MORE ON MODIFIED NEWTON METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

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บทคัดย่อ

เนื้อหาของวิทยานิพนธ์ฉบับนี้ เป็นการเสนอแนวทางใหม่ซึ่งจัดอยู่ในประเภทเดียวกันกับการปรับปรุงวิธีนิวตันสำหรับการหาผลเฉลยของระบบสมการไม่เชิงเส้นอยู่ในรูป

$$f(x) = 0 \tag{1}$$

เมื่อ $x \in R^n$, f เป็นฟังก์ชันจากเซตย่อยของ R^n ไปยังเซตย่อยของ R^n

สมการ (1) อาจเขียนให้อยู่ในรูป

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned} \tag{2}$$

พร้อมทั้งแสดงตัวอย่างปัญหาทางด้านวิทยาศาสตร์และเทคโนโลยี และทำการหาผลเฉลยของตัวอย่างโดยวิธีการใหม่ที่นำเสนอ

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ABSTRACT

In this thesis we will find more formulas of the same type as the Modified Newton method for solving systems of nonlinear equations of the form

$$f(x) = 0 \tag{1}$$

where $x \in R^n$, f is a function from a subset of R^n to a subset of R^n

The equation (1) may be written as

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned} \tag{2}$$

and we will find some applications in science and technology and solve for their solution from new methods.

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CHAPTER 1

INTRODUCTION

A system of n equations in n unknowns $x_1, x_2, \ldots, x_n$ is called nonlinear if one or more of the equations is nonlinear. By bringing all nonzero terms to the left hand side of all equations, any nonlinear $n \times n$ system can be put in the general form

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned} \tag{1.1}$$

or simply

$$\begin{aligned} f_1(x) &= 0 \\ f_2(x) &= 0 \; , \\ &\vdots \\ f_n(x) &= 0 \end{aligned} \qquad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The system (1.1) will be abbreviated as simply $f(x) = 0$. A vector $x = [x_1 \; x_2 \; \cdots \; x_n]^T$ that satisfies $f(x) = 0$ will be called a root or solution of the nonlinear system (1.1).

For this purpose we use iterative methods that produce the desired solution as the limit of a sequence of approximations. We shall formulate the Newton's method that is theoretical basis for the study of the convergence behavior of the sequence of approximated values towards the solution is importance for the usefulness and the efficiency of a method. With this point of view, some methods for determining a solution of a nonlinear equation are developed and analyzed.

The method for solving the systems of nonlinear equations shown as above is very important in the study of applied mathematics. Many problems in science and technology require the method for solving the systems of nonlinear equations.

The method of finding the solutions of systems of nonlinear equations is used in many applications of various branches in science and technology. For example; a turbulent flow of fluids in an interconnected network [1], the flow rate V from one node to another is about proportional to the square root of the difference in pressures at the nodes (thus fluid flow differs from flow of electrical current in a network in that nonlinear equations result). For the conduits in Figure 1.1 the pressure at each node. The values of b represents conductance factors in the relation $v_{ij} = b_{ij}(p_i - p_j)^{1/2}$.

These equations can be set up for the pressures at each node :

At node 1: $0.3\sqrt{500 - p_1} = 0.2\sqrt{p_1 - p_2} + 0.1\sqrt{p_1 - p_3}$;

node 2: $0.2\sqrt{p_1 - p_2} = 0.1\sqrt{p_2 - p_4} + 0.2\sqrt{p_2 - p_3}$;

node 3: $0.1\sqrt{p_1 - p_3} + 0.2\sqrt{p_2 - p_3} = 0.1\sqrt{p_3 - p_4}$;

node 4: $0.1\sqrt{p_2 - p_4} + 0.1\sqrt{p_3 - p_4} = 0.2\sqrt{p_4 - 0}$.

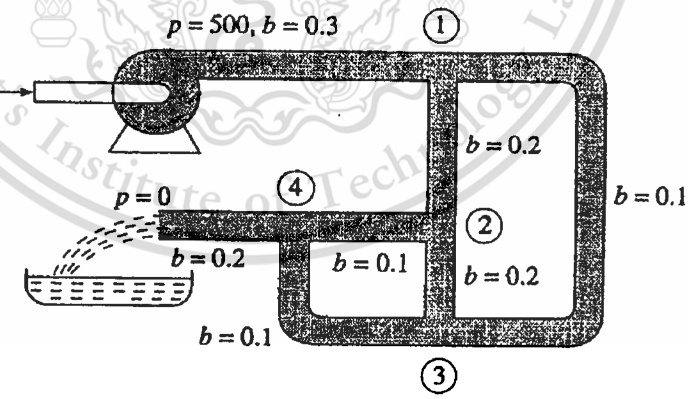

Figure 1.1 A turbulent flow of fluids in an interconnected network.

Another one is the problem of the hanging chain [2], which has been used to test the minimizer. Given a chain of 8 sticks of unit weight and unit length, we are required to find the shape it will assume when suspended between 2 points spaced 6 required to find the shape it will assume when suspended between 2 points spaced 6 units horizontally apart. In other words, we must find the angles α, β, γ and δ see Figure 1.2. Solving this by the 'energy method' involves finding an expression for the chain's energy, depending on α, β and γ let the nonlinear minimizer do the rest. A different approach comes from statics. We introduce a 5 th unknown, the horizontal component f of the force going through the chain, which is the same in all joints. Then there is this equations system for α, β, γ, δ and f :

$$2f\ sin(\alpha) - 7\ cos(\alpha) = 0$$

$$2f\ sin(\beta) - 5\ cos(\beta) = 0$$

$$2f\ sin(\gamma) - 3\ cos(\gamma) = 0$$

$$2f\ sin(\delta) - \ cos(\delta) = 0$$

$$cos(\alpha) + cos(\beta) + cos(\gamma) + cos(\delta) = 3$$

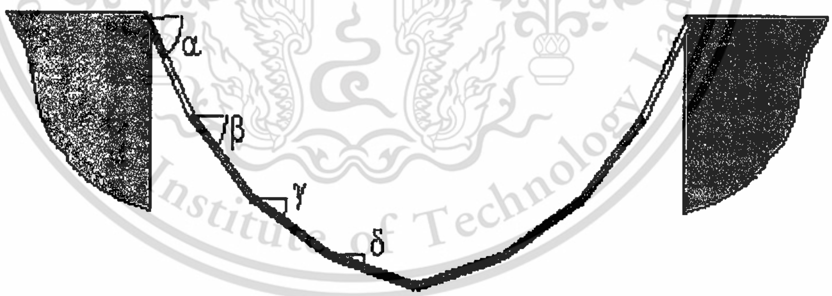

Figure 1.2 The problem of the hanging chain.

In order to find the solutions of these problems, we need to solve the systems of nonlinear equations and their numerical solution that will be stated in chapter 4.

There are several methods for solving systems of nonlinear equations that satisfied (1.1), one of the basic iterative procedures for approximating a solution of (1.1) is **Newton's method**:

$$x^{k+1} = x^k - \left[f'(x^k)\right]^{-1} f(x^k) \qquad , \quad k = 0,1,2,\ldots \tag{1.2}$$

where $f'(x)$ is the Jacobian matrix of $f(x)$ denoted by $J(x)$

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{1.3}$$

or

$$J(x) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}, \tag{1.4}$$

and f_{ij} denotes the partial derivative of $f_i(x)$ with respect to the j^{th} variable and evaluated at x. So, we can write the new form for an approximation solution of (1.1) by using the Newton's method as below

$$x^{k+1} = x^k - \left[J(x^k)\right]^{-1} f(x^k) \qquad , \quad k = 0,1,2,\ldots . \tag{1.5}$$

There are many mathematicians who gave their best efforts to study the systems of nonlinear equations base on Newton's method by letting matrix $A(x)$ be an invertible operator to replace the Jacobian matrix in Newton's method, this new method is called **Newton-like method**. The form of an approximated solution of (1.1) by using Newton-like method is

$$x^{k+1} = x^k - \left[A(x^k)\right]^{-1} f(x^k) \qquad , \quad k = 0,1,2,\ldots . \tag{1.6}$$

Podisuk, M. [3] is one of many mathematicians who studied the systems of nonlinear equations. He published his work in 1991, which is a special kind of the Newton-like method, which modified from Newton's method by letting the matrix $H(x)$ to replace the Jacobian matrix in Newton's method. Then the approximating solution of (1.1) by this **Modified Newton method** is

$$x^{k+1} = x^k - \left[H(x^k)\right]^{-1} f(x^k) \qquad , \; k = 0,1,2,\ldots \tag{1.7}$$

where $H(x)$ denotes the diagonal matrix

$$H(x) = \begin{bmatrix} f_{11} & 0 & \cdots & 0 \\ 0 & f_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & f_{nn} \end{bmatrix}, \tag{1.8}$$

and f_{ii} is the partial derivative of $f_i(x)$ with respect to the i^{th} variable.

In this thesis, we will find more formulas of the same type as the Modified Newton method for solving systems of nonlinear equations. For solving systems of nonlinear equations, in chapter 2, we will discuss the important theorems concern with the Modified Newton method. We then will introduce new methods that are developed from the Modified Newton method for solving the systems in chapter 3. In chapter 4, we will find the approximating solutions of examples including application problems. Furthermore, for the purpose of comparison result from these methods, we then also give tables of results in that section. Finally, we will make a conclusion and suggestions in chapter 5.

CHAPTER 2

LITERATURE REVIEWS

In this chapter, we will state about definitions, The Kantorovich Theorem, The Banach Lemma and other theorems and lemmas, which concern with the Modified Newton method for solving systems of nonlinear equations, which will be used in the later chapter.

We shall begin our consideration of multivariable problems by discussing the systems of nonlinear equations in real variables. It is

$$\text{Given } f : X \to X$$
$$\text{find } x \in X \text{ for which } f(x) = 0 \tag{2.1}$$

where $X \equiv R^n$ is n-dimensional real vector space, and f is assumed to be continuously differentiable.

Of course, (2.1) is just the standard way of denoting a system of n-nonlinear equations in n-unknowns, with the convention that right-hand side of each equation is zero, and throughout this study we let

$D \subset X$ denotes the domain of $f(x)$,

D_0 denotes closed convex subset of D,

$N(x,t)$ denotes the open neighborhood of radius t around x, i.e.

$$N(x,t) = \{y \in X : \|x - y\| < t\}$$

$L(X,Y)$ denotes a real Banach space,

$f \in Lip_k(D)$ if $\|f(x) - f(y)\| \le K\|x - y\|$ for some constant K,

$\{x_k\}$ denotes sequence of vectors in R^n,

$B(x) = f'(x) - A(x)$,

and $M(x) = f'(x) - H(x)$.

In the analysis of Newton's method, it will be necessary to assume that the Jacobian matrix,

$$J(x) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \qquad (2.2)$$

where f_{ij} denotes the partial derivative of $f_i(x\ )$ with respect to the j^{th} variable and evaluated at x, is at least continuous at the solution x^* ; that is

$$\left\|f'(x^* + h) - f'(x^*)\right\| \to 0 \quad \text{as} \quad h \to 0 .$$

On occasion, it will also be useful to assume, instead of continuity of f', that the derivative satisfies the following property.

Definition 2.1 The mapping $f : R^n \to R^n$ is (Totally or Frechet) differentiable at x if the Jacobian matrix (2.2) exists at x and

$$\lim_{h \to 0} \frac{\left\|f(x+h) - f(x) - f'(x)h\right\|}{\|h\|} = 0 . \qquad (2.3)$$

Note that if $n = 1$, Definition 2.1 reduces to the usual definition of differentiability. Note also that if f is differentiable at x, then f is continuous at x; this follows from the inequality

$$\left\|f(x+h) - f(x)\right\| \le \left\|f(x+h) - f(x) - f'(x)h\right\| + \left\|f'(x)h\right\| . \qquad (2.4)$$

Finally, we note that it is possible to say that if the Jacobian matrix is continuous at x then f is differentiable at x.

One of the basic tools of nonlinear analysis is the mean value theorem. If f is a differentiable function from R^1 to R^1, this state that

$$f(x) - f(y) = f'(z)(x - y) \tag{2.5}$$

for some point z between x and y. Unfortunately, this result does not extend verbatim to mappings from R^n to R^n. However, we are able to prove some results that are often just as useful. First, we define the integral $\int_a^b G(t)dt$ of a mapping $G : [a, b] \subset R^1 \to R^n$ as the vector with components $\int_a^b g_i(t)dt,\ i = 1,2,\ldots,n$ where $g_1, g_2, \ldots, g_n$ are the components of G. Thus, for example, if $f : R^n \to R^n$ the relation

$$f(y) - f(x) = \int_0^1 f'(x + t(y - x))(y - x)dt \tag{2.6}$$

is equivalent to

$$f_i(y) - f_i(x) = \int_0^1 \sum_{j=1}^{n} f_{ij}(x + t(y - x))(y_j - x_j)dt \tag{2.7}$$

for $i = 1,2,\ldots,n$.

For $n = 1$, (2.6) is simply the fundamental theorem of the integral calculus. Hence the next result is a natural extension of that theorem to n-dimensions.

Lemma 2.1 Assume that $f : R^n \to R^n$ is continuously differentiable on a convex set $D \subset R^n$. Then for any $x, y \in D$, (2.6) holds.

Proof: For fixed $x, y \in D$ define the functions $g_i : [0,1] \subset R^1 \to R^1$ by

$$g_i(t) = f_i(x + t(y - x)), \qquad t \in [0,1],\ i = 1,2,\ldots,n. \tag{2.8}$$

By the convexity of D, it follows that g_i is continuously differentiable on $[0,1]$ and thus the fundamental theorem of the integral calculus implies that

$$g_i(1) - g_i(0) = \int_0^1 g_i'(t)dt. \tag{2.9}$$

But a simple calculation shown that

$$g_i'(t) = \sum_{j=1}^{n} f_{ij}(x + t(y - x))(y_j - x_j) \tag{2.10}$$

so that (2.9) is equivalent to (2.7).

For the next result, we first need a lemma on integration.

Lemma 2.2 Assume that $G : [\,a, b\,] \subset R^1 \to R^n$ is continuous. Then

$$\left\| \int_a^b G(t)dt \right\| \le \int_a^b \|G(t)\| dt. \tag{2.11}$$

proof: Since any norm is a continuous function on R^n, both integrals of (2.11) exist and therefore any $\varepsilon > 0$ there is a partition $a < t_0 < \ldots < t_p < b$ of $[\,a, b\,]$ such that

$$\left\| \int_a^b G(t)dt - \sum_{i=1}^{p} G(t_i)(t_i - t_{i-1}) \right\| < \varepsilon \tag{2.12}$$

and

$$\left| \int_a^b \|G(t)\|dt - \sum_{i=1}^{p} \|G(t_i)\|(t_i - t_{i-1}) \right| < \varepsilon .$$

Hence

$$\left\| \int_a^b G(t)dt \right\| \le \left\| \sum_{i=1}^{p} G(t_i)(t_i - t_{i-1}) \right\| + \varepsilon$$

$$\le \sum_{i=1}^{p} \|G(t_i)\|(t_i - t_{i-1}) + \varepsilon \le \int_a^b \|G(t)\|dt + 2\varepsilon$$

and, since ε was arbitrary, (2.11) must be valid.

By means of lemma 2.1 and 2.2 we can prove the following useful alternative of the mean value theorem.

Theorem 2.3 Assume that $f : R^n \to R^n$ is continuously differentiable on the convex set $D \subset R^n$. Then for any $x, y \in D$,

$$\|f(x) - f(y)\| \le \sup_{0 \le t \le 1} \|f'(x + t(y - x))\| \|y - x\| . \tag{2.13}$$

proof: By lemma 2.1 and 2.2 we have

$$\|f(x) - f(y)\| = \left\| \int_0^1 f'(x + t(y - x))(y - x)dt \right\|$$

$$\le \int_0^1 \|f'(x = t(y - x))\| \|y - x\|dt$$

$$\le \sup_{0 \le t \le 1} \|f'(x + t(y - x))\| \int_0^1 \|y - x\|dt$$

which is (2.13).

Lemma 2.4 Let $f' \in Lip_k(D_0)$ and let $x, y \in D_0$, then

$$\|f(y) - f(x) - f'(x)(y - x)\| \leq \frac{1}{2} K \|x - y\|^2 . \tag{2.14}$$

proof: From $f : R^n \to R^n$ is differentiable and satisfied $f' \in Lip_k(D_0)$, that means

$\|f'(x) - f'(y)\| \leq K\|x - y\|$ for all x, y in closed convex subset D_0 of D.

Since $f'(x)(y - x)$ is constant with respect to the integration, we have

$$f(y) - f(x) - f'(x)(y - x) = \int_0^1 \left[(f'(x + t(y - x))(y - x)) - f'(x)(y - x) \right] dt$$

$$= \int_0^1 [f'(x + t(y - x) - f'(x)](y - x)\, dt$$

Hence, the result follows by taking norms of both sides,

$$\|f(y) - f(x) - f'(x)(y - x)\| = \left\| \int_0^1 [f'(x + t(y - x) - f'(x)](y - x) dt \right\|$$

$$\leq \int_0^1 \|f'(x + t(y - x) - f'(x)\| \|y - x\| dt$$

$$\leq \int_0^1 K \|x + t(y - x) - x\| \|y - x\| dt$$

$$= \int_0^1 K \|t(y - x)\| \|y - x\| dt$$

$$= \int_0^1 K \|t\| \|y - x\|^2 dt$$

$$= K\|y - x\|^2 \int_0^1 t dt$$

$$= \frac{1}{2} K\|x - y\|^2 .$$

Definition 2.2 Let t_0 and t' be non-negative real numbers, g be a continuously differentiable real function on $[\, t_0, t_0 + t' \,]$ and G a continuously differentiable operator on $\overline{N}(x_0, t) \subset X$ into X. Then the equation $t = g(t)$ will be said to majorize the equation $x = G(x)$, or g majorizes G, on $N(x_0, t')$ if

$$\|G(x_0) - x_0\| \le g(t_0) - t_0 \qquad (2.15)$$

and

$$\|G'(x)\| \le g'(t) \text{ where } \|x - x_0\| \le t - t_0 < t' . \qquad (2.16)$$

Definition 2.3 Let t_0 and t' be non-negative real numbers, g be a real function on $[\, t_0, t_0 + t' \,]$, and G an operator sending $N(x_0, t')$ into X. Then the equation $t = g(t)$ will be said to weakly majorize the equation $x = G(x)$, or g weakly majorizes G, if (2.15) holds and in addition

$$\|G(G(x)) - G(x)\| \le g(g(t)) - g(t) \qquad (2.17)$$

when $\|x - x_0\| \le t - t_0 < t'$

and $\|G(x) - x\| \le g(t) - t$.

We remark that Lemma 2.4 and Lemma 2.5 are known results in mathematical analysis and Lemma 2.6 is given by Ortega [4].

Lemma 2.5 (The Banach Lemma) Let $M \in L(X,X)$ and $\|I - J\| \leq \delta < 1$, then J^{-1} exists in $L(X,X)$ and $J^{-1} \leq (1-\delta)^{-1}$.

Lemma 2.6 Let $\{x_k\}$ be a sequence in X and $\{t_k\}$ a sequence of non-negative real numbers such that

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \quad , \quad k = 0,1,2,.. \quad , \tag{2.18}$$

and $t_k \to t^* < \infty$. By these conditions, there exists a point $s \in X$ such that $x_k \to s$ and

$$\|s - x_k\| \leq t^* - t_k \quad , \quad k = 0,1,2,... \quad . \tag{2.19}$$

proof: The proof is immediate from

$$\|x_{k+p} - x_k\| \leq \sum_{i=1}^{p} \|x_{k+i} - x_{k+i-1}\|$$

$$\leq t_{k+p} - t_k$$

$$\leq t^* - t_k \, ,$$

which show that $\{x_k\}$ is a Cauchy Sequence.

We shall say that $\{t_k\}$ majorizes $\{x_k\}$ if $\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \quad , \quad k = 0,1,2,...$ holds.

The following theorem is The Kantorovich Theorem. It is one of the fundamental theorems in numerical mathematics. The ideas for proof this theorem in [5].

Theorem 2.7 (The Kantorovich Theorem) Let x_0 be in D_0 and let $\Gamma_0 = [f'(x_0)]^{-1}$ exist with $\Gamma_0 \in Lip_k(D_0)$,

$$\|\Gamma_0 f(x_0)\| \leq \eta \quad \text{and} \quad h = K\eta \leq \frac{1}{2}. \tag{2.20}$$

Define
$$r_0(h) = \frac{1}{h}(1 - \sqrt{1-2h})\eta \tag{2.21}$$

$$r_1(h) = \frac{1}{h}(1 + \sqrt{1-2h})\eta. \tag{2.22}$$

Then if $N(x_0, r_0(h)) \subset D_0$, the sequence of iterates defined by Newton's method exists, remains in $N(x_0, r_0(h))$ and converges to s in $N(x_0, r_0(h))$ such that $f(s) = 0$. If $h < \frac{1}{2}$, s is the only root in $N(x_0, r_1(h)) \cap D_0$, and if $h = \frac{1}{2}$, s is unique in $N(x_0, r_1(h)) \cap D_0$. Furthermore, the sequence of the iterates satisfy the error bounds

$$\|s - x_m\| \leq \frac{1}{h}\frac{1}{2^m}(1 - \sqrt{1-2h})^{2m}\eta. \tag{2.23}$$

The following theorem is given by Kantorovich [5]. This theorem together with Lemma 2.6 and Definition 2.2 give the convergence of the sequence $\{x_k\}$ in X when the sequence of $\{t_k\}$ converges.

Theorem 2.8 If g majorizes G on $\overline{N}(x_0, t')$ and g has a fixed point in $[\ t_0, t_0 + t'\ ]$, then G has a fixed point s in $\overline{N}(x_0, t')$. Furthermore, $x_{k+1} = G(x_k)$ and $t_{k+1} = g(t_k)$, $k = 0,1,2,\ldots$, converges to s and t^* respectively with the real sequence majorizing the vector sequence.

Next lemma uses the weakly majorizing property and it is given by Dennis [6].

Lemma 2.9 If $g(t) \in (t, t_0 + t')$ when $t \in (t_0, t_0 + t')$ and g weakly majorizing G on $N(x_0, t')$, then there are elements $t^* \in [t_0, t_0 + t']$, $s \in \overline{N}(x_0, t')$ such that

$$x_{k+1} = G(x_k) \tag{2.24}$$

and
$$t_{k+1} = g(t_k), \qquad k = 0,1,2,\ldots \tag{2.25}$$

converge to s and t^* respectively with the t sequence majorizing in x sequence.

Theorems 2.10-2.12 in this section are given by Dennis [6]. These theorems give the convergence of Newton-like method.

Theorem 2.10 Let A be a function on $N(x_0, r)$ such that $A(x) \in L(X, Y)$ for each x and $A(x)$ is invertible for each x in $N(x_0, r)$ and that there is a real, nonvanishing, nonincreasing function $a(t)$ on $[0, r)$ such that

$$\|A^{-1}(x)\| \le a(\|x - x_0\|)^{-1} \tag{2.26}$$

If $f' \in (Lip_k(\overline{N}(x_0, r))$ then if $\sigma \ge 1$ and $\delta > 0$ are real numbers such that

$$a(t) + \sigma kt \tag{2.27}$$

is isotonic on $(0, r)$, and

$$\|B(x)\| \le a(\|x - x_0\|) + \sigma k\|x - x_0\| - \delta \tag{2.28}$$

for every $x \in N(x_0, r)$

then $$g(t) = t + (a(t))^{-1}(0.5\sigma kt^2 - \delta t - a(0)\left\|(A(x_0))^{-1} - f(x_0)\right\|) \tag{2.29}$$

weakly majorizes $G(x) = x - (A(x))^{-1} f(x)$ on $N(x_0, r)$.

Theorem 2.11 Let $f' \in Lip_k\left(\overline{N}(x_0, r)\right)$ and $[A(x_0)]^{-1}$ exist and be bounded in the norm by $[a(0)]^{-1}$, If $\|B(x_0)\| < a(0)$ and

$$h' = \frac{K}{a(0)}\left\|[A(x_0)]^{-1} f(x_0)\right\| a(0) - (\|B(x_0)\|)^2 \le \frac{1}{2} \tag{2.30}$$

and

$$r_0' = \frac{1}{K}\left(1 - \sqrt{1 - 2h'}\right)\left[a(0) - \|B(x_0)\|\right] \le r. \tag{2.31}$$

Then if f has a unique zero $s \in \overline{N}(x_0, r_0')$, and

$$x_{m+1}' = x_m' - [A(x_0)]^{-1} f(x_m') \quad , \quad m = 0,1,2,..$$

converges to s from any $x_0' \in \overline{N}(x_0, r)$ such that

$$\|x_0' - x_0\| < r_1' = \frac{1}{K}\left(1 - \sqrt{1 - 2h}\right)\left[a(0) - \|B(x_0)\|\right].$$

If, in addition, σ, δ and a satisfy the conditions of Theorem 2.10 and

$$h = \frac{1}{\delta^2}\sigma K \left\| [A(x_0)]^{-1} f(x_0) \right\| a(0) \le \frac{1}{2} \tag{2.32}$$

and

$$r_0 = \frac{1}{\sigma.K}\left(1 - \sqrt{1 - 2h}\right)\delta < r, \tag{2.33}$$

then

$$x_{m+1} = x_m' - [A(x_m)]^{-1} f(x_m') \quad , \quad m = 0,1,2,\ldots \tag{2.34}$$

converges to s.

In the following theorem, we impose one more condition on $A(x)$ and one condition on $B(x)$ instead of $B(x_0)$.

Theorem 2.12 Let $f' \in Lip_k(\overline{D})$ where $x_0 \in D$ and D is an open convex subset of X. Assume that

$$\left\|[A(x_0)]^{-1} f(x_0)\right\| \le \alpha \tag{2.35}$$

$$\left\|[A(x_0)]^{-1}\right\| \le \beta \tag{2.36}$$

$$\|A(x) - A(x_0)\| \le \eta_0 + \eta_1\|x - x_0\|, \quad \forall x \in D \tag{2.37}$$

$$\|B(x)\| \le \delta_0 + \delta_1\|x - x_0\|, \quad \forall x \in D. \tag{2.38}$$

Then

$$\beta.\delta_0 < 1 \ , \ h' = \frac{\beta.K.\alpha}{(1-\beta.\delta_0)^2} \le \frac{1}{2}$$

and $N(x_0, r_0') \subset D$ where

$$r_0' = \frac{1-\sqrt{1-2h'}}{\beta.K}(1-\beta.\delta_0)$$

implied that f has a solution $r \in \overline{N}(x_0, r_0')$ which is unique in $D \cap N(x_0, r_0')$ where

$$r_1' = \frac{1-\sqrt{1+2h'}}{\beta.K}(1-\beta.\delta_0).$$

Furthermore $$x_{m+1}' = x_m' - [A(x_0)]^{-1}(f(x_m'))$$

converges to s from any $x_0' \in \overline{D} \cap N(x_0, r_1')$.

If, in addition, $\beta(\delta_0 + \eta_0) < 1$ and $h = \dfrac{\sigma.\beta.K.\alpha}{(1-\beta.\eta_0 - \beta.\delta_0)^2} \le \dfrac{1}{2}$ where $\sigma = max\left(1, \dfrac{\delta_1 + \eta_1}{K}\right)$, and $N(x_0, r_0) \subset D$,

$$r_0' = \frac{1-\sqrt{1-2h}}{\sigma.\beta.K}(1-\beta.\eta_0-\beta.\delta_0)$$

then

$$x_{m+1} = x_m - [A(x_m)]^{-1} f(x_m)$$

converges to s.

