_0^y \int_0^x u^m(s,t)\,dsdt \quad \text{for all } (x,y) \in Z.$$

Theorem 3.1.3 Suppose that $L + aK \le K$, $K \le N$ and $aT_m < 1$ then T has a unique fixed point, that is, there is a unique solution to the problem (3.1) – (3.2).

Proof We have $\quad \| (Tu)(x,y) \| \le \| c \| + \int_0^y \int_0^x \| u^m(,t) \| dsdt$

and hence $\quad 0 \le \| Tu \| \le L + aK = \rho$

and

$\| (Tu)(x,y) - (Tu)(\bar{x},\bar{y}) \|$

$$\le \left\| c + \int_0^y \int_0^x u^m(s,t)\,dsdt - c - \int_0^{\bar{y}} \int_0^{\bar{x}} u^m(s,t)\,dsdt \right\|$$

$$\le \int_y^{\bar{y}} \int_x^{\bar{x}} \| u^m(s,t) \| dsdt$$

$$\le K(x-\bar{x})(y-\bar{y})$$

$$\le K\| x-\bar{x} \| \le N\| x-\bar{x} \|.$$

Thus, we have $T : S(\rho) \to S(\rho)$.

Now, for all $u, v \in S(\rho)$ we have

$$\|(Tu)(x,y) - (Tv)(x,y)\| \le \int_0^y \int_0^x \| u^m(s,t) - v^m(s,t) \| dsdt$$

$$\leq N^{m-1}\int_0^y\int_0^x \| u(s,t)-v(s,t)\| dsdt + N^{m-2}\int_0^y\int_0^x \| u(v(s,t))-v(v(t))\| dt$$

$$+N^{m-3}\int_0^y\int_0^x \left\| u(v^2(s,t))-v(v^2(s,t))\right\| dsdt + \ldots$$

$$+N\int_0^y\int_0^x \left\| u(v^{m-2}(s,t))-v(v^{m-2}(s,t))\right\| dsdt$$

$$+\int_0^y\int_0^x \left\| u(v^{m-1}(s,t))-v(v^{m-1}(s,t))\right\| dsdt$$

$$\leq a_1a_2N^{m-1}\| u-v\| + a_1a_2N^{m-2}\| u-v\| + a_1a_2N^{m-3}\| u-v\|$$

$$+\ldots + a_1a_2N\| u-v\| + a_1a_2\| u-v\|$$

$$\leq a\| u-v\| (N^{m-1}+N^{m-2}+N^{m-3}+\ldots+N+1)$$

$$= aT_m\| u-v\|$$

$$< \| u-v\|.$$

Hence, by the Banach Contraction Principle, T has a unique fixed point.

Case 2 : $g(x) \neq c$.

Choose the Banach space $B = C[Z]$ equipped with norm $\| u\| = \max_{(x,y)\in Z}\| u(x,y)\|$.

Set $S(\rho) = \left\{u\in B : \| u\| \leq \rho, \left\| u(x)-u(\bar{x})\right\| \leq N\left\| x-\bar{x}\right\|\right\}$

where $\rho = L + aK$ and $\| g(x)\| \leq L \leq K$

and $\left\| g(x)-g(\bar{x})\right\| \leq M\left\| x-\bar{x}\right\|$ for all x and $\bar{x}$ in Z where $M\in R^+$

and $T_m = N^{m-1}+N^{m-2}+N^{m-3}+\ldots+N+1$ where $N\in R^+$.

Define $T : S(\rho) \to S(\rho)$ by

$$(Tu)(x,y) = g(x,y) + \int_0^y\int_0^x u^m(s,t)\,dsdt \quad \text{for all } (x,y)\in Z.$$

Theorem 3.1.4 Suppose that $L + aK \leq K$, $M + K \leq N$ and $aT_m < 1$ then T has a unique fixed point, that is, there is a unique solution to the problem (3.1) – (3.2).

Proof We have $\quad \| (Tu)(x,y) \| \leq \|g(x,y)\| + \int_0^y \int_0^x \| u^m(s,t) \| dsdt$

and hence, $\quad 0 \leq \| Tu \| \leq L + aK = \rho$

and

$$\| (Tu)(x,y) - (Tu)(\bar{x},\bar{y}) \| \leq \left\| g(x,y) - g(\bar{x},\bar{y}) + \int_y^{\bar{y}} \int_x^{\bar{x}} u^m(s,t)\, dsdt \right\|$$

$$\leq \| g(x,y) - g(\bar{x},\bar{y}) \| + \left\| \int_y^{\bar{y}} \int_x^{\bar{x}} u^m(s,t)\, dsdt \right\|$$

$$\leq \| g(x,y) - g(\bar{x},\bar{y}) \| + \int_y^{\bar{y}} \int_x^{\bar{x}} \| u^m(s,t) \| dsdt$$

$$\leq \| g(x,y) - g(\bar{x},\bar{y}) \| + K \int_y^{\bar{y}} \int_x^{\bar{x}} dsdt$$

$$= \| g(x,y) - g(\bar{x},\bar{y}) \| + K(x - \bar{x})(y - \bar{y})$$

$$\leq \| g(x,y) - g(\bar{x},\bar{y}) \| + K \| x - \bar{x} \|$$

$$\leq M \| x - \bar{x} \| + K \| x - \bar{x} \|$$

$$\leq N \| x - \bar{x} \|.$$

Thus, we have $T : S(\rho) \to S(\rho)$.

Now, for all $u, v \in S(\rho)$ we have

$$\|(Tu)(x,y) - (Tv)(x,y)\| \leq \int_0^y \int_0^x \| u^m(s,t) - v^m(s,t) \| dsdt$$

$$\leq N^{m-1}\int_0^y\int_0^x \| u(s,t)-v(s,t) \| dsdt + N^{m-2}\int_0^y\int_0^x \| u(v(s,t))-v(v(t)) \| dt$$

$$+N^{m-3}\int_0^y\int_0^x \left\| u(v^2(s,t))-v(v^2(s,t)) \right\| dsdt + \ldots$$

$$+N\int_0^y\int_0^x \left\| u(v^{m-2}(s,t))-v(v^{m-2}(s,t)) \right\| dsdt$$

$$+\int_0^y\int_0^x \left\| u(v^{m-1}(s,t))-v(v^{m-1}(s,t)) \right\| dsdt$$

$$\leq a_1a_2N^{m-1}\| u-v \| + a_1a_2N^{m-2}\| u-v \| + a_1a_2N^{m-3}\| u-v \|$$

$$+\ldots + a_1a_2N\| u-v \| + a_1a_2\| u-v \|$$

$$\leq a\| u-v \|(N^{m-1}+N^{m-2}+N^{m-3}+\ldots+N+1)$$

$$= aT_m \| u-v \|$$

$$< \| u-v \|.$$

Since $aT_m < 1$, so T is a contraction on $S(\delta)$. By the Banach Contraction Principle, T has a unique fixed point. This means that $u \equiv v$.

The preceding theorems show that there exists a unique solution to the problem (3.1) – (3.2). However, it does not tell us how to find this solution. To find the power series solution of the problem (3.1) – (3.2), we will define the following approximating sequence

$$u_{k+1}(x,y) = g(x,y)+\int_0^y\int_0^x u_k^m(s,\ t)\, dsdt \qquad (3.5)$$

where $k = 0,1,2,\ldots$ and $u_0(x,y)$ are fixed functions of the class C^1 mapping Z to Z such that $\left\| \dfrac{\partial^2 u_0(x,y)}{\partial x \partial y} \right\| \leq K$.

Hence we have the following theorem.

Theorem 3.1.5 If the conditions of theorem 3.1.4 are satisfied then sequences defined in (3.5) converges uniformly to the (unique) solution $u = u(x, y)$ of the problem (3.1) – (3.2).

Proof We put $U_k = \max_{(x,y)\in Z} \| u_k(x,y) - u_{k-1}(x,y) \|$.

$$\text{Then} \quad U_1 = \max_{(x,y)\in Z} \| u_1(x,y) - u_0(x,y) \|$$

$$= \max_{(x,y)\in Z} \left\| g(x,y) + \int_0^y \int_0^x u_0^m(s,t)\, dsdt - u_0(x,y) \right\|.$$

Since $u_0(x)$ maps from Z to Z then, we have

$$U_1 \le L + aK \le K$$

and

$$U_2 = \max_{(x,y)\in Z} \| u_2(x,y) - u_1(x,y) \|$$

$$= \max_{(x,y)\in Z} \left\| g(x,y) + \int_0^y \int_0^x u_1^m(s,t)\, dsdt - g(x,y) + \int_0^y \int_0^x u_0^m(s,t)\, dsdt \right\|$$

$$= \max_{(x,y)\in Z} \left\| \int_0^y \int_0^x u_1^m(s,t)\, dsdt - \int_0^y \int_0^x u_0^m(s,t)\, dsdt \right\|$$

$$\le \max_{(x,y)\in Z} \int_0^y \int_0^x \left\| u_1^m(s,t) - u_0^m(s,t) \right\| dsdt$$

$$\le aU_1 \le aK$$

and

$$U_3 = \max_{(x,y)\in Z} \| u_3(x,y) - u_2(x,y) \|$$

$$= \max_{(x,y)\in Z} \left\| g(x,y) + \int_0^y \int_0^x u_2^m(s,t)\, dsdt - g(x,y) + \int_0^y \int_0^x u_1^m(s,t)\, dsdt \right\|$$

$$= \max_{(x,y)\in Z} \left\| \int_0^y \int_0^x u_2^m(s,t)\, dsdt - \int_0^y \int_0^x u_1^m(s,t)\, dsdt \right\|$$

$$\le \max_{(x,y)\in Z} \int_0^y \int_0^x \left\| u_2^m(s,t) - u_1^m(s,t) \right\| dsdt$$

$$\le aU_2 \le a^2 K$$

and $$U_4 = \max_{(x,y)\in Z} \| u_4(x,y) - u_3(x,y) \|$$

$$= \max_{(x,y)\in Z} \left\| g(x,y) + \int_0^y \int_0^x u_3^m(s,t)\,dsdt - g(x,y) + \int_0^y \int_0^x u_2^m(s,t)\,dsdt \right\|$$

$$= \max_{(x,y)\in Z} \left\| \int_0^y \int_0^x u_3^m(s,t)\,dsdt - \int_0^y \int_0^x u_2^m(s,t)\,dsdt \right\|$$

$$\le \max_{(x,y)\in Z} \int_0^y \int_0^x \| u_3^m(s,t) - u_2^m(s,t) \| \,dsdt$$

$$\le aU_3 \le a^3 K.$$

Thus, we have $U_k \le a^{k-1}K$. Since $L + aK \le K$ then $a < 1$ and $K = \sqrt{a_1^2 + a_2^2} \ge 0$. Hence U_k tends to zero as k tends to infinity. Since the family $\{u_k\}$ is an Arzela-Ascoli family, thus for every subsequence $\{u_{k_j}\}$ of $\{u_k\}$ there exists a subsequence $\{u_{l_j}\}$ uniformly convergent and the limit needs to be a solution of the problem (3.1) – (3.2). Thus the sequence $\{u_k\}$ tends uniformly to the (unique) solution of the problem (3.1) – (3.2).

3.2 Existence and Uniqueness Theorems for the Third Order.

In this section we will show the steps and conditions in proving of the existence and uniqueness theorems for the solutions of the third order simple iterative partial differential equations degree m

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} = u^m(x,y,z) \tag{3.6}$$

with the initial conditions

$$\begin{aligned}
u(0,y,z) &= g_{1,1}(y,z) \\
u(x,0,z) &= g_{1,2}(x,z) \\
u(x,y,0) &= g_{1,3}(x,y) \\
u(x,0,0) &= g_{2,2,3}(x) \\
u(0,y,0) &= g_{2,1,3}(y) \\
u(0,0,z) &= g_{2,1,2}(z) \\
u(0,0,0) &= c = [c_1 \quad c_2 \quad c_3]^T
\end{aligned} \tag{3.7}$$

or the compatibility initial condition

$$g(x,y,z) = \begin{bmatrix} g_1(x,y,z) \\ g_2(x,y,z) \\ g_3(x,y,z) \end{bmatrix}$$

$$= (-1)^2\left(g_{1,1}(y,z)+g_{1,2}(x,z)+g_{1,3}(x,y)\right) + (-1)^3\left(g_{2,2,3}(x)+g_{2,1,3}(y)+g_{2,1,2}(z)\right)+(-1)^4 c \tag{3.8}$$

where m is a positive integer greater than 1 and

$$u(x,y,z) = \begin{bmatrix} u_1(x,y,z) \\ u_2(x,y,z) \\ u_3(x,y,z) \end{bmatrix}$$

and

$$u^2(x,y,z) = u(u(x,y,z)) = \begin{bmatrix} u_1(u_1(x,y,z),u_2(x,y,z),u_3(x,y,z)) \\ u_2(u_1(x,y,z),u_2(x,y,z),u_3(x,y,z)) \\ u_n(u_1(x,y,z),u_2(x,y,z),u_3(x,y,z)) \end{bmatrix}$$

$$u^3(x,y,z) = u(u^2(x,y,z))$$

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$$u^m(x,y,z) = u(u^{m-1}(x,y,z))$$

and $Z = [0,a_1]\times[0,a_2]\times[0,a_3]$

$$u_1 : Z \to R\ , \quad u_2 : Z \to R\ , \quad u_3 : Z \to R$$
$$u : Z \to R^3$$

and $Y = [0,b_1]\times[0,b_2]\times[0,b_3] \subset Z$ where $b_i \le a_i$ for $i = 1,2,3$

$$g_i : Y \to R\ , \qquad i = 1,2,3$$
$$g : Y \to R^3 .$$

When g and $u_x, u_y, \ldots, u_{xyz}$ are continuous on Z then the problem (3.6) – (3.7) is equivalent to the problem of the continuous solution of the integral equation

$$u(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u^m(r,s,t)drdsdt. \tag{3.9}$$

Remark 3.2.1 We will use the following graph to consider the boundary of the function $u(x,y,z)$.

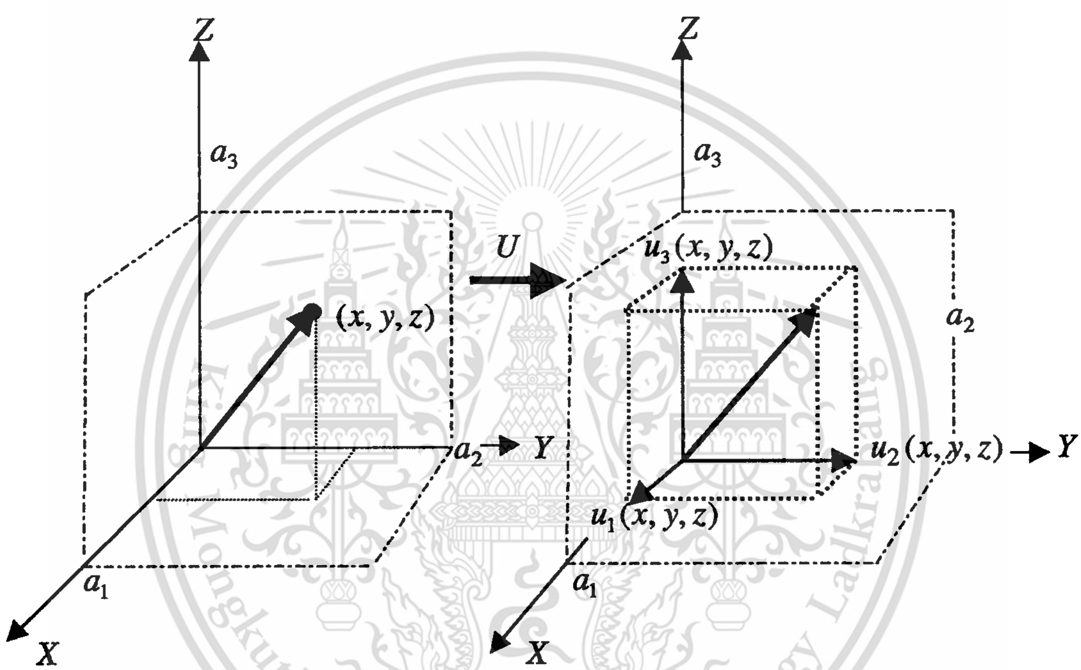

Figure 3.2 Graph of the boundary for the vector function $u(x,y,z)$.

Remark 3.2.2 We will consider the boundary of the integral equation as follow:

Let $a = a_1.a_2.a_3$ and $K = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

We will consider a vector function $u(x,y) = [u_1(x,y),\ u_2(x,y),\ u_3(x,y)]^T$ which u_1, u_2, u_3 satisfy the integral equations:

$$u_1(x,y,z) = g_1(x,y,z) + \int_0^z\int_0^y\int_0^x u_1^m(r,s,t)\,drdsdt \quad \Rightarrow \quad b_1 + aa_1 \le a_1$$

$$u_2(x,y,z) = g_2(x,y,z) + \int_0^z\int_0^y\int_0^x u_2^m(r,s,t)\,drdsdt \;\Rightarrow\; b_2 + aa_2 \le a_2$$

$$u_3(x,y,z) = g_3(x,y,z) + \int_0^z\int_0^y\int_0^x u_3^m(r,s,t)\,drdsdt \;\Rightarrow\; b_3 + aa_3 \le a_3$$

we obtain that $a_1, a_2, a_3 < 1$ and also $a < 1$.

Next, we will apply the fixed – point technique to prove the existence and uniqueness theorems for the second order simple iterative partial differential equations. By using the integral equation in (3.9) and considering $g(x,y,z)$ into 2 cases which are $g(x,y,z) = c$ and $g(x,y,z) \neq c$ respectively:

Case 1 : $g(x,y,z) = c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

Choose the Banach space $B = C[Z]$ equipped with norm $\| u \|_s = \max\limits_{(x,y,z)\in Z} \| u(x,y,z) \|$.

Set $\quad S(\rho) = \left\{ u \in B : \| u \| \le \rho, \; \| u(x) - u(\bar{x}) \| \le N \| (x) - (\bar{x}) \| \right\}$

where $\quad \rho = L + aK$ and $\| c \| \le L \le K$ and

and $\quad T_m = N^{m-1} + N^{m-2} + N^{m-3} + \ldots + N + 1$ where $N \in R^+$.

Define $\quad T : S(\rho) \to S(\rho)$ by

$$(Tu)(x,y,z) = c + \int_0^z\int_0^y\int_0^x u^m(r,s,t)\,drdsdt \quad \text{for all } (x,y,z) \in Z.$$

Theorem 3.2.3 Suppose that $L + aK \le K$, $K \le N$ and $aT_m < 1$ then T has a unique fixed point, that is, there is a unique solution to the problem (3.6) – (3.7).