Main Results

The following theorem will ensure that the convergence of the Modified Newton method for solving system of nonlinear equations which is a special kind of the Newton-like method. The proof of this theorem in [7]

Theorem 2.13 Let D be an open convex subset of the space X and $f' \in Lip_k(\overline{D})$. Assuming that $f(x)$ and $H(x)$ satisfy all the conditions of the previous theorems, then there exists a unique zero s in D so that for any point x_0 in D the sequence $\{x_m\}$ where

$$x_{m+1} = x_m - [H(x_m)]^{-1} f(x_m)$$

converges to s.

The last theorem of this section, Theorem 2.14, shows that the order of the convergence of the Modified Newton method for solving a system of nonlinear equations which is of second order. The proof of this theorem in [7]

Theorem 2.14 Let the conditions of Theorem 2.12 be satisfied and $\delta_0 = 0$, that is

$$\|M(x)\| \le \delta_1 \|x - x_0\|, \quad \forall x \in D. \tag{2.39}$$

Then the order of the convergence of the method is equal to 2.

CHAPTER 3

MORE ON MODIFIED NEWTON METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

In this chapter, we will introduce iterative methods for solving systems of nonlinear equations. These methods are developed from the Modified Newton method for solving systems of nonlinear equations.

First, considering the Newton's method for solving a systems of nonlinear equations which is expressed in the form

$$f(x) = 0 \tag{3.1}$$

we can find the solutions of this system from the following iteration equation

$$x^{k+1} = x^k - (J(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots . \tag{3.2}$$

where x is in R^n, f is a function from subset of R^n to subset of R^n and $J(x)$ is the Jacobian matrix of $f(x)$

$$J(x) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}, \tag{3.3}$$

where we have used f_{ij} to denote the partial derivative of $f_i(x)$ with respect to the j^{th} variable and evaluated at x, could be written as follows :

$$x^{k+1} = x^k - \left[J(x_1{}^k, x_2{}^k, \ldots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \ldots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \ldots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \ldots, x_n{}^k) \end{bmatrix} \quad ; k = 0,1,2,\ldots . \qquad (3.4)$$

Second, the Modified Newton method which is one of the Newton-like method for solving (3.1) which has the form

$$x^{k+1} = x^k - (H(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\ldots . \qquad (3.5)$$

where $H(x)$ is the diagonal matrix

$$H(x) = \begin{bmatrix} f_{11} & 0 & \cdots & 0 \\ 0 & f_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ f_{n1} & \cdots & 0 & f_{nn} \end{bmatrix},$$

and f_{ii} is the partial derivative of $f_i(x)$ with respect to the i^{th} variable and evaluated at x, could be written as follows :

$$x^{k+1} = x^k - \left[H(x_1{}^k, x_2{}^k, \ldots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \ldots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \ldots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \ldots, x_n{}^k) \end{bmatrix} \quad ; k = 0,1,2,\ldots . \qquad (3.6)$$

Next, an emphasis will be placed on definitions and theorem, which we will use for guarantee that our methods are convergence.

3.1 Definitions and Theorem

Definition 3.1.1 We define the matrix $D(x)$ as the matrix $H(x)$ above.

Definition 3.1.2 We define the matrices $U_2(x)$, $U_3(x)$, ..., $U_n(x)$ of $f(x)$ as

$$U_2(x)=\begin{bmatrix} f_{11} & f_{12} & 0 & & \cdots & & 0 \\ 0 & f_{22} & f_{23} & 0 & & & \vdots \\ & & & \ddots & \ddots & & \\ \vdots & & & & & & \\ & & & & f_{n-2n-2} & f_{n-2n-1} & 0 \\ & & & & & f_{n-1n-1} & f_{n-1n} \\ 0 & & & \cdots & & 0 & f_{nn} \end{bmatrix}, \tag{3.7}$$

$$U_3(x)=\begin{bmatrix} f_{11} & f_{12} & f_{13} & 0 & \cdots & & 0 \\ 0 & f_{22} & f_{23} & f_{24} & & & \vdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & & & & & & 0 \\ & & & & f_{n-2n-2} & f_{n-2n-1} & f_{n-2n} \\ & & & & & f_{n-1n-1} & f_{n-1n} \\ 0 & & & \cdots & & 0 & f_{nn} \end{bmatrix}, \tag{3.8}$$

$$\vdots$$

$$U_n(x)=\begin{bmatrix} f_{11} & f_{12} & f_{13} & & \cdots & & f_{1n} \\ 0 & f_{22} & f_{23} & f_{24} & & & f_{2n} \\ & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & & \\ & & & & f_{n-2n-2} & f_{n-2n-1} & f_{n-2n} \\ & & & & & f_{n-1n-1} & f_{n-1n} \\ 0 & & & \cdots & & 0 & f_{nn} \end{bmatrix}. \tag{3.9}$$

Definition 3.1.3 We define the matrices $L_2(x)$, $L_3(x)$, ... , $L_n(x)$ of $f(x)$ as

$$L_2(x)=\begin{bmatrix} f_{11} & 0 & & & \cdots & & 0 \\ f_{21} & f_{22} & & & & & \\ 0 & f_{32} & f_{33} & & & & \\ & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & & \\ & & & & f_{n-1n-2} & f_{n-1n-1} & 0 \\ 0 & & \cdots & & 0 & f_{nn-1} & f_{nn} \end{bmatrix}, \tag{3.10}$$

$$L_3(x)=\begin{bmatrix} f_{11} & 0 & & & \cdots & & 0 \\ f_{21} & f_{22} & & & & & \\ f_{31} & f_{32} & f_{33} & & & & \\ 0 & & & & & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & & & & \\ & & & f_{n-1n-3} & f_{n-1n-2} & f_{n-1n-1} & 0 \\ 0 & & \cdots & 0 & f_{nn-2} & f_{nn-1} & f_{nn} \end{bmatrix}, \tag{3.11}$$

$$\vdots$$

$$L_n(x)=\begin{bmatrix} f_{11} & 0 & & & \cdots & & 0 \\ f_{21} & f_{22} & & & & & \\ f_{31} & f_{32} & f_{33} & & & & \\ & & & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \\ f_{n-11} & & & f_{n-1n-3} & f_{n-1n-2} & f_{n-1n-1} & 0 \\ f_{n1} & & \cdots & & & f_{nn-1} & f_{nn} \end{bmatrix}. \tag{3.12}$$

First method

We call this method the D-method.

The first iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (H(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots \ . \quad (3.13)$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[H(x_1{}^k, x_2{}^k, \dots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \end{bmatrix} \quad ; k = 0,1,2,\dots \ . \quad (3.14)$$

Second method

From preceding method, replacing matrix $H(x)$ by matrix $U_2(x)$ in equation (3.13) we then obtain new method and call this method the U_2-method.

The second iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (U_2(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots \ . \quad (3.15)$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[U_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \end{bmatrix} \quad ; k = 0,1,2,\dots \ . \ (3.16)$$

Third method

We call this method the U_3-method.

The third iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (U_3(x^k))^{-1} f(x^k) \quad , \qquad k = 0,1,2,\dots \quad (3.17)$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[U_3({x_1}^k, {x_2}^k, \dots, {x_n}^k)\right]^{-1} \begin{bmatrix} f_1({x_1}^k, {x_2}^k, \dots, {x_n}^k) \\ f_2({x_1}^k, {x_2}^k, \dots, {x_n}^k) \\ \vdots \\ f_n({x_1}^k, {x_2}^k, \dots, {x_n}^k) \end{bmatrix} \quad ; k = 0,1,2,\dots \quad (3.18)$$

Fourth method

We call this method the U_4-method.

The fourth iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (U_4(x^k))^{-1} f(x^k) \quad , \qquad k = 0,1,2,\dots \quad (3.19)$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[U_4({x_1}^k, {x_2}^k, \dots, {x_n}^k)\right]^{-1} \begin{bmatrix} f_1({x_1}^k, {x_2}^k, \dots, {x_n}^k) \\ f_2({x_1}^k, {x_2}^k, \dots, {x_n}^k) \\ \vdots \\ f_n({x_1}^k, {x_2}^k, \dots, {x_n}^k) \end{bmatrix} \quad ; k = 0,1,2,\dots \quad (3.20)$$

n^{th} method

We may continue this same manner and call this method the U_n-method. The fifth iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (U_n(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots \ . \tag{3.21}$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[U_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \end{bmatrix} \quad ; k = 0,1,2,\dots \ . \tag{3.22}$$

$(n+1)^{th}$ method

We call this method the L_2-method. The sixth iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (L_2(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots \ . \tag{3.23}$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[L_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \end{bmatrix} \quad ; k = 0,1,2,\dots \ . \tag{3.24}$$

$(n+2)^{\text{th}}$ method

We call this method the L_3-method.

The seventh iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (L_3(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots \ . \qquad (3.25)$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[L_3(x_1{}^k, x_2{}^k, \dots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \end{bmatrix} \ ; k = 0,1,2,\dots \ . \quad (3.26)$$

$(n+3)^{\text{th}}$ method

We call this method the L_4-method.

The eighth iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (L_4(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots \ . \qquad (3.27)$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[L_4(x_1{}^k, x_2{}^k, \dots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \end{bmatrix} \ ; k = 0,1,2,\dots \ . \quad (3.28)$$

$(2n-1)^{\text{th}}$ method

We may continue this same manner and call this method the L_n-method. The ninth iterative method to solve problem (3.1) is the iterative equations

$$x^{k+1} = x^k - (L_n(x^k))^{-1} f(x^k) \ , \qquad k = 0,1,2,\dots \qquad (3.29)$$

Evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[L_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k)\right]^{-1} \begin{bmatrix} f_1(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ f_2(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \\ \vdots \\ f_n(x_1{}^k, x_2{}^k, \dots, x_n{}^k) \end{bmatrix} \quad ; k = 0,1,2,\dots \quad (3.30)$$

Theorem 3.1 If the matrices $D(x)$, $U_2(x)$, $U_3(x)$, ... , $U_n(x)$, $L_2(x)$, $L_3(x)$, ... , $L_n(x)$ of the D-method, U_2-method, U_3-method, ... , U_n-method, L_2-method, L_3-method, ... , L_n-method respectively satisfy the conditions of the theorem 2.13 and the theorem 2.14 then there exists a unique zero in D so that for any point x^0 in D the sequence $\{x^k\}$ converges to this zero and the order of convergence is equal to 2.

CHAPTER 4

SOME EXAMPLES AND APPLICATION PROBLEMS OF SYSTEMS OF NONLINEAR EQUATIONS

In this chapter, we shall find the solutions by using the Modified Newton method and new methods obtained from chapter 3 for solving systems of nonlinear equations. The examples are illustrated and the results of methods are compared.

Systems of nonlinear algebraic equations arise routinely in abroad range of theoretical and practical applications. Their frequent occurrence is a reflection of the predominantly nonlinear nature of the physical laws.

Example 4.1 Find a solution of the system [8]

$$f(x,y) = 0.5\,sin(xy) - 0.5x - \frac{0.25y}{\pi} = 0$$

$$g(x,y) = (1 - \frac{0.25}{\pi})(e^{2x} - e) - 2ex + \frac{ey}{\pi} = 0 \tag{4.1}$$

First we compute $f_x = 0.5y\cos xy - 0.5$ $\qquad f_y = 0.5x\cos xy - \dfrac{0.25}{\pi}$

$$g_x = \left(2 - \frac{0.5}{\pi}\right)e^{2x} - 2e \qquad g_y = \frac{e}{\pi}$$

Next we select initial values of x and y. Repeat the methods until the error less than 0.0000005.

The matrix H of Modified Newton method (D-method) for system (4.1) is

$$H(x,y)=\begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix}=\begin{bmatrix} 0.5y\cos(xy)-0.5 & 0 \\ 0 & \dfrac{e}{\pi} \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix}=\begin{bmatrix} x^{k} \\ y^{k} \end{bmatrix}-\left[H(x^k,y^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k) \\ g(x^k,y^k) \end{bmatrix} \qquad ;k=0,1,2,\dots .$$

The matrix U_2 of U_2-method for system (4.1) is

$$U_2(x,y)=\begin{bmatrix} f_x & f_y \\ 0 & g_y \end{bmatrix}=\begin{bmatrix} 0.5y\cos(xy)-0.5 & 0.5x\cos(xy)-\dfrac{0.25}{\pi} \\ 0 & \dfrac{e}{\pi} \end{bmatrix}.$$

Therefore U_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix}=\begin{bmatrix} x^{k} \\ y^{k} \end{bmatrix}-\left[U_2(x^k,y^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k) \\ g(x^k,y^k) \end{bmatrix} \qquad ;k=0,1,2,\dots .$$

The matrix L_2 of L_2-method for system (4.1) is

$$L_2(x,y)=\begin{bmatrix} f_x & 0 \\ g_x & g_y \end{bmatrix}=\begin{bmatrix} 0.5y\cos(xy)-0.5 & 0 \\ (2-\dfrac{0.5}{\pi})e^{2x}-2e & \dfrac{e}{\pi} \end{bmatrix}.$$

Therefore L_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix}=\begin{bmatrix} x^{k} \\ y^{k} \end{bmatrix}-\left[L_2(x^k,y^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k) \\ g(x^k,y^k) \end{bmatrix} \qquad ;k=0,1,2,\dots .$$

We obtain the results from D-method, U_2-method and L_2- method as follows in the table 4.1.

Table 4.1 Give the results with different initial points of example 4.1

Initial point (x^0, y^0)	Number of iterations (k) from D-method	Number of iterations (k) from U_2-method	Number of iterations (k) from L_2-method	Solutions
(0,3.5)	16	11	5	x=0.2994487 y=2.8369276
(0.75,2.5)	12	6	7	x=0.5 $y=\pi$
(0.8,4)	12	8	10	x=0.5 $y=\pi$

Showing the iterations of table 4.1 in appendix.

Example 4.2 Find a solution of the system

$$f(x,y,z) = xyz + x + y - z - 1.25 = 0$$

$$g(x,y,z) = x^2 + z^2 + y^2 - 1.5 = 0 \qquad (4.2)$$

$$h(x,y,z) = xz^2 + y^2 z + x - y + z - 0.625 = 0$$

First we compute $f_x = yz + 1$ $\quad f_y = xz + 1$ $\quad f_z = xy - 1$

$g_x = 2x$ $\quad g_y = 2y$ $\quad g_z = 2z$

$h_x = z^2 + 1$ $\quad h_y = 2yz - 1$ $\quad h_z = 2xz + y^2 + 1$

Next we select initial values for x, y and z. Repeat the methods until the error less than 0.0000005.

The matrix H of D-method for system (4.2) is

$$H(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ 0 & g_y & 0 \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} yz+1 & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 2xz + y^2 + 1 \end{bmatrix}.$$

Therefore D-method method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} - \left[H(x^k, y^k, z^k)\right]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

The matrix U_2 of U_2-method for system (4.2) is

$$U_2(x,y,z)=\begin{bmatrix} f_x & f_y & 0 \\ 0 & g_y & g_z \\ 0 & 0 & h_z \end{bmatrix}=\begin{bmatrix} yz+1 & xz+1 & 0 \\ 0 & 2y & 2z \\ 0 & 0 & 2xz+y^2+1 \end{bmatrix}.$$

Therefore U_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix}=\begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix}-\left[U_2(x^k,y^k,z^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k,z^k) \\ g(x^k,y^k,z^k) \\ h(x^k,y^k,z^k) \end{bmatrix} \qquad ; k=0,1,2,\ldots.$$

The matrix U_3 of U_3-method for system (4.2) is

$$U_3(x,y,z)=\begin{bmatrix} f_x & f_y & f_z \\ 0 & g_y & g_z \\ 0 & 0 & h_z \end{bmatrix}=\begin{bmatrix} yz+1 & xz+1 & xy-1 \\ 0 & 2y & 2z \\ 0 & 0 & 2xz+y^2+1 \end{bmatrix}.$$

Therefore U_3-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix}=\begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix}-\left[U_3(x^k,y^k,z^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k,z^k) \\ g(x^k,y^k,z^k) \\ h(x^k,y^k,z^k) \end{bmatrix} \qquad ; k=0,1,2,\ldots.$$

The matrix L_2 of L_2-method for system (4.2) is

$$L_2(x,y,z)=\begin{bmatrix} f_x & 0 & 0 \\ g_x & g_y & 0 \\ 0 & h_y & h_z \end{bmatrix}=\begin{bmatrix} yz+1 & 0 & 0 \\ 2x & 2y & 0 \\ 0 & 2yz-1 & 2xz+y^2+1 \end{bmatrix}.$$

Therefore L_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} - \left[L_2(x^k, y^k, z^k)\right]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

The matrix L_3 of L_3-method for system (4.2) is

$$L_3(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ g_x & g_y & 0 \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} yz+1 & 0 & 0 \\ 2x & 2y & 0 \\ z^2+1 & 2yz-1 & 2xz+y^2+1 \end{bmatrix}.$$

Therefore L_3-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} - \left[L_3(x^k, y^k, z^k)\right]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

We obtain the results from D-method, U_2-method, U_3-method, L_2-method and L_3-method follows in the table 4.2.

Table 4.2 Give the results with different initial points of example 4.2

Initial point (x^0, y^0, z^0)	Number of iterations(k) from D-method	Number of iterations(k) from U_2-method	Number of iterations(k) from U_3-method	Number of iterations(k) from L_2-method	Number of iterations(k) from L_3-method
(0,1,0)	47	18	8	47	20
(1,1.5,1)	52	18	8	42	21
(1,2,1)	50	17	8	45	21

The above methods solution with different 3 initial points always converges to $x = 0.5$, $y=1$ and $z = 0.5$.

Showing the iterations of table 4.2 in appendix.

Example 4.3 Find the solution of the system

$$
\begin{aligned}
f(x,y,z,w) &= -2.0625x + 2y - 0.0625x^3 + 0.5 &= 0 \\
g(x,y,z,w) &= z - 2y + x - 0.0625y^3 + 0.125y(z-x) &= 0 \\
h(x,y,z,w) &= w - 2z + y - 0.0625z^3 + 0.125z(w-y) &= 0 \\
j(x,y,z,w) &= -1.9375w + z - 0.0625w^3 - 0.125zw + 0.5 &= 0
\end{aligned}
\qquad (4.3)
$$

Next we select initial values for x , y, z and w. Repeat the methods until the error less than 0.0000005.

Since, in this case the matrices $U_2(x,y,z,w) = U_3(x,y,z,w) = U_4(x,y,z,w)$ and the matrices $L_2(x,y,z,w) = L_3(x,y,z,w) = L_4(x,y,z,w)$ so we will consider only the methods from D-method, U_2-method and L_2-method.

The matrix H of D-method for system (4.3) is

$$
H(x,y,z,w) = \begin{bmatrix} f_x & 0 & 0 & 0 \\ 0 & g_y & 0 & 0 \\ 0 & 0 & h_z & 0 \\ 0 & 0 & 0 & j_w \end{bmatrix}.
$$

Therefore D-method algorithm could be written as follows:

$$
\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \\ w^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ z^k \\ w^k \end{bmatrix} - \left[H(x^k,y^k,z^k,w^k)\right]^{-1} \begin{bmatrix} f(x^k,y^k,z^k,w^k) \\ g(x^k,y^k,z^k,w^k) \\ h(x^k,y^k,z^k,w^k) \\ j(x^k,y^k,z^k,w^k) \end{bmatrix} \qquad ;k = 0,1,2,\ldots.
$$

The matrix U_2 of U_2-method for system (4.3) is

$$U_2(x,y,z,w)=\begin{bmatrix} f_x & f_y & 0 & 0 \\ 0 & g_y & g_z & 0 \\ 0 & 0 & h_z & h_w \\ 0 & 0 & 0 & j_w \end{bmatrix}.$$

Therefore U_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \\ w^{k+1} \end{bmatrix}=\begin{bmatrix} x^k \\ y^k \\ z^k \\ w^k \end{bmatrix}-\left[U_2(x^k,y^k,z^k,w^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k,z^k,w^k) \\ g(x^k,y^k,z^k,w^k) \\ h(x^k,y^k,z^k,w^k) \\ j(x^k,y^k,z^k,w^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots.$$

The matrix L_2 of L_2-method for system (4.3) is

$$L_2(x,y,z,w)=\begin{bmatrix} f_x & 0 & 0 & 0 \\ g_x & g_y & 0 & 0 \\ 0 & h_y & h_z & 0 \\ 0 & 0 & j_z & j_w \end{bmatrix}.$$

Therefore L_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \\ w^{k+1} \end{bmatrix}=\begin{bmatrix} x^k \\ y^k \\ z^k \\ w^k \end{bmatrix}-\left[L_2(x^k,y^k,z^k,w^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k,z^k,w^k) \\ g(x^k,y^k,z^k,w^k) \\ h(x^k,y^k,z^k,w^k) \\ j(x^k,y^k,z^k,w^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots.$$

We obtain the results from D-method, U_2-method and L_2-method follows in the table 4.3.

Table 4.3 Give the results with different initial points of example 4.3

Initial point (x^0, y^0, z^0, w^0)	Number of iterations (k) from D-method	Number of iterations (k) from U_2-method	Number of iterations (k) from L_2-method
(0,0,0,0)	85	43	43
(0.5,0.5,0.5,0.5)	77	39	38
(1,1,0,0)	89	32	41

The above methods solution with different 3 initial points always converges to $x = 0.9789103$, $y = 0.7888155$, $z = 0.660767$ and $w = 0.5689128$.

Showing the iterations of table 4.3 in appendix.

Example 4.4 Position of a Two-Link Robot Arm [9]

The position of a two-link robot arm can be described in terms of the angle that the first link makes with the horizontal axis and the angle that the second link makes with the first link. In this example, we assume that the lengths of the two links are d_1 and d_2; the first link makes an angle α with the horizontal axis, and the second link makes an angle β with the direction defined by the first link. Our problem is to find the angles α and β that allow the end of the second link to be at a specified point, with coordinates (p_1, p_2). The arrangement is illustrated in Figure 4.1. The equations for these requirements are as follows:

Location of the end of the first link (x_1, y_1)

$$x_1 = d_1 \cos(\alpha),$$

$$y_1 = d_1 \sin(\alpha).$$

Location of the end of the second link (x_2, y_2)

$$x_2 = x_1 + d_2 \cos(\alpha + \beta),$$

$$y_2 = y_1 + d_2 \sin(\alpha + \beta).$$

Thus, we need to solve

$$p_1 = d_1 \cos(\alpha) + d_2 \cos(\alpha + \beta),$$

$$p_2 = d_1 \sin(\alpha) + d_2 \sin(\alpha + \beta),$$

for the unknown angles α and β.

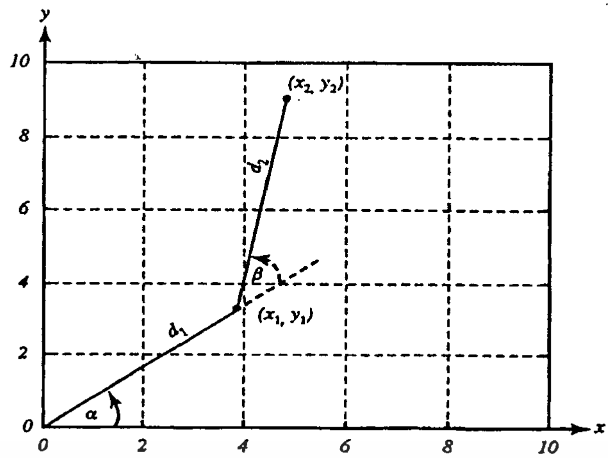

Figure 4.1 Initial position two-link robot arm.

Consider a two-link robot arm, as introduced in above. Let the length of the first link be 5 and the length of the second link be 6. We wish to find the angles so that the arm will move to the point (10,4), starting from initial angles of α=0.7 ≈40.11° and β=0.7≈40.11°.

The system of equations in this case is

$$f(\alpha,\beta) = 5\cos(\alpha) + 6\cos(\alpha+\beta) - 10 = 0,$$

$$g(\alpha,\beta) = 5\sin(\alpha) + 6\sin(\alpha+\beta) - 4 = 0. \tag{4.4}$$

Repeat the methods until the error less than 0.0000005.

The matrix H of Modified Newton method (D-method) for system (4.4) is

$$H(\alpha,\beta)=\begin{bmatrix} f_\alpha & 0 \\ 0 & g_\beta \end{bmatrix}=\begin{bmatrix} -5\sin(\alpha)-6\sin(\alpha+\beta) & 0 \\ 0 & 6\cos(\alpha+\beta) \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \end{bmatrix}-\left[H(\alpha^k,\beta^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k) \\ g(\alpha^k,\beta^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots .$$

The matrix U_2 of U_2-method for system (4.4) is

$$U_2(\alpha,\beta)=\begin{bmatrix} f_\alpha & f_\beta \\ 0 & g_\beta \end{bmatrix}=\begin{bmatrix} -5\sin(\alpha)-6\sin(\alpha+\beta) & -6\sin(\alpha+\beta) \\ 0 & 6\cos(\alpha+\beta) \end{bmatrix}.$$

Therefore U_2-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \end{bmatrix}-\left[U_2(\alpha^k,\beta^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k) \\ g(\alpha^k,\beta^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots .$$

The matrix L_2 of L_2-method for system (4.4) is

$$L_2(\alpha,\beta)=\begin{bmatrix} f_\alpha & 0 \\ g_\alpha & g_\beta \end{bmatrix}=\begin{bmatrix} -5\sin(\alpha)-6\sin(\alpha+\beta) & 0 \\ 5\cos(\alpha)+6\cos(\alpha+\beta) & 6\cos(\alpha+\beta) \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \end{bmatrix}-\left[L_2(\alpha^k,\beta^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k) \\ g(\alpha^k,\beta^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots .$$

Table 4.4 Contains the results from D-method, U_2 - method and L_2 - method of example 4.4

Number of iterations(k) from D-method	Number of iterations(k) from U_2-method	Number of iterations(k) from L_2-method
75	38	32

The above methods solution converges to $\alpha = 0.1559838 \approx 8.94°$ and $\beta = 0.4111379 \approx 23.56°$.

Showing the iterations of table 4.4 in appendix.

The initial position and final position of the arm are illustrated in Figure 4.2

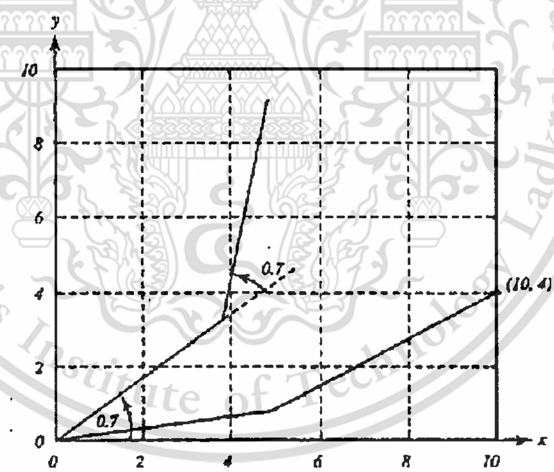

Figure 4.2 Initial and final positions of two-link robot arm.

Example 4.5 We seek a common solution of the following three equations, which represent the unit sphere centered at the origin, a cylinder with radius $\frac{1}{2}$ and axis along the y-axis, and a paraboloid of revolution around the z-axis. [9]

$$
\begin{aligned}
f(x,y,z) &= x^2 \qquad + z^2 - \frac{1}{4} = 0 \\
g(x,y,z) &= x^2 + y^2 + z^2 - 1 = 0 \qquad (4.5) \\
h(x,y,z) &= x^2 + y^2 - 4z \qquad = 0
\end{aligned}
$$

Next we select initial values for $x=1$, $y=1$ and $z=0$. Repeat the methods until the error less than 0.0000005.

The matrix H of D-method for system (4.5) is

$$
H(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ 0 & g_y & 0 \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} 2x & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & -4 \end{bmatrix}.
$$

Therefore D-method method algorithm could be written as follows:

$$
\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} - \left[H(x^k,y^k,z^k)\right]^{-1} \begin{bmatrix} f(x^k,y^k,z^k) \\ g(x^k,y^k,z^k) \\ h(x^k,y^k,z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .
$$

The matrix U_2 of U_2-method for system (4.5) is

$$
U_2(x,y,z) = \begin{bmatrix} f_x & f_y & 0 \\ 0 & g_y & g_z \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} 2x & 0 & 0 \\ 0 & 2y & 2z \\ 0 & 0 & -4 \end{bmatrix}.
$$

Therefore U_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} - \left[U_2(x^k, y^k, z^k)\right]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

The matrix U_3 of U_3-method for system (4.5) is

$$U_3(x,y,z) = \begin{bmatrix} f_x & f_y & f_z \\ 0 & g_y & g_z \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} 2x & 0 & 2z \\ 0 & 2y & 2z \\ 0 & 0 & -4 \end{bmatrix}.$$

Therefore U_3-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} - \left[U_3(x^k, y^k, z^k)\right]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

The matrix L_2 of L_2-method for system (4.5) is

$$L_2(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ g_x & g_y & 0 \\ 0 & h_y & h_z \end{bmatrix} = \begin{bmatrix} 2x & 0 & 0 \\ 2x & 2y & 0 \\ 0 & 2y & -4 \end{bmatrix}.$$

Therefore L_2-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} - \left[L_2(x^k, y^k, z^k)\right]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

The matrix L_3 of L_3-method for system (4.5) is

$$L_3(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ g_x & g_y & 0 \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} 2x & 0 & 0 \\ 2x & 2y & 0 \\ 2x & 2y & -4 \end{bmatrix}.$$

Therefore L_3-method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} - \left[L_3(x^k,y^k,z^k)\right]^{-1} \begin{bmatrix} f(x^k,y^k,z^k) \\ g(x^k,y^k,z^k) \\ h(x^k,y^k,z^k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

Table 4.5 Contains the results from D-method, U_2- method, U_3- method, L_2-method and L_3-method of example 4.5

Number of iterations(k) from D-method	Number of iterations(k) from U_2-method	Number of iterations(k) from U_3-method	Number of iterations(k) from L_2-method	Number of iterations(k) from L_3-method
31	27	22	15	7

The above methods solution converges to $x = 0.4407629$, $y = 0.2360679$ and $z = 0.8660254$.

Showing the iterations of table 4.5 in appendix.

Example 4.6 Astronomy (dynamical parallaxes) [10]

A binary or double star system is a pair of stars that are orbiting around each other. If one star is much more massive than the other, we can say that the smaller star is in orbit around the larger one. The nonlinear system for this system is as follows;

$$f(p,\mu_1,\mu_2) = \frac{\alpha^3}{T^2(\mu_1+\mu_2)} - p^3 = 0$$

$$g(p,\mu_1,\mu_2) = 5\log p + 8.75\log \mu_1 + m_1 + 0.2 = 0 \qquad (4.6)$$

$$h(p,\mu_1,\mu_2) = 5\log p + 8.75\log \mu_2 + m_2 + 0.2 = 0$$

where

α is the angle separation

p is the parallax of the star

m_1 is visual magnitude of the first star

m_2 is visual magnitude of the second star

μ_1 is the mass of the first star

μ_2 is the mass of the second star

T is time in year on the earth.

Given values $T = 88$, $\alpha = 4.6$, m_1=4.2, m_2=6.0 for a certain double star system, find the dynamical parallax.

Next we select initial values for p=0.5, μ_1=0.5 and μ_2=0.5. Repeat the methods until the error less than 0.0000005.

The matrix H of D-method for system (4.6) is

$$H(p,\mu_1,\mu_2)=\begin{bmatrix} f_p & 0 & 0 \\ 0 & g_{\mu_1} & 0 \\ 0 & 0 & h_{\mu_2} \end{bmatrix}=\begin{bmatrix} -3p^2T^2(\mu_1+\mu_2)-\alpha^3 & 0 & 0 \\ 0 & \dfrac{8.75}{\mu_1\, ln10} & 0 \\ 0 & 0 & \dfrac{8.75}{\mu_2\, ln10} \end{bmatrix}.$$

Therefore D-method method algorithm could be written as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix}=\begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix}-\left[H(x^k,y^k,z^k)\right]^{-1}\begin{bmatrix} f(x^k,y^k,z^k) \\ g(x^k,y^k,z^k) \\ h(x^k,y^k,z^k) \end{bmatrix} \qquad ;k=0,1,2,\dots .$$

The matrix U_2 of U_2-method for system (4.6) is

$$U_2(p,\mu_1,\mu_2)=\begin{bmatrix} f_p & f_{\mu_1} & 0 \\ 0 & g_{\mu_1} & g_{\mu_2} \\ 0 & 0 & h_{\mu_2} \end{bmatrix}=\begin{bmatrix} -3p^2T^2(\mu_1+\mu_2)-\alpha^3 & p^3T^2 & 0 \\ 0 & \dfrac{8.75}{\mu_1\, ln10} & 0 \\ 0 & 0 & \dfrac{8.75}{\mu_2\, ln10} \end{bmatrix}.$$

Therefore U_2-method algorithm could be written as follows:

$$\begin{bmatrix} p^{k+1} \\ \mu_1^{k+1} \\ \mu_2^{k+1} \end{bmatrix}=\begin{bmatrix} p^k \\ \mu_1^k \\ \mu_2^k \end{bmatrix}-\left[U_2(p^k,\mu_1^k,\mu_2^k)\right]^{-1}\begin{bmatrix} f(p^k,\mu_1^k,\mu_2^k) \\ g(p^k,\mu_1^k,\mu_2^k) \\ h(p^k,\mu_1^k,\mu_2^k) \end{bmatrix} \qquad ;k=0,1,2,\dots .$$

The matrix U_3 of U_3-method for system (4.6) is

$$U_3(p,\mu_1,\mu_2)=\begin{bmatrix} f_p & f_{\mu_1} & f_{\mu_2} \\ 0 & g_{\mu_1} & g_{\mu_2} \\ 0 & 0 & h_{\mu_2} \end{bmatrix}=\begin{bmatrix} -3p^2T^2(\mu_1+\mu_2)-\alpha^3 & p^3T^2 & p^3T^2 \\ 0 & \dfrac{8.75}{\mu_1 \ln 10} & 0 \\ 0 & 0 & \dfrac{8.75}{\mu_2 \ln 10} \end{bmatrix}.$$

Therefore U_3-method algorithm could be written as follows:

$$\begin{bmatrix} p^{k+1} \\ \mu_1{}^{k+1} \\ \mu_2{}^{k+1} \end{bmatrix}=\begin{bmatrix} p^k \\ \mu_1{}^k \\ \mu_2{}^k \end{bmatrix}-\left[U_3(p^k,\mu_1{}^k,\mu_2{}^k)\right]^{-1}\begin{bmatrix} f(p^k,\mu_1{}^k,\mu_2{}^k) \\ g(p^k,\mu_1{}^k,\mu_2{}^k) \\ h(p^k,\mu_1{}^k,\mu_2{}^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots.$$

The matrix L_2 of L_2-method for system (4.6) is

$$L_2(p,\mu_1,\mu_2)=\begin{bmatrix} f_p & 0 & 0 \\ g_p & g_{\mu_1} & 0 \\ 0 & h_{\mu_1} & h_{\mu_2} \end{bmatrix}=\begin{bmatrix} -3p^2T^2(\mu_1+\mu_2)-\alpha^3 & 0 & 0 \\ \dfrac{5}{p \ln 10} & \dfrac{8.75}{\mu_1 \ln 10} & 0 \\ 0 & 0 & \dfrac{8.75}{\mu_2 \ln 10} \end{bmatrix}.$$

Therefore L_2-method algorithm could be written as follows:

$$\begin{bmatrix} p^{k+1} \\ \mu_1{}^{k+1} \\ \mu_2{}^{k+1} \end{bmatrix}=\begin{bmatrix} p^k \\ \mu_1{}^k \\ \mu_2{}^k \end{bmatrix}-\left[L_2(p^k,\mu_1{}^k,\mu_2{}^k)\right]^{-1}\begin{bmatrix} f(p^k,\mu_1{}^k,\mu_2{}^k) \\ g(p^k,\mu_1{}^k,\mu_2{}^k) \\ h(p^k,\mu_1{}^k,\mu_2{}^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots.$$

The matrix L_3 of L_3 -method for system (4.6) is

$$L_3(p,\mu_1,\mu_2)=\begin{bmatrix} f_p & 0 & 0 \\ g_p & g_{\mu_1} & 0 \\ h_p & h_{\mu_1} & h_{\mu_2} \end{bmatrix}=\begin{bmatrix} -3p^2T^2(\mu_1+\mu_2)-\alpha^3 & 0 & 0 \\ \dfrac{5}{p\,ln10} & \dfrac{8.75}{\mu_1\,ln10} & 0 \\ \dfrac{5}{p\,ln10} & 0 & \dfrac{8.75}{\mu_2\,ln10} \end{bmatrix}.$$

Therefore L_3 -method algorithm could be written as follows:

$$\begin{bmatrix} p^{k+1} \\ \mu_1^{k+1} \\ \mu_2^{k+1} \end{bmatrix}=\begin{bmatrix} p^{k} \\ \mu_1^{k} \\ \mu_2^{k} \end{bmatrix}-\left[L_3(p^k,\mu_1^k,\mu_2^k)\right]^{-1}\begin{bmatrix} f(p^k,\mu_1^k,\mu_2^k) \\ g(p^k,\mu_1^k,\mu_2^k) \\ h(p^k,\mu_1^k,\mu_2^k) \end{bmatrix} \qquad ;k=0,1,2,\dots .$$

Table 4.6 Contains the results from D-method, U_2-method, U_3-method, L_2- method and L_3-method of example 4.6

Number of iterations(k) from D-method	Number of iterations(k) from U_2-method	Number of iterations(k) from U_3-method	Number of iterations(k) from L_2-method	Number of iterations(k) from L_3-method
25	19	12	20	14

The above methods solution converges to $p = 0.2176964$, $\mu_1 = 0.7507815$ and $\mu_2 = 0.467519$.

Showing the iterations of table 4.6 in appendix.

Example 4.7 The hydrogen spectrum in atomic physics. [10]

This application is taken from the Bohr theory of the hydrogen spectrum in atomic physics. A certain constant known as the Rydberg constant, usually denoted by R_∞, plays an important part in that theory, but it cannot be measured directly. Values R_H and R_{He} can be determined from measurements of the spectrum of hydrogen and helium, respectively. If we knew the exact mass M_H of the nucleus of a hydrogen atom and the exact mass m of an electron, then theoretically, we could determine R_∞ from the equation

$$R_\infty = (1 + \frac{m}{M_H})R_H .$$

Suppose, for the moment, that m and M_H are unknown (because even our best values are inferred from other measured variables). The atomic weight A_H of hydrogen can be measured. Also, $A_H = M_H + m$. Similar equations hold for the helium atom. So we have a system of four equations:

$$R_\infty = (1 + \frac{m}{M_H})R_H \qquad R_\infty = (1 + \frac{m}{M_{He}})R_{He}$$

$$A_H = M_H + m \qquad A_{He} = M_{He} + m$$

In this system, the numbers A_H, A_{He}, R_H, and R_{He} can be taken determined from laboratory measurements. The number R_∞ and masses m, M_H and M_{He} will be regarded as unknown. Their values can be determined by solving the system of four equations.

The work is simplified if we first make some algebraic changes in the system. Let $x = R_\infty$, $y = m$, $z = M_H$, and $w = M_{He}$. Then our system becomes

$$x = (1 + \frac{y}{w})R_{He}$$

$$x = (1 + \frac{y}{z})R_H$$

$$z + y = A_H$$

$$2y + w = A_{He}$$

The result is the following system:

$$\begin{aligned} f(x,y,z,w) &= x - (1 + \frac{y}{w})R_{He} = 0 \\ g(x,y,z,w) &= x - (1 + \frac{y}{z})R_H = 0 \\ h(x,y,z,w) &= z + y - A_H = 0 \\ j(x,y,z,w) &= 2y + w - A_{He} = 0 \end{aligned} \qquad (4.7)$$

Since $x = R_\infty$ and R_∞ is approximated by the measured number R_H, we have an initial estimate of x. Similarly, since y is the mass of the electron and we know that the value is somewhere around $\frac{1}{2000}$ of a unit of atomic mass, an initial estimate of 0.0005 for y seems reasonable. We need values of A_H, A_{He}, R_H, and R_{He} to perform the calculations. Suppose laboratory measurements produced the following values: A_H=1.00812, A_{He}=4.00388, R_H=109677.68 and R_{He}=109722.34. What are the corresponding values for Rydberg constant R_∞ and masses m, M_H and M_{He}? A machine implementation of the iterations suggested here produced the following results. Next we select initial values for x= 109677, y= 0.0005, z= 1 and w= 5. Repeat the methods until the error less than 0.0000005.

The following matrix is the matrix H for Modified Newton method

$$H(x,y,z,w)=\begin{bmatrix} f_x & 0 & 0 & 0 \\ 0 & g_y & 0 & 0 \\ 0 & 0 & h_z & 0 \\ 0 & 0 & 0 & j_w \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{-R_H}{z} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ w_{k+1} \end{bmatrix}=\begin{bmatrix} x_k \\ y_k \\ z_k \\ w_k \end{bmatrix}-[H(x_k,y_k,z_k,w_k)]^{-1}\begin{bmatrix} f(x_k,y_k,z_k,w_k) \\ g(x_k,y_k,z_k,w_k) \\ h(x_k,y_k,z_k,w_k) \\ k(x_k,y_k,z_k,w_k) \end{bmatrix} \quad ;k=0,1,2,\dots .$$

The following matrix is the matrix U_2 for U_2-method

$$U_2(x,y,z,w)=\begin{bmatrix} f_x & f_y & 0 & 0 \\ 0 & g_y & g_z & 0 \\ 0 & 0 & h_z & h_w \\ 0 & 0 & 0 & j_w \end{bmatrix}=\begin{bmatrix} 1 & \dfrac{-R_{He}}{w} & 0 & 0 \\ 0 & \dfrac{-R_H}{z} & \dfrac{yR_H}{z^2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the U_2-method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ w_{k+1} \end{bmatrix}=\begin{bmatrix} x_k \\ y_k \\ z_k \\ w_k \end{bmatrix}-[U_2(x_k,y_k,z_k,w_k)]^{-1}\begin{bmatrix} f(x_k,y_k,z_k,w_k) \\ g(x_k,y_k,z_k,w_k) \\ h(x_k,y_k,z_k,w_k) \\ k(x_k,y_k,z_k,w_k) \end{bmatrix} \quad ;k=0,1,2,\dots .$$

The following matrix is the matrix U_3 for U_3-method

$$U_3(x,y,z,w)=\begin{bmatrix} f_x & f_y & f_z & 0 \\ 0 & g_y & g_z & g_w \\ 0 & 0 & h_z & h_w \\ 0 & 0 & 0 & j_w \end{bmatrix}=\begin{bmatrix} 1 & \dfrac{-R_{He}}{w} & 0 & 0 \\ 0 & \dfrac{-R_H}{z} & \dfrac{yR_H}{z^2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the U_3-method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \\ w_k \end{bmatrix} - [U_3(x_k, y_k, z_k, w_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k, w_k) \\ g(x_k, y_k, z_k, w_k) \\ h(x_k, y_k, z_k, w_k) \\ k(x_k, y_k, z_k, w_k) \end{bmatrix} \qquad ; k = 0,1,2,\ldots .$$

The following matrix is the matrix U_4 for U_4-method

$$U_4(x,y,z,w) = \begin{bmatrix} f_x & f_y & f_z & f_w \\ 0 & g_y & g_z & g_w \\ 0 & 0 & h_z & h_w \\ 0 & 0 & 0 & j_w \end{bmatrix} = \begin{bmatrix} 1 & \frac{-R_{He}}{w} & 0 & \frac{yR_{He}}{w^2} \\ 0 & \frac{-R_H}{z} & \frac{yR_H}{z^2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the U_4-method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \\ w_k \end{bmatrix} - [U_4(x_k, y_k, z_k, w_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k, w_k) \\ g(x_k, y_k, z_k, w_k) \\ h(x_k, y_k, z_k, w_k) \\ k(x_k, y_k, z_k, w_k) \end{bmatrix} \qquad ; k = 0,1,2,\ldots .$$

The following matrix is the matrix L_2 for L_2-method

$$L_2(x,y,z,w) = \begin{bmatrix} f_x & 0 & 0 & 0 \\ g_x & g_y & 0 & 0 \\ 0 & h_y & h_z & 0 \\ 0 & 0 & j_z & j_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{-R_H}{z} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the L_2-method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \\ w_k \end{bmatrix} - [L_2(x_k, y_k, z_k, w_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k, w_k) \\ g(x_k, y_k, z_k, w_k) \\ h(x_k, y_k, z_k, w_k) \\ k(x_k, y_k, z_k, w_k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

The following matrix is the matrix L_3 for L_3-method

$$L_3(x, y, z, w) = \begin{bmatrix} f_x & 0 & 0 & 0 \\ g_x & g_y & 0 & 0 \\ h_x & h_y & h_z & 0 \\ 0 & j_y & j_z & j_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{-R_H}{z} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}.$$

Therefore the L_3-method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \\ w_k \end{bmatrix} - [L_3(x_k, y_k, z_k, w_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k, w_k) \\ g(x_k, y_k, z_k, w_k) \\ h(x_k, y_k, z_k, w_k) \\ k(x_k, y_k, z_k, w_k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

The following matrix is the matrix L_4 for L_4-method

$$L_4(x, y, z, w) = \begin{bmatrix} f_x & 0 & 0 & 0 \\ g_x & g_y & 0 & 0 \\ h_x & h_y & h_z & 0 \\ j_x & j_y & j_z & j_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{-R_H}{z} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}.$$

Therefore the L_4-method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \\ w_k \end{bmatrix} - [L_4(x_k, y_k, z_k, w_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k, w_k) \\ g(x_k, y_k, z_k, w_k) \\ h(x_k, y_k, z_k, w_k) \\ k(x_k, y_k, z_k, w_k) \end{bmatrix} \qquad ; k = 0,1,2,\dots .$$

Table 4.7 Contains the results from D-method, U_2-method, U_3-method, U_4-method L_2-method, L_3-method, and L_4-method of example 4.7

Number of iterations(k) from D-method	Number of iterations(k) from U_2-method	Number of iterations(k) from U_3-method	Number of iterations(k) from U_4-method	Number of iterations(k) from L_2-method	Number of iterations(k) from L_3-method	Number of iterations(k) from L_4-method
28	14	14	15	13	13	13

For this problem, we obtain that

$$R_\infty = 109737.37$$

electron mass : $m = 0.00054837$

mass of helium nucleus : $M_{He} = 4.00278327$

mass of hydrogen nucleus : $M_H = 1.00757163$

hydrogen mass / electron mass $\frac{M_H}{m} = 1837.41$.

The value of one atomic mass unit is 1.6599×10^{-24} g. So the computed mass of the electron in this case is equivalent to

$$0.00054837 \times 1.6599 \times 10^{-24} = 9.103 \times 10^{-28} \text{ g.}$$

Showing the iterations of table 4.7 in appendix.

Example 4.8

For turbulent flow of fluids in an interconnected network, the flow rate V from one node to another is about proportional to the square root of the difference in pressures at the nodes. (Thus fluid flow differs from flow of electrical current in a network in that nonlinear equations result.) For the conduits in Figure 4.3, find the pressure at each node. The values of b represent conductance factors in the relation $v_{ij} = b_{ij}(p_i - p_j)^{1/2}$.

These equations can be set up for the pressures at each node:

At node 1: $0.3\sqrt{500 - p_1} = 0.2\sqrt{p_1 - p_2} + 0.1\sqrt{p_1 - p_3}$;

node 2: $0.2\sqrt{p_1 - p_2} = 0.1\sqrt{p_2 - p_4} + 0.2\sqrt{p_2 - p_3}$;

node 3: $0.1\sqrt{p_1 - p_3} + 0.2\sqrt{p_2 - p_3} = 0.1\sqrt{p_3 - p_4}$;

node 4: $0.1\sqrt{p_2 - p_4} + 0.1\sqrt{p_3 - p_4} = 0.2\sqrt{p_4 - 0}$.

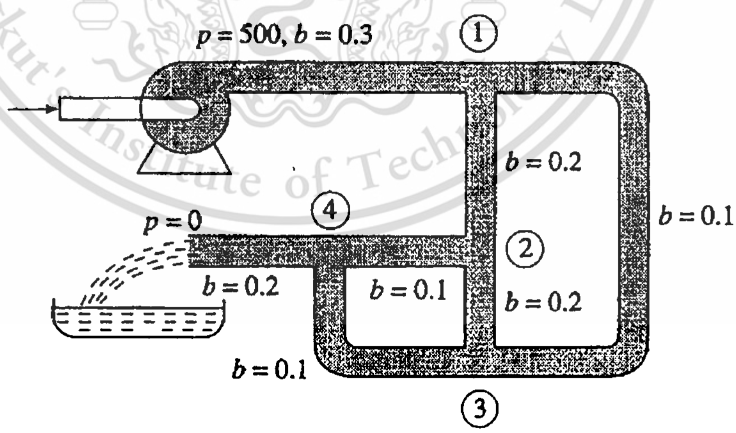

Figure 4.3 A turbulent flow of fluids in an interconnected network.

The result is the following system of four equations with four unknowns:

$$f(p_1, p_2, p_3, p_4) = 0.2\sqrt{p_1 - p_2} + 0.1\sqrt{p_1 - p_3} - 0.3\sqrt{500 - p_1} = 0$$

$$g(p_1, p_2, p_3, p_4) = 0.2\sqrt{p_1 - p_2} - 0.1\sqrt{p_2 - p_4} - 0.2\sqrt{p_2 - p_3} = 0 \quad (4.8)$$

$$h(p_1, p_2, p_3, p_4) = 0.1\sqrt{p_1 - p_3} + 0.2\sqrt{p_2 - p_3} - 0.1\sqrt{p_3 - p_4} = 0$$

$$j(p_1, p_2, p_3, p_4) = 0.1\sqrt{p_2 - p_4} + 0.1\sqrt{p_3 - p_4} - 0.2\sqrt{p_4} = 0$$

Next we select the initial values for p_1=400, p_2=350, p_3=340 and p_4=180. Repeat the methods until the error less than 0.0000005.

The matrix H of D-method for system (4.8) is

$$H(p_1, p_2, p_3, p_4) = \begin{bmatrix} f_{p_1} & 0 & 0 & 0 \\ 0 & g_{p_2} & 0 & 0 \\ 0 & 0 & h_{p_3} & 0 \\ 0 & 0 & 0 & j_{p_4} \end{bmatrix}.$$

Therefore D-method algorithm could be written as follows:

$$\begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \\ p_4^{k+1} \end{bmatrix} = \begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \\ p_4^{k} \end{bmatrix} - \left[H(p_1^k, p_2^k, p_3^k, p_4^k)\right]^{-1} \begin{bmatrix} f(p_1^k, p_2^k, p_3^k, p_4^k) \\ g(p_1^k, p_2^k, p_3^k, p_4^k) \\ h(p_1^k, p_2^k, p_3^k, p_4^k) \\ j(p_1^k, p_2^k, p_3^k, p_4^k) \end{bmatrix} \quad ; k = 0,1,2,\ldots .$$

The matrix U_2 of U_2-method for system (4.8) is

$$U_2(p_1, p_2, p_3, p_4) = \begin{bmatrix} f_{p_1} & f_{p_2} & 0 & 0 \\ 0 & g_{p_2} & g_{p_3} & 0 \\ 0 & 0 & h_{p_3} & h_{p_4} \\ 0 & 0 & 0 & j_{p_4} \end{bmatrix}.$$

Therefore U_2 -method algorithm could be written as follows:

$$\begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \\ p_4^{k+1} \end{bmatrix} = \begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \\ p_4^{k} \end{bmatrix} - \left[U_2(p_1^k, p_2^k, p_3^k, p_4^k)\right]^{-1} \begin{bmatrix} f(p_1^k, p_2^k, p_3^k, p_4^k) \\ g(p_1^k, p_2^k, p_3^k, p_4^k) \\ h(p_1^k, p_2^k, p_3^k, p_4^k) \\ j(p_1^k, p_2^k, p_3^k, p_4^k) \end{bmatrix} \quad ;k = 0,1,2,\ldots .$$

The matrix U_3 of U_3 -method for system (4.8) is

$$U_3(p_1, p_2, p_3, p_4) = \begin{bmatrix} f_{p_1} & f_{p_2} & f_{p_3} & 0 \\ 0 & g_{p_2} & g_{p_3} & g_{p_4} \\ 0 & 0 & h_{p_3} & h_{p_4} \\ 0 & 0 & 0 & j_{p_4} \end{bmatrix}.$$

Therefore U_3 -method algorithm could be written as follows:

$$\begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \\ p_4^{k+1} \end{bmatrix} = \begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \\ p_4^{k} \end{bmatrix} - \left[U_3(p_1^k, p_2^k, p_3^k, p_4^k)\right]^{-1} \begin{bmatrix} f(p_1^k, p_2^k, p_3^k, p_4^k) \\ g(p_1^k, p_2^k, p_3^k, p_4^k) \\ h(p_1^k, p_2^k, p_3^k, p_4^k) \\ j(p_1^k, p_2^k, p_3^k, p_4^k) \end{bmatrix} \quad ;k = 0,1,2,\ldots .$$

The matrix U_4 of U_4 -method for system (4.8) is

$$U_4(p_1, p_2, p_3, p_4) = \begin{bmatrix} f_{p_1} & f_{p_2} & f_{p_3} & f_{p_4} \\ 0 & g_{p_2} & g_{p_3} & g_{p_4} \\ 0 & 0 & h_{p_3} & h_{p_4} \\ 0 & 0 & 0 & j_{p_4} \end{bmatrix}.$$

Therefore U_4 -method algorithm could be written as follows:

$$\begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \\ p_4^{k+1} \end{bmatrix} = \begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \\ p_4^{k} \end{bmatrix} - \left[U_4(p_1^k, p_2^k, p_3^k, p_4^k)\right]^{-1} \begin{bmatrix} f(p_1^k, p_2^k, p_3^k, p_4^k) \\ g(p_1^k, p_2^k, p_3^k, p_4^k) \\ h(p_1^k, p_2^k, p_3^k, p_4^k) \\ j(p_1^k, p_2^k, p_3^k, p_4^k) \end{bmatrix} \quad ;k = 0,1,2,\ldots .$$

The matrix L_2 of L_2-method for system (4.8) is

$$L_2(p_1,p_2,p_3,p_4)=\begin{bmatrix} f_{p_1} & 0 & 0 & 0 \\ g_{p_1} & g_{p_2} & 0 & 0 \\ 0 & h_{p_2} & h_{p_3} & 0 \\ 0 & 0 & j_{p_3} & j_{p_4} \end{bmatrix}.$$

Therefore L_2-method algorithm could be written as follows:

$$\begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \\ p_4^{k+1} \end{bmatrix}=\begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \\ p_4^{k} \end{bmatrix}-\left[L_2(p_1^k,p_2^k,p_3^k,p_4^k)\right]^{-1}\begin{bmatrix} f(p_1^k,p_2^k,p_3^k,p_4^k) \\ g(p_1^k,p_2^k,p_3^k,p_4^k) \\ h(p_1^k,p_2^k,p_3^k,p_4^k) \\ j(p_1^k,p_2^k,p_3^k,p_4^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots .$$

The matrix L_3 of L_3-method for system (4.8) is

$$L_3(p_1,p_2,p_3,p_4)=\begin{bmatrix} f_{p_1} & 0 & 0 & 0 \\ g_{p_1} & g_{p_2} & 0 & 0 \\ h_{p_1} & h_{p_2} & h_{p_3} & 0 \\ 0 & j_{p_2} & j_{p_3} & j_{p_4} \end{bmatrix}.$$

Therefore L_3-method algorithm could be written as follows:

$$\begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \\ p_4^{k+1} \end{bmatrix}=\begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \\ p_4^{k} \end{bmatrix}-\left[L_3(p_1^k,p_2^k,p_3^k,p_4^k)\right]^{-1}\begin{bmatrix} f(p_1^k,p_2^k,p_3^k,p_4^k) \\ g(p_1^k,p_2^k,p_3^k,p_4^k) \\ h(p_1^k,p_2^k,p_3^k,p_4^k) \\ j(p_1^k,p_2^k,p_3^k,p_4^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots .$$

The matrix L_4 of L_4-method for system (4.8) is

$$L_4(p_1,p_2,p_3,p_4)=\begin{bmatrix} f_{p_1} & 0 & 0 & 0 \\ g_{p_1} & g_{p_2} & 0 & 0 \\ h_{p_1} & h_{p_2} & h_{p_3} & 0 \\ j_{p_1} & j_{p_2} & j_{p_3} & j_{p_4} \end{bmatrix}.$$

Therefore L_4-method algorithm could be written as follows:

$$\begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \\ p_4^{k+1} \end{bmatrix} = \begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \\ p_4^{k} \end{bmatrix} - \left[L_4(p_1^k, p_2^k, p_3^k, p_4^k)\right]^{-1} \begin{bmatrix} f(p_1^k, p_2^k, p_3^k, p_4^k) \\ g(p_1^k, p_2^k, p_3^k, p_4^k) \\ h(p_1^k, p_2^k, p_3^k, p_4^k) \\ j(p_1^k, p_2^k, p_3^k, p_4^k) \end{bmatrix} \qquad ; k = 0,1,2,\ldots\,.$$

Table 4.8 Contains the results from D-method, U_2-method, U_3-method, U_4-method L_2- method, L_3- method and L_4- method of example 4.8

Number of iterations(k) from D-method	Number of iterations(k) from U_2-method	Number of iterations(k) from U_3-method	Number of iterations(k) from U_4-method	Number of iterations(k) from L_2-method	Number of iterations(k) from L_3-method	Number of iterations(k) from L_4-method
59	31	43	43	54	49	49

The above methods solution converges to p_1= 423.1897384, p_2=347.7936349, p_3=343.5117615 and p_4 = 172.8230339.