Proof We have $\|(Tu)(x,y,z)\| \le \|c\| + \int_0^z\int_0^y\int_0^x \| u^m(r,s,t)\| drdsdt$

and hence $0 \le \|Tu\| \le L + aK = \rho$

and $\| (Tu)(x,y,z) - (Tu)(\bar{x},\bar{y},\bar{z}) \| \le \left\| c - c + \int_z^{\bar{z}}\int_y^{\bar{y}}\int_x^{\bar{x}} u^m(r,s,t)\, drdsdt \right\|$

$$\le \int_z^{\bar{z}}\int_y^{\bar{y}}\int_x^{\bar{x}} \|u^m(r,s,t)\| drdsdt$$

$$\le K(x-\bar{x})(y-\bar{y})(z-\bar{z})$$

$$\le K\| x-\bar{x} \| \le N\| x-\bar{x} \|.$$

Thus, we have $T : S(\rho) \to S(\rho)$.

Now, for all $u, v \in S(\rho)$ we have

$$\|(Tu)(x,y,z)-(Tv)(x,y,z)\|$$

$$\le \int_0^z\int_0^y\int_0^x \| u^m(r,s,t) - v^m(r,s,t) \| drdsdt$$

$$\le N^{m-1}\int_0^z\int_0^y\int_0^x \| u(r,s,t) - v(r,s,t) \| drdsdt +$$

$$N^{m-2}\int_0^z\int_0^y\int_0^x \| u(v(r,s,t)) - v(v(r,s,t)) \| drdsdt +$$

$$N^{m-3}\int_0^z\int_0^y\int_0^x \| u(v^2(r,s,t)) - v(v^2(r,s,t)) \| drdsdt + \ldots$$

$$+ N\int_0^z\int_0^y\int_0^x \| u(v^{m-2}(r,s,t)) - v(v^{m-2}(r,s,t)) \| drdsdt$$

$$+ \int_0^z\int_0^y\int_0^x \| u(v^{m-1}(r,s,t)) - v(v^{m-1}(r,s,t)) \| drdsdt$$

$$\le a_1a_2a_3N^{m-1}\| u-v \| + a_1a_2a_3N^{m-2}\| u-v \| + a_1a_2a_3N^{m-3}\| u-v \|$$

$$+ \ldots + a_1a_2a_3N\| u-v \| + a_1a_2a_3\| u-v \|$$

$$\le a\| u-v \|(N^{m-1} + N^{m-2} + N^{m-3} + \ldots + N + 1)$$

$$= aT_m\| u-v \|$$

$$< \| u-v \|.$$

Hence, by the Banach Contraction Principle, T has a unique fixed point.

Case 2 : $g(x, y, z) \neq c$.

Choose the Banach space $B = C[Z]$ equipped with norm $\| u \|_s = \max\limits_{(x,y,z) \in Z} \|u(x, y, z)\|$.

Set $\quad S(\rho) = \left\{u \in B : \| u \| \leq \rho, \left\| u(x) - u(\bar{x}) \right\| \leq N \left\| x - \bar{x} \right\| \right\}$

where $\quad \rho = L + aK$ and $\| g(x) \| \leq L \leq K$

and $\quad \left\| g(x) - g(\bar{x}) \right\| \leq M \left\| x - \bar{x} \right\|$ for all x and $\bar{x}$ in Z where $M \in R^+$

and $\quad T_m = N^{m-1} + N^{m-2} + N^{m-3} + \ldots + N + 1$ where $N \in R^+$.

Define $\quad T : S(\rho) \to S(\rho)$ by

$$(Tu)(x, y, z) = g(x, y, z) + \int_0^z \int_0^y \int_0^x u^m(r, s, t)\, drdsdt \quad \text{for all } (x, y, z) \in Z.$$

Theorem 3.2.4 Suppose that $L + aK \leq K$, $M + K \leq N$ and $aT_m < 1$ then T has a unique fixed point, that is, there is a unique solution to the problem (3.6) – (3.7).

Proof We have $\quad \| (Tu)(x, y, z) \| \leq \|g(x, y, z)\| + \int_0^z \int_0^y \int_0^x \left\| u^m(r, s, t) \right\| drdsdt$

and hence, $\quad 0 \leq \| Tu \| \leq L + aK = \rho$

and

$$\left\| (Tu)(x, y, z) - (Tu)(\bar{x}, \bar{y}, \bar{z}) \right\| \leq \left\| g(x, y, z) - g(\bar{x}, \bar{y}, \bar{z}) + \int_z^{\bar{z}} \int_y^{\bar{y}} \int_x^{\bar{x}} u^m(r, s, t)\, drdsdt \right\|$$

$$\leq \left\| g(x, y, z) - g(\bar{x}, \bar{y}, \bar{z}) \right\| + \left\| \int_z^{\bar{z}} \int_y^{\bar{y}} \int_x^{\bar{x}} u^m(r, s, t)\, drdsdt \right\|$$

$$\leq \left\| g(x,y,z) - g(\bar{x},\bar{y},\bar{z}) \right\| + \left\| \int_z^{\bar{z}} \int_y^{\bar{y}} \int_x^{\bar{x}} u^m(r,s,t)\, drdsdt \right\|$$

$$\leq \left\| g(x) - g(\bar{x}) \right\| + K \int_z^{\bar{z}} \int_y^{\bar{y}} \int_x^{\bar{x}} drdsdt$$

$$= \left\| g(x) - g(\bar{x}) \right\| + K(x-\bar{x})(y-\bar{y})(z-\bar{z})$$

$$\leq \left\| g(x) - g(\bar{x}) \right\| + K\left\| x-\bar{x} \right\|$$

$$\leq M\left\| x-\bar{x} \right\| + K\left\| x-\bar{x} \right\|$$

$$\leq N\left\| x-\bar{x} \right\|.$$

Thus, we have $T : S(\rho) \to S(\rho)$.

Now, for all $u, v \in S(\rho)$ we have

$$\left\|(Tu)(x,y,z) - (Tv)(x,y,z)\right\|$$

$$\leq \int_0^z \int_0^y \int_0^x \left\| u^m(r,s,t) - v^m(r,s,t) \right\| drdsdt$$

$$\leq N^{m-1} \int_0^z \int_0^y \int_0^x \left\| u(r,s,t) - v(r,s,t) \right\| drdsdt + N^{m-2} \int_0^z \int_0^y \int_0^x \left\| u(v(r,s,t)) - v(v(r,s,t)) \right\| drdsdt +$$

$$N^{m-3} \int_0^z \int_0^y \int_0^x \left\| u(v^2(r,s,t)) - v(v^2(r,s,t)) \right\| drdsdt + \ldots$$

$$+ N \int_0^z \int_0^y \int_0^x \left\| u(v^{m-2}(r,s,t)) - v(v^{m-2}(r,s,t)) \right\| drdsdt$$

$$+ \int_0^z \int_0^y \int_0^x \left\| u(v^{m-1}(r,s,t)) - v(v^{m-1}(r,s,t)) \right\| drdsdt$$

$$\leq a_1a_2a_3N^{m-1}\left\| u-v \right\| + a_1a_2a_3N^{m-2}\left\| u-v \right\| + a_1a_2a_3N^{m-3}\left\| u-v \right\|$$

$$+ \ldots + a_1a_2a_3N\left\| u-v \right\| + a_1a_2a_3\left\| u-v \right\|$$

$$\leq a\left\| u-v \right\|(N^{m-1} + N^{m-2} + N^{m-3} + \ldots + N + 1)$$

$$= a\,T_m \left\| u-v \right\|$$

$$< \left\| u-v \right\|.$$

Hence, by the Banach Contraction Principle, T has a unique fixed point.

The preceding theorems shows that there exists a unique solution to the problem (3.6) – (3.7). However, it does not tell us how to find this solution. To find the power series solution of the problem (3.6) – (3.7), we will define approximating sequence in following existence theorem

$$u_{k+1}(x,y,z) \;=\; g(x,y,z)+\int_0^z\int_0^y\int_0^x u_k^m(r,s,\;t)\,drdsdt \tag{3.10}$$

where $\mathrm{k}=0,1,2,\ldots$ and $u_0(x,y,z)$ are fixed functions of the class C^1 mapping Z to Z such that $\left\|\dfrac{\partial^3 u_0(x,y,z)}{\partial x\partial y\partial z}\right\| \le K$.

Hence we have the following theorem.

Theorem 3.2.5 If the conditions of theorem 3.2.4 are satisfied then sequences defined in (3.10) converges uniformly to the (unique) solution $\mathrm{u}=\mathrm{u}(\mathrm{x},\mathrm{y},\mathrm{z})$ of the problem (3.6) - (3.7).

Proof We put $U_k \;=\; \max\limits_{(x,y,z)\in Z} \| u_k(x,y,z)-u_{k-1}(x,y,z)\|$.

Then

$$U_1 \;=\; \max_{(x,y,z)\in Z} \| u_1(x,y,z)-u_0(x,y,z)\|$$

$$= \max_{(x,y,z)\in Z}\left\| g(x,y,z)+\int_0^z\int_0^y\int_0^x u_0^m(r,s,t)drdsdt - u_0(x,y,z)\right\|.$$

Since $u_0(x,y,z)$ maps from Z to Z then, we have

$$U_1 \;\le\; L + aK \;\le\; K$$

and

$$U_2 \;=\; \max_{(x,y,z)\in Z} \| u_2(x,y,z)-u_1(x,y,z)\|$$

$$= \max_{(x,y,z)\in Z}\left\| g(x,y,z)+\int_0^z\int_0^y\int_0^x u_1^m(r,s,t)\,drdsdt - g(x,y,z)+\int_0^z\int_0^y\int_0^x u_0^m(r,s,t)\,drdsdt\right\|$$

$$= \max_{(x,y,z)\in Z} \left\| \int_0^z\int_0^y\int_0^x u_1^m(r,s,t)\,drdsdt + \int_0^z\int_0^y\int_0^x u_0^m(r,s,t)\,drdsdt \right\|$$

$$\leq \max_{(x,y,z)\in Z} \int_0^z\int_0^y\int_0^x \left\| u_1^m(r,s,t) - u_0^m(r,s,t) \right\| drdsdt$$

$$\leq aU_1 \leq aK$$

and $$U_3 = \max_{(x,y,z)\in Z} \| u_3(x,y,z) - u_2(x,y,z) \|$$

$$= \max_{(x,y,z)\in Z} \left\| g(x,y,z) + \int_0^z\int_0^y\int_0^x u_2^m(r,s,t)\,drdsdt - g(x,y,z) + \int_0^z\int_0^y\int_0^x u_1^m(r,s,t)\,drdsdt \right\|$$

$$= \max_{(x,y,z)\in Z} \left\| \int_0^z\int_0^y\int_0^x u_2^m(r,s,t)\,drdsdt + \int_0^z\int_0^y\int_0^x u_1^m(r,s,t)\,drdsdt \right\|$$

$$\leq \max_{(x,y,z)\in Z} \int_0^z\int_0^y\int_0^x \left\| u_2^m(r,s,t) - u_1^m(r,s,t) \right\| drdsdt$$

$$\leq aU_2 \leq a^2K$$

and $$U_4 = \max_{(x,y,z)\in Z} \| u_4(x,y,z) - u_3(x,y,z) \|$$

$$= \max_{(x,y,z)\in Z} \left\| g(x,y,z) + \int_0^z\int_0^y\int_0^x u_3^m(r,s,t)\,drdsdt - g(x,y,z) + \int_0^z\int_0^y\int_0^x u_2^m(r,s,t)\,drdsdt \right\|$$

$$= \max_{(x,y,z)\in Z} \left\| \int_0^z\int_0^y\int_0^x u_3^m(r,s,t)\,drdsdt + \int_0^z\int_0^y\int_0^x u_2^m(r,s,t)\,drdsdt \right\|$$

$$\leq \max_{(x,y,z)\in Z} \int_0^z\int_0^y\int_0^x \left\| u_3^m(r,s,t) - u_2^m(r,s,t) \right\| drdsdt$$

$$\leq aU_3 \leq a^3K.$$

Thus, we have $U_k \leq a^{k-1}K$. Since $L + aK \leq K$ then $a < 1$ and $K = \sqrt{a_1^2 + a_2^2 + a_3^2} \geq 0$. Hence U_k tends to zero as k tends to infinity. Since the family $\{u_k\}$ is an Arzela-Ascoli family, thus for every subsequence $\{u_{k_j}\}$ of $\{u_k\}$

there exists a subsequence $\{u_{l_j}\}$ uniformly convergent and the limit needs to be a solution of the problem (3.6) – (3.7). Thus the sequence $\{u_k\}$ tends uniformly to the (unique) solution of the problem (3.6) – (3.7).

3.3 Existence and Uniqueness Theorem for the nth Order.

In this section we will study the existence and uniqueness theorems for the solutions of the n^{th}- order m-degree simple iterative partial differential equations in the form

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = u^m(x) \tag{3.11}$$

with the initial conditions

$$\begin{aligned}
&u(x_1,\ x_2,\ldots,\ x_{i-1},\ 0,\ x_{i+1},\ldots,\ x_n) = g_{1,i}(x_1,\ x_2,\ldots,\ x_{i-1},\ x_{i+1},\ldots,x_n) \quad i=1,2,\ldots n\\
&u(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)\\
&\qquad\qquad = g_{2,i,j}(x_1,\ldots,\ x_{i-1},\ x_{i+1},\ldots,x_{j-1},\ x_{j+1},\ldots,\ x_n)\\
&\qquad\qquad\qquad i\neq j,\ i,\ j=1,2,\ldots n\\
&\qquad\vdots\\
&u(x_1,0,\ldots,0) = g_{n-1,2,3,\ldots,n}(x_1)\\
&u(0,\ x_2,0,\ldots,\ 0) = g_{n-1,1,3,\ldots,n}(x_2)\\
&\qquad\vdots\\
&u(0,\ldots,0,x_n) = g_{n-1,1,2,\ldots,n-1}(x_n)\\
&u(0,\ldots,0) = c = [c_1 \quad c_2 \ \ldots \ c_n]^T
\end{aligned} \tag{3.12}$$

or the compatibility initial condition

$$g(x)=\begin{bmatrix} g_1(x)\\ g_2(x)\\ \vdots\\ g_n(x)\end{bmatrix}$$

$$\begin{aligned}
&= (-1)^2\left(g_{1,1}(x_2,x_3,\ldots,x_n)+\ldots+g_{1,n}(x_1,x_2,\ldots,x_{n-1})\right)\\
&\quad +(-1)^3\left(g_{2,1,2}(x_3,x_4,\ldots,x_n)+\ldots+g_{2,n-1,n}(x_1,x_2,\ldots,x_{n-2})\right)\\
&\qquad\vdots\\
&\quad +(-1)^n\left(g_{n-1,2,3,\ldots,n}(x_1)+\ldots+g_{n-1,1,2,\ldots,n-1}(x_n)\right)\\
&\quad +(-1)^{n+1}c
\end{aligned} \tag{3.13}$$

where m is a positive integer greater than 1 and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}$$

and

$$u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x), u_2(x), \ldots, u_n(x)) \\ u_2(u_1(x), u_2(x), \ldots, u_n(x)) \\ \vdots \\ u_n(u_1(x), u_2(x), \ldots, u_n(x)) \end{bmatrix}$$

$$\begin{aligned} u^3(x) &= u(u^2(x)) \\ &\vdots \\ u^m(x) &= u(u^{m-1}(x)) \end{aligned}$$

and $\quad Z = [0,a_1]\times[0,a_2]\times\ldots\times[0,a_n]$

$$u_i : Z \to R, \qquad i = 1,2,\ldots,n$$
$$u : Z \to R^n,$$

and $\quad Y = [0,b_1]\times[0,b_2]\times\ldots\times[0,b_n] \subset Z$ where $b_i \le a_i$ for $i = 1,2,\ldots,n$

$$g_i : Y \to R, \qquad i = 1,2,\ldots,n$$
$$g : Y \to R^n.$$

Remark 3.3.1 We will consider the boundary of function u_i and g_i for $i = 1, 2, \ldots, n$ as follow:

$$\begin{array}{ll} 0 \le u_1(x) \le a_1 & 0 \le g_1(x) \le b_1 \le a_1 \\ 0 \le u_2(x) \le a_2 & 0 \le g_2(x) \le b_2 \le a_2 \\ \vdots & \vdots \\ 0 \le u_n(x) \le a_n & 0 \le g_n(x) \le b_n \le a_n . \end{array}$$

When $g(x)$ and $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^n u}{\partial x_1 \partial x_2 \ldots \partial x_n}$ are continuous on Z, then problem (3.11) – (3.12) is equivalent to the problem of the continuous solution of the integral equation

$$u(x) = g(x) + \int_0^x u^m(t)dt. \qquad (3.14)$$

Remark 3.3.2 We will consider the boundary of the integral equation as follow:
Let $a = a_1 . a_2 . a_3 \ldots a_n$ and $K = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.
First, we will consider a vector function $u(x) = [u_1(x) \;\; u_2(x) \ldots u_n(x)]^T$ which $u_1, u_2, \ldots, u_n$ satisfy the integral equations:

$$\begin{array}{lll} u_1 = g_1(x) + \int_0^x u_1(t)\,dt & \Rightarrow & b_1 + aa_1 \le a_1 \\ u_2 = g_2(x) + \int_0^x u_2(t)\,dt & \Rightarrow & b_2 + aa_2 \le a_2 \\ \vdots & & \vdots \\ u_n = g_n(x) + \int_0^x u_n(t)\,dt & \Rightarrow & b_n + a\,a_n \le a_n \end{array}$$ respectively.

Since $b_i + a.a_i \le a_i$ we obtain that for $a_i < 1$ where $i = 1, 2, \ldots, n$.

Next, we will apply fixed – point technique to prove the existence and uniqueness theorems for the n^{th} order simple iterative partial differential equations. By using the integral equation in (3.14) and considering $g(x)$ into 2 cases which are $g(x) = c$ and $g(x) \neq c$ respectively :

Case 1 : $g(x) = c$.

Choose the Banach space $B = C[Z]$ equipped with norm $\| u \| = \max_{x \in Z} \| u(x) \|$.

Set $\qquad S(\rho) = \left\{ u \in B : \| u \| \leq \rho, \| u(x) - u(\bar{x}) \| \leq N \| x - \bar{x} \| \right\}$

where $\qquad \rho = c + aK$ and $\| c \| \leq L \leq K$

and $\qquad T_m = N^{m-1} + N^{m-2} + N^{m-3} + \ldots + N + 1$ where $N \in R^+$.

Define $\qquad T : S(\rho) \to S(\rho)$ by

$$(Tu)(x) = c + \int_0^x u^m(t)dt \quad \text{for all } x \in Z.$$

Theorem 3.3.3 Suppose that $L + aK \leq K$, $K \leq N$ and $aT_m < 1$ then T has a unique fixed point, that is, there is a unique solution to the problem (3.11) – (3.12).