Showing the iterations of table 4.8 in appendix.

Example 4.9

Here is the problem of the hanging chain, which has been used to test the minimizer. Given a chain of 8 sticks of unit weight and unit length, we are required to find the shape it will assume when suspended between 2 points spaced 6 units horizontally apart. In other words, we must find the angles α, β, γ and δ. Solving this by the 'energy method' involves finding an expression for the chain's energy, depending on α, β and γ let the nonlinear minimizer do the rest. A different approach comes from statics. We introduce a 5 th unknown, the horizontal component x of the force going through the chain, which is the same in all joints. Then there is this equation system for α, β, γ, δ and x:

$$2x\,sin(\alpha) - 7\,cos(\alpha) = 0 \tag{4.9}$$

$$2x\,sin(\beta) - 5\,cos(\beta) = 0 \tag{4.10}$$

$$2x\,sin(\gamma) - 3\,cos(\gamma) = 0 \tag{4.11}$$

$$2x\,sin(\delta) - \,cos(\delta) = 0 \tag{4.12}$$

$$cos(\alpha) + cos(\beta) + cos(\gamma) + cos(\delta) = 3 \tag{4.13}$$

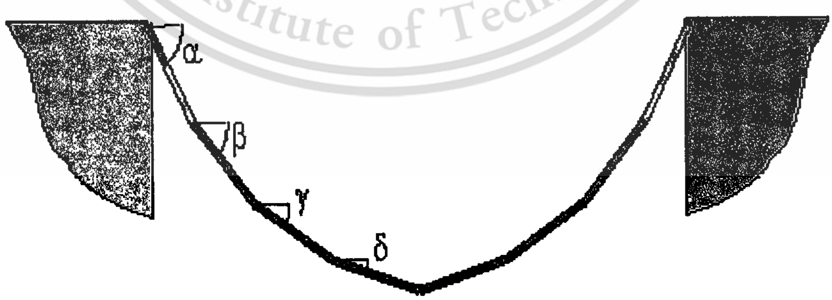

Figure 4.4 The problem hanging chain.

Equations (4.9) to (4.12) arise from the requirement that, in the state of equilibrium, the torque on each stick equals 0. Equation (4.13) is purely geometric, stating that the horizontal length of the chain is 6 units.

The result is the following system of five equations with five unknowns:

$$\begin{aligned}
f(\alpha,\beta,\gamma,\delta,x) &= 2x\sin(\alpha) - 7\cos(\alpha) = 0 \\
g(\alpha,\beta,\gamma,\delta,x) &= 2x\sin(\beta) - 5\cos(\beta) = 0 \\
h(\alpha,\beta,\gamma,\delta,x) &= 2x\sin(\gamma) - 3\cos(\gamma) = 0 \\
j(\alpha,\beta,\gamma,\delta,x) &= 2x\sin(\delta) - \cos(\delta) = 0 \\
q(\alpha,\beta,\gamma,\delta,x) &= \cos(\alpha) + \cos(\beta) + \cos(\gamma) + \cos(\delta) - 3 = 0
\end{aligned} \tag{4.14}$$

Next we select the initial values for $\alpha = 0.5 \approx 28.65^\circ$, $\beta = 0.5 \approx 28.65^\circ$, $\gamma = 0.5 \approx 28.65^\circ$, $\delta = 0.5 \approx 28.65^\circ$ and $x = 2$. Repeat the methods until the error less than 0.0000005.

The matrix H of D-method for system (4.14) is

$$H(\alpha,\beta,\gamma,\delta,x) = \begin{bmatrix} f_\alpha & 0 & 0 & 0 & 0 \\ 0 & g_\beta & 0 & 0 & 0 \\ 0 & 0 & h_\gamma & 0 & 0 \\ 0 & 0 & 0 & j_\delta & 0 \\ 0 & 0 & 0 & 0 & q_x \end{bmatrix}.$$

Therefore D-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix} = \begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix} - \left[H(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1} \begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \quad ;k = 0,1,2,\ldots .$$

The matrix U_2 of U_2-method for system (4.14) is

$$U_2(\alpha,\beta,\gamma,\delta,x)=\begin{bmatrix} f_\alpha & f_\beta & 0 & 0 & 0 \\ 0 & g_\beta & g_\gamma & 0 & 0 \\ 0 & 0 & h_\gamma & h_\delta & 0 \\ 0 & 0 & 0 & j_\delta & j_x \\ 0 & 0 & 0 & 0 & q_x \end{bmatrix}.$$

Therefore U_2-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix}-\left[U_2(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots$$

The matrix U_3 of U_3-method for system (4.14) is

$$U_3(\alpha,\beta,\gamma,\delta,x)=\begin{bmatrix} f_\alpha & f_\beta & f_\gamma & 0 & 0 \\ 0 & g_\beta & g_\gamma & g_\delta & 0 \\ 0 & 0 & h_\gamma & h_\delta & h_x \\ 0 & 0 & 0 & j_\delta & j_x \\ 0 & 0 & 0 & 0 & q_x \end{bmatrix}.$$

Therefore U_3-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix}-\left[U_3(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \qquad ;k=0,1,2,\ldots\,.$$

The matrix U_4 of U_4-method for system (4.14) is

$$U_4(\alpha,\beta,\gamma,\delta,x) = \begin{bmatrix} f_\alpha & f_\beta & f_\gamma & f_\delta & 0 \\ 0 & g_\beta & g_\gamma & g_\delta & g_x \\ 0 & 0 & h_\gamma & h_\delta & h_x \\ 0 & 0 & 0 & j_\delta & j_x \\ 0 & 0 & 0 & 0 & q_x \end{bmatrix}.$$

Therefore U_4-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix} = \begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix} - \left[U_4(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1} \begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \quad ;k = 0,1,2,\ldots\ .$$

The matrix U_5 of U_5-method for system (4.14) is

$$U_5(\alpha,\beta,\gamma,\delta,x) = \begin{bmatrix} f_\alpha & f_\beta & f_\gamma & f_\delta & f_x \\ 0 & g_\beta & g_\gamma & g_\delta & g_x \\ 0 & 0 & h_\gamma & h_\delta & h_x \\ 0 & 0 & 0 & j_\delta & j_x \\ 0 & 0 & 0 & 0 & q_x \end{bmatrix}.$$

Therefore U_5-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix} = \begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix} - \left[U_5(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1} \begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \quad ;k = 0,1,2,\ldots$$

The matrix L_2 of L_2-method for system (4.14) is

$$L_2(\alpha,\beta,\gamma,\delta,x)=\begin{bmatrix} f_\alpha & 0 & 0 & 0 & 0 \\ g_\alpha & g_\beta & 0 & 0 & 0 \\ 0 & h_\beta & h_\gamma & 0 & 0 \\ 0 & 0 & j_\gamma & j_\delta & 0 \\ 0 & 0 & 0 & q_\delta & q_x \end{bmatrix}.$$

Therefore L_2-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix}-\left[L_2(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \quad ;k=0,1,2,\ldots\,.$$

The matrix L_3 of L_3-method for system (4.14) is

$$L_3(\alpha,\beta,\gamma,\delta,x)=\begin{bmatrix} f_\alpha & 0 & 0 & 0 & 0 \\ g_\alpha & g_\beta & 0 & 0 & 0 \\ h_\alpha & h_\beta & h_\gamma & 0 & 0 \\ 0 & j_\beta & j_\gamma & j_\delta & 0 \\ 0 & 0 & q_\gamma & q_\delta & q_x \end{bmatrix}.$$

Therefore L_3-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix}-\left[L_3(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \quad ;k=0,1,2,\ldots\,.$$

The matrix L_4 of L_4-method for system (4.14) is

$$L_4(\alpha,\beta,\gamma,\delta,x)=\begin{bmatrix} f_\alpha & 0 & 0 & 0 & 0 \\ g_\alpha & g_\beta & 0 & 0 & 0 \\ h_\alpha & h_\beta & h_\gamma & 0 & 0 \\ j_\alpha & j_\beta & j_\gamma & j_\delta & 0 \\ 0 & q_\beta & q_\gamma & q_\delta & q_x \end{bmatrix}.$$

Therefore L_4-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix}-\left[L_4(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \quad ;k=0,1,2,\ldots\ .$$

The matrix L_5 of L_5-method for system (4.14) is

$$L_5(\alpha,\beta,\gamma,\delta,x)=\begin{bmatrix} f_\alpha & 0 & 0 & 0 & 0 \\ g_\alpha & g_\beta & 0 & 0 & 0 \\ h_\alpha & h_\beta & h_\gamma & 0 & 0 \\ j_\alpha & j_\beta & j_\gamma & j_\delta & 0 \\ q_\alpha & q_\beta & q_\gamma & q_\delta & q_x \end{bmatrix}.$$

Therefore L_5-method algorithm could be written as follows:

$$\begin{bmatrix} \alpha^{k+1} \\ \beta^{k+1} \\ \gamma^{k+1} \\ \delta^{k+1} \\ x^{k+1} \end{bmatrix}=\begin{bmatrix} \alpha^{k} \\ \beta^{k} \\ \gamma^{k} \\ \delta^{k} \\ x^{k} \end{bmatrix}-\left[L_5(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k)\right]^{-1}\begin{bmatrix} f(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ g(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ h(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ j(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \\ q(\alpha^k,\beta^k,\gamma^k,\delta^k,x^k) \end{bmatrix} \quad ;k=0,1,2,\ldots\ .$$

Table 4.9 Contains the results from D-method, U_2-method, U_3-method, U_4-method, U_5-method, L_2- method, L_3- method, L_4- method and L_5- method of example 4.9

Methods	Number of iterations (k)
D-method	80
U_2-method	62
U_3-method	62
U_4-method	56
U_5-method	64
L_2-method	55
L_3-method	42
L_4-method	37
L_5-method	39

The above methods solution converges to $\alpha = 1.0071539 \approx 57.71°$, $\beta = 0.8464150 \approx 48.50°$, $\gamma = 0.5958635 \approx 34.14°$, $\delta = 0.2222919 \approx 12.74°$, and $x = 2.2121238$.

Showing the iterations of table 4.9 in appendix.

Example 4.10 Flash vaporization

We end this section with yet another example from the multifaceted field of chemical engineering [11]. A liquid containing 50 mol % of benzene, denoted by B, 25 mol % of toluene, denoted by T, and 25 mol % of o-xylem, denoted by O, is supplied into a flash evaporator operating at 1 atm or 760 mm Hg, and 100°C. We want to compute the fraction of the total moles that end up in the gas and in the liquid phase, as well as the compositions of the gas and the liquid phases.

It is reasonable to assume that the gas in the evaporator is in equilibrium with the liquid. Furthermore, for lack of a better alternative, we assume that the equilibrium molar fraction are given by Henry's law, relating the mole fraction of the i^{th} species in the gas phase $f_{G,i}$ to the corresponding mole fraction in the liquid phase $f_{L,i}$ at equilibrium, by means of the equation

$$f_{G,i} = m_i f_{L,i} \tag{4.15}$$

where m_i are constants. Raoult's law says that

$$m_i = \frac{P_i}{P_{Tot}} \tag{4.16}$$

where P_i is the vapor pressure of the i^{th} species, and P_{Tot} is the total pressure of the mixture. The vapor pressures of the benzene, toluene, and o-xylene at 100°C are, respectively, equal to 1370, 550, and 200 mm Hg.

Proceeding with the mathematical formulation, we denote the feed flow rate by F, the liquid flow rate by L, and the gas flow rate by G, all measured in mol/min. The mole fractions of the feed stream are

$$f_{F,B} = 0.50, \quad f_{F,T} = 0.25, \quad f_{F,O} = 0.25,$$

Next, we introduce the unknown fractions of total moles that end up in the liquid and in the gas phase,

$$x_1 = L / F, \quad x_2 = G / F$$

And the unknown mole fractions of the species in the gas and liquid phases,

$$x_3 = f_{G,B} \qquad x_4 = f_{G,T} \qquad x_5 = f_{G,O}$$

$$x_6 = f_{L,B} \qquad x_7 = f_{L,T} \qquad x_8 = f_{L,O}$$

We have a total of eight unknown and require an equal number of equations.

Two equations arise by requiring that the sun of the mole fractions of all species add up to unity:

$$x_3 + x_4 + x_5 = 1 \tag{4.17}$$

and

$$x_6 + x_7 + x_8 = 1 \tag{4.18}$$

Three additional equations arise by performing mass balances. An over all mole balance gives $L + G = F$. Dividing both sides by F, we obtain

$$x_1 + x_2 = 1 \tag{4.19}$$

A mole balance for benzene gives $Ff_{F,B} = Lf_{L,B} + Gf_{G,B}$. Dividing both sides by F, we obtain

$$0.50 = x_1 x_6 + x_2 x_3 \tag{4.20}$$

Working similarly with the toluene, we find

$$0.25 = x_1 x_7 + x_2 x_4 \tag{4.21}$$

A mole balance for o-xylene is implicit in equations (4.17)-(4.21); that is, it can be derived from them and does not have to be written down explicitly.

Equations (4.15) and (4.16) provide us with three additional equations,

$$x_3 = \frac{1370}{760} x_6, \tag{4.22}$$

$$x_4 = \frac{550}{760} x_7, \tag{4.23}$$

$$x_5 = \frac{200}{760} x_8. \tag{4.24}$$

We have obtained the desired system of eight linear and nonlinear equations. The result is the following system:

$$\begin{aligned}
f_1(x_1, x_2, x_3, x_4, x_5, x_6 x_7, x_8) &= x_3 + x_4 + x_5 - 1 = 0 \\
f_2(x_1, x_2, x_3, x_4, x_5, x_6 x_7, x_8) &= x_6 + x_7 + x_8 - 1 = 0 \\
f_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= x_1 + x_2 - 1 = 0 \\
f_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= x_1 x_6 + x_2 x_3 - 0.50 = 0 \\
f_5(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= x_1 x_7 + x_2 x_4 - 0.25 = 0 \\
f_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= \frac{1370}{760} x_6 - x_3 = 0 \\
f_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= \frac{550}{760} x_7 - x_4 = 0 \\
f_8(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= \frac{200}{760} x_8 - x_5 = 0
\end{aligned} \tag{4.25}$$

Next we select the initial values for $x_1 = 0$, $x_2 = 0.5$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 1$, $x_7 = 0.5$, and $x_8 = 1$. Repeat the methods until the error less than 0.0000005.

Table 4.10 Contains the results from D-method, U_2-method, U_3-method, U_4-method, L_2- method, L_3- method and L_4- method of example 4.10

Number of iterations(k) from D -method	Number of iterations(k) from U_2-method	Number of iterations(k) from U_3-method	Number of iterations(k) from U_4-method	Number of iterations(k) from L_2-method	Number of iterations(k) from L_3-method	Number of iterations(k) from L_4-method
36	26	14	14	30	28	18

The above methods solution converges to $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 = 0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746482$ and $x_8 = 0.32884$.

Showing the iterations of table 4.10 in appendix.

CHAPTER 5

CONCLUSION AND SUGGESTIONS

In this chapter, we shall give the conclusion and suggestions. We could notice from some examples and application problems in chapter 4.

5.1 Conclusion

In this thesis, we found the methods that can solve systems of nonlinear equations $f(x) = 0$ in the iterative fashion. We used the 2n-1 numerical methods, where the first method is D -method. The second method is U_2-method. The third method is U_3-method. The fourth method is U_4-method. The n^{th} method is U_n-method. The $(n+1)^{\text{th}}$ method is L_2-method. The $(n+2)^{\text{th}}$ method is L_3-method. The $(n+3)^{\text{th}}$ method is L_4- method and the $(2n-1)^{\text{th}}$ method is L_n- method.

An important question to be asked in using any iterative method is: When do we stop? For our algorithms, we set an absolute tolerance or absolute error; ε , and iterated until either $\sum_{i,j=1}^{n}\left|f_i(x_j) - f_i(x_{j-1})\right| < \varepsilon$, or we reached a maximum number of iterations. It may happen that if we set tolerance too small then we may need initial value that closed to its solution. In this thesis we will consider the number of iterations of our methods from assuming that the absolute error is ε = 0.0000005 for all examples.

The results from examples in this thesis are somewhat as good as we expected. We can see from the example 4.2 at the initial point (1,1.5,1), the U_2-method has the number of iteration 86 % less than the Modified Newton method. And for the example 4.5 with the initial point (1,1,0), the L_3-method has the number of iteration 77 % less than the Modified Newton method. By these results, we can notice that the solution of our method converges faster than the Modified Newton method. For the example 4.9, we cannot solve the system of nonlinear equation immediately because we have a singular matrix, which cannot find the inverse matrix. For this problem, we have to rearrange our system in order to get a nonsingular matrix. In some problems, in order to get the solutions of the system of equations the arrangement of the equation is very important. The other important thing is that technique to choose the initial value. We have to choose the initial value, which is close to the root. Sometime, if we choose the initial point far from the root we cannot obtain the solution for the problem though in actually this problem has a solution. In order to solve the systems of nonlinear equation, guessing an initial value is easier than using theorem to find an initial value.

For efficiently guessing an initial value we need to understand the mathematical modeling of a real problem. In applied area, for examples in engineering and science, we roughly know the solution; hence we can easily guess an initial value. However, an efficient guess of an initial value requires deep understanding of the mathematical model, which is the task of the mathematician. It is easier for engineers or scientists who model the problem to derive an initial value. Thus the work for mathematicians is to guess an initial value until a solution is obtained.

It is possible for nonlinear systems to have more than one solution in which all obtained solutions do not satisfy the real world problem. In this case, engineers or scientists who model the problem need to resolve or improve upon a better model. Thus the duty of mathematicians is to create new methods and guess an initial value to solve the systems of nonlinear equations.

5.2 Suggestions

For solving systems of nonlinear equations, in order to solve complicated functions or the functions that are difficult to find its derivative, we can modify our methods by using

$$f_{ij} = \frac{f_i(x^{k+1}) - f_i(x^k)}{x_j^{k+1} - x_j^k} \quad , \quad i, j = 1,2,\dots \quad , \quad k = 0,1,2,\dots .$$

For solving systems of linear equations, we can also use the new matrices as introduced in chapter 3.

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APPENDIX

The iterations of table 4.1 at the initial point (0,3.5)

The result from D-method

k	x^k	y^k	ER
1	0.2228169	1.8278352	0.7533696
2	0.3962134	2.6305570	0.3712523
3	0.2471582	3.0315003	0.3105540
4	0.2869561	2.7006336	0.0998936
5	0.3053382	2.8062103	0.0406569
6	0.3007790	2.8509919	0.0102464
7	0.2988231	2.8401280	0.0042876
8	0.2993070	2.8354180	0.0011017
9	0.2995154	2.8365860	0.0004572
10	0.2994638	2.8370885	0.0001172
11	0.2994416	2.8369642	0.0000487
12	0.2994471	2.8369107	0.0000125
13	0.2994494	2.8369239	0.0000052
14	0.2994489	2.8369296	0.0000013
15	0.2994486	2.8369282	0.0000006
16	0.2994487	2.8369276	0.0000001

Estimates values are x = 0.2994487 and y = 2.8369276 after 16 iterations.

The result from U_2-method

k	x^k	y^k	ER
1	0.1163636	1.8278352	0.4895228
2	0.3918671	2.2802252	0.6562725
3	0.1847447	3.0245273	0.5104800
4	0.2708351	2.5131255	0.2385511
5	0.3008405	2.7648333	0.0661521
6	0.2990659	2.8402758	0.0037990
7	0.2994882	2.8360043	0.0008820
8	0.2994444	2.8370229	0.0000912
9	0.2994491	2.8369175	0.0000098
10	0.2994486	2.8369289	0.0000010
11	0.2994487	2.8369277	0.0000001

Estimates values are x = 0.2994487 and y = 2.8369277 after 11 iterations.

The result from L_2-method

k	x^k	y^k	ER
1	0.2228169	2.7537888	0.1492716
2	0.2909687	2.8323622	0.0178771
3	0.2994093	2.8371055	0.0002498
4	0.2994408	2.8369088	0.0000038
5	0.2994495	2.8369298	0.0000004

Estimates values are x = 0.2994495 and y = 2.8369298 after 5 iterations.

The iterations of table 4.1 at the initial point (0.75,2.5)

The result from D-method

k	x^k	y^k	ER
1	0.6391824	2.8365513	0.2774809
2	0.5678718	3.0879928	0.0896106
3	0.5191945	3.1476403	0.0122882
4	0.5007137	3.1490317	0.0070882
5	0.4988336	3.1419466	0.0013697
6	0.4999498	3.1410016	0.0005618
7	0.5000942	3.1415676	0.0001075
8	0.5000040	3.1416397	0.0000447
9	0.4999925	3.1415947	0.0000086
10	0.4999997	3.1415889	0.0000036
11	0.5000006	3.1415925	0.0000007
12	0.5000000	3.1415930	0.0000003

Estimates values are x = 0.5 and y = π after 12 iterations.

The result from the U_2-method

k	x^k	y^k	ER
1	0.5653217	2.8365513	0.2785413
2	0.4987763	3.1484665	0.0065493
3	0.5000988	3.1409722	0.0005796
4	0.4999921	3.1416420	0.0000461
5	0.5000006	3.1415887	0.0000037
6	0.5000000	3.1415930	0.0000003

Estimates values are x = 0.5 and y = π after 6 iterations.

The result from the L_2-method

k	x^k	y^k	ER
1	0.6391824	3.1968942	0.2230262
2	0.5342652	3.2302833	0.0981148
3	0.4939834	3.1481609	0.0108909
4	0.4990834	3.1412787	0.0006100
5	0.5000546	3.1416254	0.0000346
6	0.4999948	3.1415901	0.0000028
7	0.5000004	3.1415929	0.0000002

Estimates values are $x = 0.5$ and $y = \pi$ after 7 iterations.

The iterations of table 4.1 at the initial point (0.8,4)

The result from D-method

k	x^k	y^k	ER
1	0.5005928	2.6493156	0.4501992
2	0.6344765	3.1418870	0.1525165
3	0.5372529	3.0942064	0.0669484
4	0.5108286	3.1519902	0.0115312
5	0.4990337	3.1463239	0.0046230
6	0.4992472	3.1411041	0.0005078
7	0.5000812	3.1412130	0.0003740
8	0.5000604	3.1416332	0.0000424
9	0.4999936	3.1416228	0.0000297
10	0.4999952	3.1415894	0.0000034
11	0.5000005	3.1415903	0.0000024
12	0.5000004	3.1415929	0.0000003

Estimates values are $x = 0.5$ and $y = \pi$ after 12 iterations.

The result from U_2-method

k	x^k	y^k	ER
1	0.7596808	2.6493156	0.2726739
2	0.5944956	2.8041730	0.3079126
3	0.5025575	3.1337871	0.0084894
4	0.4998108	3.1428335	0.0011599
5	0.5000151	3.1414978	0.0000886
6	0.4999988	3.1416002	0.0000071
7	0.5000001	3.1415921	0.0000006
8	0.5000000	3.1415927	0.0000000

Estimates values are $x = 0.5$ and $y = \pi$ after 8 iterations.

The result from the L_2-method

k	x^k	y^k	ER
1	0.5005928	3.9231332	0.7766036
2	0.4201809	3.1022327	0.0560110
3	0.7001690	3.4456295	0.6978205
4	0.5370245	3.3586141	0.2280705
5	0.4815022	3.1488960	0.0239360
6	0.5006353	3.1439744	0.0022978
7	0.4996260	3.1414107	0.0002062
8	0.5000298	3.1416085	0.0000170
9	0.4999975	3.1415914	0.0000014
10	0.5000002	3.1415928	0.0000001

Estimates values are $x = 0.5$ and $y = \pi$ after 10 iterations.

The iterations of table 4.2 at the initial point (1,1.5,1)

The result from the D-method

k	x^k	y^k	z^k	ER
1	0.3000000	0.5833333	0.4047619	1.9231755
2	0.8667737	1.3598105	0.6048019	3.4729453
3	0.2716127	0.8207036	0.3682043	1.5819653
4	0.6124320	1.1966601	0.6463132	1.7296618
5	0.3945225	0.8938218	0.4544717	0.8200116
6	0.5764758	1.0833954	0.5588768	0.7814497
7	0.4518769	0.9364438	0.4655535	0.4701876
8	0.5425688	1.0443735	0.5255347	0.3922719
9	0.4720658	0.9671580	0.4797663	0.2659800
10	0.5209038	1.0248443	0.5144751	0.2060206
11	0.4842871	0.9827251	0.4898197	0.1438700
12	0.5110814	1.0131477	0.5080085	0.1093059
13	0.4917587	0.9905720	0.4945378	0.0774303
14	0.5060597	1.0069125	0.5041627	0.0579557
15	0.4956398	0.9949208	0.4969931	0.0417323
16	0.5032372	1.0037012	0.5021921	0.0308235
17	0.4976500	0.9972946	0.4983879	0.0223994
18	0.5017191	1.0019860	0.5011785	0.0164592
19	0.4987389	0.9985539	0.4991423	0.0119903
20	0.5009197	1.0010608	0.5006316	0.0087980
21	0.4993268	0.9992251	0.4995407	0.0064176
22	0.5004928	1.0005667	0.5003369	0.0047022
23	0.4996401	0.9995854	0.4997538	0.0034347
24	0.5002635	1.0003032	0.5001800	0.0025139
25	0.4998074	0.9997783	0.4998683	0.0018376
26	0.5001409	1.0001622	0.5000963	0.0013444
27	0.4998970	0.9998814	0.4999296	0.0009829
28	0.5000753	1.0000867	0.5000515	0.0007191
29	0.4999449	0.9999366	0.4999623	0.0005258

The result from the D-method (cont.)