Proof We have $\qquad \| (Tu)(x) \| \leq \|c\| + \int_0^x \| u^m(t) \| dt$

and hence $\qquad 0 \leq \| Tu \| \leq L + aK = \rho$

and

$$\| (Tu)(x) - (Tu)(\bar{x}) \| \leq \left\| c - c + \int_x^{\bar{x}} u^m(t)\, dt \right\|$$

$$\leq \int_x^{\bar{x}} \| u^m(t) \| dt$$

$$\leq K \| x - \bar{x} \| \leq N \| x - \bar{x} \|.$$

Thus, we have $T : S(\rho) \to S(\rho)$.

Now, for all $u, v \in S(\rho)$ we have

$$\|(Tu)(x)-(Tv)(x)\| \le \int_0^x \| u^m(t) - v^m(t) \| dt$$

$$\le N^{m-1}\int_0^x \| u(t)-v(t) \| dt + N^{m-2}\int_0^x \| u(v(t)) - v(v(t)) \| dt$$

$$+ N^{m-3}\int_0^x \| u(v^2(t)) - v(v^2(t)) \| dt + \ldots$$

$$+ N\int_0^x \| u(v^{m-2}(t)) - v(v^{m-2}(t)) \| dt$$

$$+ \int_0^x \| u(v^{m-1}(t)) - v(v^{m-1}(t)) \| dt$$

$$\|(Tu)(x)-(Tv)(x)\| \le a_1a_2\ldots a_n N^{m-1}\| u-v \| + a_1a_2\ldots a_n N^{m-2}\| u-v \|$$

$$+ a_1a_2\ldots a_n N^{m-3}\| u-v \| + \ldots$$

$$+ a_1a_2\ldots a_n N\| u-v \| + a_1a_2\ldots a_n\| u-v \|$$

$$\le a\| u-v \| \,(N^{m-1}+N^{m-2}+N^{m-3}+\ldots+N+1)$$

$$= a\,T_m \| u-v \|$$

$$< \| u-v \|.$$

Hence, by the Banach Contraction Principle, T has a unique fixed point.

Case 2 : $g(x) \neq c$.

Choose the Banach space $B = C[Z]$ equipped with norm $\| u \| = \max_{x\in Z} \| u(x) \|$.

Set $\quad S(\rho) = \{ u \in B : \| u \| \le \rho,\ \| u(x) - u(\bar{x}) \| \le N \| x - \bar{x} \| \}$

where $\quad \rho = L + aK$ and $\| g(x) \| \le L \le K$

and $\quad \| g(x) - g(\bar{x}) \| \le M \| x - \bar{x} \|$ for all x and $\bar{x}$ in Z where $M \in R^+$

and $\quad T_m = N^{m-1} + N^{m-2} + N^{m-3} + \ldots + N + 1$ where $N \in R^+$.

Define $\quad T : S(\rho) \to S(\rho)$ by

$$(Tu)(x) = g(x) + \int_0^x u^m(t)dt \text{ for all } x \in Z.$$

Theorem 3.3.4 Suppose that $L + aK \leq K$, $M + K \leq N$ and $aT_m < 1$ then T has a unique fixed point, that is, there is a unique solution to the problem (3.11) – (3.12).

Proof We have $\| (Tu)(x) \| \leq \| g(x) \| + \int_0^x \| u^m(t) \| dt$

and hence $\quad 0 \leq \| Tu \|_S \leq L + aK = \rho$

and

$$\| (Tu)(x) - (Tu)(\bar{x}) \| \leq \left\| g(x) - g(\bar{x}) + \int_x^{\bar{x}} u^m(t)\,dt \right\|$$

$$\leq \| g(x) - g(\bar{x}) \| + \left\| \int_x^{\bar{x}} u^m(t)\,dt \right\|$$

$$\leq \| g(x) - g(\bar{x}) \| + \int_x^{\bar{x}} \| u^m(t) \|\,dt$$

$$\leq \| g(x) - g(\bar{x}) \| + K \int_x^{\bar{x}} dt$$

$$= \| g(x) - g(\bar{x}) \| + K(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\ldots(x_n - \bar{x}_n)$$

$$\leq \| g(x) - g(\bar{x}) \| + K \| x - \bar{x} \|$$

$$\leq M \| x - \bar{x} \| + K \| x - \bar{x} \|$$

$$\leq N \| x - \bar{x} \|.$$

Thus, we have $T : S(\rho) \to S(\rho)$.

Now, for all $u, v \in S(\rho)$ we have

$$\| (Tu)(x) - (Tv)(x) \| \leq \int_0^x \| u^m(t) - v^m(t) \|\,dt$$

$$\leq N^{m-1}\int_0^x \| u(t)-v(t) \| dt + N^{m-2}\int_0^x \| u(v(t))-v(v(t)) \| dt$$

$$+N^{m-3}\int_0^x \left\| u(v^2(t))-v(v^2(t)) \right\| dt$$

$$+\ldots+N\int_0^x \left\| u(v^{m-2}(t))-v(v^{m-2}(t)) \right\| dt$$

$$+\int_0^x \left\| u(v^{m-1}(t))-v(v^{m-1}(t)) \right\| dt$$

$$\leq a_1 a_2 \ldots a_n N^{m-1} \| u-v \| + a_1 a_2 \ldots a_n N^{m-2} \| u-v \| +$$

$$a_1 a_2 \ldots a_n N^{m-3} \| u-v \| + \ldots + a_1 a_2 \ldots a_n N \| u-v \| + a_1 a_2 \ldots a_n \| u-v \|$$

$$\leq a \| u-v \| (N^{m-1}+N^{m-2}+N^{m-3}+\ldots+N+1)$$

$$= a T_m \| u-v \|$$

$$< \| u-v \|.$$

Hence, by the Banach Contraction Principle, T has a unique fixed point.

The preceding theorems shows that there exists a unique solution to the problem (3.11) – (3.12). However, it does not tell us how to find this solution. To find the power series solution of the problem (3.11) – (3.12), we will define approximating sequence in following existence theorem

$$u_{k+1}(x) = g(x)+\int_0^x u_k^m(t)dt \tag{3.15}$$

where $k=0,1,2,\ldots$ and $u_0(x)$ are fixed functions of the class C^1 mapping Z to Z

such that $\left\| \dfrac{\partial^n u_0(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} \right\| \leq K.$

Hence we have the following theorem.

Theorem 3.3.5 If the conditions of theorem 3.3.4 are satisfied then sequences defined in (3.15) converges uniformly to the (unique) solution $u = u(x)$ of the problem (3.11) – (3.12).

Proof We put $U_k = \max_{x\in Z} \| u_k(x) - u_{k-1}(x) \|$.

Then
$$U_1 = \max_{x\in Z} \| u_1(x) - u_0(x) \| = \max_{x\in Z} \left\| g(x) + \int_0^x u_0^m(t)dt - u_0(x) \right\|.$$

Since $u_0(x)$ maps from Z to Z then, we have

$$U_1 \le L + aK \le K$$

and
$$\begin{aligned} U_2 &= \max_{x\in Z} \| u_2(x) - u_1(x) \| \\ &= \max_{x\in Z} \left\| g(x) + \int_0^x u_1^m(t)\,dt - g(x) + \int_0^x u_0^m(t)\,dt \right\| \\ &= \max_{x\in Z} \left\| \int_0^x u_1^m(t)dt - \int_0^x u_0^m(t)\,dt \right\| \\ &\le \max_{x\in Z} \int_0^x \left\| u_1^m(t) - u_0^m(t) \right\| dt \\ &\le aU_1 \le aK \end{aligned}$$

and
$$\begin{aligned} U_3 &= \max_{x\in Z} \| u_3(x) - u_2(x) \| \\ &= \max_{x\in Z} \left\| g(x) + \int_0^x u_2^m(t)\,dt - g(x) + \int_0^x u_1^m(t)\,dt \right\| \\ &= \max_{x\in Z} \left\| \int_0^x u_2^m(t)dt - \int_0^x u_1^m(t)\,dt \right\| \\ &\le \max_{x\in Z} \int_0^x \left\| u_2^m(t) - u_1^m(t) \right\| dt \\ &\le aU_2 \le a^2K \end{aligned}$$

and
$$
\begin{aligned}
U_4 &= \max_{x \in Z} \| u_4(x) - u_3(x) \| \\
&= \max_{x \in Z} \left\| g(x) + \int_0^x u_3^m(t)\, dt - g(x) + \int_0^x u_2^m(t)\, dt \right\| \\
&= \max_{x \in Z} \left\| \int_0^x u_3^m(t) dt - \int_0^x u_2^m(t)\, dt \right\| \\
&\le \max_{x \in Z} \int_0^x \| u_3^m(t) - u_2^m(t) \|\, dt \\
&\le aU_3 \le a^3 K .
\end{aligned}
$$

Thus, we have $U_k \le a^{k-1}K$. Since $L + aK \le K$ then $a < 1$ and $K = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} \ge 0$. Hence U_k tends to zero as k tends to infinity. Since the family $\{u_k\}$ is an Arzela-Ascoli family, thus for every subsequence $\{u_{k_j}\}$ of $\{u_k\}$ there exists a subsequence $\{u_{l_j}\}$ uniformly convergent and the limit needs to be a solution of the problem (3.11) – (3.12). Thus the sequence $\{u_k\}$ tends uniformly to the (unique) solution of the problem (3.11) – (3.12).

CHAPTER 4

SOME EXAMPLES OF THE SIMPLE ITERATIVE PRATIAL DIFFERNTIAL EQUATIONS

In this chapter, we will give some examples for finding the power series solutions of the second order and third order of the simple iterative partial differential equations with various initial conditions. We will see that the domains and initial conditions of the problems need to be satisfied the condition of the existence theorem.

4.1 The Second Order Simple Iterative Partial Differential Equations.

Example 4.1 Find the solution on the domain $Z=[0,\frac{1}{2}]\times[0,\frac{1}{2}]$ of the equation

$$\frac{\partial^2 u}{\partial x\partial y}=u^2(x,y)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix}\frac{1}{4} & 0\end{bmatrix}^T$$

or $$g(x,y) = \begin{bmatrix}\frac{1}{4} & 0\end{bmatrix}^T$$

and $$u_0(x,y) = g(x,y) = \begin{bmatrix}\frac{1}{4} & 0\end{bmatrix}^T.$$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering, $\|g(x,y)\| = \sqrt{\left(\frac{1}{4}\right)^2} = 0.25$, $K=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} \approx 0.707107$.

We obtain

$$\|g(x,y)\| + aK \approx \sqrt{\left(\frac{1}{4}\right)^2} + \frac{1}{2}.\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} \approx 0.43 \le K.$$

By the equation (3.2.5), we obtain

$$\begin{aligned} u_1(x,y) &= g(x,y) + \int_0^y\int_0^x u^2(x,y)dxdy \\ &= g(x,y) + \int_0^y\int_0^x u_0(u_0(x,y))dxdy \\ &= \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} dxdy = \begin{bmatrix} \frac{1}{4}+\frac{xy}{4} \\ 0 \end{bmatrix}. \end{aligned}$$

Hence $\quad u_2(x,y) = g(x,y) + \int_0^y\int_0^x u_1(u_1(x,y))dxdy.$

Since

$$u_1(u_1(x,y)) = \begin{bmatrix} \frac{1}{4} + \frac{\left(\frac{1}{4}+\frac{xy}{4}\right)(0)}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} u_2(x,y) &= \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} dx \\ &= \begin{bmatrix} \frac{1}{4}+\frac{xy}{4} \\ 0 \end{bmatrix}. \end{aligned}$$

Hence $\quad u_3(x,y) = g(x,y) + \int_0^y\int_0^x u_2(u_2(x,y))dxdy.$

Since

$$u_2(u_2(x,y)) = \begin{bmatrix} \frac{1}{4}+\frac{1}{4}\left(\frac{1}{4}+\frac{xy}{4}\right)(0) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}.$$

Thus

$$u_3(x,y) = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \frac{1}{4}+\frac{xy}{4} \\ 0 \end{bmatrix}.$$

Hence $$u_4(x,y) = g(x,y) + \int_0^y\int_0^x u_3(u_3(x,y))dxdy.$$

Since

$$u_3(u_3(x,y)) = \begin{bmatrix} \frac{1}{4}+\frac{1}{4}\left(\frac{1}{4}+\frac{xy}{4}\right)(0) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}.$$

We will obtain

$$u_4(x,y) = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \frac{1}{4}+\frac{xy}{4} \\ 0 \end{bmatrix}.$$

Thus we obtain $u_k(x,y)$ tends to $\begin{bmatrix} \frac{1}{4}+\frac{xy}{4} \\ 0 \end{bmatrix}$ as k tends to ∞.

Thus the solution of the problem is

$$u(x,y) = \begin{bmatrix} \frac{1}{4}+\frac{xy}{4} \\ 0 \end{bmatrix}.$$

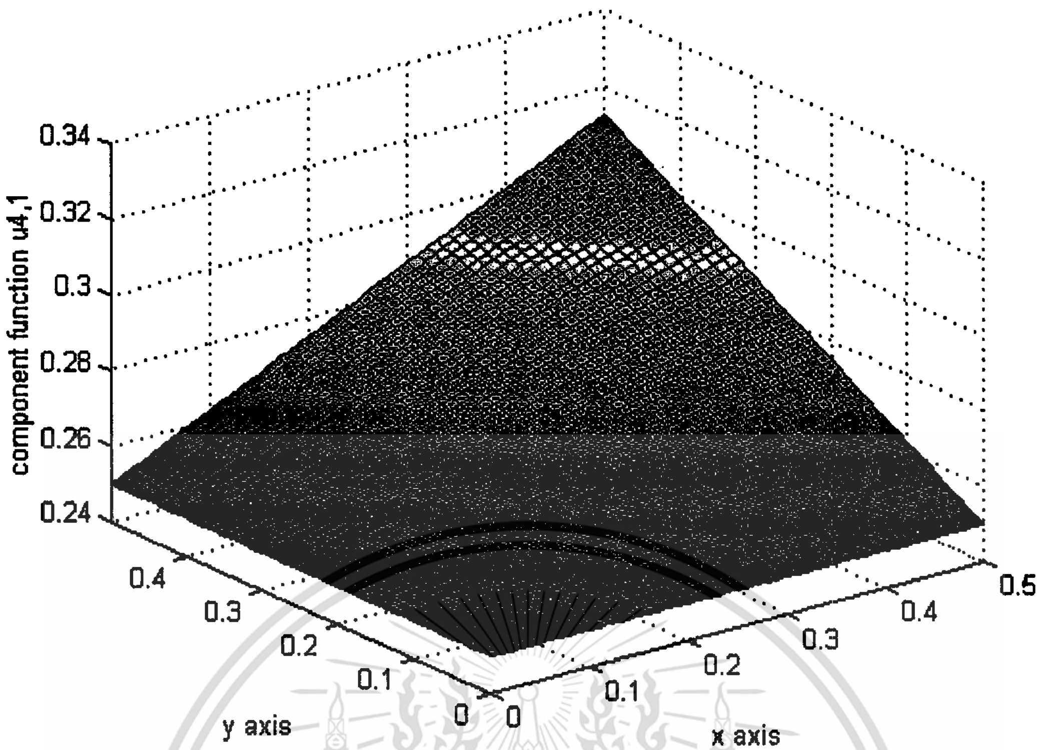

Figure 4.1 Graph of the component function $u_{4,1}$ of the power series solution u_4 .

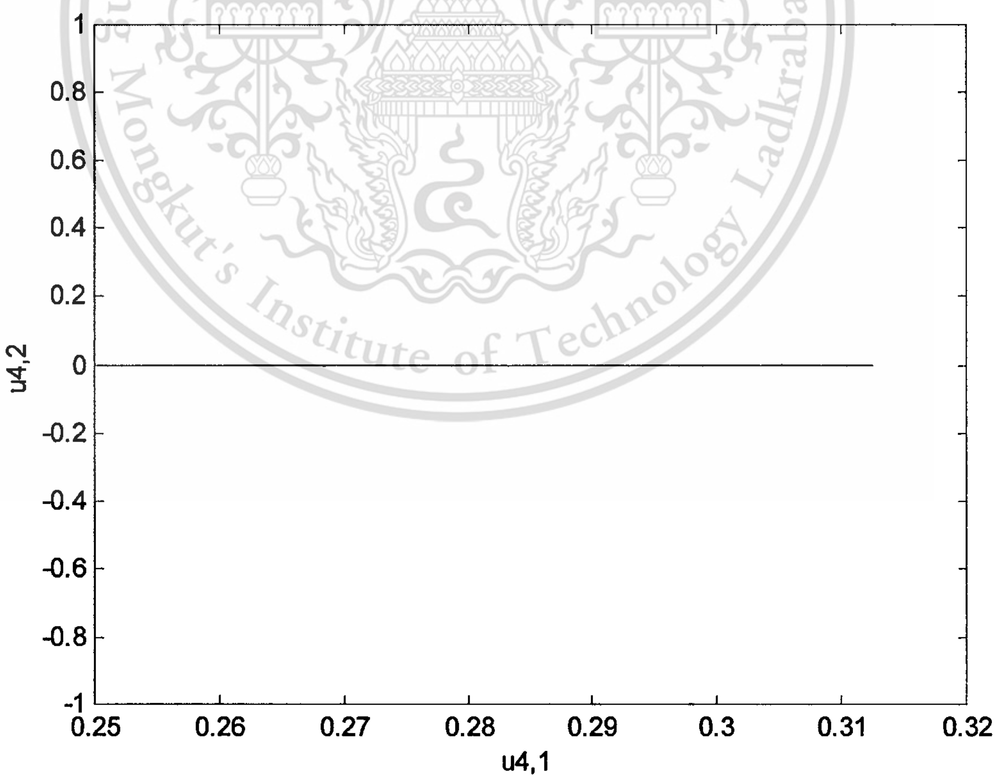

Figure 4.2 Graph of the power series solution u_4 in the example 4.1.