30	0.5000403	1.0000464	0.5000276	0.0003846
31	0.4999705	0.9999661	0.4999798	0.0002812
32	0.5000216	1.0000248	0.5000147	0.0002057
33	0.4999842	0.9999819	0.4999892	0.0001504
34	0.5000115	1.0000133	0.5000079	0.0001100
35	0.4999916	0.9999903	0.4999942	0.0000805
36	0.5000062	1.0000071	0.5000042	0.0000588
37	0.4999955	0.9999948	0.4999969	0.0000430
38	0.5000033	1.0000038	0.5000023	0.0000315
39	0.4999976	0.9999972	0.4999984	0.0000230
40	0.5000018	1.0000020	0.5000012	0.0000168
41	0.4999987	0.9999985	0.4999991	0.0000123
42	0.5000009	1.0000011	0.5000006	0.0000090
43	0.4999993	0.9999992	0.4999995	0.0000066
44	0.5000005	1.0000006	0.5000003	0.0000048
45	0.4999996	0.9999996	0.4999997	0.0000035
46	0.5000003	1.0000003	0.5000002	0.0000026
47	0.4999998	0.9999998	0.4999999	0.0000019
48	0.5000001	1.0000002	0.5000001	0.0000014
49	0.4999999	0.9999999	0.4999999	0.0000010
50	0.5000001	1.0000001	0.5000001	0.0000007
51	0.4999999	0.9999999	0.5000000	0.0000005
52	0.5000000	1.0000000	0.5000000	0.0000004

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 52 iterations.

The result from the U_2-method

k	x^k	y^k	z^k	ER
1	0.7158730	0.9801587	0.4047619	0.4839148
2	0.5442843	0.9137776	0.3962514	0.4025530
3	0.4654145	0.9948350	0.4765198	0.2128664
4	0.4850367	1.0085136	0.5178671	0.0677331
5	0.5028125	1.0036654	0.5074709	0.0451000
6	0.5030346	0.9993312	0.4986121	0.0050478
7	0.5001747	0.9992351	0.4984832	0.0065037
8	0.4995292	0.9999575	0.4999129	0.0021648
9	0.4998730	1.0001176	0.5002355	0.0009350
10	0.5000520	1.0000318	0.5000635	0.0004888
11	0.5000320	0.9999870	0.4999740	0.0000897
12	0.4999980	0.9999920	0.4999840	0.0000815
13	0.4999942	1.0000005	0.5000010	0.0000169
14	0.4999991	1.0000014	0.5000029	0.0000119
15	0.5000008	1.0000002	0.5000004	0.0000049
16	0.5000003	0.9999998	0.4999996	0.0000015
17	0.4999999	0.9999999	0.4999998	0.0000010
18	0.4999999	1.0000000	0.5000000	0.0000001

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 18 iterations.

The result from the U_3-method

k	x^k	y^k	z^k	ER
1	0.8349206	0.9801587	0.4047619	0.9733621
2	0.6082238	0.8411719	0.3440132	0.5340764
3	0.4855186	0.9837688	0.4373984	0.2839950
4	0.4992643	1.0056862	0.5083061	0.0408962
5	0.4999439	1.0002277	0.5003723	0.0016472
6	0.4999976	1.0000141	0.5000280	0.0001210
7	0.4999999	1.0000006	0.5000012	0.0000052
8	0.5000000	1.0000000	0.5000000	0.0000002

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 8 iterations.

The result from the L_2-method

k	x^k	y^k	z^k	ER
1	0.3000000	1.0500000	0.5761905	0.3831468
2	0.4836078	0.9858761	0.6078163	0.4237888
3	0.5452245	0.9174773	0.5154646	0.2371531
4	0.5757157	0.9512726	0.4758823	0.1352364
5	0.5332230	0.9965253	0.4628127	0.1229491
6	0.4902034	1.0237665	0.4844433	0.0970611
7	0.4750650	1.0197418	0.5047758	0.0547444
8	0.4852539	1.0050128	0.5124704	0.0431998
9	0.4999590	0.9938590	0.5075101	0.0349137
10	0.5076140	0.9923821	0.5000295	0.0189512
11	0.5063742	0.9967831	0.4961832	0.0127925
12	0.5014095	1.0012168	0.4968280	0.0120747
13	0.4979264	1.0026192	0.4993001	0.0073284
14	0.4975856	1.0015534	0.5010350	0.0043657
15	0.4990511	0.9999583	0.5012084	0.0040937
16	0.5004372	0.9991773	0.5004755	0.0030772
17	0.5008442	0.9993396	0.4997812	0.0017526
18	0.5004775	0.9998707	0.4995780	0.0014264
19	0.4999671	1.0002276	0.4997614	0.0011758
20	0.4997308	1.0002538	0.5000165	0.0006451
21	0.4997940	1.0000948	0.5001346	0.0004550
22	0.4999659	0.9999498	0.5001030	0.0004119
23	0.5000762	0.9999104	0.5000171	0.0002301
24	0.5000804	0.9999513	0.4999619	0.0001391
25	0.5000279	1.0000051	0.4999598	0.0001360
26	0.4999824	1.0000289	0.4999861	0.0000998
27	0.4999713	1.0000213	0.5000088	0.0000575
28	0.4999852	1.0000030	0.5000144	0.0000486
29	0.5000023	0.9999917	0.5000074	0.0000391
30	0.5000094	0.9999916	0.4999989	0.0000216
31	0.5000066	0.9999973	0.4999953	0.0000159
32	0.5000007	1.0000020	0.4999967	0.0000140

The result from the L_2-method (cont.)

33	0.4999972	1.0000030	0.4999996	0.0000075
34	0.4999973	1.0000015	0.5000014	0.0000047
35	0.4999992	0.9999997	0.5000013	0.0000046
36	0.5000007	0.9999990	0.5000004	0.0000032
37	0.5000010	0.9999993	0.4999997	0.0000019
38	0.5000005	0.9999999	0.4999995	0.0000016
39	0.4999999	1.0000003	0.4999998	0.0000013
40	0.4999997	1.0000003	0.5000001	0.0000007
41	0.4999998	1.0000001	0.5000002	0.0000006
42	0.5000000	0.9999999	0.5000001	0.0000005

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 42 iterations.

The result from the L_3-method

k	x^k	y^k	z^k	ER
1	0.3000000	1.0500000	0.8428571	1.4904949
2	0.5532399	0.7857847	0.5208256	0.6455324
3	0.6989790	0.8773819	0.3824371	0.2981676
4	0.5653543	1.0381859	0.4824157	0.3205733
5	0.4625617	1.0314665	0.5169554	0.0747406
6	0.4797014	1.0019039	0.5104949	0.0397540
7	0.5018906	0.9940098	0.4990510	0.0151990
8	0.5046859	0.9981308	0.4976538	0.0072260
9	0.5007758	1.0007930	0.4996156	0.0043342
10	0.4992111	1.0005873	0.5003938	0.0014284
11	0.4996419	0.9999823	0.5001792	0.0008635
12	0.5000745	0.9998732	0.4999627	0.0002444
13	0.5000932	0.9999720	0.4999534	0.0001375
14	0.5000078	1.0000194	0.4999961	0.0000806
15	0.4999825	1.0000107	0.5000087	0.0000299
16	0.4999940	0.9999986	0.5000030	0.0000179
17	0.5000021	0.9999974	0.4999989	0.0000046
18	0.5000018	0.9999996	0.4999991	0.0000029
19	0.5000000	1.0000004	0.5000000	0.0000014
20	0.4999996	1.0000002	0.5000002	0.0000006
21	0.4999999	1.0000000	0.5000000	0.0000004

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 21 iterations.

The iterations of table 4.2 at the initial point (0,1,0)

The result from the D-method

k	x^k	y^k	z^k	ER
1	0.2500000	1.2500000	0.8125000	1.7158203
2	0.4031008	0.9359375	0.6029605	0.5042648
3	0.5862069	0.9882751	0.5522188	0.4607108

The result from the D-method (cont.)

4	0.5265708	0.9248961	0.4595182	0.2499196
5	0.5506095	1.0093020	0.4849657	0.1763112
6	0.4871937	0.9810384	0.4752499	0.1803712
7	0.5075651	1.0189286	0.5066487	0.1109849
8	0.4865461	0.9931511	0.4962339	0.0835255
9	0.5044647	1.0085947	0.5068041	0.0655152
10	0.4951225	0.9944174	0.4977875	0.0429276
11	0.5039236	1.0035661	0.5024513	0.0337491
12	0.4978485	0.9968195	0.4980474	0.0242095
13	0.5020022	1.0020594	0.5010790	0.0174623
14	0.4986447	0.9984621	0.4990012	0.0130627
15	0.5009492	1.0011787	0.5006789	0.0094344
16	0.4992444	0.9991869	0.4995260	0.0068941
17	0.5005197	1.0006152	0.5003781	0.0050851
18	0.4996134	0.9995513	0.4997403	0.0036857
19	0.5002874	1.0003233	0.5001934	0.0027090
20	0.4997951	0.9997597	0.4998564	0.0019800
21	0.5001524	1.0001743	0.5001025	0.0014456
22	0.4998889	0.9998726	0.4999238	0.0010591
23	0.5000808	1.0000936	0.5000556	0.0007740
24	0.4999405	0.9999318	0.4999596	0.0005659
25	0.5000433	1.0000500	0.5000298	0.0004142
26	0.4999683	0.9999635	0.4999783	0.0003027
27	0.5000232	1.0000267	0.5000159	0.0002214
28	0.4999830	0.9999805	0.4999884	0.0001620
29	0.5000124	1.0000143	0.5000085	0.0001184
30	0.4999909	0.9999896	0.4999938	0.0000866
31	0.5000066	1.0000076	0.5000045	0.0000633
32	0.4999951	0.9999944	0.4999967	0.0000463
33	0.5000036	1.0000041	0.5000024	0.0000339
34	0.4999974	0.9999970	0.4999982	0.0000248
35	0.5000019	1.0000022	0.5000013	0.0000181
36	0.4999986	0.9999984	0.4999991	0.0000133
37	0.5000010	1.0000012	0.5000007	0.0000097
38	0.4999993	0.9999991	0.4999995	0.0000071
39	0.5000005	1.0000006	0.5000004	0.0000052
40	0.4999996	0.9999995	0.4999997	0.0000038
41	0.5000003	1.0000003	0.5000002	0.0000028
42	0.4999998	0.9999998	0.4999999	0.0000020
43	0.5000002	1.0000002	0.5000001	0.0000015
44	0.4999999	0.9999999	0.4999999	0.0000011
45	0.5000001	1.0000001	0.5000001	0.0000008
46	0.4999999	0.9999999	0.5000000	0.0000006
47	0.5000000	1.0000001	0.5000000	0.0000004

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 47 iterations.

The result from the U_2-method

k	x^k	y^k	z^k	ER
1	0.0000000	1.2500000	0.8125000	1.7421875
2	0.5204576	1.0134527	0.7317073	1.0390989
3	0.5542009	1.0156873	0.5008176	0.2605446
4	0.5133133	0.9849315	0.4737607	0.1040315
5	0.4936785	0.9968280	0.4933230	0.0537767
6	0.4964362	1.0015984	0.5031921	0.0113368
7	0.5003246	1.0008841	0.5017814	0.0094462
8	0.5006590	0.9999208	0.4998384	0.0017293
9	0.5000837	0.9998350	0.4996705	0.0013785
10	0.4999076	0.9999791	0.4999582	0.0005400
11	0.4999668	1.0000231	0.5000462	0.0001773
12	0.5000085	1.0000083	0.5000166	0.0001086
13	0.5000073	0.9999979	0.4999958	0.0000131
14	0.5000001	0.9999982	0.4999963	0.0000165
15	0.4999988	1.0000000	0.4999999	0.0000047
16	0.4999997	1.0000003	0.5000006	0.0000024
17	0.5000001	1.0000001	0.5000001	0.0000012
18	0.5000001	1.0000000	0.4999999	0.0000003

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 18 iterations.

The result from the U_3-method

k	x^k	y^k	z^k	ER
1	0.8125000	1.2500000	0.8125000	3.7639160
2	0.6472538	0.9573409	0.4117832	0.2391266
3	0.5174276	0.9483790	0.4265418	0.3039601
4	0.4981010	0.9988067	0.4913173	0.0369170
5	0.4999157	1.0005105	0.5009692	0.0042536
6	0.4999962	1.0000216	0.5000422	0.0001826
7	0.4999998	1.0000010	0.5000019	0.0000082
8	0.5000000	1.0000000	0.5000001	0.0000003

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 8 iterations.

The result from the L_2-method

k	x^k	y^k	z^k	ER
1	0.2500000	1.2500000	0.9375000	2.3955078
2	0.4316547	0.8121066	0.8026873	1.1421341
3	0.7510170	0.6484222	0.5667568	0.8595112
4	0.8543594	0.6785607	0.3396643	0.4798531
5	0.7404440	0.9651254	0.3440936	0.5529040
6	0.4721652	1.1201135	0.4053194	0.5747263
7	0.3680911	1.1006522	0.5113832	0.3243067
8	0.4227718	1.0331040	0.5650564	0.2455084
9	0.4937310	0.9724481	0.5424104	0.1790570

The result from the L_2-method (cont.)

10	0.5368121	0.9589895	0.5036611	0.0957779
11	0.5358520	0.9795988	0.4812436	0.0600695
12	0.5108276	1.0043397	0.4824589	0.0631350
13	0.4904637	1.0134976	0.4947619	0.0426624
14	0.4870420	1.0089762	0.5047187	0.0249843
15	0.4941149	1.0006145	0.5065039	0.0221691
16	0.5016471	0.9959331	0.5029742	0.0170772
17	0.5043819	0.9963049	0.4991735	0.0095790
18	0.5028077	0.9990095	0.4978089	0.0073803
19	0.5000945	1.0010511	0.4986005	0.0063370
20	0.4986573	1.0013694	0.4999532	0.0034273
21	0.4988438	1.0006010	0.5006710	0.0022711
22	0.4997229	0.9998035	0.5005786	0.0021418
23	0.5003566	0.9995323	0.5001387	0.0013514
24	0.5004360	0.9997125	0.4998217	0.0008000
25	0.5001802	0.9999991	0.4997820	0.0007373
26	0.4999281	1.0001450	0.4999100	0.0005613
27	0.4998492	1.0001204	0.5000360	0.0003183
28	0.4999116	1.0000262	0.5000754	0.0002550
29	0.5000033	0.9999606	0.5000442	0.0002121
30	0.5000475	0.9999541	0.4999983	0.0001159
31	0.5000377	0.9999820	0.4999762	0.0000803
32	0.5000071	1.0000083	0.4999812	0.0000736
33	0.4999868	1.0000160	0.4999965	0.0000426
34	0.4999855	1.0000090	0.5000066	0.0000256
35	0.4999947	0.9999994	0.5000073	0.0000246
36	0.5000030	0.9999949	0.5000027	0.0000182
37	0.5000051	0.9999961	0.4999985	0.0000105
38	0.5000028	0.9999994	0.4999974	0.0000087
39	0.4999997	1.0000014	0.4999986	0.0000071
40	0.4999983	1.0000015	0.5000002	0.0000039
41	0.4999988	1.0000005	0.5000008	0.0000028
42	0.4999998	0.9999997	0.5000006	0.0000025
43	0.5000005	0.9999995	0.5000001	0.0000013
44	0.5000005	0.9999997	0.4999998	0.0000008
45	0.5000002	1.0000000	0.4999998	0.0000008
46	0.4999999	1.0000002	0.4999999	0.0000006
47	0.4999998	1.0000001	0.5000001	0.0000003

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 47 iterations.

The result from the L_3-method

k	x^k	y^k	z^k	ER
1	0.2500000	1.2500000	0.8125000	1.7158203
2	0.4031008	0.9053173	0.6370768	0.6538244
3	0.6226449	0.8694441	0.4233131	0.2420405
4	0.5876033	0.9964351	0.4635000	0.2110204
5	0.4905196	1.0270947	0.5045525	0.0674813
6	0.4791507	1.0081320	0.5103721	0.0330445

The result from the L_3 -method (cont.)

7	0.4966846	0.9966266	0.5017215	0.0185184
8	0.5033869	0.9974582	0.4982945	0.0061517
9	0.5015513	1.0000795	0.4992276	0.0037595
10	0.4996761	1.0005495	0.5001617	0.0010578
11	0.4995960	1.0001211	0.5002020	0.0005962
12	0.4999664	0.9999159	0.5000168	0.0003491
13	0.5000757	0.9999537	0.4999621	0.0001294
14	0.5000259	1.0000060	0.4999870	0.0000777
15	0.4999907	1.0000111	0.5000046	0.0000200
16	0.4999923	1.0000015	0.5000039	0.0000124
17	0.5000000	0.9999981	0.5000000	0.0000063
18	0.5000016	0.9999992	0.4999992	0.0000026
19	0.5000004	1.0000002	0.4999998	0.0000016
20	0.4999998	1.0000002	0.5000001	0.0000005

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 20 iterations.

The iterations of table 4.2 at the initial point (1,2,1)

The result from the D-method

k	x^k	y^k	z^k	ER
1	0.3000000	0.5833333	0.4047619	1.9231755
2	0.8667737	1.3598105	0.6048019	3.4729453
3	0.2716127	0.8207036	0.3682043	1.5819653
4	0.6124320	1.1966601	0.6463132	1.7296618
5	0.3945225	0.8938218	0.4544717	0.8200116
6	0.5764758	1.0833954	0.5588768	0.7814497
7	0.4518769	0.9364438	0.4655535	0.4701876
8	0.5425688	1.0443735	0.5255347	0.3922719
9	0.4720658	0.9671580	0.4797663	0.2659800
10	0.5209038	1.0248443	0.5144751	0.2060206
11	0.4842871	0.9827251	0.4898197	0.1438700
12	0.5110814	1.0131477	0.5080085	0.1093059
13	0.4917587	0.9905720	0.4945378	0.0774303
14	0.5060597	1.0069125	0.5041627	0.0579557
15	0.4956398	0.9949208	0.4969931	0.0417323
16	0.5032372	1.0037012	0.5021921	0.0308235
17	0.4976500	0.9972946	0.4983879	0.0223994
18	0.5017191	1.0019860	0.5011785	0.0164592
19	0.4987389	0.9985539	0.4991423	0.0119903
20	0.5009197	1.0010608	0.5006316	0.0087980
21	0.4993268	0.9992251	0.4995407	0.0064176
22	0.5004928	1.0005667	0.5003369	0.0047022
23	0.4996401	0.9995854	0.4997538	0.0034347
24	0.5002635	1.0003032	0.5001800	0.0025139
25	0.4998074	0.9997783	0.4998683	0.0018376
26	0.5001409	1.0001622	0.5000963	0.0013444
27	0.4998970	0.9998814	0.4999296	0.0009829
28	0.5000753	1.0000867	0.5000515	0.0007191

The result from the D-method (cont.)

29	0.4999449	0.9999366	0.4999623	0.0005258
30	0.5000403	1.0000464	0.5000276	0.0003846
31	0.4999705	0.9999661	0.4999798	0.0002812
32	0.5000216	1.0000248	0.5000147	0.0002057
33	0.4999842	0.9999819	0.4999892	0.0001504
34	0.5000115	1.0000133	0.5000079	0.0001100
35	0.4999916	0.9999903	0.4999942	0.0000805
36	0.5000062	1.0000071	0.5000042	0.0000588
37	0.4999955	0.9999948	0.4999969	0.0000430
38	0.5000033	1.0000038	0.5000023	0.0000315
39	0.4999976	0.9999972	0.4999984	0.0000230
40	0.5000018	1.0000020	0.5000012	0.0000168
41	0.4999987	0.9999985	0.4999991	0.0000123
42	0.5000009	1.0000011	0.5000006	0.0000090
43	0.4999993	0.9999992	0.4999995	0.0000066
44	0.5000005	1.0000006	0.5000003	0.0000048
45	0.4999996	0.9999996	0.4999997	0.0000035
46	0.5000003	1.0000003	0.5000002	0.0000026
47	0.4999998	0.9999998	0.4999999	0.0000019
48	0.5000001	1.0000002	0.5000001	0.0000014
49	0.4999999	0.9999999	0.4999999	0.0000010
50	0.5000001	1.0000001	0.5000001	0.0000007
51	0.4999999	0.9999999	0.5000000	0.0000005
52	0.5000000	1.0000000	0.5000000	0.0000004

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 52 iterations.

The result from the U_2-method

k	x^k	y^k	z^k	ER
1	0.7158730	0.9801587	0.4047619	0.4839148
2	0.5442843	0.9137776	0.3962514	0.4025530
3	0.4654145	0.9948350	0.4765198	0.2128664
4	0.4850367	1.0085136	0.5178671	0.0677331
5	0.5028125	1.0036654	0.5074709	0.0451000
6	0.5030346	0.9993312	0.4986121	0.0050478
7	0.5001747	0.9992351	0.4984832	0.0065037
8	0.4995292	0.9999575	0.4999129	0.0021648
9	0.4998730	1.0001176	0.5002355	0.0009350
10	0.5000520	1.0000318	0.5000635	0.0004888
11	0.5000320	0.9999870	0.4999740	0.0000897
12	0.4999980	0.9999920	0.4999840	0.0000815
13	0.4999942	1.0000005	0.5000010	0.0000169
14	0.4999991	1.0000014	0.5000029	0.0000119
15	0.5000008	1.0000002	0.5000004	0.0000049
16	0.5000003	0.9999998	0.4999996	0.0000015
17	0.4999999	0.9999999	0.4999998	0.0000010
18	0.4999999	1.0000000	0.5000000	0.0000001

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 18 iterations.

The result from the U_3-method

k	x^k	y^k	z^k	ER
1	0.8349206	0.9801587	0.4047619	0.9733621
2	0.6082238	0.8411719	0.3440132	0.5340764
3	0.4855186	0.9837688	0.4373984	0.2839950
4	0.4992643	1.0056862	0.5083061	0.0408962
5	0.4999439	1.0002277	0.5003723	0.0016472
6	0.4999976	1.0000141	0.5000280	0.0001210
7	0.4999999	1.0000006	0.5000012	0.0000052
8	0.5000000	1.0000000	0.5000000	0.0000002

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 8 iterations.

The result from the L_2-method

k	x^k	y^k	z^k	ER
1	0.3000000	1.0500000	0.5761905	0.3831468
2	0.4836078	0.9858761	0.6078163	0.4237888
3	0.5452245	0.9174773	0.5154646	0.2371531
4	0.5757157	0.9512726	0.4758823	0.1352364
5	0.5332230	0.9965253	0.4628127	0.1229491
6	0.4902034	1.0237665	0.4844433	0.0970611
7	0.4750650	1.0197418	0.5047758	0.0547444
8	0.4852539	1.0050128	0.5124704	0.0431998
9	0.4999590	0.9938590	0.5075101	0.0349137
10	0.5076140	0.9923821	0.5000295	0.0189512
11	0.5063742	0.9967831	0.4961832	0.0127925
12	0.5014095	1.0012168	0.4968280	0.0120747
13	0.4979264	1.0026192	0.4993001	0.0073284
14	0.4975856	1.0015534	0.5010350	0.0043657
15	0.4990511	0.9999583	0.5012084	0.0040937
16	0.5004372	0.9991773	0.5004755	0.0030772
17	0.5008442	0.9993396	0.4997812	0.0017526
18	0.5004775	0.9998707	0.4995780	0.0014264
19	0.4999671	1.0002276	0.4997614	0.0011758
20	0.4997308	1.0002538	0.5000165	0.0006451
21	0.4997940	1.0000948	0.5001346	0.0004550
22	0.4999659	0.9999498	0.5001030	0.0004119
23	0.5000762	0.9999104	0.5000171	0.0002301
24	0.5000804	0.9999513	0.4999619	0.0001391
25	0.5000279	1.0000051	0.4999598	0.0001360
26	0.4999824	1.0000289	0.4999861	0.0000998
27	0.4999713	1.0000213	0.5000088	0.0000575
28	0.4999852	1.0000030	0.5000144	0.0000486
29	0.5000023	0.9999917	0.5000074	0.0000391
30	0.5000094	0.9999916	0.4999989	0.0000216
31	0.5000066	0.9999973	0.4999953	0.0000159
32	0.5000007	1.0000020	0.4999967	0.0000140
33	0.4999972	1.0000030	0.4999996	0.0000075
34	0.4999973	1.0000015	0.5000014	0.0000047
35	0.4999992	0.9999997	0.5000013	0.0000046
36	0.5000007	0.9999990	0.5000004	0.0000032

The result from the L_2 -method (cont.)

37	0.5000010	0.9999993	0.4999997	0.0000019
38	0.5000005	0.9999999	0.4999995	0.0000016
39	0.4999999	1.0000003	0.4999998	0.0000013
40	0.4999997	1.0000003	0.5000001	0.0000007
41	0.4999998	1.0000001	0.5000002	0.0000006
42	0.5000000	0.9999999	0.5000001	0.0000005

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 42 iterations.

The result from the L_3 -method

k	x^k	y^k	z^k	ER
1	0.3000000	1.0500000	0.8428571	1.4904949
2	0.5532399	0.7857847	0.5208256	0.6455324
3	0.6989790	0.8773819	0.3824371	0.2981676
4	0.5653543	1.0381859	0.4824157	0.3205733
5	0.4625617	1.0314665	0.5169554	0.0747406
6	0.4797014	1.0019039	0.5104949	0.0397540
7	0.5018906	0.9940098	0.4990510	0.0151990
8	0.5046859	0.9981308	0.4976538	0.0072260
9	0.5007758	1.0007930	0.4996156	0.0043342
10	0.4992111	1.0005873	0.5003938	0.0014284
11	0.4996419	0.9999823	0.5001792	0.0008635
12	0.5000745	0.9998732	0.4999627	0.0002444
13	0.5000932	0.9999720	0.4999534	0.0001375
14	0.5000078	1.0000194	0.4999961	0.0000806
15	0.4999825	1.0000107	0.5000087	0.0000299
16	0.4999940	0.9999986	0.5000030	0.0000179
17	0.5000021	0.9999974	0.4999989	0.0000046
18	0.5000018	0.9999996	0.4999991	0.0000029
19	0.5000000	1.0000004	0.5000000	0.0000014
20	0.4999996	1.0000002	0.5000002	0.0000006
21	0.4999999	1.0000000	0.5000000	0.0000004

Estimates values are $x = 0.5$, $y = 1$ and $z = 0.5$ after 21 iterations.

The iterations of table 4.3 at the initial point (0.5,0.5,0.5,0.5)

The result from the D -method

k	x^k	y^k	z^k	w^k	ER
1	0.7185185	0.4961832	0.4961832	0.4961832	0.2248892
2	0.7126081	0.5930564	0.4924519	0.4944325	0.2967030
3	0.8023983	0.5880837	0.5357264	0.4927204	0.1879637
4	0.7973288	0.6498718	0.5325099	0.5125298	0.2139929
5	0.8539693	0.6458141	0.5705724	0.5110458	0.1429448
6	0.8500559	0.6899095	0.5679225	0.5283609	0.1568961
7	0.8901782	0.6867581	0.5965232	0.5271458	0.1053227

The result from the D-method (cont.)