Example 4.2 Find the solution on the domain $Z = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = u^2(x, y)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T$$

or $$g(x,y) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T$$

and $$u_0(x,y) = g(x,y) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T.$$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering,

$$\|g(x,y)\| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} \approx 0.35355 , \quad K = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.707107 .$$

We obtain

$$\|g(x,y)\| + aK \approx 0.35355 + \frac{1}{2} \cdot \frac{1}{2} \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.530327 \leq K .$$

By the equation (3.5.2), we obtain

$$\begin{aligned} u_1(x,y) &= g(x,y) + \int_0^y\int_0^x u^2(x,y)dxdy \\ &= g(x,y) + \int_0^y\int_0^x u_0(u_0(x,y))dxdy \\ &= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} dxdy \\ &= \begin{bmatrix} \frac{1}{4}+\frac{xy}{4} \\ \frac{1}{4}+\frac{xy}{4} \end{bmatrix}. \end{aligned}$$

Hence $\quad u_2(x,y) = g(x,y) + \int_0^y\int_0^x u_1(u_1(x,y))dxdy.$

Since

$$u_1(u_1(x,y)) = \begin{bmatrix} \frac{1}{4}+\frac{\left(\frac{1}{4}+\frac{xy}{4}\right)\left(\frac{1}{4}+\frac{xy}{4}\right)}{4} \\ \frac{1}{4}+\frac{\left(\frac{1}{4}+\frac{xy}{4}\right)\left(\frac{1}{4}+\frac{xy}{4}\right)}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}+\frac{\left(\frac{1}{16}+\frac{xy}{8}+\frac{x^2y^2}{16}\right)}{4} \\ \frac{1}{4}+\frac{\left(\frac{1}{16}+\frac{xy}{8}+\frac{x^2y^2}{16}\right)}{4} \end{bmatrix}.$$

Thus

$$u_2(x,y) = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{17}{64}+\frac{xy}{32}+\frac{x^2y^2}{64} \\ \frac{17}{64}+\frac{xy}{32}+\frac{x^2y^2}{64} \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576} \\ \frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576} \end{bmatrix}.$$

Hence $\quad u_3(x,y) = g(x,y) + \int_0^y\int_0^x u_2(u_2(x,y))dxdy.$

Since

$$u_2(u_2(x,y)) = \left[\frac{\begin{array}{l} \frac{1}{4}+\frac{17}{64}\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right) \\ +\frac{1}{128}\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^2\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^2 \\ +\frac{1}{576}\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^3\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^3 \end{array}}{\begin{array}{l} \frac{1}{4}+\frac{17}{64}\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right) \\ +\frac{1}{128}\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^2\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^2 \\ +\frac{1}{256}\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^3\left(\frac{1}{4}+\frac{17xy}{64}+\frac{x^2y^2}{128}+\frac{x^3y^3}{576}\right)^3 \end{array}}\right].$$

We will obtain

$$u_3(x,y) = \left[\frac{\begin{array}{l} 0.25 + 0.2666xy + 8.8527\times10^{-3}x^2y^2 + 2.2219\times10^{-3}x^3y^3 \\ + 9.3941\times10^{-5}x^4y^4 + 1.2999\times10^{-5}x^5y^5 + 5.4825\times10^{-7}x^6y^6 \\ + 7.3125\times10^{-8}x^7y^7 + 6.0289\times10^{-9}x^8y^8 + 7.6298\times10^{-10}x^9y^9 \\ + 6.6786\times10^{-11}x^{10}y^{10} + 6.6834\times10^{-12}x^{11}y^{11} + 4.6664\times10^{-13}x^{12}y^{12} \\ + 3.6757\times10^{-14}x^{13}y^{13} + 1.9413\times10^{-15}x^{14}y^{14} + 1.1887\times10^{-16}x^{15}y^{15} \\ + 4.3344\times10^{-14}x^{16}y^{16} + 2.0097\times10^{-19}x^{17}y^{17} + 3.9615\times10^{-21}x^{18}y^{18} \\ + 1.3169\times10^{-22}x^{19}y^{19} \end{array}}{\begin{array}{l} 0.25 + 0.2666xy + 8.8527\times10^{-3}x^2y^2 + 2.2219\times10^{-3}x^3y^3 \\ + 9.3941\times10^{-5}x^4y^4 + 1.2999\times10^{-5}x^5y^5 + 5.4825\times10^{-7}x^6y^6 \\ + 7.3125\times10^{-8}x^7y^7 + 6.0289\times10^{-9}x^8y^8 + 7.6298\times10^{-10}x^9y^9 \\ + 6.6786\times10^{-11}x^{10}y^{10} + 6.6834\times10^{-12}x^{11}y^{11} + 4.6664\times10^{-13}x^{12}y^{12} \\ + 3.6757\times10^{-14}x^{13}y^{13} + 1.9413\times10^{-15}x^{14}y^{14} + 1.1887\times10^{-16}x^{15}y^{15} \\ + 4.3344\times10^{-14}x^{16}y^{16} + 2.0097\times10^{-19}x^{17}y^{17} + 3.9615\times10^{-21}x^{18}y^{18} \\ + 1.3169\times10^{-22}x^{19}y^{19} \end{array}}\right].$$

Thus we may use $u_3(x,y)$ as the approximating solution for the problem.

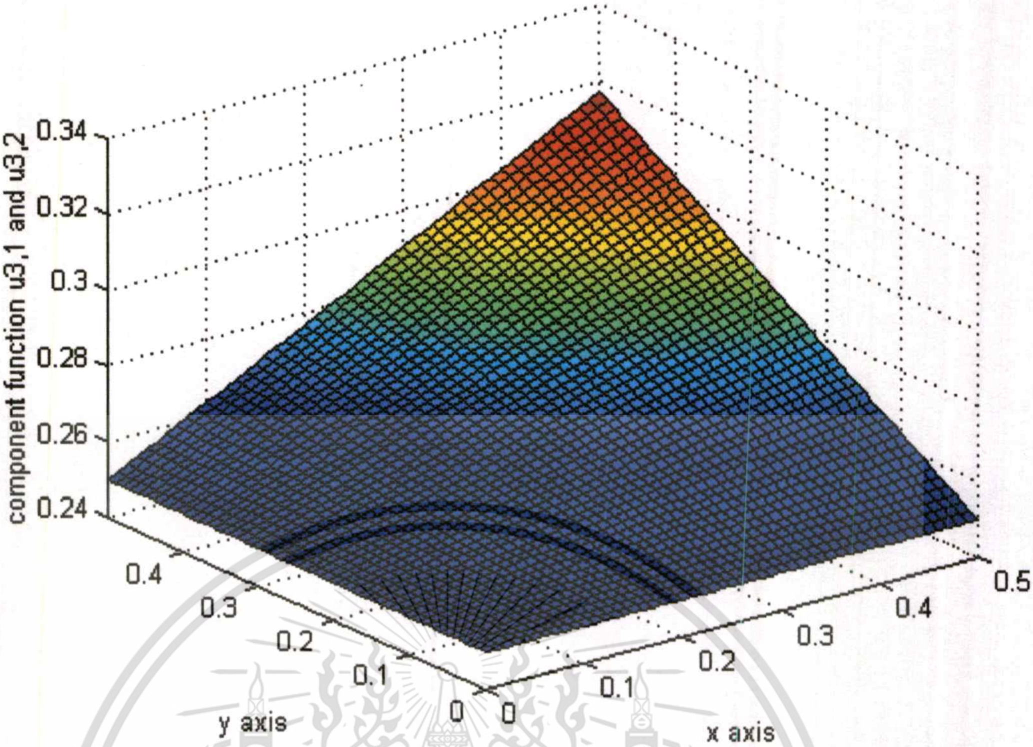

Figure 4.3 Graph of component function $u_{3,1}$ and $u_{3,2}$ of the power series solution u_3.

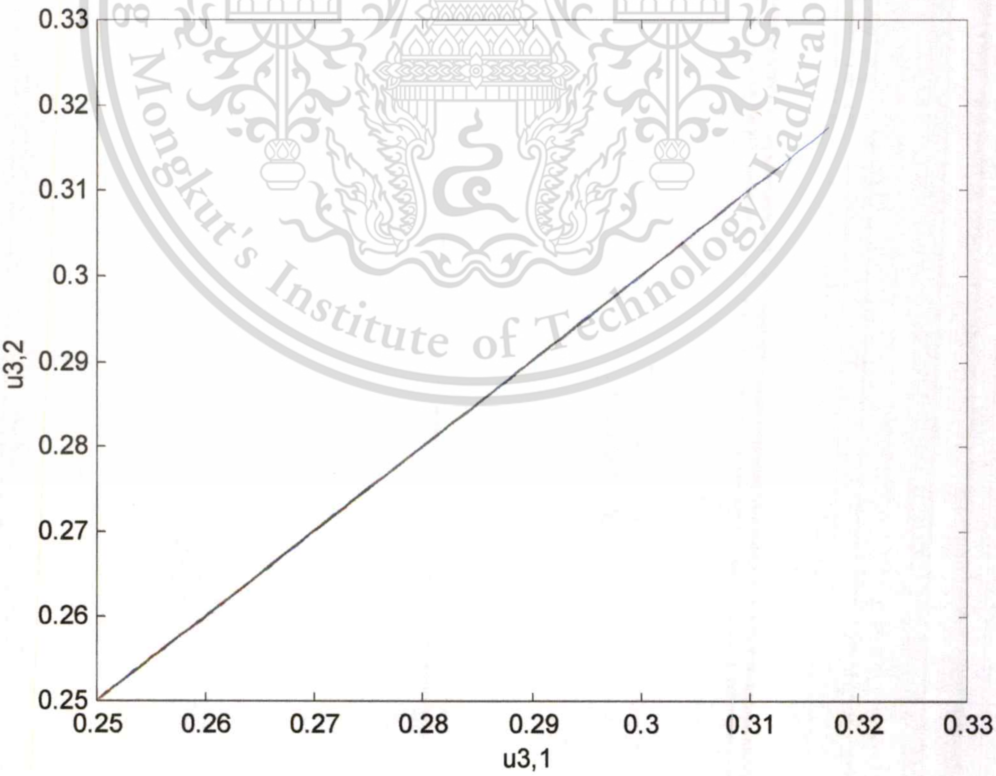

Figure 4.4 Graph of the power series solution u_3 in the example 4.2.

Example 4.3 Find the solution on the domain $Z = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = u^2(x, y)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{1}{8} & \frac{1}{16} \end{bmatrix}^T$$

or $$g(x,y) = \begin{bmatrix} \frac{1}{8} & \frac{1}{16} \end{bmatrix}^T$$

and $$u_0(x,y) = g(x,y) = \begin{bmatrix} \frac{1}{8} & \frac{1}{16} \end{bmatrix}^T.$$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering,

$$\|g(x,y)\| = \sqrt{\left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2} \approx 0.13975\,, \quad K = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.707107\,.$$

We obtain

$$\|g(x,y)\| + aK \approx 0.13975 + \frac{1}{2}.\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.31653 \le K\,.$$

By the equation (3.5.2), we obtain

$$
\begin{aligned}
u_1(x,y) &= g(x,y) + \int_0^y\int_0^x u^2(x,y)\,dxdy \\
&= g(x,y) + \int_0^y\int_0^x u_0(u_0(x,y))\,dxdy \\
&= \begin{bmatrix} \frac{1}{8} \\ \frac{1}{16} \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{1}{8} \\ \frac{1}{16} \end{bmatrix} dxdy \\
&= \begin{bmatrix} \frac{1}{8} + \frac{xy}{8} \\ \frac{1}{16} + \frac{xy}{16} \end{bmatrix}.
\end{aligned}
$$

Hence $\quad u_2(x,y) = g(x,y) + \int_0^y\int_0^x u_1(u_1(x,y))\,dxdy$.

Since

$$
u_1(u_1(x,y)) = \begin{bmatrix} \frac{1}{8} + \dfrac{\left(\frac{1}{8}+\frac{xy}{8}\right)\left(\frac{1}{16}+\frac{xy}{16}\right)}{8} \\ \frac{1}{16} + \dfrac{\left(\frac{1}{8}+\frac{xy}{8}\right)\left(\frac{1}{16}+\frac{xy}{16}\right)}{16} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} + \dfrac{\left(\frac{1}{128}+\frac{xy}{64}+\frac{x^2y^2}{128}\right)}{8} \\ \frac{1}{16} + \dfrac{\left(\frac{1}{128}+\frac{xy}{64}+\frac{x^2y^2}{128}\right)}{16} \end{bmatrix}.
$$

Thus

$$
u_2(x,y) = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{16} \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{129}{1024} + \frac{xy}{512} + \frac{x^2y^2}{1024} \\ \frac{129}{2048} + \frac{xy}{1024} + \frac{x^2y^2}{2048} \end{bmatrix} dxdy
$$

$$
= \begin{bmatrix} \frac{1}{8} + \frac{129xy}{1024} + \frac{x^2y^2}{2048} + \frac{x^3y^3}{9216} \\ \frac{1}{16} + \frac{129xy}{2048} + \frac{x^2y^2}{4096} + \frac{x^3y^3}{18432} \end{bmatrix}.
$$

Hence $\quad u_3(x,y) = g(x,y) + \int_0^y\int_0^x u_2(u_2(x,y))\,dxdy$.

Since

$$u_2(u_2(x,y))=\left[\frac{\begin{aligned}&\frac{1}{8}+\frac{129}{1024}\left(\frac{1}{8}+\frac{129xy}{1024}+\frac{x^2y^2}{2048}+\frac{x^3y^3}{9216}\right)\left(\frac{1}{16}+\frac{129xy}{2048}+\frac{x^2y^2}{4096}+\frac{x^3y^3}{18432}\right)\\&+\frac{1}{2048}\left(\frac{1}{8}+\frac{129xy}{1024}+\frac{x^2y^2}{2048}+\frac{x^3y^3}{9216}\right)^2\left(\frac{1}{16}+\frac{129xy}{2048}+\frac{x^2y^2}{4096}+\frac{x^3y^3}{18432}\right)^2\\&+\frac{1}{9216}\left(\frac{1}{8}+\frac{129xy}{1024}+\frac{x^2y^2}{2048}+\frac{x^3y^3}{9216}\right)^3\left(\frac{1}{16}+\frac{129xy}{2048}+\frac{x^2y^2}{4096}+\frac{x^3y^3}{18432}\right)^3\end{aligned}}{\begin{aligned}&\frac{1}{16}+\frac{129}{2048}\left(\frac{1}{8}+\frac{129xy}{1024}+\frac{x^2y^2}{2048}+\frac{x^3y^3}{9216}\right)\left(\frac{1}{16}+\frac{129xy}{2048}+\frac{x^2y^2}{4096}+\frac{x^3y^3}{18432}\right)\\&+\frac{1}{4096}\left(\frac{1}{8}+\frac{129xy}{1024}+\frac{x^2y^2}{2048}+\frac{x^3y^3}{9216}\right)^2\left(\frac{1}{16}+\frac{129xy}{2048}+\frac{x^2y^2}{4096}+\frac{x^3y^3}{18432}\right)^2\\&+\frac{1}{18432}\left(\frac{1}{8}+\frac{129xy}{1024}+\frac{x^2y^2}{2048}+\frac{x^3y^3}{9216}\right)^3\left(\frac{1}{16}+\frac{129xy}{2048}+\frac{x^2y^2}{4096}+\frac{x^3y^3}{18432}\right)^3\end{aligned}}\right].$$

We obtain

$$u_3(x,y)=\left[\frac{\begin{aligned}&0.125+0.1259xy+4.9597\times10^{-4}x^2y^2+1.1195\times10^{-4}x^3y^3\\&+5.9889\times10^{-7}x^4y^4+7.0813\times10^{-8}x^5y^5+2.1698\times10^{-10}x^6y^6\\&+1.8698\times10^{-11}x^7y^7+6.6392\times10^{-14}x^8y^8+5.6853\times10^{-15}x^9y^9\\&+9.1865\times10^{-17}x^{10}y^{10}+6.2519\times10^{-18}x^{11}y^{11}+8.3334\times10^{-20}x^{12}y^{12}\\&+4.6215\times10^{-21}x^{13}y^{13}+4.6512\times10^{-23}x^{14}y^{14}+2.1439\times10^{-24}x^{15}y^{15}\\&+1.4308\times10^{-26}x^{16}y^{16}+5.5685\times10^{-28}x^{17}y^{17}+1.8447\times10^{-30}x^{18}y^{18}\\&+6.1321\times10^{-32}x^{19}y^{19}\end{aligned}}{\begin{aligned}&0.0625+0.0629xy+2.4799\times10^{-4}x^2y^2+5.5972\times10^{-5}x^3y^3\\&+2.9945\times10^{-7}x^4y^4+3.5407\times10^{-8}x^5y^5+1.0849\times10^{-10}x^6y^6\\&+9.3491\times10^{-12}x^7y^7+3.3196\times10^{-14}x^8y^8+2.8427\times10^{-15}x^9y^9\\&+4.5932\times10^{-17}x^{10}y^{10}+3.1259\times10^{-18}x^{11}y^{11}+4.1667\times10^{-20}x^{12}y^{12}\\&+2.3108\times10^{-21}x^{13}y^{13}+2.3256\times10^{-23}x^{14}y^{14}+1.0719\times10^{-24}x^{15}y^{15}\\&+7.1542\times10^{-27}x^{16}y^{16}+2.7842\times10^{-28}x^{17}y^{17}+9.2236\times10^{-31}x^{18}y^{18}\\&+3.066\times10^{-32}x^{19}y^{19}\end{aligned}}\right].$$

Thus we may use $u_3(x,y)$ as the approximating solution for the problem.

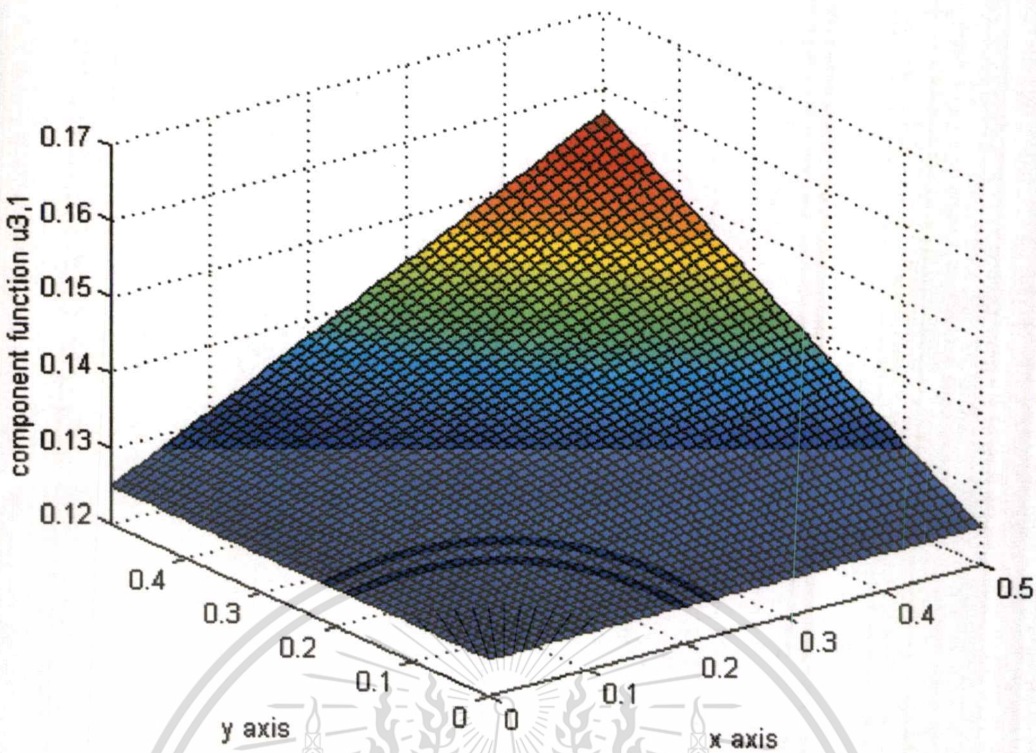

Figure 4.5 Graph of the component function $u_{3,1}$ of the power series solution u_3.

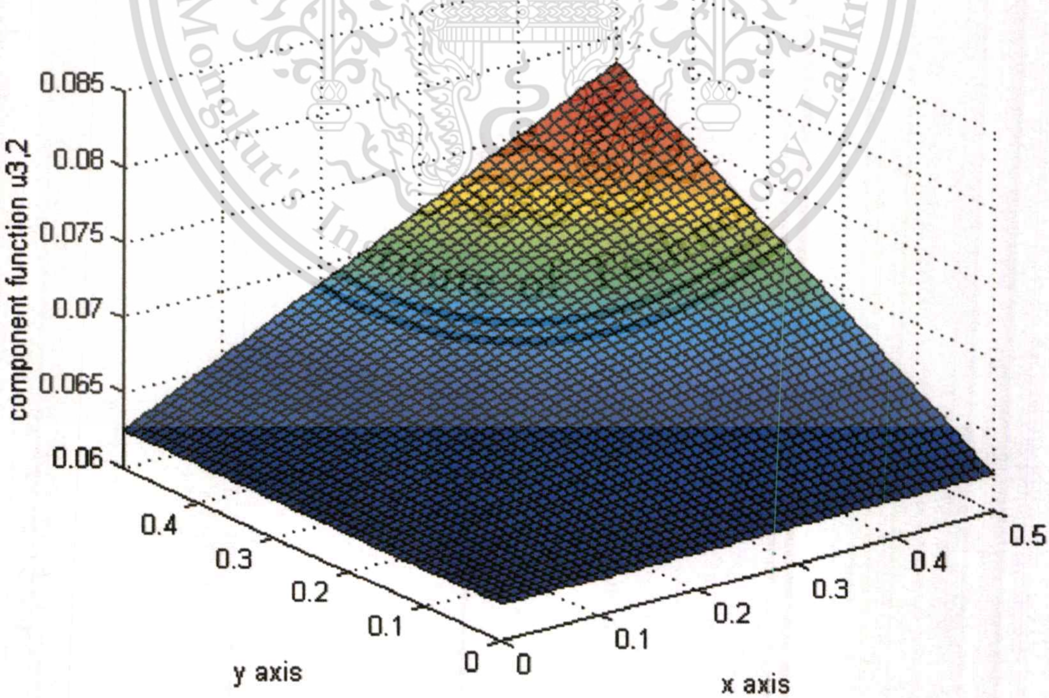

Figure 4.6 Graph of component function $u_{3,2}$ of the power series solution u_3.

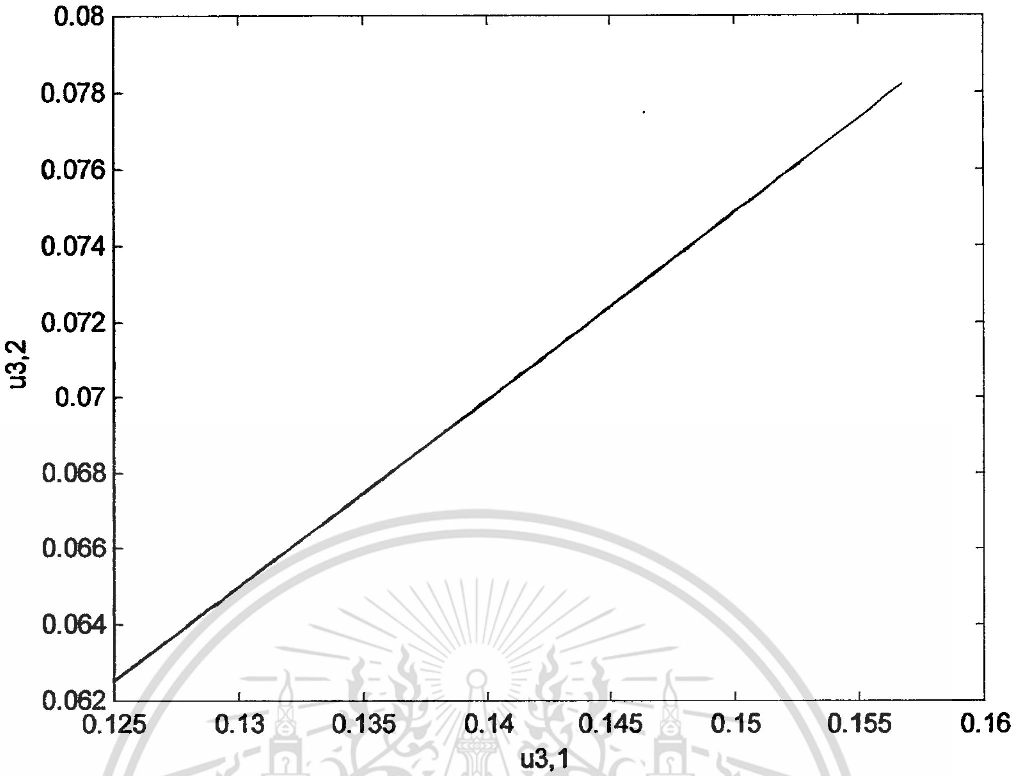

Figure 4.7 Graph of the power series solution u_3 in the example 4.3.

Example 4.4 Find the solution on the domain $Z = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = u^2(x, y)$$

with the initial conditions

$$u(x,0) = \begin{bmatrix} \frac{x}{4} & 0 \end{bmatrix}^T \qquad u(0,y) = \begin{bmatrix} 0 & \frac{y}{4} \end{bmatrix}^T \qquad u(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

or $\quad g(x,y) = \begin{bmatrix} \frac{x}{4} & \frac{y}{4} \end{bmatrix}^T$

and $\quad u_0(x,y) = g(x,y) = \begin{bmatrix} \frac{x}{4} & \frac{y}{4} \end{bmatrix}^T.$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering,

$$\|g(x,y)\| = \sqrt{\left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2} \le 0.17678 \;, \quad K = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.707107 .$$

We obtain

$$\|g(x,y)\| + aK \approx 0.17678 + \frac{1}{2}.\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.35355 \le K .$$

By the equation (3.5.2), we obtain

$$\begin{aligned} u_1(x,y) &= g(x,y) + \int_0^y\int_0^x u^2(x,y)dxdy \\ &= g(x,y) + \int_0^y\int_0^x u_0(u_0(x,y))dxdy \\ &= \begin{bmatrix} \frac{x}{4} \\ \frac{y}{4} \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{x}{4} \\ \frac{y}{4} \end{bmatrix} dxdy \\ &= \begin{bmatrix} \frac{x}{4} + \frac{x^2y}{8} \\ \frac{y}{4} + \frac{xy^2}{8} \end{bmatrix}. \end{aligned}$$

Hence $\quad u_2(x,y) = g(x,y) + \int_0^y\int_0^x u_1(u_1(x,y))dxdy .$

Since $$u_1(u_1(x,y)) = \begin{bmatrix} \dfrac{\left(\frac{x}{4} + \frac{x^2y}{8}\right)}{4} + \dfrac{\left(\frac{x}{4} + \frac{x^2y}{8}\right)^2\left(\frac{y}{4} + \frac{xy^2}{8}\right)}{8} \\ \dfrac{\left(\frac{y}{4} + \frac{xy^2}{8}\right)}{4} + \dfrac{\left(\frac{x}{4} + \frac{x^2y}{8}\right)\left(\frac{y}{4} + \frac{xy^2}{8}\right)^2}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{16} + \frac{17x^2y}{512} + \frac{3x^3y^2}{1024} + \frac{3x^4y^3}{2048} + \frac{x^5y^4}{4096} \\ \frac{17xy^2}{512} + \frac{3x^2y^3}{1024} + \frac{x^2y^4}{2048} + \frac{y}{16} + \frac{x^3y^4}{1024} + \frac{x^3y^5}{4096} \end{bmatrix}.$$

Thus

$$u_2(x,y) = \begin{bmatrix} \frac{x}{4} \\ \frac{y}{4} \end{bmatrix} + \int_0^y \int_0^x \begin{bmatrix} \frac{x}{16} + \frac{17x^2y}{512} + \frac{3x^3y^2}{1024} + \frac{3x^4y^3}{2048} + \frac{x^5y^4}{4096} \\ \frac{17xy^2}{512} + \frac{3x^2y^3}{1024} + \frac{y}{16} + \frac{3x^3y^4}{2048} + \frac{x^4y^5}{4096} \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \frac{x}{4} + \frac{x^2y}{32} + \frac{17x^3y^2}{3072} + \frac{x^4y^3}{4096} + \frac{3x^5y^4}{40960} + \frac{x^6y^5}{122880} \\ \frac{y}{4} + \frac{xy^2}{32} + \frac{17x^2y^3}{3072} + \frac{x^3y^4}{4096} + \frac{3x^4y^5}{2048} + \frac{x^5y^6}{4096} \end{bmatrix}.$$

Hence $u_3(x,y) = g(x,y) + \int_0^y \int_0^x u_2(u_2(x,y))dxdy$.

Since

$$u_2(u_2(x,y)) = \begin{bmatrix} \frac{\left(\frac{x}{4} + \frac{x^2y}{32} + \frac{x^4y^3}{4096} + \frac{17x^3y^2}{3072} + \frac{3x^5y^4}{40960} + \frac{x^6y^5}{122880}\right)}{4} \\ + \frac{\left(\frac{x}{4} + \frac{x^2y}{32} + \frac{x^4y^3}{4096} + \frac{17x^3y^2}{3072} + \frac{3x^5y^4}{40960} + \frac{x^6y^5}{122880}\right)^2}{32} \\ \\ \frac{\left(\frac{y}{4} + \frac{xy^2}{32} + \frac{17x^2y^3}{3072} + \frac{x^3y^4}{4096} + \frac{x^3y^5}{30720} + \frac{x^4y^5}{20480} + \frac{x^4y^6}{98304}\right)}{4} \\ + \frac{\left(\frac{x}{4} + \frac{x^2y}{32} + \frac{x^4y^3}{4096} + \frac{17x^3y^2}{3072} + \frac{3x^5y^4}{40960} + \frac{x^6y^5}{122880}\right)}{32} \end{bmatrix}.$$

Thus

$$u_3(x,y)=\begin{bmatrix} 0.25x+0.03125\,x^2y+1.3835\times10^{-3}x^3y^2+1.3055\times10^{-4}x^4y^3 \\ +5.8174\times10^{-6}x^5y^4+9.6003\times10^{-7}x^6y^5+1.6104\times10^{-7}x^7y^6 \\ +2.6264\times10^{-8}x^8y^7+4.5870\times10^{-9}x^9y^8+6.1003\times10^{-10}x^{10}y^9 \\ +7.1524\times10^{-11}x^{11}y^{10}+9.0884\times10^{-12}x^{12}y^{11}+9.8692\times10^{-13}x^{13}y^{12} \\ +9.1329\times10^{-14}x^{14}y^{13}+6.7049\times10^{-15}x^{15}y^{14}+5.2633\times10^{-16}x^{16}y^{15} \\ +4.4583\times10^{-17}x^{17}y^{16}+1.6512\times10^{-18}x^{18}y^{17} \\ \\ 0.25y+0.03125\,xy^2+1.3835\times10^{-3}x^2y^3+1.3055\times10^{-4}x^3y^4 \\ +5.8174\times10^{-6}x^4y^5+1.2557\times10^{-5}x^5y^6+1.6305\times10^{-6}x^6y^7 \\ +4.6629\times10^{-8}x^7y^8+8.4486\times10^{-9}x^8y^9+1.1429\times10^{-9}x^9y^{10} \\ +2.7305\times10^{-10}x^{10}y^{11}+7.0639\times10^{-11}x^{11}y^{12}+1.0956\times10^{-11}x^{12}y^{13} \\ +1.1814\times10^{-12}x^{13}y^{14}+1.0389\times10^{-13}x^{14}y^{15}+1.1116\times10^{-14}x^{15}y^{16} \\ +1.1703\times10^{-15}x^{16}y^{17}+4.9537\times10^{-17}x^{17}y^{18} \end{bmatrix}.$$

Thus we may use $u_3(x, y)$ as the approximating solution for the problem.

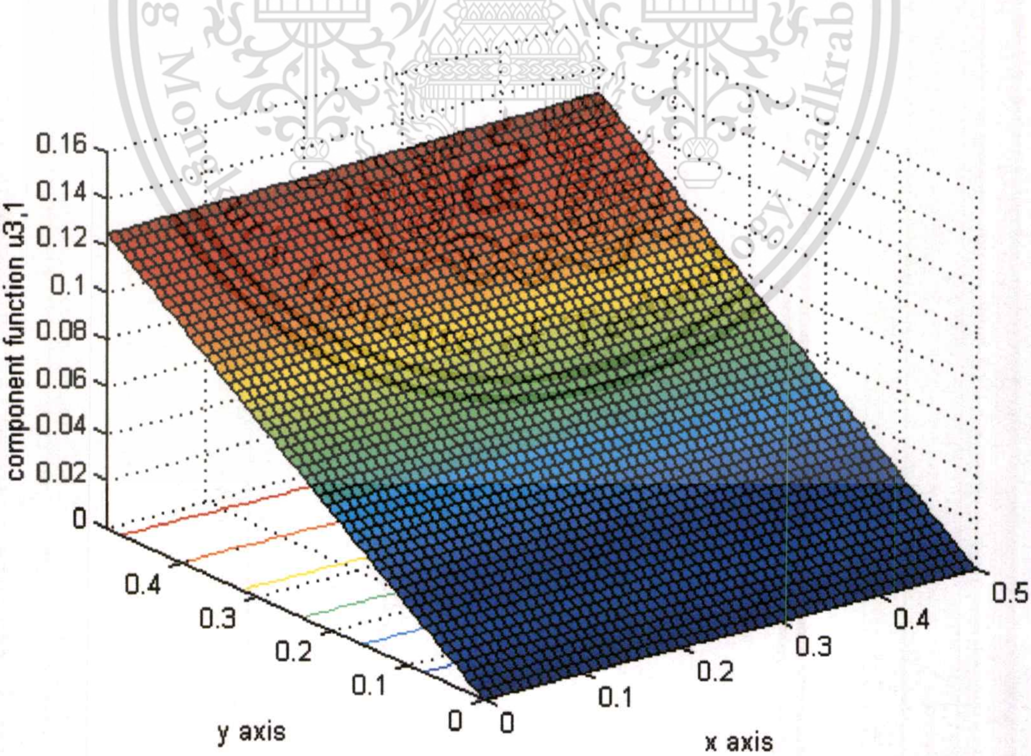

Figure 4.8 Graph of component functions $u_{3,1}$ of the power series solution u_3.

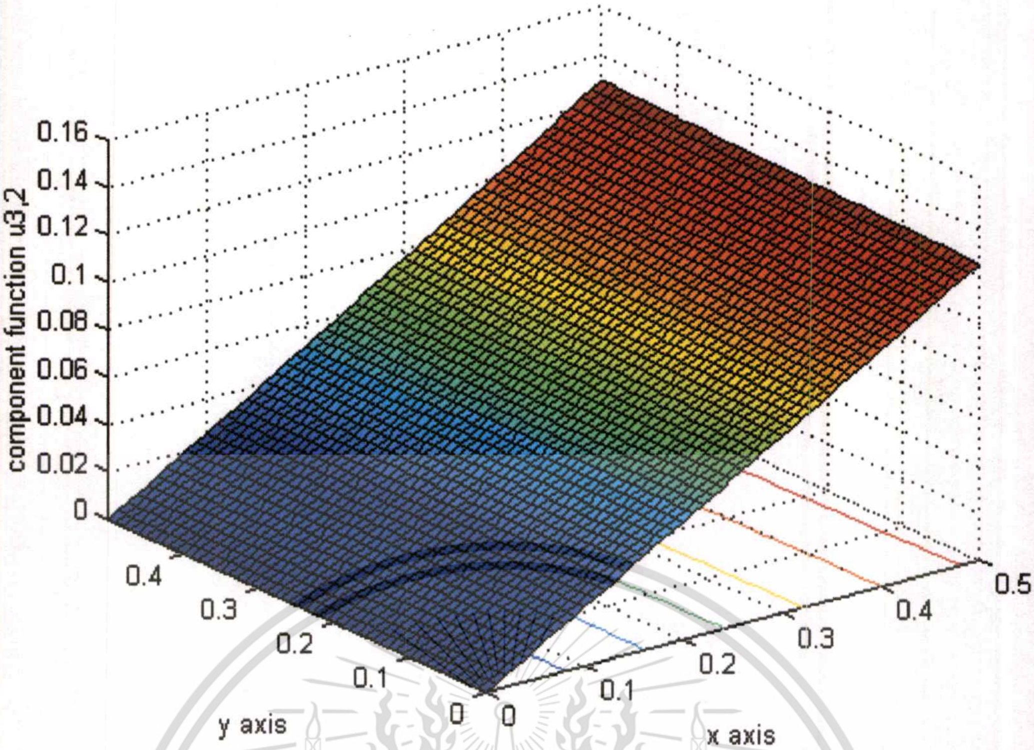

Figure 4.9 Graph of component functions $u_{3,2}$ of the power series solution u_3.

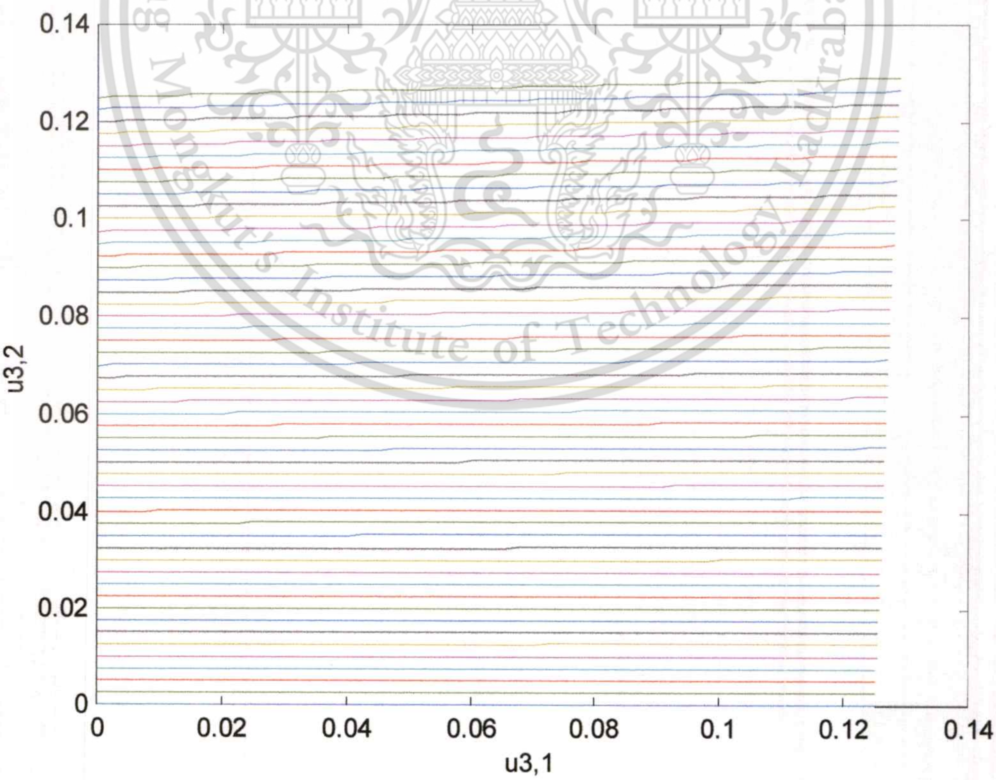

Figure 4.10 Graph of the power series solution u_3 in the example 4.4.