8	0.8872098	0.7186302	0.5944574	0.5400886	0.1141795
9	0.9160515	0.7162463	0.6152408	0.5391483	0.0764552
10	0.9138406	0.7391646	0.6136824	0.5485169	0.0823237
11	0.9344960	0.7373981	0.6286015	0.5478115	0.0549636
12	0.9328759	0.7537729	0.6274506	0.5545177	0.0589114
13	0.9475902	0.7524841	0.6380869	0.5539989	0.0392414
14	0.9464177	0.7641244	0.6372494	0.5587703	0.0419230
15	0.9568552	0.7631947	0.6447974	0.5583939	0.0278779
16	0.9560142	0.7714386	0.6441944	0.5617750	0.0297135
17	0.9633950	0.7707733	0.6495335	0.5615045	0.0197347
18	0.9627957	0.7765962	0.6491026	0.5638937	0.0209992
19	0.9680033	0.7761229	0.6528704	0.5637007	0.0139348
20	0.9675782	0.7802281	0.6525641	0.5653856	0.0148102
21	0.9712467	0.7798927	0.6552188	0.5652485	0.0098218
22	0.9709461	0.7827830	0.6550019	0.5664350	0.0104301
23	0.9735275	0.7825460	0.6568702	0.5663380	0.0069140
24	0.9733154	0.7845790	0.6567170	0.5671727	0.0073379
25	0.9751305	0.7844119	0.6580307	0.5671043	0.0048627
26	0.9749811	0.7858410	0.6579227	0.5676911	0.0051587
27	0.9762566	0.7857233	0.6588459	0.5676428	0.0034179
28	0.9761514	0.7867273	0.6587699	0.5680551	0.0036248
29	0.9770475	0.7866445	0.6594185	0.5680212	0.0024013
30	0.9769735	0.7873498	0.6593650	0.5683108	0.0025461
31	0.9776028	0.7872916	0.6598205	0.5682869	0.0016865
32	0.9775508	0.7877868	0.6597829	0.5684903	0.0017880
33	0.9779926	0.7877459	0.6601027	0.5684735	0.0011842
34	0.9779561	0.7880935	0.6600763	0.5686163	0.0012554
35	0.9782663	0.7880648	0.6603008	0.5686045	0.0008314
36	0.9782407	0.7883089	0.6602823	0.5687048	0.0008813
37	0.9784584	0.7882887	0.6604399	0.5686965	0.0005837
38	0.9784404	0.7884601	0.6604269	0.5687668	0.0006186
39	0.9785932	0.7884459	0.6605375	0.5687610	0.0004097
40	0.9785806	0.7885661	0.6605284	0.5688104	0.0004342
41	0.9786879	0.7885562	0.6606060	0.5688063	0.0002876
42	0.9786790	0.7886406	0.6605996	0.5688410	0.0003048
43	0.9787543	0.7886336	0.6606541	0.5688381	0.0002018
44	0.9787481	0.7886929	0.6606496	0.5688625	0.0002139
45	0.9788009	0.7886880	0.6606878	0.5688605	0.0001417
46	0.9787966	0.7887296	0.6606847	0.5688775	0.0001501
47	0.9788336	0.7887261	0.6607115	0.5688761	0.0000994
48	0.9788306	0.7887553	0.6607093	0.5688881	0.0001054
49	0.9788566	0.7887529	0.6607282	0.5688871	0.0000698
50	0.9788545	0.7887734	0.6607266	0.5688955	0.0000740
51	0.9788727	0.7887717	0.6607398	0.5688948	0.0000490
52	0.9788712	0.7887861	0.6607387	0.5689007	0.0000519
53	0.9788840	0.7887849	0.6607480	0.5689002	0.0000344
54	0.9788830	0.7887950	0.6607472	0.5689044	0.0000364
55	0.9788920	0.7887941	0.6607538	0.5689040	0.0000241
56	0.9788912	0.7888012	0.6607532	0.5689070	0.0000256
57	0.9788975	0.7888006	0.6607578	0.5689067	0.0000169
58	0.9788970	0.7888056	0.6607574	0.5689088	0.0000179
59	0.9789015	0.7888052	0.6607606	0.5689086	0.0000119

The result from the D-method (cont.)

60	0.9789011	0.7888087	0.6607604	0.5689100	0.0000126
61	0.9789042	0.7888084	0.6607626	0.5689099	0.0000083
62	0.9789039	0.7888108	0.6607624	0.5689109	0.0000088
63	0.9789061	0.7888106	0.6607640	0.5689108	0.0000059
64	0.9789059	0.7888123	0.6607639	0.5689115	0.0000062
65	0.9789075	0.7888122	0.6607650	0.5689115	0.0000041
66	0.9789074	0.7888134	0.6607649	0.5689120	0.0000044
67	0.9789084	0.7888133	0.6607657	0.5689119	0.0000029
68	0.9789083	0.7888141	0.6607656	0.5689123	0.0000031
69	0.9789091	0.7888141	0.6607662	0.5689122	0.0000020
70	0.9789090	0.7888147	0.6607661	0.5689125	0.0000021
71	0.9789096	0.7888146	0.6607665	0.5689125	0.0000014
72	0.9789095	0.7888150	0.6607665	0.5689126	0.0000015
73	0.9789099	0.7888150	0.6607667	0.5689126	0.0000010
74	0.9789099	0.7888153	0.6607667	0.5689127	0.0000011
75	0.9789101	0.7888153	0.6607669	0.5689127	0.0000007
76	0.9789101	0.7888155	0.6607669	0.5689128	0.0000007
77	0.9789103	0.7888155	0.6607670	0.5689128	0.0000005

Estimates values are $x = 0.9789103$, $y = 0.7888155$ and $z = 0.6607670$ $w = 0.5689128$ after 77 iterations.

The result from the U_2-method

k	x^k	y^k	z^k	w^k	ER
1	0.7120460	0.4931735	0.4942020	0.4961832	0.2156308
2	0.7966554	0.5868542	0.4896900	0.4935237	0.1706609
3	0.8494913	0.6449817	0.5313835	0.4914525	0.1421446
4	0.8868203	0.6862158	0.5673155	0.5105486	0.1061965
5	0.9135827	0.7159004	0.5940922	0.5268832	0.0769169
6	0.9327052	0.7371768	0.6134547	0.5389905	0.0550889
7	0.9463036	0.7523407	0.6273060	0.5477128	0.0392094
8	0.9559372	0.7631004	0.6371560	0.5539359	0.0277909
9	0.9627431	0.7707104	0.6441331	0.5583530	0.0196404
10	0.9675421	0.7760804	0.6490617	0.5614776	0.0138517
11	0.9709211	0.7798637	0.6525366	0.5636827	0.0097550
12	0.9732981	0.7825261	0.6549832	0.5652363	0.0068628
13	0.9749690	0.7843981	0.6567041	0.5663297	0.0048247
14	0.9761430	0.7857137	0.6579138	0.5670986	0.0033901
15	0.9769676	0.7866379	0.6587637	0.5676389	0.0023813
16	0.9775467	0.7872869	0.6593607	0.5680184	0.0016722
17	0.9779532	0.7877426	0.6597799	0.5682850	0.0011741
18	0.9782387	0.7880625	0.6600742	0.5684722	0.0008242
19	0.9784390	0.7882871	0.6602808	0.5686036	0.0005786
20	0.9785796	0.7884448	0.6604258	0.5686958	0.0004061
21	0.9786783	0.7885554	0.6605276	0.5687606	0.0002850
22	0.9787476	0.7886331	0.6605991	0.5688060	0.0002001
23	0.9787962	0.7886876	0.6606492	0.5688379	0.0001404
24	0.9788303	0.7887259	0.6606844	0.5688603	0.0000986
25	0.9788543	0.7887527	0.6607091	0.5688760	0.0000692
26	0.9788711	0.7887715	0.6607265	0.5688870	0.0000485

The result from the U_2 -method (cont.)

27	0.9788829	0.7887848	0.6607386	0.5688948	0.0000341
28	0.9788912	0.7887941	0.6607472	0.5689002	0.0000239
29	0.9788970	0.7888006	0.6607532	0.5689040	0.0000168
30	0.9789011	0.7888051	0.6607574	0.5689067	0.0000118
31	0.9789039	0.7888083	0.6607603	0.5689086	0.0000083
32	0.9789059	0.7888106	0.6607624	0.5689099	0.0000058
33	0.9789073	0.7888122	0.6607639	0.5689108	0.0000041
34	0.9789083	0.7888133	0.6607649	0.5689115	0.0000029
35	0.9789090	0.7888141	0.6607656	0.5689119	0.0000020
36	0.9789095	0.7888146	0.6607661	0.5689122	0.0000014
37	0.9789099	0.7888150	0.6607665	0.5689125	0.0000010
38	0.9789101	0.7888153	0.6607667	0.5689126	0.0000007
39	0.9789103	0.7888155	0.6607669	0.5689127	0.0000005

Estimates values are $x = 0.9789103$, $y = 0.7888155$ and $z = 0.6607669$ $w = 0.5689127$ after 39 iterations.

The result from the L_2 -method

k	x^k	y^k	z^k	w^k	ER
1	0.7185185	0.5962680	0.5402754	0.5158598	0.2460414
2	0.8053093	0.6537349	0.5742022	0.5310943	0.1676375
3	0.8574489	0.6933617	0.5996148	0.5423144	0.1193811
4	0.8932769	0.7215769	0.6178067	0.5502737	0.0853507
5	0.9186920	0.7416159	0.6306739	0.5558780	0.0606330
6	0.9366899	0.7557670	0.6397278	0.5598102	0.0428192
7	0.9493733	0.7657168	0.6460775	0.5625626	0.0301086
8	0.9582782	0.7726913	0.6505205	0.5644858	0.0211066
9	0.9645139	0.7775698	0.6536244	0.5658281	0.0147644
10	0.9688725	0.7809771	0.6557903	0.5667641	0.0103124
11	0.9719153	0.7833544	0.6573006	0.5674165	0.0071953
12	0.9740375	0.7850119	0.6583531	0.5678710	0.0050167
13	0.9755167	0.7861669	0.6590863	0.5681875	0.0034959
14	0.9765474	0.7869714	0.6595970	0.5684079	0.0024353
15	0.9772652	0.7875318	0.6599526	0.5685614	0.0016961
16	0.9777652	0.7879220	0.6602001	0.5686683	0.0011810
17	0.9781132	0.7881936	0.6603725	0.5687426	0.0008223
18	0.9783556	0.7883827	0.6604925	0.5687944	0.0005724
19	0.9785243	0.7885144	0.6605760	0.5688305	0.0003985
20	0.9786417	0.7886060	0.6606341	0.5688555	0.0002774
21	0.9787234	0.7886698	0.6606746	0.5688730	0.0001931
22	0.9787803	0.7887142	0.6607028	0.5688852	0.0001344
23	0.9788200	0.7887451	0.6607224	0.5688936	0.0000936
24	0.9788475	0.7887666	0.6607360	0.5688995	0.0000651
25	0.9788667	0.7887816	0.6607455	0.5689036	0.0000453
26	0.9788801	0.7887920	0.6607521	0.5689065	0.0000315
27	0.9788894	0.7887993	0.6607567	0.5689084	0.0000220
28	0.9788958	0.7888043	0.6607600	0.5689098	0.0000153
29	0.9789003	0.7888078	0.6607622	0.5689108	0.0000106

The result from the L_2-method (cont.)

30	0.9789035	0.7888103	0.6607637	0.5689115	0.0000074
31	0.9789057	0.7888120	0.6607648	0.5689119	0.0000052
32	0.9789072	0.7888132	0.6607656	0.5689123	0.0000036
33	0.9789082	0.7888140	0.6607661	0.5689125	0.0000025
34	0.9789090	0.7888146	0.6607665	0.5689126	0.0000017
35	0.9789095	0.7888150	0.6607667	0.5689127	0.0000012
36	0.9789098	0.7888153	0.6607669	0.5689128	0.0000008
37	0.9789101	0.7888155	0.6607670	0.5689129	0.0000006
38	0.9789103	0.7888156	0.6607671	0.5689129	0.0000004

Estimates values are $x = 0.9789103$, $y = 0.7888156$ and $z = 0.6607671$ $w = 0.5689129$ after 38 iterations.

The iterations of table 4.3 at the initial point (1,1,0,0)

The result from the D-method

k	x^k	y^k	z^k	w^k	ER
1	1.1666667	0.4864865	0.4705882	0.2580645	2.3181907
2	0.7211729	0.7749171	0.3659662	0.4842351	1.6775050
3	0.9706955	0.5309732	0.6137819	0.4341725	1.5052068
4	0.7486143	0.7643421	0.4772804	0.5484144	1.2818864
5	0.9601156	0.5979649	0.6408393	0.4859374	1.0838586
6	0.8081815	0.7725286	0.5340970	0.5602569	0.9439459
7	0.9661661	0.6522854	0.6499120	0.5118855	0.7852728
8	0.8568869	0.7789189	0.5715925	0.5641934	0.6838390
9	0.9710531	0.6920711	0.6544015	0.5288717	0.5650151
10	0.8925095	0.7828024	0.5978845	0.5661375	0.4906689
11	0.9740561	0.7205173	0.6569171	0.5407273	0.4037599
12	0.9179425	0.7850931	0.6164724	0.5672285	0.3497833
13	0.9758520	0.7406948	0.6583814	0.5490832	0.2870268
14	0.9359663	0.7864591	0.6296003	0.5678654	0.2482103
15	0.9769414	0.7549533	0.6592543	0.5549722	0.2032766
16	0.9486961	0.7872914	0.6388551	0.5682464	0.1755616
17	0.9776177	0.7650066	0.6597871	0.5591176	0.1435786
18	0.9576686	0.7878109	0.6453703	0.5684799	0.1238910
19	0.9780477	0.7720847	0.6601204	0.5620329	0.1012208
20	0.9639845	0.7881430	0.6499523	0.5686264	0.0872859
21	0.9783271	0.7770631	0.6603338	0.5640818	0.0712640
22	0.9684263	0.7883598	0.6531727	0.5687206	0.0614257
23	0.9785121	0.7805625	0.6604732	0.5655211	0.0501259
24	0.9715482	0.7885038	0.6554351	0.5687823	0.0431922
25	0.9786363	0.7830209	0.6605660	0.5665319	0.0352343
26	0.9737414	0.7886008	0.6570240	0.5688234	0.0303539
27	0.9787208	0.7847476	0.6606285	0.5672416	0.0247553
28	0.9752817	0.7886669	0.6581397	0.5688512	0.0213230
29	0.9787787	0.7859601	0.6606711	0.5677398	0.0173872
30	0.9763633	0.7887123	0.6589229	0.5688701	0.0149749
31	0.9788187	0.7868113	0.6607004	0.5680895	0.0122093
32	0.9771226	0.7887437	0.6594727	0.5688832	0.0105145

The result from the D-method (cont.)

33	0.9788465	0.7874089	0.6607206	0.5683350	0.0085720
34	0.9776556	0.7887654	0.6598587	0.5688922	0.0073817
35	0.9788658	0.7878284	0.6607347	0.5685074	0.0060176
36	0.9780298	0.7887806	0.6601296	0.5688985	0.0051818
37	0.9788792	0.7881228	0.6607445	0.5686283	0.0042241
38	0.9782924	0.7887912	0.6603197	0.5689028	0.0036373
39	0.9788886	0.7883294	0.6607513	0.5687132	0.0029649
40	0.9784768	0.7887986	0.6604532	0.5689059	0.0025530
41	0.9788952	0.7884745	0.6607561	0.5687728	0.0020810
42	0.9786061	0.7888038	0.6605468	0.5689080	0.0017919
43	0.9788998	0.7885763	0.6607594	0.5688146	0.0014606
44	0.9786969	0.7888074	0.6606126	0.5689095	0.0012577
45	0.9789031	0.7886477	0.6607618	0.5688439	0.0010251
46	0.9787607	0.7888099	0.6606587	0.5689105	0.0008827
47	0.9789053	0.7886979	0.6607634	0.5688645	0.0007195
48	0.9788054	0.7888117	0.6606911	0.5689113	0.0006195
49	0.9789069	0.7887331	0.6607646	0.5688790	0.0005050
50	0.9788368	0.7888130	0.6607138	0.5689118	0.0004348
51	0.9789080	0.7887578	0.6607654	0.5688891	0.0003544
52	0.9788588	0.7888138	0.6607297	0.5689121	0.0003052
53	0.9789088	0.7887751	0.6607660	0.5688962	0.0002487
54	0.9788743	0.7888145	0.6607409	0.5689124	0.0002142
55	0.9789094	0.7887873	0.6607664	0.5689012	0.0001746
56	0.9788851	0.7888149	0.6607488	0.5689126	0.0001503
57	0.9789098	0.7887958	0.6607666	0.5689047	0.0001225
58	0.9788927	0.7888152	0.6607543	0.5689127	0.0001055
59	0.9789100	0.7888018	0.6607668	0.5689072	0.0000860
60	0.9788981	0.7888154	0.6607582	0.5689128	0.0000740
61	0.9789102	0.7888060	0.6607670	0.5689089	0.0000603
62	0.9789018	0.7888155	0.6607609	0.5689129	0.0000520
63	0.9789103	0.7888090	0.6607671	0.5689101	0.0000424
64	0.9789045	0.7888157	0.6607628	0.5689129	0.0000365
65	0.9789104	0.7888110	0.6607671	0.5689110	0.0000297
66	0.9789063	0.7888157	0.6607641	0.5689129	0.0000256
67	0.9789105	0.7888125	0.6607672	0.5689116	0.0000209
68	0.9789076	0.7888158	0.6607651	0.5689129	0.0000180
69	0.9789106	0.7888135	0.6607672	0.5689120	0.0000146
70	0.9789085	0.7888158	0.6607657	0.5689130	0.0000126
71	0.9789106	0.7888142	0.6607672	0.5689123	0.0000103
72	0.9789092	0.7888158	0.6607662	0.5689130	0.0000088
73	0.9789106	0.7888147	0.6607672	0.5689125	0.0000072
74	0.9789096	0.7888159	0.6607665	0.5689130	0.0000062
75	0.9789106	0.7888151	0.6607673	0.5689127	0.0000051
76	0.9789099	0.7888159	0.6607668	0.5689130	0.0000044
77	0.9789106	0.7888153	0.6607673	0.5689128	0.0000036
78	0.9789101	0.7888159	0.6607669	0.5689130	0.0000031
79	0.9789106	0.7888155	0.6607673	0.5689128	0.0000025
80	0.9789103	0.7888159	0.6607670	0.5689130	0.0000021
81	0.9789106	0.7888156	0.6607673	0.5689129	0.0000017
82	0.9789104	0.7888159	0.6607671	0.5689130	0.0000015
83	0.9789107	0.7888157	0.6607673	0.5689129	0.0000012
84	0.9789105	0.7888159	0.6607672	0.5689130	0.0000011

The result from the D-method (cont.)

85	0.9789107	0.7888158	0.6607673	0.5689129	0.0000009
86	0.9789105	0.7888159	0.6607672	0.5689130	0.0000007
87	0.9789107	0.7888158	0.6607673	0.5689130	0.0000006
88	0.9789106	0.7888159	0.6607672	0.5689130	0.0000005
89	0.9789107	0.7888158	0.6607673	0.5689130	0.0000004

Estimates values are $x = 0.9789107$, $y = 0.7888158$ and $z = 0.6607673$ $w = 0.5689130$ after 89 iterations.

The result from the U_2-method

k	x^k	y^k	z^k	w^k	ER
1	0.9662233	0.7745013	0.5920304	0.2580645	0.8307644
2	0.9644190	0.7725884	0.6393326	0.5406046	0.0408203
3	0.9680400	0.7766385	0.6487802	0.5593446	0.0157583
4	0.9711885	0.7801633	0.6527152	0.5635574	0.0097366
5	0.9734760	0.7827254	0.6551546	0.5653162	0.0066833
6	0.9750928	0.7845369	0.6568302	0.5664063	0.0046780
7	0.9762299	0.7858110	0.6580031	0.5671549	0.0032845
8	0.9770286	0.7867062	0.6588265	0.5676788	0.0023067
9	0.9775895	0.7873349	0.6594048	0.5680465	0.0016198
10	0.9779833	0.7877763	0.6598109	0.5683047	0.0011372
11	0.9782597	0.7880862	0.6600959	0.5684860	0.0007984
12	0.9784538	0.7883037	0.6602961	0.5686133	0.0005604
13	0.9785900	0.7884564	0.6604366	0.5687026	0.0003934
14	0.9786856	0.7885636	0.6605352	0.5687654	0.0002761
15	0.9787527	0.7886388	0.6606044	0.5688094	0.0001938
16	0.9787998	0.7886916	0.6606529	0.5688403	0.0001360
17	0.9788329	0.7887287	0.6606870	0.5688620	0.0000955
18	0.9788561	0.7887547	0.6607110	0.5688772	0.0000670
19	0.9788723	0.7887729	0.6607278	0.5688879	0.0000470
20	0.9788838	0.7887857	0.6607395	0.5688954	0.0000330
21	0.9788918	0.7887947	0.6607478	0.5689006	0.0000232
22	0.9788974	0.7888010	0.6607577	0.5689069	0.0000114
24	0.9789041	0.7888086	0.6607606	0.5689087	0.0000080
25	0.9789061	0.7888108	0.6607626	0.5689100	0.0000056
26	0.9789074	0.7888123	0.6607640	0.5689109	0.0000039
27	0.9789084	0.7888134	0.6607650	0.5689115	0.0000028
28	0.9789091	0.7888141	0.6607657	0.5689120	0.0000019
29	0.9789096	0.7888147	0.6607661	0.5689123	0.0000014
30	0.9789099	0.7888150	0.6607665	0.5689125	0.0000010
31	0.9789101	0.7888153	0.6607667	0.5689126	0.0000007
32	0.9789103	0.7888155	0.6607669	0.5689127	0.0000005

Estimates values are $x = 0.9789103$, $y = 0.7888155$ and $z = 0.6607669$ $w = 0.5689127$ after 32 iterations.

The result from the L_2-method

k	x^k	y^k	z^k	w^k	ER
1	1.1666667	0.5495495	0.2586116	0.3915414	1.5939149
2	0.7755913	0.4985009	0.4402434	0.4742121	0.4188691
3	0.7149949	0.5623615	0.5116534	0.5037528	0.2389804
4	0.7739292	0.6253473	0.5552766	0.5228093	0.1942997
5	0.8316324	0.6726315	0.5861753	0.5364201	0.1431826
6	0.8745456	0.7067565	0.6082605	0.5461042	0.1032385
7	0.9053528	0.7311058	0.6239329	0.5529443	0.0736690
8	0.9272558	0.7483544	0.6349887	0.5577531	0.0521894
9	0.9427322	0.7605094	0.6427560	0.5611234	0.0367802
10	0.9536190	0.7690433	0.6481974	0.5634805	0.0258246
11	0.9612530	0.7750192	0.6520020	0.5651266	0.0180850
12	0.9665940	0.7791962	0.6546584	0.5662750	0.0126417
13	0.9703251	0.7821121	0.6565115	0.5670757	0.0088253
14	0.9729285	0.7841458	0.6578032	0.5676335	0.0061555
15	0.9747439	0.7855634	0.6587033	0.5680222	0.0042907
16	0.9760089	0.7865511	0.6593302	0.5682928	0.0029895
17	0.9768902	0.7872391	0.6597668	0.5684812	0.0020823
18	0.9775040	0.7877181	0.6600708	0.5686124	0.0014501
19	0.9779314	0.7880517	0.6602825	0.5687038	0.0010097
20	0.9782290	0.7882839	0.6604298	0.5687674	0.0007029
21	0.9784361	0.7884456	0.6605324	0.5688116	0.0004894
22	0.9785804	0.7885581	0.6606038	0.5688424	0.0003406
23	0.9786807	0.7886365	0.6606535	0.5688639	0.0002371
24	0.9787506	0.7886910	0.6606881	0.5688788	0.0001651
25	0.9787993	0.7887290	0.6607121	0.5688892	0.0001149
26	0.9788331	0.7887554	0.6607289	0.5688964	0.0000800
27	0.9788567	0.7887738	0.6607406	0.5689015	0.0000557
28	0.9788731	0.7887866	0.6607487	0.5689050	0.0000387
29	0.9788845	0.7887955	0.6607543	0.5689074	0.0000270
30	0.9788925	0.7888017	0.6607583	0.5689091	0.0000188
31	0.9788980	0.7888060	0.6607610	0.5689103	0.0000131
32	0.9789018	0.7888090	0.6607629	0.5689111	0.0000091
33	0.9789045	0.7888111	0.6607642	0.5689117	0.0000063
34	0.9789064	0.7888126	0.6607652	0.5689121	0.0000044
35	0.9789077	0.7888136	0.6607658	0.5689124	0.0000031
36	0.9789086	0.7888143	0.6607663	0.5689126	0.0000021
37	0.9789092	0.7888148	0.6607666	0.5689127	0.0000015
38	0.9789097	0.7888151	0.6607668	0.5689128	0.0000010
39	0.9789100	0.7888154	0.6607669	0.5689128	0.0000007
40	0.9789102	0.7888155	0.6607670	0.5689129	0.0000005
41	0.9789103	0.7888156	0.6607671	0.5689129	0.0000003

Estimates values are $x = 0.9789103$, $y = 0.7888156$ and $z = 0.6607671$ $w = 0.5689129$ after 41 iterations.

The iterations of table 4.3 at the initial point(0,0,0,0)

The result from the D-method

k	x^k	y^k	z^k	w^k	ER
1	0.2424242	0.0000000	0.0000000	0.2580645	0.5024534
2	0.2419948	0.1194030	0.1311475	0.2575137	0.6150522
3	0.3571664	0.1851474	0.1899325	0.3220634	0.4896971
4	0.4198534	0.2702529	0.2553552	0.3504376	0.4718503
5	0.5009454	0.3331320	0.3110433	0.3818095	0.4065439
6	0.5602979	0.3994089	0.3571984	0.4082423	0.3713151
7	0.6226210	0.4502805	0.4020864	0.4299710	0.3207812
8	0.6700735	0.5016111	0.4370102	0.4509551	0.2872331
9	0.7177711	0.5407959	0.4715634	0.4671693	0.2457485
10	0.7539329	0.5797361	0.4978663	0.4831276	0.2173920
11	0.7897552	0.6091356	0.5238098	0.4952108	0.1841420
12	0.8166496	0.6381054	0.5433117	0.5070823	0.1614354
13	0.8430861	0.6597633	0.5624466	0.5159703	0.1356238
14	0.8627638	0.6809815	0.5767010	0.5246659	0.1180904
15	0.8820069	0.6967228	0.5906255	0.5311244	0.0985839
16	0.8962356	0.7120805	0.6009285	0.5374202	0.0854010
17	0.9100993	0.7234085	0.6109599	0.5420685	0.0709592
18	0.9203002	0.7344275	0.6183452	0.5465874	0.0612392
19	0.9302134	0.7425211	0.6255185	0.5499091	0.0507085
20	0.9374817	0.7503770	0.6307802	0.5531320	0.0436428
21	0.9445318	0.7561297	0.6358821	0.5554933	0.0360482
22	0.9496877	0.7617049	0.6396145	0.5577813	0.0309644
23	0.9546822	0.7657786	0.6432291	0.5594537	0.0255305
24	0.9583282	0.7697223	0.6458685	0.5610725	0.0218992
25	0.9618567	0.7725994	0.6484223	0.5622538	0.0180333
26	0.9644291	0.7753826	0.6502845	0.5633965	0.0154530
27	0.9669171	0.7774108	0.6520853	0.5642293	0.0127136
28	0.9687293	0.7793718	0.6533971	0.5650345	0.0108868
29	0.9704811	0.7807997	0.6546652	0.5656209	0.0089512
30	0.9717563	0.7821798	0.6555883	0.5661876	0.0076611
31	0.9729886	0.7831841	0.6564803	0.5666000	0.0062962
32	0.9738852	0.7841545	0.6571293	0.5669986	0.0053869
33	0.9747514	0.7848605	0.6577564	0.5672885	0.0044258
34	0.9753815	0.7855425	0.6582125	0.5675686	0.0037857
35	0.9759902	0.7860385	0.6586531	0.5677723	0.0031096
36	0.9764328	0.7865176	0.6589735	0.5679690	0.0026594
37	0.9768603	0.7868660	0.6592830	0.5681121	0.0021840
38	0.9771711	0.7872024	0.6595080	0.5682503	0.0018676
39	0.9774713	0.7874470	0.6597253	0.5683508	0.0015336
40	0.9776895	0.7876833	0.6598833	0.5684478	0.0013113
41	0.9779003	0.7878550	0.6600359	0.5685183	0.0010768
42	0.9780535	0.7880209	0.6601468	0.5685865	0.0009206
43	0.9782015	0.7881415	0.6602539	0.5686360	0.0007559
44	0.9783090	0.7882579	0.6603318	0.5686838	0.0006463
45	0.9784129	0.7883425	0.6604070	0.5687186	0.0005306
46	0.9784884	0.7884242	0.6604616	0.5687521	0.0004536
47	0.9785613	0.7884837	0.6605144	0.5687765	0.0003724
48	0.9786143	0.7885410	0.6605527	0.5688001	0.0003184

The result from the D-method (cont.)

49	0.9786655	0.7885827	0.6605898	0.5688172	0.0002614
50	0.9787027	0.7886230	0.6606167	0.5688338	0.0002235
51	0.9787386	0.7886522	0.6606427	0.5688458	0.0001835
52	0.9787647	0.7886805	0.6606616	0.5688574	0.0001569
53	0.9787899	0.7887010	0.6606799	0.5688658	0.0001288
54	0.9788082	0.7887209	0.6606931	0.5688740	0.0001101
55	0.9788259	0.7887353	0.6607059	0.5688799	0.0000904
56	0.9788388	0.7887492	0.6607152	0.5688856	0.0000773
57	0.9788512	0.7887593	0.6607242	0.5688898	0.0000634
58	0.9788602	0.7887691	0.6607308	0.5688938	0.0000542
59	0.9788689	0.7887762	0.6607371	0.5688967	0.0000445
60	0.9788752	0.7887830	0.6607416	0.5688995	0.0000381
61	0.9788814	0.7887880	0.6607461	0.5689015	0.0000312
62	0.9788858	0.7887928	0.6607493	0.5689035	0.0000267
63	0.9788901	0.7887963	0.6607524	0.5689050	0.0000219
64	0.9788932	0.7887997	0.6607547	0.5689063	0.0000187
65	0.9788962	0.7888022	0.6607568	0.5689074	0.0000154
66	0.9788984	0.7888045	0.6607584	0.5689083	0.0000132
67	0.9789005	0.7888063	0.6607600	0.5689090	0.0000108
68	0.9789021	0.7888079	0.6607611	0.5689097	0.0000092
69	0.9789036	0.7888091	0.6607621	0.5689102	0.0000076
70	0.9789046	0.7888103	0.6607629	0.5689107	0.0000065
71	0.9789057	0.7888112	0.6607637	0.5689110	0.0000053
72	0.9789064	0.7888120	0.6607642	0.5689114	0.0000045
73	0.9789072	0.7888126	0.6607648	0.5689116	0.0000037
74	0.9789077	0.7888131	0.6607651	0.5689119	0.0000032
75	0.9789082	0.7888136	0.6607655	0.5689120	0.0000026
76	0.9789086	0.7888140	0.6607658	0.5689122	0.0000022
77	0.9789089	0.7888143	0.6607660	0.5689123	0.0000018
78	0.9789092	0.7888145	0.6607662	0.5689124	0.0000016
79	0.9789095	0.7888147	0.6607664	0.5689125	0.0000013
80	0.9789096	0.7888149	0.6607665	0.5689126	0.0000011
81	0.9789098	0.7888151	0.6607667	0.5689127	0.0000009
82	0.9789099	0.7888152	0.6607668	0.5689127	0.0000008
83	0.9789101	0.7888153	0.6607669	0.5689128	0.0000006
84	0.9789102	0.7888154	0.6607669	0.5689128	0.0000005
85	0.9789102	0.7888155	0.6607670	0.5689128	0.0000004

Estimates values are $x = 0.9789102$, $y = 0.7888155$ and $z = 0.6607670$ $w = 0.5689128$ after 85 iterations.

The result from the U_2-method

k	x^k	y^k	z^k	w^k	ER
1	0.3049853	0.0645161	0.1290323	0.2580645	0.4966220
2	0.4802704	0.2476939	0.1954742	0.3210292	0.4153508
3	0.6091694	0.3844552	0.3015647	0.3531140	0.3580077
4	0.7078302	0.4904465	0.3926911	0.4038219	0.2813047
5	0.7823798	0.5714133	0.4641499	0.4466287	0.2136540
6	0.8376900	0.6320078	0.5182804	0.4797464	0.1588210

The result from the U_2-method (cont.)

7	0.8781074	0.6765768	0.5584214	0.5045774	0.1162174
8	0.9073063	0.7089293	0.5877348	0.5228509	0.0840536
9	0.9282253	0.7321874	0.6089012	0.5361209	0.0602718
10	0.9431227	0.7487912	0.6240598	0.5456640	0.0429508
11	0.9536864	0.7605853	0.6348519	0.5524787	0.0304713
12	0.9611544	0.7689334	0.6425031	0.5573204	0.0215492
13	0.9664226	0.7748274	0.6479113	0.5607479	0.0152052
14	0.9701332	0.7789814	0.6517260	0.5631681	0.0107118
15	0.9727440	0.7819054	0.6544127	0.5648740	0.0075378
16	0.9745796	0.7839618	0.6563029	0.5660748	0.0053001
17	0.9758695	0.7854071	0.6576318	0.5669193	0.0037246
18	0.9767755	0.7864225	0.6585657	0.5675130	0.0026164
19	0.9774118	0.7871357	0.6592216	0.5679300	0.0018374
20	0.9778585	0.7876365	0.6596822	0.5682229	0.0012901
21	0.9781722	0.7879880	0.6600056	0.5684286	0.0009057
22	0.9783923	0.7882348	0.6602327	0.5685730	0.0006358
23	0.9785469	0.7884081	0.6603921	0.5686743	0.0003133
25	0.9787315	0.7886150	0.6605824	0.5687954	0.0002199
26	0.9787849	0.7886749	0.6606376	0.5688305	0.0001543
27	0.9788224	0.7887169	0.6606762	0.5688551	0.0001083
28	0.9788487	0.7887464	0.6607034	0.5688724	0.0000760
29	0.9788672	0.7887672	0.6607224	0.5688845	0.0000533
30	0.9788801	0.7887817	0.6607358	0.5688930	0.0000374
31	0.9788892	0.7887919	0.6607452	0.5688989	0.0000263
32	0.9788956	0.7887990	0.6607518	0.5689031	0.0000184
33	0.9789001	0.7888041	0.6607564	0.5689061	0.0000129
34	0.9789033	0.7888076	0.6607597	0.5689081	0.0000091
35	0.9789055	0.7888101	0.6607619	0.5689096	0.0000064
36	0.9789070	0.7888118	0.6607635	0.5689106	0.0000045
37	0.9789081	0.7888130	0.6607646	0.5689113	0.0000031
38	0.9789089	0.7888139	0.6607654	0.5689118	0.0000022
39	0.9789094	0.7888145	0.6607660	0.5689122	0.0000015
40	0.9789098	0.7888149	0.6607664	0.5689124	0.0000011
41	0.9789100	0.7888152	0.6607666	0.5689126	0.0000008
42	0.9789102	0.7888154	0.6607668	0.5689127	0.0000005
43	0.9789104	0.7888156	0.6607670	0.5689128	0.0000004

Estimates values are $x = 0.9789104$, $y = 0.7888156$ and $z = 0.6607670$ $w = 0.5689128$ after 43 iterations.

The result from the L_2-method

k	x^k	y^k	z^k	w^k	ER
1	0.2424242	0.2063573	0.2488547	0.3834623	0.4674482
3	0.4401756	0.3403168	0.3615451	0.4351283	0.4427358
4	0.5675995	0.4571157	0.4430951	0.4723993	0.3591741
5	0.6769442	0.5478697	0.5040951	0.4999548	0.2758280
6	0.7607981	0.6158695	0.5490686	0.5200257	0.2060944
7	0.8230130	0.6657620	0.5816928	0.5344471	0.1510474
8	0.8683282	0.7018198	0.6050707	0.5447089	0.1091139

The result from the L_2-method (cont.)

9	0.9009041	0.7275942	0.6216770	0.5519615	0.0779848
10	0.9241010	0.7458732	0.6334008	0.5570633	0.0553067
11	0.9405079	0.7587642	0.6416420	0.5606404	0.0390067
12	0.9520569	0.7678196	0.6474178	0.5631430	0.0274026
13	0.9601588	0.7741630	0.6514572	0.5648910	0.0191972
14	0.9658291	0.7785981	0.6542783	0.5661107	0.0134227
15	0.9697910	0.7816948	0.6562464	0.5669611	0.0093723
16	0.9725560	0.7838549	0.6576184	0.5675538	0.0065379
17	0.9744842	0.7853607	0.6585745	0.5679666	0.0045576
18	0.9758280	0.7864099	0.6592406	0.5682541	0.0031757
19	0.9767642	0.7871407	0.6597044	0.5684543	0.0022121
20	0.9774163	0.7876496	0.6600273	0.5685937	0.0015405
21	0.9778703	0.7880040	0.6602522	0.5686907	0.0010726
22	0.9781864	0.7882507	0.6604087	0.5687583	0.0007468
23	0.9784065	0.7884225	0.6605177	0.5688053	0.0005199
24	0.9785597	0.7885421	0.6605936	0.5688380	0.0003619
25	0.9786664	0.7886253	0.6606464	0.5688608	0.0002519
26	0.9787406	0.7886832	0.6606831	0.5688767	0.0001754
27	0.9787923	0.7887235	0.6607087	0.5688877	0.0001221
28	0.9788283	0.7887516	0.6607265	0.5688954	0.0000850
29	0.9788533	0.7887712	0.6607389	0.5689007	0.0000591
30	0.9788708	0.7887848	0.6607475	0.5689045	0.0000412
31	0.9788829	0.7887942	0.6607535	0.5689071	0.0000287
32	0.9788913	0.7888008	0.6607577	0.5689089	0.0000199
33	0.9788972	0.7888054	0.6607606	0.5689101	0.0000139
34	0.9789013	0.7888086	0.6607626	0.5689110	0.0000097
35	0.9789041	0.7888108	0.6607641	0.5689116	0.0000067
36	0.9789061	0.7888124	0.6607650	0.5689120	0.0000047
37	0.9789075	0.7888134	0.6607657	0.5689123	0.0000033
38	0.9789085	0.7888142	0.6607662	0.5689125	0.0000023
39	0.9789091	0.7888147	0.6607665	0.5689127	0.0000016
40	0.9789096	0.7888151	0.6607668	0.5689128	0.0000011
41	0.9789099	0.7888153	0.6607669	0.5689128	0.0000008
42	0.9789101	0.7888155	0.6607670	0.5689129	0.0000005
43	0.9789103	0.7888156	0.6607671	0.5689129	0.0000004

Estimates values are $x = 0.9789103$, $y = 0.7888156$ and $z = 0.6607671$ $w = 0.5689129$ after 43 iterations.

The iterations of table 4.4

The result from the D-method

k	α^k	β^k	ER
1	1.2109755	1.3987772	17.1332397
2	0.7315968	0.1328928	6.2901647
3	0.2187899	1.1811254	7.0967465
4	1.2815707	0.4878798	16.4323737
5	0.3995267	0.3997076	3.4560856
6	0.1440196	0.1183178	2.4693111

The result from the D-method (cont.)

7	0.3047163	0.5957850	3.7049785
8	0.0455716	0.2761146	2.5622513
9	0.2210385	0.6382815	2.8445408
10	0.0345225	0.3733544	1.9522655
11	0.1723121	0.5854628	1.6969877
12	0.0666326	0.4118765	1.2193446
13	0.1543055	0.5258934	0.9368773
14	0.0978620	0.4212949	0.7200837
15	0.1503465	0.4836094	0.5251829
16	0.1193656	0.4210717	0.4268465
17	0.1507744	0.4564854	0.3041719
18	0.1330718	0.4186795	0.2569685
19	0.1520999	0.4394873	0.1810307
20	0.1416545	0.4163602	0.1568423
21	0.1533157	0.4288773	0.1098160
22	0.1470140	0.4146020	0.0966852
23	0.1542211	0.4222501	0.0674598
24	0.1503634	0.4133844	0.0600006
25	0.1548432	0.4181047	0.0417778
26	0.1524594	0.4125762	0.0373972
27	0.1552545	0.4155083	0.0260069
28	0.1537726	0.4120519	0.0233740
29	0.1555207	0.4138807	0.0162425
30	0.1545960	0.4117161	0.0146351
31	0.1556910	0.4128596	0.0101651
32	0.1551126	0.4115027	0.0091737
33	0.1557992	0.4122189	0.0063699
34	0.1554368	0.4113676	0.0057545
35	0.1558675	0.4118166	0.0039950
36	0.1556404	0.4112824	0.0036112
37	0.1559107	0.4115641	0.0025068
38	0.1557682	0.4112287	0.0022669
39	0.1559379	0.4114055	0.0015735
40	0.1558484	0.4111950	0.0014232
41	0.1559550	0.4113060	0.0009878
42	0.1558988	0.4111737	0.0008937
43	0.1559657	0.4112434	0.0006203
44	0.1559304	0.4111604	0.0005612
45	0.1559725	0.4112042	0.0003895
46	0.1559503	0.4111520	0.0003524
47	0.1559767	0.4111795	0.0002446
48	0.1559628	0.4111468	0.0002213
49	0.1559794	0.4111640	0.0001536
50	0.1559706	0.4111434	0.0001390
51	0.1559810	0.4111543	0.0000965
52	0.1559755	0.4111414	0.0000873
53	0.1559821	0.4111482	0.0000606
54	0.1559786	0.4111401	0.0000548
55	0.1559827	0.4111443	0.0000380
56	0.1559806	0.4111392	0.0000344
57	0.1559832	0.4111419	0.0000239
58	0.1559818	0.4111387	0.0000216

The result from the D-method (cont.)

59	0.1559834	0.4111404	0.0000150
60	0.1559826	0.4111384	0.0000136
61	0.1559836	0.4111395	0.0000094
62	0.1559830	0.4111382	0.0000085
63	0.1559837	0.4111389	0.0000059
64	0.1559834	0.4111381	0.0000054
65	0.1559838	0.4111385	0.0000037
66	0.1559835	0.4111380	0.0000034
67	0.1559838	0.4111383	0.0000023
68	0.1559837	0.4111379	0.0000021
69	0.1559838	0.4111381	0.0000015
70	0.1559837	0.4111379	0.0000013
71	0.1559838	0.4111380	0.0000009
72	0.1559838	0.4111379	0.0000008
73	0.1559838	0.4111380	0.0000006
74	0.1559838	0.4111379	0.0000005
75	0.1559838	0.4111379	0.0000004

Estimates values are $\alpha = 0.1559838$ and $\beta = 0.4111379$ after 75 iterations.

The result from the U_2-method

k	α^k	β^k	ER
1	1.2109755	1.4651653	16.9739398
2	0.5767498	0.1335735	3.8988815
3	0.2748100	0.3239077	0.9695848
4	0.1992298	0.3614238	0.1974691
5	0.1811972	0.3795779	0.0928906
6	0.1719872	0.3911385	0.0589162
7	0.1661177	0.3985065	0.0374543
8	0.1623813	0.4031768	0.0237024
9	0.1600147	0.4061270	0.0149566
10	0.1585205	0.4079865	0.0094211
11	0.1575790	0.4091570	0.0059277
12	0.1569864	0.4098931	0.0037271
13	0.1566138	0.4103559	0.0023424
14	0.1563796	0.4106466	0.0014718
15	0.1562325	0.4108293	0.0009246
16	0.1561400	0.4109441	0.0005808
17	0.1560819	0.4110162	0.0003648
18	0.1560455	0.4110614	0.0002291
19	0.1560225	0.4110899	0.0001439
20	0.1560082	0.4111077	0.0000904
21	0.1559991	0.4111189	0.0000568
22	0.1559934	0.4111260	0.0000356
23	0.1559899	0.4111304	0.0000224
24	0.1559876	0.4111332	0.0000141
25	0.1559862	0.4111349	0.0000088
26	0.1559854	0.4111360	0.0000055
27	0.1559848	0.4111367	0.0000035
28	0.1559844	0.4111371	0.0000022

The result from the U_2 -method (cont.)

29	0.1559842	0.4111374	0.0000014
30	0.1559841	0.4111376	0.0000009
31	0.1559840	0.4111377	0.0000005
32	0.1559840	0.4111377	0.0000003

Estimates values are α = 0.155984 and β = 0.4111377 after 32 iterations.

The result from the L_2 -method

k	α^k	β^k	ER
1	1.3945603	1.3987772	17.7332942
2	1.1304847	0.4717857	14.5778288
3	0.7845440	0.6989656	11.4480618
4	1.1267613	1.2762476	16.8431799
5	1.1602021	1.3756912	16.9372957
6	1.0290545	0.7301739	14.7236213
7	0.3726444	0.8510101	6.7643833
8	0.0340576	0.2657856	2.7869199
9	0.0059357	0.6773577	0.5293212
10	0.0689810	0.5857281	0.2543379
11	0.1029151	0.5166065	0.1439754
12	0.1232922	0.4759858	0.0843167
13	0.1357014	0.4513141	0.0506838
14	0.1433436	0.4361537	0.0309661
15	0.1480838	0.4267636	0.0191139
16	0.1510375	0.4209178	0.0118742
17	0.1528834	0.4172668	0.0074066
18	0.1540390	0.4149818	0.0046316
19	0.1547634	0.4135499	0.0029010
20	0.1552177	0.4126519	0.0018188
21	0.1555029	0.4120884	0.0011410
22	0.1556818	0.4117347	0.0007161
23	0.1557942	0.4115126	0.0004496
24	0.1558648	0.4113732	0.0002823
25	0.1559091	0.4112857	0.0001772
26	0.1559369	0.4112307	0.0001113
27	0.1559544	0.4111962	0.0000699
28	0.1559653	0.4111745	0.0000439
29	0.1559722	0.4111609	0.0000276
30	0.1559760	0.4111569	0.0000020
31	0.1559793	0.4111469	0.0000109
32	0.1559810	0.4111436	0.0000068
33	0.1559821	0.4111414	0.0000043
34	0.1559827	0.4111401	0.0000027
35	0.1559831	0.4111393	0.0000017
36	0.1559834	0.4111387	0.0000011
37	0.1559836	0.4111384	0.0000007
38	0.1559837	0.4111382	0.0000004

Estimates values are α = 0.1559837 and β = 0.4111382 after 38 iterations.

The iterations of table 4.5

The result from the D-method

k	x^k	y^k	z^k	ER
1	0.6250000	0.5000000	0.5000000	1.8593750
2	0.3125000	0.6093750	0.1601563	0.8036804
3	0.5152100	1.0240259	0.1172485	1.2020825
4	0.4868831	0.8639628	0.3285176	0.5169733
5	0.3893453	0.8110608	0.2458717	0.3421724
6	0.4380904	0.8912876	0.2023524	0.2213010
7	0.4576415	0.8759933	0.2465792	0.0673552
8	0.4355313	0.8545312	0.2442000	0.0780230
9	0.4363106	0.8664999	0.2299778	0.0339424
10	0.4440382	0.8699166	0.2352973	0.0245603
11	0.4411840	0.8645772	0.2384812	0.0142940
12	0.4394652	0.8651495	0.2355343	0.0048280
13	0.4410512	0.8668315	0.2354033	0.0057063
14	0.4411182	0.8660599	0.2364807	0.0023544
15	0.4405419	0.8657319	0.2361613	0.0018857
16	0.4407129	0.8661125	0.2358922	0.0009606
17	0.4408570	0.8660987	0.2360947	0.0004213
18	0.4407486	0.8659702	0.2361205	0.0004136
19	0.4407348	0.8660184	0.2360409	0.0001585
20	0.4407774	0.8660471	0.2360587	0.0001416
21	0.4407678	0.8660206	0.2360806	0.0000666
22	0.4407561	0.8660195	0.2360670	0.0000353
23	0.4407634	0.8660291	0.2360639	0.0000295
24	0.4407650	0.8660262	0.2360697	0.0000104
25	0.4407619	0.8660238	0.2360688	0.0000105
26	0.4407624	0.8660256	0.2360671	0.0000048
27	0.4407633	0.8660259	0.2360680	0.0000029
28	0.4407629	0.8660252	0.2360683	0.0000021
29	0.4407627	0.8660253	0.2360679	0.0000007
30	0.4407629	0.8660255	0.2360679	0.0000008
31	0.4407629	0.8660254	0.2360680	0.0000003

Estimates values are $x = 0.4407629$, $y = 0.8660254$ and $z = 0.2360680$ after 31 iterations.

The result from the U_2-method

k	x^k	y^k	z^k	ER
1	0.6250000	0.5000000	0.5000000	1.8593750
2	0.3125000	0.9492188	0.1601563	0.5090637
3	0.5152100	0.9213041	0.2496681	0.3699221
4	0.4397306	0.8176446	0.2785606	0.3337941
5	0.4158989	0.8761320	0.2154764	0.1222622
6	0.4526843	0.8787084	0.2351448	0.0790331
7	0.4414003	0.8578637	0.2442629	0.0603569
8	0.4363045	0.8667371	0.2326911	0.0205836

The result from the U_2-method (cont.)

9	0.4425997	0.8684677	0.2353987	0.0153860
10	0.4411231	0.8646979	0.2375327	0.0101360
11	0.4399768	0.8659799	0.2355730	0.0031390
12	0.4410289	0.8664777	0.2358752	0.0028602
13	0.4408661	0.8658209	0.2363225	0.0016355
14	0.4406265	0.8659909	0.2360022	0.0004455
15	0.4407981	0.8661071	0.2360230	0.0005137
16	0.4407870	0.8659957	0.2361111	0.0002542
17	0.4407398	0.8660152	0.2360604	0.0000734
18	0.4407669	0.8660398	0.2360585	0.0000913
19	0.4407680	0.8660214	0.2360751	0.0000396
20	0.4407591	0.8660230	0.2360674	0.0000166
21	0.4407632	0.8660279	0.2360661	0.0000163
22	0.4407639	0.8660249	0.2360691	0.0000065
23	0.4407623	0.8660249	0.2360680	0.0000035
24	0.4407629	0.8660258	0.2360676	0.0000028
25	0.4407631	0.8660254	0.2360682	0.0000010
26	0.4407628	0.8660253	0.2360680	0.0000007
27	0.4407629	0.8660255	0.2360679	0.0000005

Estimates values are $x = 0.4407629$, $y = 0.8660255$ and $z = 0.2360679$ after 27 iterations.

The result from the U_3-method

k	x^k	y^k	z^k	ER
1	0.6250000	0.5000000	0.5000000	1.8593750
2	0.5843750	0.9492188	0.1601563	0.9871899
3	0.4429059	0.7825774	0.3106276	0.5714901
4	0.4708331	0.8862801	0.2021482	0.2591712
5	0.4361931	0.8478545	0.2517941	0.1291975
6	0.4461430	0.8713390	0.2272804	0.0597852
7	0.4390984	0.8624347	0.2395688	0.0278807
8	0.4418263	0.8674217	0.2341502	0.0135241
9	0.4403233	0.8652610	0.2369077	0.0063933
10	0.4409940	0.8663671	0.2356403	0.0031025
11	0.4406568	0.8658539	0.2362669	0.0014834
12	0.4408153	0.8661061	0.2359703	0.0007164
13	0.4407380	0.8659861	0.2361145	0.0003440
14	0.4407749	0.8660442	0.2360455	0.0001658
15	0.4407571	0.8660163	0.2360788	0.0000797
16	0.4407657	0.8660298	0.2360628	0.0000384
17	0.4407615	0.8660233	0.2360705	0.0000185
18	0.4407635	0.8660264	0.2360668	0.0000089
19	0.4407626	0.8660249	0.2360686	0.0000043
20	0.4407630	0.8660256	0.2360677	0.0000021
21	0.4407628	0.8660253	0.2360681	0.0000010
22	0.4407629	0.8660255	0.2360679	0.0000005

Estimates values are $x = 0.4407629$, $y = 0.8660255$ and $z = 0.2360679$ after 22 iterations.

The result from the L_2 -method

k	x^k	y^k	z^k	ER
1	0.6250000	0.8750000	0.4375000	1.2734375
2	0.3593750	0.8660714	0.2851563	0.3403862
3	0.4143809	0.8660254	0.2197876	0.1025249
4	0.4505576	0.8660254	0.2304279	0.0434889
5	0.4437891	0.8660254	0.2382505	0.0134776
6	0.4396068	0.8660254	0.2367372	0.0050973
7	0.4404045	0.8660254	0.2358135	0.0015736
8	0.4408993	0.8660254	0.2359890	0.0006022
9	0.4408052	0.8660254	0.2360981	0.0001860
10	0.4407468	0.8660254	0.2360773	0.0000711
11	0.4407579	0.8660254	0.2360644	0.0000219
12	0.4407648	0.8660254	0.2360669	0.0000084
13	0.4407635	0.8660254	0.2360684	0.0000026
14	0.4407626	0.8660254	0.2360681	0.0000010
15	0.4407628	0.8660254	0.2360679	0.0000003

Estimates values are $x = 0.4407628$, $y = 0.8660254$ and $z = 0.2360679$ after 15 iterations.

The result from the L_3 -method

k	x^k	y^k	z^k	ER
1	0.6250000	0.8750000	0.2500000	0.5781250
2	0.4625000	0.8660714	0.2343750	0.0642415
3	0.4421347	0.8660254	0.2362671	0.0030252
4	0.4406586	0.8660254	0.2360445	0.0002081
5	0.4407755	0.8660254	0.2360708	0.0000249
6	0.4407614	0.8660254	0.2360676	0.0000029
7	0.4407630	0.8660254	0.2360680	0.0000003

Estimates values are $x = 0.440763$, $y = 0.8660254$ and $z = 0.236068$ after 7 iterations.

The iterations of table 4.6

The result from the D-method

k	p^k	$\mu_1{}^k$	$\mu_2{}^k$	ER
1	0.3500923	0.4656800	0.2288426	135.9110891
2	0.2826142	0.5616720	0.3302022	59.8563918
3	0.2472254	0.6409040	0.3957813	24.6545412
4	0.2309401	0.6957332	0.4329424	10.6434886
5	0.2235616	0.7252334	0.4515972	4.6407020
6	0.2202736	0.7393242	0.4603842	2.0250266
7	0.2188248	0.7457215	0.4643681	0.8841805
8	0.2181895	0.7485613	0.4661365	0.3858327
9	0.2179118	0.7498104	0.4669143	0.1684408
10	0.2177904	0.7503572	0.4672548	0.0734975
11	0.2177374	0.7505963	0.4674037	0.0320850
12	0.2177143	0.7507007	0.4674687	0.0139997

The result from the D-method (cont.)