Example 4.5 Find the solution on the domain $Z=[0,\frac{1}{2}]\times[0,\frac{1}{2}]$ of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = u^2(x,y)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{x}{4} & 0 \end{bmatrix}^T$$

or $$g(x,y) = \begin{bmatrix} \frac{x}{4} & 0 \end{bmatrix}^T$$

and $$u_{1,0}(x,y) = g(x,y) = \begin{bmatrix} \frac{x}{4} & 0 \end{bmatrix}^T .$$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Consider, $\|g(x,y)\| = \sqrt{\left(\frac{x}{4}\right)^2} \le 0.125$, $K=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} \approx 0.707107$.

We obtain $\|g(x,y)\| + aK \approx 0.125 + \frac{1}{2}.\frac{1}{2}\sqrt{\frac{1}{2}^2+\frac{1}{2}^2} \approx 0.30178 \le K$.

By the equation (3.5.2), we obtain

$$\begin{aligned} u_1(x,y) &= g(x,y) + \int_0^y\int_0^x u^2(x,y)dxdy \\ &= g(x,y) + \int_0^y\int_0^x u_0(u_0(x,y))dxdy \\ &= \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix} dxdy = \begin{bmatrix} \frac{x}{4}+\frac{x^2y}{8} \\ 0 \end{bmatrix}. \end{aligned}$$

Hence $$u_2(x,y) = g(x,y) + \int_0^y \int_0^x u_1(u_1(x,y))dxdy.$$

Since $$u_1(u_1(x,y)) = \begin{bmatrix} \frac{x}{4} + \frac{\left(\frac{x}{4} + \frac{x^2 y}{8}\right)^2 (0)}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix}.$$

Thus

$$u_2(x,y) = \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix} + \int_0^y \int_0^x \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \frac{x}{4} + \frac{x^2 y}{8} \\ 0 \end{bmatrix}.$$

Hence $$u_3(x,y) = g(x,y) + \int_0^y \int_0^x u_2(u_2(x,y))dxdy.$$

Since

$$u_2(u_2(x,y)) = \begin{bmatrix} \frac{x}{4} + \left(\frac{x}{4} + \frac{x^2 y}{8}\right)^2 (0) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix}.$$

Thus

$$u_3(x,y) = \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix} + \int_0^y \int_0^x \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \frac{x}{4} + \frac{x^2 y}{8} \\ 0 \end{bmatrix}.$$

Hence $$u_4(x,y) = g(x,y) + \int_0^y \int_0^x u_3(u_3(x,y))dxdy.$$

Since $$u_3(u_3(x,y)) = \begin{bmatrix} \dfrac{x}{4} + \left(\dfrac{x}{4} + \dfrac{x^2 y}{8}\right)^2 (0) \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{x}{4} \\ 0 \end{bmatrix}.$$

We obtain

$$u_3(x,y) = \begin{bmatrix} \dfrac{x}{4} \\ 0 \end{bmatrix} + \int_0^y \int_0^x \begin{bmatrix} \dfrac{x}{4} \\ 0 \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \dfrac{x}{4} + \dfrac{x^2 y}{8} \\ 0 \end{bmatrix}.$$

Thus we obtain $u_k(x,y)$ tends to $\begin{bmatrix} \dfrac{x}{4} + \dfrac{x^2 y}{8} \\ 0 \end{bmatrix}$ as k tends to ∞.

Thus solution of the equations is $u(x,y) = \begin{bmatrix} \dfrac{x}{4} + \dfrac{x^2 y}{8} \\ 0 \end{bmatrix}$.

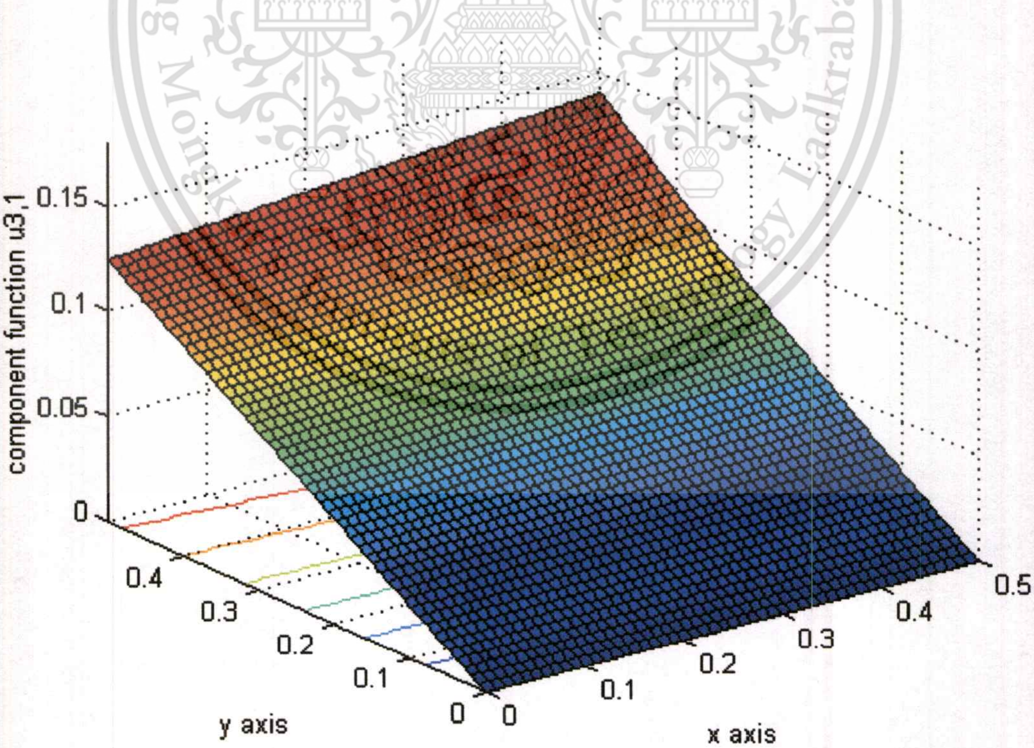

Figure 4.11 Graph of component functions $u_{3,1}$ of the power series solution u_3.

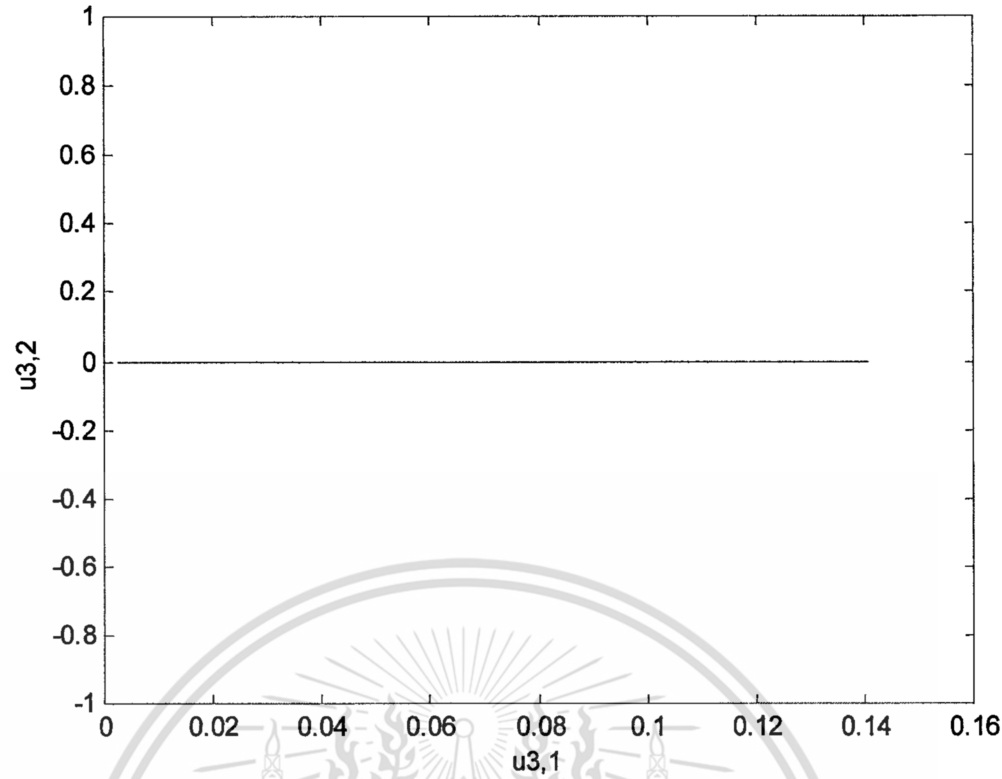

Figure 4.12 Graph of the power series solution u_3 in the example 4.5.

4.2 The Third Order Simple Iterative Partial Differential Equations.

Example 4.6 Find the solution on the domain $Z=[0,\frac{1}{2}]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}]$ of the equation

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=u^2(x,y,z)$$

with the initial conditions

$$u(x,0,0) = \begin{bmatrix}\frac{x}{4} & 0 & 0\end{bmatrix}^T,\ u(0,y,0) = \begin{bmatrix}0 & \frac{y}{4} & 0\end{bmatrix}^T,\ u(0,0,z) = \begin{bmatrix}0 & 0 & \frac{z}{4}\end{bmatrix}^T$$

$$u(x,y,0) = u(x,0,z) = u(0,y,z) = u(0,0,0) = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$$

or $$g(x,y,z) = \begin{bmatrix}\frac{x}{4} & \frac{y}{4} & \frac{z}{4}\end{bmatrix}^T$$

and $\quad u_0(x,y,z) = g(x,y,z) = \begin{bmatrix} \frac{x}{4} & \frac{y}{4} & \frac{z}{4} \end{bmatrix}^T .$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering, $\quad \|g(x,y,z)\| = \sqrt{\left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{4}\right)^2} \leq 0.21651 .$

$$K = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0{,}86603 .$$

We obtain

$$\|g(x,y,z)\| + aK \approx 0.21651 + \frac{1}{2}.\frac{1}{2}.\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.32476 \leq K .$$

By the equation (3.5.2), we obtain

$$u_1(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u^2(x,y,z)dxdydz$$

$$= g(x,y,z) + \int_0^z\int_0^y\int_0^x u_0(u_0(x,y,z))dxdydz$$

$$= \begin{bmatrix} \frac{x}{4} \\ \frac{y}{4} \\ \frac{z}{4} \end{bmatrix} + \int_0^z\int_0^y\int_0^x \begin{bmatrix} \frac{x}{4} \\ \frac{y}{4} \\ \frac{z}{4} \end{bmatrix} dxdydz$$

$$= \begin{bmatrix} \dfrac{x}{4} + \dfrac{x^2yz}{8} \\ \dfrac{y}{4} + \dfrac{xy^2z}{8} \\ \dfrac{z}{4} + \dfrac{xyz^2}{8} \end{bmatrix}.$$

Hence $\quad u_2(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_1(u_1(x,y))dxdydz.$

Since

$$u_1(u_1(x,y,z)) = \begin{bmatrix} \dfrac{\left(\dfrac{x}{4}+\dfrac{x^2yz}{8}\right)}{4} + \dfrac{\left(\dfrac{x}{4}+\dfrac{x^2yz}{8}\right)^2\left(\dfrac{y}{4}+\dfrac{xy^2z}{8}\right)\left(\dfrac{z}{4}+\dfrac{xyz^2}{8}\right)}{8} \\ \dfrac{\left(\dfrac{y}{4}+\dfrac{xy^2z}{8}\right)}{4} + \dfrac{\left(\dfrac{x}{4}+\dfrac{x^2yz}{8}\right)\left(\dfrac{y}{4}+\dfrac{xy^2z}{8}\right)^2\left(\dfrac{z}{4}+\dfrac{xyz^2}{8}\right)}{8} \\ \dfrac{\left(\dfrac{z}{4}+\dfrac{xyz^2}{8}\right)}{4} + \dfrac{\left(\dfrac{x}{4}+\dfrac{x^2yz}{8}\right)\left(\dfrac{y}{4}+\dfrac{xy^2z}{8}\right)\left(\dfrac{z}{4}+\dfrac{xyz^2}{8}\right)^2}{8} \end{bmatrix}.$$

Thus

$$u_2(x,y,z) = \begin{bmatrix} \dfrac{x}{4}+\dfrac{x^2yz}{32}+\dfrac{65x^3y^2z^2}{24576}+\dfrac{x^4y^3z^3}{36864}+\dfrac{x^5y^3z^4}{491520}+\dfrac{x^5y^4z^4}{131072}+\dfrac{x^6y^4z^5}{983040} \\ +\dfrac{x^6y^5z^5}{1228800}+\dfrac{x^7y^5z^6}{6881280} \\ \\ \dfrac{y}{4}+\dfrac{xy^2z}{32}+\dfrac{x^3y^4z^3}{36864}+\dfrac{3x^4y^5z^4}{655360}+\dfrac{65x^2y^3z^2}{24576}+\dfrac{x^5y^6z^5}{1228800} \\ +\dfrac{x^4y^6z^5}{983040}+\dfrac{x^5y^7z^6}{6881280}+\dfrac{x^4y^5z^4}{327680}+\dfrac{x^3y^5z^4}{491520} \\ \\ \dfrac{z}{4}+\dfrac{xyz^2}{32}+\dfrac{65x^2y^2z^3}{24576}+\dfrac{x^3y^3z^4}{36864}+\dfrac{x^3y^3z^5}{368640}+\dfrac{x^4y^4z^5}{131072}+\dfrac{x^4y^4z^6}{786432} \\ +\dfrac{x^5y^5z^6}{1228800}+\dfrac{x^5y^5z^7}{5734400} \end{bmatrix}.$$

Hence $u_3(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_2(u_2(x,y,z))dxdydz$.

Since

$$u_2(u_2(x,y,z)) = \begin{bmatrix} \frac{x^2yz}{32}+\frac{x^3y^2z^2}{1536}+\frac{65x^4y^3z^3}{3538944}+\frac{x^5y^4z^4}{11796480}+\frac{x^6y^4z^5}{235929600}+\frac{x^6y^5z^5}{78643200} \\ +\frac{x^7y^5z^6}{825753600}+\frac{x^7y^6z^6}{1238630400}+\frac{x^8y^6z^7}{9248440320} \\ \\ \frac{xy^2z}{32}+\frac{x^2y^3z^2}{1536}+\frac{65x^3y^4z^3}{3538944}+\frac{x^4y^5z^4}{11796480}+\frac{x^4y^6z^5}{235929600} \\ +\frac{x^5y^6z^5}{78643200}+\frac{x^5y^7z^6}{825753600}+\frac{x^6y^7z^6}{1238630400}+\frac{x^6y^8z^7}{9248440320} \\ \\ \frac{xyz^2}{128}+\frac{x^2y^2z^3}{1536}+\frac{65x^3y^3z^4}{3538944}+\frac{x^4y^4z^5}{11796480}+\frac{x^4y^4z^6}{141557760}+\frac{x^5y^5z^6}{78643200} \\ +\frac{x^5y^5z^7}{550502400}+\frac{x^6y^6z^7}{1238630400}+\frac{x^6y^6z^8}{6606028800} \end{bmatrix}.$$

$$u_3(x,y,z) = \begin{bmatrix} 0.25x+1.6276\times10^{-4}x^3y^2z^2+1.80845\times10^{-5}x^4y^3z^3 \\ +2.29588\times10^{-7}x^5y^4z^4+5.6514\times10^{-10}x^6y^5z^5+2.01836\times10^{-11}x^7y^5z^6 \\ +5.0459\times10^{-11}x^7y^6z^6+3.60421\times10^{-12}x^8y^6z^7+2.05955\times10^{-12}x^8y^7z^7 \\ +2.14536\times10^{-13}x^9y^7z^8 \\ \\ 0.25y+2.60417\times10^{-3}x^2y^3z^2+1.80845\times10^{-5}x^3y^4z^3 \\ +2.29588\times10^{-7}x^4y^5z^4+5.6514\times10^{-10}x^5y^6z^5+2.01836\times10^{-11}x^5y^7z^6 \\ +5.0459\times10^{-11}x^6y^7z^6+3.60421\times10^{-12}x^6y^8z^7+2.05955\times10^{-12}x^7y^8z^7 \\ +2.14536\times10^{-13}x^7y^9z^8 \\ \\ 0.25z+2.60417\times10^{-3}x^2y^2z^3+1.80845\times10^{-5}x^3y^3z^4 \\ +2.29588\times10^{-7}x^4y^4z^5+5.6514\times10^{-10}x^5y^5z^6+4.03672\times10^{-11}x^5y^5z^7 \\ +5.0459\times10^{-11}x^6y^6z^7+6.30737\times10^{-12}x^6y^6z^8+2.05955\times10^{-12}x^7y^7z^8 \\ +3.43258\times10^{-13}x^7y^7z^9 \end{bmatrix}.$$

Thus we may use $u_3(x,y)$ as the approximating solution for the problem.

Example 4.7 Find the solution on the domain $Z = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ of the equation

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = u^2(x, y, z)$$

with the initial conditions

$$u(x,0,0) = u(0, y, 0) = u(x, y, 0) = (x, 0, z)$$

$$= u(0, y, z) = u(0, 0', 0) = [0 \quad 0 \quad 0]^T$$

$$u(0,0,z) = \begin{bmatrix} 0 & 0 & \frac{z}{4} \end{bmatrix}^T$$

or $$g(x,y,z) = \begin{bmatrix} 0 & 0 & \frac{z}{4} \end{bmatrix}^T$$

and $$u_0(x,y,z) = g(x,y,z) = \begin{bmatrix} 0 & 0 & \frac{z}{4} \end{bmatrix}^T.$$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering,

$$\|g(x,y,z)\| = \sqrt{\left(\frac{z}{4}\right)^2} \leq 0.125, \quad K = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.86603.$$

We will obtain

$$\|g(x,y,z)\| + aK \approx 0.125 + \frac{1}{2}.\frac{1}{2}.\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.341506 \leq K.$$

By the equation (3.5.2), we obtain

$$u_1(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u^2(x,y,z)dxdydz$$

$$= g(x,y,z) + \int_0^z\int_0^y\int_0^x u_0(u_0(x,y,z))dxdydz$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{z}{4} \end{bmatrix} + \int_0^z\int_0^y\int_0^x \begin{bmatrix} 0 \\ 0 \\ \frac{z}{4} \end{bmatrix} dxdydz$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{z}{4} + \frac{xyz^2}{8} \end{bmatrix}.$$

Hence $u_2(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_1(u_1(x,y))dxdydz$.

Since

$$u_1(u_1(x,y,z)) = \begin{bmatrix} 0 \\ 0 \\ \frac{\left(\frac{z}{4}+\frac{xyz^2}{8}\right)}{4} + \frac{(0)\,(0)\left(\frac{z}{4}+\frac{xyz^2}{8}\right)^2}{8} \end{bmatrix}.$$

Thus $$u_2(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ \left(\frac{z}{4}+\frac{xyz^2}{32}+\frac{x^2y^2z^3}{384}\right) \end{bmatrix}.$$

Hence $$u_3(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_2(u_2(x,y,z))\,dxdydz.$$

Since

$$u_2\big(u_2(x,y,z)\big) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\left(\dfrac{z}{4} + \dfrac{xyz^2}{32} + \dfrac{x^2y^2z^3}{384}\right)}{4} + 0 \end{bmatrix}.$$

Thus

$$u_{1,3}(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{z}{4} + \dfrac{xyz^2}{32} + \dfrac{x^2y^2z^3}{1536} + \dfrac{x^3y^3z^4}{55296} \end{bmatrix}.$$

Hence $$u_4(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_3(u_3(x,y,z))\,dxdydz.$$

Since

$$u_3\big(u_3(x,y,z)\big) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\left(\dfrac{z}{4} + \dfrac{xyz^2}{32} + \dfrac{x^2y^2z^3}{1536} + \dfrac{x^3y^3z^4}{55296}\right)}{4} + 0 \end{bmatrix}.$$

Thus

$$u_4(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ 0.25z + 6.51042\times10^{-4}x^2y^2z^3 + 4.52112x^3\times10^{-6}x^3y^3z^4 \\ +5.6514\times10^{-8}x^4y^4z^5 \end{bmatrix}.$$

Thus we may use $u_4(x,y)$ as the approximating solution for the problem.

Example 4.8 Find the solution on the domain $Z=[0,\frac{1}{2}]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}]$ of the equation

$$\frac{\partial^3 u}{\partial x\partial y\partial z} = u^3(x,y,z)$$

with the initial conditions

$$u(x,0,0) = u(0,y,0) = u(0,0,z) = u(x,y,0) = u(x,0,z)$$

$$= u(0,y,z) = u(0,0,0) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \end{bmatrix}^T$$

or $\quad g(x,y,z) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \end{bmatrix}^T$

and $\quad u_0(x,y,z) = g(x,y,z) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \end{bmatrix}^T.$

Sol^n^. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering,

$$\|g(x,y,z)\| = \sqrt{\left(\frac{1}{4}\right)^2} \approx 0.25\ , \quad K = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.866025.$$

We obtain

$$\|g(x,y,z)\| + aK \approx 0.25 + \frac{1}{2}.\frac{1}{2}.\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \approx 0.35825 \le K .$$

By the equation (3.5.2), we obtain

$$\begin{aligned} u_1(x,y,z) &= g(x,y,z) + \int_0^z\int_0^y\int_0^x u^2(x,y,z)dxdydz \\ &= g(x,y,z) + \int_0^z\int_0^y\int_0^x u_0(u_0(u_0(x,y,z)))dxdydz \\ &= \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \end{bmatrix} + \int_0^z\int_0^y\int_0^x \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \end{bmatrix} dxdydz \\ &= \begin{bmatrix} \frac{1}{4} + \frac{xyz}{4} \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Hence $u_2(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_1(u_1(u_1(x,y,z)))dxdydz$.

Since

$$u_1(u_1(u_1(x,y,z))) = \begin{bmatrix} \frac{1}{4} + \dfrac{\left(\frac{1}{4} + \frac{xyz}{4}\right)(0)(0)}{4} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \end{bmatrix}.$$

Thus

$$= \begin{bmatrix} \frac{1}{4} + \frac{xyz}{4} \\ 0 \\ 0 \end{bmatrix}.$$

Hence $\quad u_3(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_2(u_2(u_2(x,y,z)))dxdydz$.

Since

$$u_2(u_2(u_2(x,y,z))) = \begin{bmatrix} \frac{1}{4} + \frac{\left(\frac{1}{4} + \frac{xyz}{4}\right)(0)(0)}{4} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \end{bmatrix}.$$

We will obtain

$$u_3(x,y,z) = \begin{bmatrix} \frac{1}{4} + \frac{xyz}{4} \\ 0 \\ 0 \end{bmatrix}.$$

Thus we will obtain $u_k(x,y,z)$ tends $u_3(x,y,z)$ to as k tends to ∞.

Thus the solution of the problem is

$$\begin{bmatrix} \frac{1}{4} + \frac{xyz}{4} \\ 0 \\ 0 \end{bmatrix}.$$

Example 4.9 Find the solution on the domain $Z=[0,\frac{1}{2}]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}]$ of the equation

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=u^3(x,y,z)$$

with the initial conditions

$$u(x,0,0)=u(0,y,0)=u(0,0,z)=u(x,y,0)=u(x,0,z)$$

$$=u(0,y,z)=u(0,0,0)=\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}^T$$

or $$g(x,y,z)=\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}^T$$

and $$u_0(x,y,z)=g(x,y,z)=\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}^T.$$

Sol^n. We can consider the domain and the initial conditions of the problem satisfied the condition of existence theorem.

Considering, $$\|g(x,y,z)\|=\sqrt{\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2}\approx 0.4330127,$$

$$K=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\approx 0.866025.$$

We obtain

$$\|g(x,y,z)\| + aK \approx 0.4330127 + \frac{1}{2}.\frac{1}{2}.\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} \approx 0.5412658 \leq K.$$

By the equation (3.5.2), we obtain

$$\begin{aligned} u_1(x,y,z) &= g(x,y,z) + \int_0^z\int_0^y\int_0^x u^2(x,y,z)dxdydz \\ &= g(x,y,z) + \int_0^z\int_0^y\int_0^x u_0(u_0(u_0(x,y,z)))dxdydz \\ &= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + \int_0^z\int_0^y\int_0^x \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} dxdydz \\ &= \begin{bmatrix} \frac{1}{4}+\frac{xyz}{4} \\ \frac{1}{4}+\frac{xyz}{4} \\ \frac{1}{4}+\frac{xyz}{4} \end{bmatrix}. \end{aligned}$$

Hence $\quad u_2(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_1(u_1(u_1(x,y,z)))dxdydz.$

Since

$$u_1(u_1(u_1(x,y,z)))=\begin{bmatrix} \dfrac{1}{4}+\dfrac{\left(\dfrac{1}{4}+\dfrac{\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)}{4}\right)^3}{4} \\[2ex] \dfrac{1}{4}+\dfrac{\left(\dfrac{1}{4}+\dfrac{\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)}{4}\right)^3}{4} \\[2ex] \dfrac{1}{4}+\dfrac{\left(\dfrac{1}{4}+\dfrac{\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)\left(\dfrac{1}{4}+\dfrac{\left(\frac{1}{4}+\frac{xyz}{4}\right)^3}{4}\right)}{4}\right)^3}{4} \end{bmatrix}.$$

We obtain

$$u_2(x,y,z)=\begin{bmatrix} 0.25+0.2541\times10^{-7}xyz+3.4296\times10^{-6}x^2y^2z^2+1.0653\times10^{-6}x^3y^3z^3 \\ +1.8478\times10^{-7}x^4y^4z^4+1.8306\times10^{-8}x^5y^5z^5+4.5676\times10^{-9}x^6y^6z^6 \\ +6.5506\times10^{-10}x^7y^7z^7+6.9153\times10^{-11}x^8y^8z^8+1.335\times10^{-11}x^9y^9z^9 \\ +1.6006\times10^{-12}x^{10}y^{10}z^{10}+1.7441\times10^{-13}x^{11}y^{11}z^{11}+3.3679\times10^{-14}x^{12}y^{12}z^{12} \\ +5.4109\times10^{-15}x^{13}y^{13}z^{13}+8.7502\times10^{-16}x^{14}y^{14}z^{14}+1.3757\times10^{-16}x^{15}y^{15}z^{15} \\ +1.8024\times10^{-17}x^{16}y^{16}z^{16}+2.4425\times10^{-18}x^{17}y^{17}z^{17}+3.3936\times10^{-19}x^{18}y^{18}z^{18} \\ +4.2292\times10^{-20}x^{19}y^{19}z^{19}+5.9628\times10^{-21}x^{20}y^{20}z^{20}+9.0258\times10^{-22}x^{21}y^{21}z^{21} \\ +1.2502\times10^{-22}x^{22}y^{22}z^{22}+1.6297\times10^{-23}x^{23}y^{23}z^{23}+1.8773\times10^{-24}x^{24}y^{24}z^{24} \\ +1.8534\times10^{-25}x^{25}y^{25}z^{25}+1.6519\times10^{-26}x^{26}y^{26}z^{26}+1.1347\times10^{-27}x^{27}y^{27}z^{27} \\ +3.7681\times10^{-29}x^{28}y^{28}z^{28} \\ \\ 0.25+0.2541\times10^{-7}xyz+3.4296\times10^{-6}x^2y^2z^2+1.0653\times10^{-6}x^3y^3z^3 \\ +1.8478\times10^{-7}x^4y^4z^4+1.8306\times10^{-8}x^5y^5z^5+4.5676\times10^{-9}x^6y^6z^6 \\ +6.5506\times10^{-10}x^7y^7z^7+6.9153\times10^{-11}x^8y^8z^8+1.335\times10^{-11}x^9y^9z^9 \\ +1.6006\times10^{-12}x^{10}y^{10}z^{10}+1.7441\times10^{-13}x^{11}y^{11}z^{11}+3.3679\times10^{-14}x^{12}y^{12}z^{12} \\ +5.4109\times10^{-15}x^{13}y^{13}z^{13}+8.7502\times10^{-16}x^{14}y^{14}z^{14}+1.3757\times10^{-16}x^{15}y^{15}z^{15} \\ +1.8024\times10^{-17}x^{16}y^{16}z^{16}+2.4425\times10^{-18}x^{17}y^{17}z^{17}+3.3936\times10^{-19}x^{18}y^{18}z^{18} \\ +4.2292\times10^{-20}x^{19}y^{19}z^{19}+5.9628\times10^{-21}x^{20}y^{20}z^{20}+9.0258\times10^{-22}x^{21}y^{21}z^{21} \\ +1.2502\times10^{-22}x^{22}y^{22}z^{22}+1.6297\times10^{-23}x^{23}y^{23}z^{23}+1.8773\times10^{-24}x^{24}y^{24}z^{24} \\ +1.8534\times10^{-25}x^{25}y^{25}z^{25}+1.6519\times10^{-26}x^{26}y^{26}z^{26}+1.1347\times10^{-27}x^{27}y^{27}z^{27} \\ +3.7681\times10^{-29}x^{28}y^{28}z^{28} \\ \\ 00.25+0.2541\times10^{-7}xyz+3.4296\times10^{-6}x^2y^2z^2+1.0653\times10^{-6}x^3y^3z^3 \\ +1.8478\times10^{-7}x^4y^4z^4+1.8306\times10^{-8}x^5y^5z^5+4.5676\times10^{-9}x^6y^6z^6 \\ +6.5506\times10^{-10}x^7y^7z^7+6.9153\times10^{-11}x^8y^8z^8+1.335\times10^{-11}x^9y^9z^9 \\ +1.6006\times10^{-12}x^{10}y^{10}z^{10}+1.7441\times10^{-13}x^{11}y^{11}z^{11}+3.3679\times10^{-14}x^{12}y^{12}z^{12} \\ +5.4109\times10^{-15}x^{13}y^{13}z^{13}+8.7502\times10^{-16}x^{14}y^{14}z^{14}+1.3757\times10^{-16}x^{15}y^{15}z^{15} \\ +1.8024\times10^{-17}x^{16}y^{16}z^{16}+2.4425\times10^{-18}x^{17}y^{17}z^{17}+3.3936\times10^{-19}x^{18}y^{18}z^{18} \\ +4.2292\times10^{-20}x^{19}y^{19}z^{19}+5.9628\times10^{-21}x^{20}y^{20}z^{20}+9.0258\times10^{-22}x^{21}y^{21}z^{21} \\ +1.2502\times10^{-22}x^{22}y^{22}z^{22}+1.6297\times10^{-23}x^{23}y^{23}z^{23}+1.8773\times10^{-24}x^{24}y^{24}z^{24} \\ +1.8534\times10^{-25}x^{25}y^{25}z^{25}+1.6519\times10^{-26}x^{26}y^{26}z^{26}+1.1347\times10^{-27}x^{27}y^{27}z^{27} \\ +3.7681\times10^{-29}x^{28}y^{28}z^{28} \end{bmatrix}.$$

Thus we may use $u_2(x,y)$ as the approximating solution for the problem.

CHAPTER 5

CONCLUSION AND SUGGESTIONS

In this chapter, we will make a conclusion and offer some suggestions about the methods of proving the existence and uniqueness theorems in relation to the solution of simple iterative differential equations. Our findings were based on information presented in previous chapters. Chapter 3 provided information on the proof which allowed us to formulate our conclusion and offer suggestions. The results of the solutions in some examples of the simple iterative partial differential equations in chapter 4 also aided in our findings.

5.1 Conclusion

In chapter 3, we applied the fixed - point technique in proving the existence and uniqueness theorems of simple iterative partial differential equations. First, we noticed that the boundaries of functions and in sections 2.1 thorough 2.3 were used in determining the boundary of the integral equation. We obtained that such boundary was $\|g(x)\| + aK \le K$ and it was the essential condition to be used in proving both the existence and uniqueness theorems. We began the proof by defining $s(\rho)$ which was the set of continuous functions and these continuous functions satisfied the Lipschitz condition $\| u(x) - u(\bar{x}) \| \le N\| x - \bar{x} \|$. This was the essential property in the method of proof. We then defined function T in the form of integral function mapping continuous function u and v from $s(\rho)$ to $s(\rho)$. We then proved to obtain that $\| Tu - Tv\| \le aT_m \| u - v\|$ where $aT_m < 1$ and used the Banach contraction principle to conclude that T have unique fixed point. This completed the proof of the existence and uniqueness theorems.

In order to find the power series solutions of the problems several steps were taken. First, we defined the approximating sequence. Following the presentation of this sequence we proved that such sequence $u_{k+1}(x) = g(x) + \int_0^x u_k^m(t)\,dt$, $k = 0,1,2,\ldots$ converged uniformly to the (unique) solutions of the problems. Consideration of examples in chapter 4 allowed us to notice that if the initial conditions and domains of the problems satisfied the conditions of the existence theorem, we then could find the solutions of problems. These solutions of the problems came about by approximating sequences in the equation (3.3.5) which converged uniformly to the solutions.

5.2 Suggestions

The examples provided in chapter 4, such as examples 4.1, 4.5 and 4.8, allowed us to see that solutions of some problems converge uniformly to the first sequence. Based on this finding we got the idea to extend the domains of the problems to infinity, that is, we may study the global existence and uniqueness for the solutions of simple iterative partial differential equations. Consideration must also be given in finding how large the values $\|g(x)\|$ are possible. This also offers us the opportunity to study more about applications related to these problems.

We will give you some examples in finding solution of problems on extending domain.

Example 5.1 Find the solution on the domain $Z = [0,2]\times[0,2]$ of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = u^2(x,y)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = [1 \;\; 1]^T$$

or $$g(x,y) = [1 \;\; 1]^T$$

and $$u_0(x,y) = g(x,y) = [1 \;\; 1]^T.$$

Sol^n. By the equation (3.10), we obtain

$$u_1(x,y) = g(x,y) + \int_0^y\int_0^x u^2(x,y)dxdy$$

$$= g(x,y) + \int_0^y\int_0^x u_0(u_0(x,y))dxdy$$

$$= \begin{bmatrix}1\\1\end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix}1\\1\end{bmatrix}dxdy$$

$$= \begin{bmatrix}1+xy\\1+xy\end{bmatrix}.$$

Hence $$u_2(x,y) = g(x,y) + \int_0^y\int_0^x u_1(u_1(x,y))dxdy.$$

Since

$$u_1(u_1(x,y)) = \begin{bmatrix}1+(1+xy)(1+xy)\\1+(1+xy)(1+xy)\end{bmatrix}$$

$$= \begin{bmatrix}2+2xy+x^2y^2\\2+2xy+x^2y^2\end{bmatrix}.$$

Thus

$$u_2(x,y) = \begin{bmatrix}1\\1\end{bmatrix} + \int_0^y\int_0^x \begin{bmatrix}2+2xy+x^2y^2\\2+2xy+x^2y^2\end{bmatrix}dx$$

$$= \begin{bmatrix}1+2xy+\dfrac{x^2y^2}{2}+\dfrac{x^3y^3}{9}\\[2ex]1+2xy+\dfrac{x^2y^2}{2}+\dfrac{x^3y^3}{9}\end{bmatrix}.$$

Hence $$u_3(x,y) = g(x,y) + \int_0^y\int_0^x u_2(u_2(x,y))dxdy.$$

Since

$$u_2(u_2(x,y)) = \begin{bmatrix} 1+2(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})+ \\ \frac{(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^2(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^2}{2} \\ \frac{(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^3(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^3}{9} \\ \\ 1+2(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9}) \\ \frac{(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^2(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^2}{2} \\ \frac{(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^3(1+2xy+\frac{x^2y^2}{2}+\frac{x^3y^3}{9})^3}{9} \end{bmatrix}.$$

We obtain

$$u_3(x,y) = \begin{bmatrix} 3.61111xy+3.33333x^2y^2+3.33333x^3y^3+2.99074x^4y^4+2.58519x^5y^5 \\ +1.90844x^6y^6+1.18176x^7y^7+0.602109x^8y^8+0.254276x^9y^9 \\ +8.92966\times10^{-2}x^{10}y^{10}+2.62504\times10^{-2}x^{11}y^{11}+6.461\times10^{-3}x^{12}y^{12} \\ +1.33008\times10^{-3}x^{13}y^{13}+2.26955\times10^{-4}x^{14}y^{14}+3.16827\times10^{-5}x^{15}y^{15} \\ +3.51712\times10^{-6}x^{16}y^{16}+2.97878\times10^{-7}x^{17}y^{17}+1.74229\times10^{-8}x^{18}y^{18} \\ +5.79156\times10^{-10}x^{19}y^{19} \\ \\ 3.61111xy+3.33333x^2y^2+3.33333x^3y^3+2.99074x^4y^4+2.58519x^5y^5 \\ +1.90844x^6y^6+1.18176x^7y^7+0.602109x^8y^8+0.254276x^9y^9 \\ +8.92966\times10^{-2}x^{10}y^{10}+2.62504\times10^{-2}x^{11}y^{11}+6.461\times10^{-3}x^{12}y^{12} \\ +1.33008\times10^{-3}x^{13}y^{13}+2.26955\times10^{-4}x^{14}y^{14}+3.16827\times10^{-5}x^{15}y^{15} \\ +3.51712\times10^{-6}x^{16}y^{16}+2.97878\times10^{-7}x^{17}y^{17}+1.74229\times10^{-8}x^{18}y^{18} \\ +5.79156\times10^{-10}x^{19}y^{19} \end{bmatrix}.$$

Thus we may use $u_3(x,y)$ as the approximating solution for the problem.

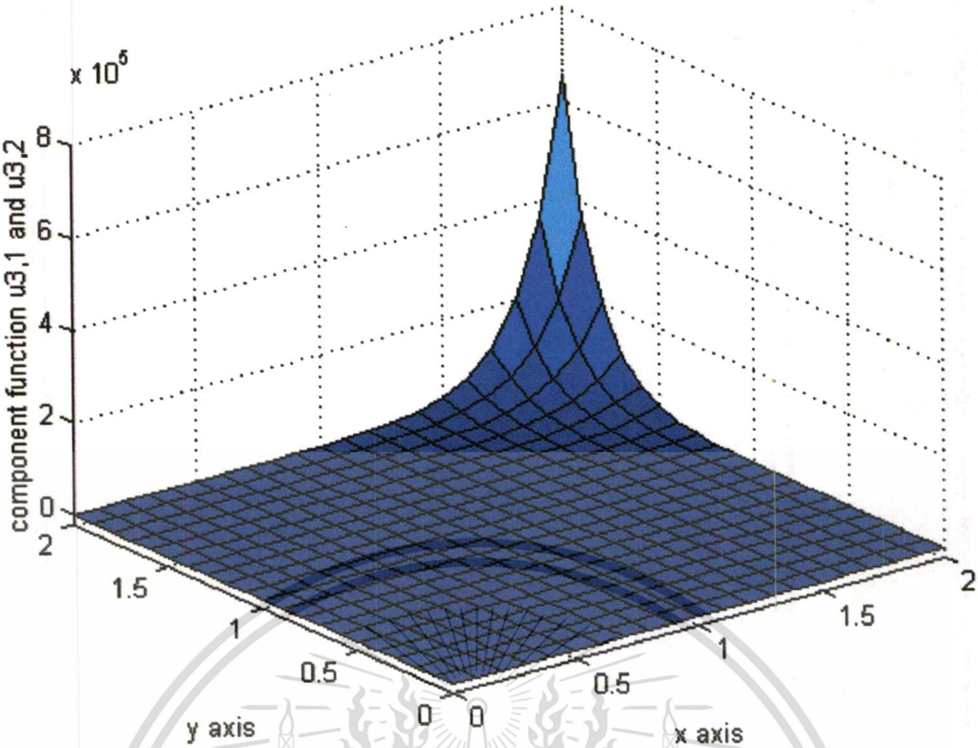

Figure 4.13 Graph of component functions $u_{3,1}$ and $u_{3,2}$ of the power series solution u_3.

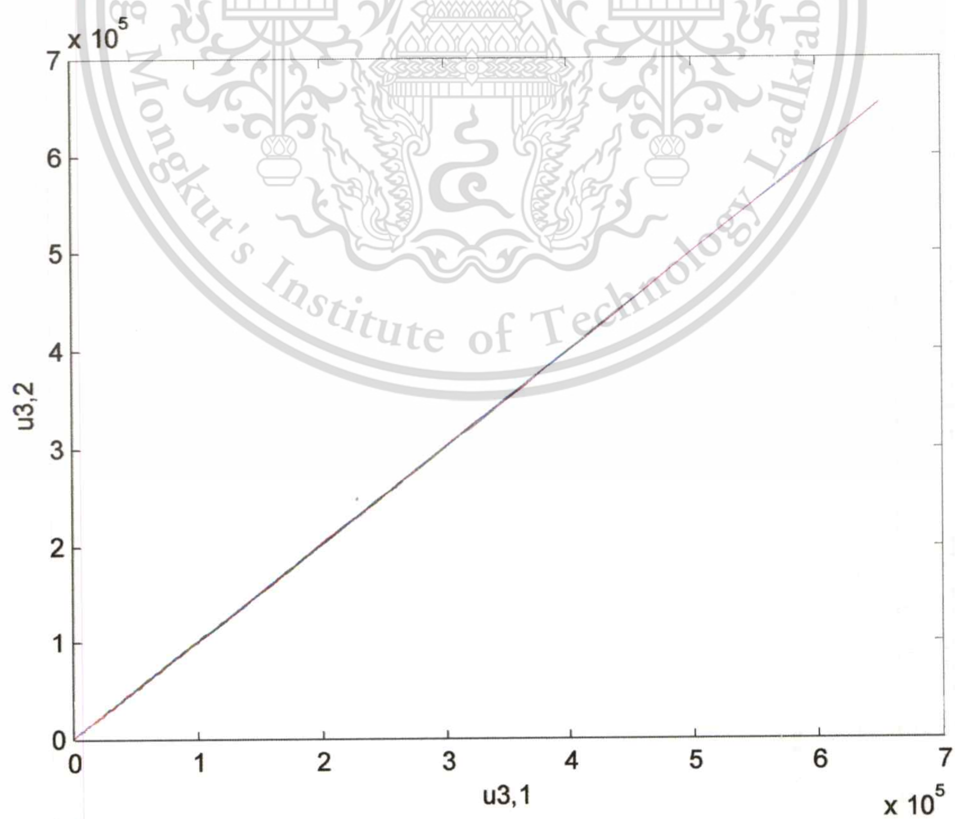

Figure 4.14 Graph of the power series solution u_3 in the example 5.1.

Example 5.2 Find the solution on the domain $Z = [0,3]\times[0,3]\times[0,3]$ of the equation

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = u^2(x,y,z)$$

with the initial conditions

$$u(x,0,0) = u(0,y,0) = u(0,0,z) = u(x,y,0) = u(x,0,z)$$

$$= u(0,\ y,z) = u(0,0,0) = [2\ \ 2\ \ 2]^T$$

or $\qquad g(x,y,z) = [2\ \ 2\ \ 2]^T$

and let $\qquad u_0(x,y,z) = g(x,y,z) = [2\ \ 2\ \ 2]^T.$

Sol^n.

By the equation (3.5.2), we obtain

$$u_1(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u^2(x,y,z)dxdydz$$

$$= g(x,y,z) + \int_0^z\int_0^y\int_0^x u_0(u_0(x,y,z))dxdydz$$

$$= \begin{bmatrix}2\\2\\2\end{bmatrix} + \int_0^z\int_0^y\int_0^x \begin{bmatrix}2\\2\\2\end{bmatrix} dxdydz$$

$$= \begin{bmatrix}2+2xyz\\2+2xyz\\2+2xyz\end{bmatrix}.$$

Hence $\qquad u_2(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_1(u_1(x,y,z))dxdydz.$

Since

$$u_1(u_1(x,y,z)) = \begin{bmatrix} 2+(2+xyz)(2+xyz)(2+xyz) \\ 2+(2+xyz)(2+xyz)(2+xyz) \\ 2+(2+xyz)(2+xyz)(2+xyz) \end{bmatrix}$$

$$= \begin{bmatrix} 10+12xyz+6x^2y^2z^2+x^3y^3z^3 \\ 10+12xyz+6x^2y^2z^2+x^3y^3z^3 \\ 10+12xyz+6x^2y^2z^2+x^3y^3z^3 \end{bmatrix}.$$

We will obtain

$$u_2(x,y,z) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \int_0^z\int_0^y\int_0^x \begin{bmatrix} 10+12xyz+6x^2y^2z^2+x^3y^3z^3 \\ 10+12xyz+6x^2y^2z^2+x^3y^3z^3 \\ 10+12xyz+6x^2y^2z^2+x^3y^3z^3 \end{bmatrix} dxdydz$$

$$= \begin{bmatrix} 2+10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64} \\ 2+10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64} \\ 2+10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64} \end{bmatrix}.$$

Hence $\quad u_3(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x u_2(u_2(x,y,z))dxdydz.$

Since

$$u_2(u_2(x,y,z))=\begin{bmatrix} \begin{array}{l} 2+10\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right) \\ \left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)+\frac{3}{2}\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)^2 \\ \left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)^2\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)^2 \end{array} \\ \\ \begin{array}{l} 2+10\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right) \\ \left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)+\frac{3}{2}\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)^2 \\ \left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)^2\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right) \end{array} \\ \\ \begin{array}{l} 2+10\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right) \\ \left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)+\frac{3}{2}\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)^2 \\ \left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right)^2\left(10xyz+\frac{3x^2y^2z^2}{2}+\frac{2x^3y^3z^3}{9}+\frac{x^4y^4z^4}{64}\right) \end{array} \end{bmatrix}.$$

We will obtain

$$u_3(x,y)=\begin{bmatrix} 2+10000x^3y^3z^3+4500x^4y^4z^4+1341.67x^5y^5z^5+1.50028\times10^6x^6y^6z^6 \\ +1.35004\times10^6x^7y^7z^7+706255x^8y^8z^8+265313x^9y^9z^9+78048.6x^{10}y^{10}z^{10} \\ +18826.7x^{11}y^{11}z^{11}+3819.6x^{12}y^{12}z^{12}+662.272x^{13}y^{13}z^{13}+99.0444x^{14}y^{14}z^{14} \\ +12.8322x^{15}y^{15}z^{15}+1.44123x^{16}y^{16}z^{16}+0.139847x^{17}y^{17}z^{17}+1.16352\times10^{-2}x^{18}y^{18}z^{18} \\ +8.19334\times10^{-4}x^{19}y^{19}z^{19}+4.78052\times10^{-5}x^{20}y^{20}z^{20}+2.23376\times10^{-6}x^{21}y^{21}z^{21} \\ +7.88002\times10^{-8}x^{22}y^{22}z^{22}+1.86265\times10^{-9}x^{23}y^{23}z^{23}+2.18279\times10^{-11}x^{24}y^{24}z^{24} \\ \\ 2+10000x^3y^3z^3+4500x^4y^4z^4+1341.67x^5y^5z^5+1.50028\times10^6x^6y^6z^6 \\ +1.35004\times10^6x^7y^7z^7+706255x^8y^8z^8+265313x^9y^9z^9+78048.6x^{10}y^{10}z^{10} \\ +18826.7x^{11}y^{11}z^{11}+3819.6x^{12}y^{12}z^{12}+662.272x^{13}y^{13}z^{13}+99.0444x^{14}y^{14}z^{14} \\ +12.8322x^{15}y^{15}z^{15}+1.44123x^{16}y^{16}z^{16}+0.139847x^{17}y^{17}z^{17}+1.16352\times10^{-2}x^{18}y^{18}z^{18} \\ +8.19334\times10^{-4}x^{19}y^{19}z^{19}+4.78052\times10^{-5}x^{20}y^{20}z^{20}+2.23376\times10^{-6}x^{21}y^{21}z^{21} \\ +7.88002\times10^{-8}x^{22}y^{22}z^{22}+1.86265\times10^{-9}x^{23}y^{23}z^{23}+2.18279\times10^{-11}x^{24}y^{24}z^{24} \\ \\ 2+10000x^3y^3z^3+4500x^4y^4z^4+1341.67x^5y^5z^5+1.50028\times10^6x^6y^6z^6 \\ +1.35004\times10^6x^7y^7z^7+706255x^8y^8z^8+265313x^9y^9z^9+78048.6x^{10}y^{10}z^{10} \\ +18826.7x^{11}y^{11}z^{11}+3819.6x^{12}y^{12}z^{12}+662.272x^{13}y^{13}z^{13}+99.0444x^{14}y^{14}z^{14} \\ +12.8322x^{15}y^{15}z^{15}+1.44123x^{16}y^{16}z^{16}+0.139847x^{17}y^{17}z^{17}+1.16352\times10^{-2}x^{18}y^{18}z^{18} \\ +8.19334\times10^{-4}x^{19}y^{19}z^{19}+4.78052\times10^{-5}x^{20}y^{20}z^{20}+2.23376\times10^{-6}x^{21}y^{21}z^{21} \\ +7.88002\times10^{-8}x^{22}y^{22}z^{22}+1.86265\times10^{-9}x^{23}y^{23}z^{23}+2.18279\times10^{-11}x^{24}y^{24}z^{24} \end{bmatrix}.$$

Thus we may use $u_3(x,y)$ as the approximating solution for the problem.

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APPENDIX

Definition (Contraction). Let $\langle M, \rho \rangle$ be a metric space. If $T : M \to M$, we say that T is a contraction on M if there exist $\alpha \in R$ with $0 \le \alpha < 1$ such that

$$\rho(Tx, Ty) \le \alpha\, \rho(x, y) \qquad\qquad (x, y) \in M.$$

Theorem (Banach Contraction Principle). Let $\langle M, \rho \rangle$ be a complete metric space. If T is a contraction on M, then there is one and only one point x in M such that $Tx = x$. (This is often stated as " T has precisely one fixed point.")

Definition (Cauchy Sequence, Completeness). A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy (or fundamental) if for every $\varepsilon > 0$ there is $N = N(\varepsilon)$ an such that $d(x_m, x_n) < \varepsilon$ for every $m, n > N$.

The space X is said to be complete if every Cauchy sequence in X converges.

Definition (Vector Space). A vector space (or linear space) over a field K is a nonempty set X of elements $x, y, \ldots$ (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of K.

Definition (Normed space, Banach space). A normed space X is a vector space with a norm defined on it. A Banach space is a complete normed space.

Definition (Converges pointwise). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Z .We say that $\{f_n\}_{n=1}^{\infty}$ converges to the function f on E if, for each $x \in E$, given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \ge N).$$

Definition (Converges uniformly). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Z. We say that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function f on E if given $\varepsilon > 0$, there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad (n \geq N \; ; \; x \in E).$$

Lemma (Arzela – Ascoli).

Let $F = \{f(t)\}$ be an infinite family of (real) continuous functions on I. Assume that :

1. F is uniformly bounded on I - that is, there exists $M > 0$ such that $|f(t)| \leq M$, for any $t \in I$ and any $f \in F$.
2. F is equicontinuous on I - that is, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(t) - f(s)| < \varepsilon \quad \text{for} \quad |t - s| < \delta, \qquad f \in F.$$

Then F is relatively compact in the sense of uniform convergence on I - that is, any infinite part of F contains a sequence that is uniformly convergent on I.

Proof. Let us consider the set $\{r_n\}$ of all rational numbers in $[a, b]$. It is well known that this set is denumerable and, consequently, we can use the sequence notation.

Assume now that G is an infinite part of F. Without loss of generality, we can assume that G is a sequence of functions : $G = \{f_n(t)\}$. Now let us consider the sequence of numbers $\{f_n(r_1)\}$. We have $|f_n(r_1)| \leq M$, and from this sequence (which is bounded!) we can extract a convergent subsequence : say, $\{f_{1n}(r_1)\}$. It is known that the sequence of functions $\{f_{1n}(t)\}$ is convergent at $t = r_1$.

Applying the same procedure as before to $\{f_{1n}(t)\}$, we can find a subsequence – say, $\{f_{2n}(t)\}$ - that converges at $t = r_2$. As a subsequence of $\{f_{1n}(t), f_{2n}(t)\}$ convergence also at $t = r_1$. We can proceed further in the same way and obtain the sequence $\{f_{kn}(t)\}$, $k > 2$, with the following properties :

(a) $\{f_{kn}(t)\}$ is a subsequence of $\{f_{hn}(t)\}$, whenever $h < k$;

(b) $\{f_{kn}(t)\}$ is convergent at $t = r_j$, $j = 1, 2, \ldots, k$.

Let us consider now the sequence of functions $\{f_{nn}(t)\}$. Excepting a finite number of terms, it is a subsequence of any sequence $\{f_{kn}(t)\}$, $k=1,2,\ldots$. This follows easily if we take into account the properties (a) and (b). Hence, $\{f_{nn}(t)\}$ converges at $t = r_k$, $k = 1,2,\ldots$; in other words, it converges for any rational $t \in I$.

We shall prove that $\{f_{nn}(t)\}$ is uniformly convergent on I. Indeed let $\varepsilon > 0$ be arbitrary and consider the corresponding number $\delta = \delta(\varepsilon/3)$. Consider also a division $a = t_0 < t_1 < \ldots < t_p = b$ of I, such that $\max(t_q - t_{q-1}) < \delta$. We can obviously assume that all t_q, $1 \le q < p$, are rational. It is clear that for any $t \in I$, there exists at least one t_q, such that $|t - t_q| < \delta$. But

$$\begin{aligned}|f_{nn}(t) - f_{mm}(t)| &\le |f_{nn}(t) - f_{mm}(t_q)| + |f_{nn}(t_q) - f_{mm}(t_q)| + |f_{mm}(t_q) - f_{mm}(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

whenever $n, m \ge N(\varepsilon)$. Indeed, from $|t - t_q| < \delta$, it follows that

$$|f_{nn}(t) - f_{nn}(t_q)| < \frac{\varepsilon}{3} \qquad \text{and} \qquad |f_{mm}(t_q) - f_{mm}(t)| < \frac{\varepsilon}{3}$$

for all n, m. On the other hand, the sequences $\{f_{nn}(t_q)\}$, $1 \le q < p$, are convergent, and we can write

$$|f_{nn}(t_q) - f_{mm}(t_q)| < \frac{\varepsilon}{3} \qquad 1 \le q < p$$

for $n, m \ge N(\varepsilon)$. Summing up the last considerations, we get

$$|f_{nn}(t) - f_{mm}(t)| < \varepsilon \qquad t \in I, \qquad n, m \ge N(\varepsilon)$$

which means that $\{f_{nn}(t)\}$ is a uniformly convergent sequence. The lemma is thus proved.

Corollary (Arzela – Ascoli).

Let $F = \{f(t)\}$ be an infinite set of continuous vector functions from $I = [a, b]$ into R^n, satisfying the following conditions :

1. It is uniformly bounded on I - that is, $\|f(t)\| \leq M$, for any $t \in I$, and $f \in F$, for some $M > 0$.
2. It is equicontinuous on I - that is, for every $\varepsilon > 0$, we can find $\delta > 0$, such that

$$|f(t) - f(s)| < \varepsilon \quad \text{for} \quad |t - s| < \delta, \qquad f \in F.$$

Then F is relatively compact in the sense of uniform convergence on I.

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