13	0.2177042	0.7507462	0.4674971	0.0061115
14	0.2176998	0.7507661	0.4675094	0.0026666
15	0.2176979	0.7507748	0.4675148	0.0011641
16	0.2176970	0.7507786	0.4675172	0.0005079
17	0.2176967	0.7507802	0.4675182	0.0002217
18	0.2176965	0.7507810	0.4675187	0.0000967
19	0.2176964	0.7507813	0.4675189	0.0000422
20	0.2176964	0.7507814	0.4675190	0.0000184
21	0.2176964	0.7507815	0.4675190	0.0000080
22	0.2176964	0.7507815	0.4675190	0.0000035
23	0.2176964	0.7507815	0.4675190	0.0000015
24	0.2176964	0.7507815	0.4675190	0.0000007
25	0.2176964	0.7507815	0.4675190	0.0000003

Estimates values are $p = 0.2176964$, $\mu_1 = 0.7507815$ and $\mu_2 = 0.467519$ after 25 iterations.

The result from the U_2 method

k	p^k	$\mu_1^{\ k}$	$\mu_2^{\ k}$	ER
1	0.3558123	0.4656800	0.2288426	147.3384061
2	0.2692016	0.5573594	0.3280829	37.9895680
3	0.2347887	0.6557648	0.4044690	9.6674594
4	0.2237520	0.7161747	0.4455929	3.6892198
5	0.2197525	0.7387431	0.4600106	1.2598774
6	0.2183896	0.7467161	0.4649873	0.4281175
7	0.2179291	0.7494140	0.4666674	0.1438075
8	0.2177745	0.7503228	0.4672334	0.0483433
9	0.2177226	0.7506276	0.4674232	0.0162055
10	0.2177052	0.7507299	0.4674869	0.0054380
11	0.2176993	0.7507642	0.4675083	0.0018231
12	0.2176974	0.7507757	0.4675154	0.0006115
13	0.2176967	0.7507796	0.4675178	0.0002050
14	0.2176965	0.7507809	0.4675186	0.0000688
15	0.2176964	0.7507813	0.4675189	0.0000231
16	0.2176964	0.7507815	0.4675190	0.0000077
17	0.2176964	0.7507815	0.4675190	0.0000026
18	0.2176964	0.7507815	0.4675190	0.0000009
19	0.2176964	0.7507815	0.4675190	0.0000003

Estimates values are $p = 0.2176964$, $\mu_1 = 0.7507815$ and $\mu_2 = 0.467519$ after 19 iterations.

The result from the U_3 method

k	p^k	$\mu_1^{\ k}$	$\mu_2^{\ k}$	ER
1	0.4010052	0.4656800	0.2288426	251.3588387
2	0.2772399	0.5255412	0.3124469	42.7843842
3	0.2280845	0.6403851	0.3951967	3.2203469
4	0.2194979	0.7251772	0.4510852	1.2382010
5	0.2180375	0.7469247	0.4651024	0.0777759

The result from the U_2 method

6	0.2177612	0.7501034	0.4670967	0.0066851
7	0.2177087	0.7506537	0.4674394	0.0010826
8	0.2176987	0.7507572	0.4675039	0.0002004
9	0.2176968	0.7507769	0.4675161	0.0000380
10	0.2176965	0.7507806	0.4675185	0.0000072
11	0.2176964	0.7507814	0.4675189	0.0000014
12	0.2176964	0.7507815	0.4675190	0.0000003

Estimates values are $p = 0.2176964$, $\mu_1 = 0.7507815$ and $\mu_2 = 0.467519$ after 25 iterations.

The result from the L_2 method

k	p^k	μ_1^k	μ_2^k	ER
1	0.3500923	0.5513415	0.2288426	163.7336189
2	0.2772100	0.6374816	0.3302022	63.1916031
3	0.2411491	0.7011177	0.3994243	22.5947301
4	0.2262310	0.7328950	0.4389476	7.9012675
5	0.2206782	0.7447411	0.4569820	2.7333424
6	0.2187107	0.7487620	0.4638469	0.9257292
7	0.2180385	0.7501048	0.4662725	0.3120377
8	0.2178113	0.7505548	0.4670989	0.1047268
9	0.2177350	0.7507055	0.4673780	0.0351508
10	0.2177093	0.7507560	0.4674717	0.0117867
11	0.2177007	0.7507730	0.4675032	0.0039535
12	0.2176978	0.7507787	0.4675137	0.0013257
13	0.2176969	0.7507806	0.4675172	0.0004446
14	0.2176966	0.7507812	0.4675184	0.0001491
15	0.2176964	0.7507814	0.4675188	0.0000500
16	0.2176964	0.7507815	0.4675190	0.0000168
17	0.2176964	0.7507815	0.4675190	0.0000056
18	0.2176964	0.7507815	0.4675190	0.0000019
19	0.2176964	0.7507815	0.4675190	0.0000006
20	0.2176964	0.7507815	0.4675190	0.0000002

Estimates values are $p = 0.2176964$, $\mu_1 = 0.7507815$ and $\mu_2 = 0.467519$ after 20 iterations.

The result from the L_3 method

k	p^k	μ_1^k	μ_2^k	ER
1	0.3500923	0.5513415	0.3145042	190.9895306
2	0.2728752	0.6413825	0.3934432	65.7633061
3	0.2362908	0.7087321	0.4406648	20.1810481
4	0.2228137	0.7394867	0.4604561	5.4686845
5	0.2188728	0.7483519	0.4660058	1.2675811
6	0.2179358	0.7503037	0.4672215	0.2593764
7	0.2177428	0.7506899	0.4674620	0.0503354
8	0.2177053	0.7507640	0.4675081	0.0096257
9	0.2176981	0.7507782	0.4675170	0.0018349

The result from the L_3 method (cont.)

10	0.2176967	0.7507809	0.4675186	0.0003496
11	0.2176965	0.7507814	0.4675190	0.0000666
12	0.2176964	0.7507815	0.4675190	0.0000127
13	0.2176964	0.7507815	0.4675190	0.0000024
14	0.2176964	0.7507815	0.4675190	0.0000005

Estimates values are p = 0.2176964, μ_1 = 0.7507815 and μ_2 = 0.467519 after 14 iterations.

The iterations of table 4.7

The result from D-method

k	x^k	y^k	z^k	w^k	ER
1	109733.3122300	-0.0000062	1.0076200	4.0028800	67.4507909
2	109722.1700500	0.0005111	1.0081262	4.0038924	25.2918606
3	109736.3461200	0.0004089	1.0076089	4.0028578	16.9501884
4	109733.5494500	0.0005390	1.0077111	4.0030621	6.3544828
5	109737.1128300	0.0005133	1.0075810	4.0028021	4.2581388
6	109736.4109400	0.0005460	1.0076067	4.0028534	1.5958060
7	109737.3062700	0.0005396	1.0075740	4.0027880	1.0695746
8	109737.1301400	0.0005478	1.0075804	4.0028009	0.4007095
9	109737.3550700	0.0005462	1.0075722	4.0027845	0.2686511
10	109737.3108800	0.0005482	1.0075738	4.0027877	0.1006159
11	109737.3673800	0.0005478	1.0075718	4.0027836	0.0674782
12	109737.3562900	0.0005483	1.0075722	4.0027844	0.0252642
13	109737.3704900	0.0005482	1.0075717	4.0027833	0.0169489
14	109737.3677100	0.0005484	1.0075718	4.0027835	0.0063435
15	109737.3712700	0.0005483	1.0075716	4.0027833	0.0042568
16	109737.3705800	0.0005484	1.0075717	4.0027833	0.0015925
17	109737.3714700	0.0005484	1.0075716	4.0027833	0.0010691
18	109737.3713000	0.0005484	1.0075716	4.0027833	0.0004000
19	109737.3715200	0.0005484	1.0075716	4.0027833	0.0002686
20	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001001
21	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000671
22	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000252
23	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000169
24	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000062
25	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000041
26	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000013
27	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000010
28	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000005

Estimates values are x = 109737.3715400, y = 0.0005484, z = 1.0075716 and w = 4.0027833 after 28 iterations.

The result from U_2-method

k	x^k	y^k	z^k	w^k	ER
1	109722.2875500	-0.0000024	1.0076200	4.0028800	44.8822702
2	109733.5733200	0.0004098	1.0081224	4.0038848	11.3121830
3	109736.4143000	0.0005136	1.0077102	4.0030604	2.8395614
4	109737.1300500	0.0005396	1.0076064	4.0028528	0.7163010
5	109737.3106000	0.0005462	1.0075804	4.0028008	0.1807504
6	109737.3561600	0.0005478	1.0075738	4.0027877	0.0456129
7	109737.3676600	0.0005482	1.0075722	4.0027844	0.0115107
8	109737.3705600	0.0005483	1.0075718	4.0027836	0.0029050
9	109737.3712900	0.0005484	1.0075717	4.0027833	0.0007331
10	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001850
11	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000466
12	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000117
13	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000033
14	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000005

Estimates values are $x = 109737.3715400$, $y = 0.0005484$, $z = 1.0075716$ and $w = 4.0027833$ after 14 iterations.

The result from U_3-method

k	x^k	y^k	z^k	w^k	ER
1	109722.2875500	-0.0000024	1.0076200	4.0028800	44.8822702
2	109733.5733200	0.0004098	1.0081224	4.0038848	11.3121830
3	109736.4143000	0.0005136	1.0077102	4.0030604	2.8395614
4	109737.1300500	0.0005396	1.0076064	4.0028528	0.7163010
5	109737.3106000	0.0005462	1.0075804	4.0028008	0.1807504
6	109737.3561600	0.0005478	1.0075738	4.0027877	0.0456129
7	109737.3676600	0.0005482	1.0075722	4.0027844	0.0115107
8	109737.3705600	0.0005483	1.0075718	4.0027836	0.0029050
9	109737.3712900	0.0005484	1.0075717	4.0027833	0.0007331
10	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001850
11	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000466
12	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000117
13	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000033
14	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000005

Estimates values are $x = 109737.3715400$, $y = 0.0005484$, $z = 1.0075716$ and $w = 4.0027833$ after 14 iterations.

The result from U_4-method

k	x^k	y^k	z^k	w^k	ER
1	109776.9907200	-0.0000024	1.0076200	4.0028800	154.2886099
2	109747.3492800	0.0009124	1.0081224	4.0038848	29.6011077
3	109740.0493600	0.0006395	1.0072076	4.0020552	7.4490001
4	109738.0095100	0.0005729	1.0074805	4.0026009	2.0781682
5	109737.5241700	0.0005542	1.0075471	4.0027341	0.4933980
6	109737.4078000	0.0005498	1.0075658	4.0027716	0.1185325
7	109737.3801600	0.0005487	1.0075702	4.0027805	0.0281491
8	109737.3735900	0.0005484	1.0075713	4.0027826	0.0066913
9	109737.3720200	0.0005484	1.0075716	4.0027831	0.0015896
10	109737.3716500	0.0005484	1.0075716	4.0027832	0.0003778
11	109737.3715700	0.0005484	1.0075716	4.0027833	0.0000897
12	109737.3715500	0.0005484	1.0075716	4.0027833	0.0000212
13	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000051
14	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000013
15	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000004

Estimates values are x = 109737.3715400, y = 0.0005484, z = 1.0075716 and w = 4.0027833 after 15 iterations.

The result from L_2-method

k	x^k	y^k	z^k	w^k	ER
1	109733.3122300	0.0005072	1.0076128	4.0028800	3.3518069
2	109736.2437100	0.0005380	1.0075820	4.0028655	0.8459680
3	109737.0878300	0.0005458	1.0075742	4.0028039	0.2128148
4	109737.3001700	0.0005477	1.0075723	4.0027885	0.0535345
5	109737.3535900	0.0005482	1.0075718	4.0027846	0.0134664
6	109737.3670200	0.0005483	1.0075717	4.0027836	0.0033879
7	109737.3704000	0.0005484	1.0075716	4.0027834	0.0008519
8	109737.3712500	0.0005484	1.0075716	4.0027833	0.0002144
9	109737.3714700	0.0005484	1.0075716	4.0027833	0.0000540
10	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000135
11	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000036
12	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000010
13	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000004

Estimates values are x = 109737.3715400, y = 0.0005484, z = 1.0075716 and w = 4.0027833 after 13 iterations.

The result from L_3-method

k	x^k	y^k	z^k	w^k	ER
1	109733.3122300	0.0005072	1.0076128	4.0028655	3.3518428
2	109736.2437600	0.0005380	1.0075820	4.0028039	0.8460959
3	109737.0880700	0.0005458	1.0075742	4.0027885	0.2126780
4	109737.3002900	0.0005477	1.0075723	4.0027846	0.0534563
5	109737.3536300	0.0005482	1.0075718	4.0027836	0.0134361
6	109737.3670400	0.0005483	1.0075717	4.0027834	0.0033772
7	109737.3704100	0.0005484	1.0075716	4.0027833	0.0008490
8	109737.3712500	0.0005484	1.0075716	4.0027833	0.0002130
9	109737.3714700	0.0005484	1.0075716	4.0027833	0.0000536
10	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000137
11	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000035
12	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000010
13	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000004

Estimates values are $x = 109737.37154$, $y = 0.0005484$, $z = 1.0075716$ and $w = 4.0027833$ after 13 iterations.

The result from L_4-method

k	x^k	y^k	z^k	w^k	ER
1	109733.3122300	0.0005072	1.0076128	4.0028655	3.3518428
2	109736.2437600	0.0005380	1.0075820	4.0028039	0.8460959
3	109737.0880700	0.0005458	1.0075742	4.0027885	0.2126780
4	109737.3002900	0.0005477	1.0075723	4.0027846	0.0534563
5	109737.3536300	0.0005482	1.0075718	4.0027836	0.0134361
6	109737.3670400	0.0005483	1.0075717	4.0027834	0.0033772
7	109737.3704100	0.0005484	1.0075716	4.0027833	0.0008490
8	109737.3712500	0.0005484	1.0075716	4.0027833	0.0002130
9	109737.3714700	0.0005484	1.0075716	4.0027833	0.0000536
10	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000137
11	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000035
12	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000010
13	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000004

Estimates values are $x = 109737.3715400$, $y = 0.0005484$, $z = 1.0075716$ and $w = 4.0027833$ after 13 iterations.

The iterations of table 4.8

The result from D-method

k	${p_1}^k$	${p_2}^k$	${p_3}^k$	${p_4}^k$	ER
1	422.7880807	345.1254021	341.8231521	171.7399530	0.1495377
2	422.0162015	346.1726212	341.9705173	172.0325769	0.0260293

The result from D-method (cont.)

3	422.3922212	346.4299345	342.2440226	172.1653089	0.0186227
4	422.5235414	346.6692933	342.4647842	172.2803128	0.0155095
5	422.6401858	346.8655114	342.6472752	172.3749708	0.0127999
6	422.7360558	347.0274751	342.7979335	172.4531105	0.0105693
7	422.8151938	347.1611299	342.9223236	172.5176087	0.0087273
8	422.8805103	347.2714385	343.0250282	172.5708514	0.0072066
9	422.9344247	347.3624879	343.1098305	172.6148055	0.0059510
10	422.9789308	347.4376472	343.1798530	172.6510937	0.0049142
11	423.0156729	347.4996943	343.2376733	172.6810546	0.0040581
12	423.0460074	347.5509201	343.2854190	172.7057926	0.0033512
13	423.0710529	347.5932143	343.3248462	172.7262189	0.0027675
14	423.0917326	347.6281357	343.3574049	172.7430856	0.0022855
15	423.1088082	347.6569707	343.3842920	172.7570134	0.0018874
16	423.1229082	347.6807810	343.4064959	172.7685147	0.0015587
17	423.1345514	347.7004426	343.4248325	172.7780124	0.0012873
18	423.1441663	347.7166789	343.4399754	172.7858556	0.0010631
19	423.1521062	347.7300868	343.4524811	172.7923327	0.0008780
20	423.1586630	347.7411591	343.4628089	172.7976817	0.0007251
21	423.1640779	347.7503029	343.4713382	172.8020991	0.0005988
22	423.1685496	347.7578542	343.4783821	172.8057472	0.0004945
23	423.1722425	347.7640903	343.4841994	172.8087600	0.0004084
24	423.1752923	347.7692404	343.4890037	172.8112481	0.0003373
25	423.1778110	347.7734936	343.4929714	172.8133029	0.0002786
26	423.1798911	347.7770061	343.4962482	172.8149999	0.0002301
27	423.1816089	347.7799070	343.4989544	172.8164014	0.0001900
28	423.1830276	347.7823027	343.5011894	172.8175588	0.0001569
29	423.1841993	347.7842812	343.5030352	172.8185147	0.0001296
30	423.1851669	347.7859152	343.5045595	172.8193042	0.0001070
31	423.1859660	347.7872647	343.5058185	172.8199561	0.0000884
32	423.1866260	347.7883792	343.5068582	172.8204946	0.0000730
33	423.1871711	347.7892996	343.5077169	172.8209393	0.0000603
34	423.1876212	347.7900597	343.5084261	172.8213065	0.0000498
35	423.1879930	347.7906875	343.5090118	172.8216099	0.0000411
36	423.1883000	347.7912060	343.5094955	172.8218603	0.0000340
37	423.1885536	347.7916342	343.5098949	172.8220672	0.0000280
38	423.1887630	347.7919878	343.5102249	172.8222381	0.0000232
39	423.1889359	347.7922799	343.5104973	172.8223792	0.0000191
40	423.1890788	347.7925211	343.5107224	172.8224957	0.0000158
41	423.1891967	347.7927203	343.5109082	172.8225920	0.0000130
42	423.1892942	347.7928848	343.5110617	172.8226714	0.0000108
43	423.1893746	347.7930207	343.5111885	172.8227371	0.0000089
44	423.1894411	347.7931329	343.5112931	172.8227913	0.0000074
45	423.1894960	347.7932255	343.5113796	172.8228361	0.0000061
46	423.1895413	347.7933021	343.5114510	172.8228731	0.0000050
47	423.1895787	347.7933653	343.5115100	172.8229036	0.0000041
48	423.1896096	347.7934175	343.5115587	172.8229288	0.0000034
49	423.1896352	347.7934606	343.5115989	172.8229496	0.0000028
50	423.1896562	347.7934962	343.5116321	172.8229668	0.0000023
51	423.1896737	347.7935256	343.5116596	172.8229811	0.0000019
52	423.1896880	347.7935499	343.5116822	172.8229928	0.0000016
53	423.1896999	347.7935700	343.5117009	172.8230025	0.0000013
54	423.1897097	347.7935865	343.5117164	172.8230105	0.0000011

The result from D-method (cont.)

55	423.1897178	347.7936002	343.5117291	172.8230171	0.0000009
56	423.1897245	347.7936115	343.5117397	172.8230226	0.0000007
57	423.1897300	347.7936208	343.5117484	172.8230271	0.0000006
58	423.1897346	347.7936285	343.5117556	172.8230308	0.0000005
59	423.1897384	347.7936349	343.5117615	172.8230339	0.0000004

Estimates values are $p_1 = 423.1897384$, $p_2 = 347.7936349$, $p_3 = 343.5117615$, $p_4 = 172.8230339$ after 59 iterations.

The result from the U_2 method

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	419.2839015	347.9709435	343.5227560	172.8728932	0.2061893
2	422.8766087	347.8297109	343.5122698	172.8321248	0.0169926
3	423.1708627	347.8126992	343.5262135	172.8314059	0.0016597
4	423.1948736	347.8079811	343.5247410	172.8298633	0.0002639
5	423.1956263	347.8050065	343.5222518	172.8284981	0.0001419
6	423.1945972	347.8027042	343.5201460	172.8273963	0.0001159
7	423.1936315	347.8008741	343.5184557	172.8265164	0.0000927
8	423.1928484	347.7994150	343.5171067	172.8258145	0.0000740
9	423.1922227	347.7982513	343.5160305	172.8252547	0.0000590
10	423.1917235	347.7973231	343.5151722	172.8248082	0.0000471
11	423.1913254	347.7965827	343.5144876	172.8244520	0.0000375
12	423.1910078	347.7959922	343.5139416	172.8241680	0.0000299
13	423.1907545	347.7955213	343.5135061	172.8239414	0.0000239
14	423.1905525	347.7951456	343.5131587	172.8237607	0.0000190
15	423.1903913	347.7948460	343.5128816	172.8236165	0.0000152
16	423.1902628	347.7946070	343.5126606	172.8235015	0.0000121
17	423.1901603	347.7944164	343.5124844	172.8234098	0.0000097
18	423.1900785	347.7942643	343.5123438	172.8233367	0.0000077
19	423.1900133	347.7941430	343.5122316	172.8232784	0.0000061
20	423.1899613	347.7940463	343.5121422	172.8232318	0.0000049
21	423.1899198	347.7939692	343.5120708	172.8231947	0.0000039
22	423.1898867	347.7939076	343.5120139	172.8231651	0.0000031
23	423.1898603	347.7938585	343.5119685	172.8231415	0.0000025
24	423.1898392	347.7938194	343.5119323	172.8231227	0.0000020
25	423.1898224	347.7937882	343.5119035	172.8231076	0.0000016
26	423.1898090	347.7937633	343.5118804	172.8230957	0.0000013
27	423.1897984	347.7937434	343.5118621	172.8230861	0.0000010
28	423.1897898	347.7937275	343.5118474	172.8230785	0.0000008
29	423.1897830	347.7937149	343.5118357	172.8230724	0.0000006
30	423.1897776	347.7937048	343.5118264	172.8230676	0.0000005
31	423.1897733	347.7936968	343.5118190	172.8230637	0.0000004

Estimates values are $p_1 = 423.1897733$, $p_2 = 347.7936968$, $p_3 = 343.511819$, $p_4 = 172.8230637$ after 31 iterations.

The result from the U_3-method

k	$p_1^{\ k}$	$p_2^{\ k}$	$p_3^{\ k}$	$p_4^{\ k}$	ER
1	419.5381790	347.4509062	343.0452044	172.6238991	0.1733695
2	422.6744304	347.4219255	343.1372620	172.6364737	0.0124028
3	422.9981159	347.4881221	343.2266320	172.6754019	0.0044013
4	423.0560524	347.5491010	343.2855547	172.7053748	0.0031826
5	423.0847569	347.5984886	343.3313987	172.7291777	0.0025106
6	423.1061391	347.6379654	343.3678855	172.7481643	0.0020004
7	423.1230690	347.6694705	343.3969954	172.7633146	0.0015955
8	423.1365656	347.6946067	343.4202229	172.7754028	0.0012727
9	423.1473320	347.7146590	343.4387547	172.7850467	0.0010151
10	423.1559204	347.7306544	343.4535386	172.7927398	0.0008096
11	423.1627711	347.7434128	343.4653317	172.7988763	0.0006457
12	423.1682353	347.7535889	343.4747383	172.8037708	0.0005150
13	423.1725934	347.7617048	343.4822410	172.8076746	0.0004107
14	423.1760692	347.7681775	343.4882249	172.8107881	0.0003275
15	423.1788412	347.7733396	343.4929973	172.8132712	0.0002612
16	423.1810519	347.7774564	343.4968033	172.8152514	0.0002083
17	423.1828149	347.7807394	343.4998387	172.8168306	0.0001661
18	423.1842209	347.7833575	343.5022593	172.8180901	0.0001325
19	423.1853421	347.7854453	343.5041897	172.8190944	0.0001056
20	423.1862362	347.7871103	343.5057291	172.8198953	0.0000842
21	423.1869492	347.7884381	343.5069567	172.8205340	0.0000672
22	423.1875178	347.7894969	343.5079357	172.8210433	0.0000536
23	423.1879712	347.7903412	343.5087163	172.8214495	0.0000427
24	423.1883328	347.7910145	343.5093389	172.8217734	0.0000341
25	423.1886211	347.7915514	343.5098353	172.8220317	0.0000272
26	423.1888511	347.7919796	343.5102312	172.8222376	0.0000217
27	423.1890344	347.7923210	343.5105469	172.8224019	0.0000173
28	423.1891807	347.7925933	343.5107986	172.8225328	0.0000138
29	423.1892972	347.7928104	343.5109994	172.8226373	0.0000110
30	423.1893902	347.7929835	343.5111595	172.8227206	0.0000088
31	423.1894644	347.7931216	343.5112871	172.8227870	0.0000070
32	423.1895235	347.7932317	343.5113889	172.8228400	0.0000056
33	423.1895706	347.7933195	343.5114701	172.8228822	0.0000044
34	423.1896082	347.7933895	343.5115348	172.8229159	0.0000035
35	423.1896382	347.7934453	343.5115865	172.8229427	0.0000028
36	423.1896621	347.7934898	343.5116276	172.8229641	0.0000023
37	423.1896812	347.7935253	343.5116605	172.8229812	0.0000018
38	423.1896964	347.7935537	343.5116866	172.8229948	0.0000014
39	423.1897085	347.7935762	343.5117075	172.8230057	0.0000011
40	423.1897182	347.7935942	343.5117241	172.8230144	0.0000009
41	423.1897259	347.7936086	343.5117374	172.8230213	0.0000007
42	423.1897320	347.7936200	343.5117480	172.8230268	0.0000006
43	423.1897370	347.7936292	343.5117564	172.8230312	0.0000005

Estimates values are $p_1 = 423.189737$, $p_2 = 347.7936292$, $p_3 = 343.5117564$, $p_4 = 172.8230312$ after 43 iterations.

The result from the U_4 method

k	$p_1^{\,k}$	$p_2^{\,k}$	$p_3^{\,k}$	$p_4^{\,k}$	ER
1	419.5381790	347.4509062	343.0452044	172.6238991	0.1733695
2	422.6744304	347.4219255	343.1372620	172.6364737	0.0124028
3	422.9981159	347.4881221	343.2266320	172.6754019	0.0044013
4	423.0560524	347.5491010	343.2855547	172.7053748	0.0031826
5	423.0847569	347.5984886	343.3313987	172.7291777	0.0025106
6	423.1061391	347.6379654	343.3678855	172.7481643	0.0020004
7	423.1230690	347.6694705	343.3969954	172.7633146	0.0015955
8	423.1365656	347.6946067	343.4202229	172.7754028	0.0012727
9	423.1473320	347.7146590	343.4387547	172.7850467	0.0010151
10	423.1559204	347.7306544	343.4535386	172.7927398	0.0008096
11	423.1627711	347.7434128	343.4653317	172.7988763	0.0006457
12	423.1682353	347.7535889	343.4747383	172.8037708	0.0005150
13	423.1725934	347.7617048	343.4822410	172.8076746	0.0004107
14	423.1760692	347.7681775	343.4882249	172.8107881	0.0003275
15	423.1788412	347.7733396	343.4929973	172.8132712	0.0002612
16	423.1810519	347.7774564	343.4968033	172.8152514	0.0002083
17	423.1828149	347.7807394	343.4998387	172.8168306	0.0001661
18	423.1842209	347.7833575	343.5022593	172.8180901	0.0001325
19	423.1853421	347.7854453	343.5041897	172.8190944	0.0001056
20	423.1862362	347.7871103	343.5057291	172.8198953	0.0000842
21	423.1869492	347.7884381	343.5069567	172.8205340	0.0000672
22	423.1875178	347.7894969	343.5079357	172.8210433	0.0000536
23	423.1879712	347.7903412	343.5087163	172.8214495	0.0000427
24	423.1883328	347.7910145	343.5093389	172.8217734	0.0000341
25	423.1886211	347.7915514	343.5098353	172.8220317	0.0000272
26	423.1888511	347.7919796	343.5102312	172.8222376	0.0000217
27	423.1890344	347.7923210	343.5105469	172.8224019	0.0000173
28	423.1891807	347.7925933	343.5107986	172.8225328	0.0000138
29	423.1892972	347.7928104	343.5109994	172.8226373	0.0000110
30	423.1893902	347.7929835	343.5111595	172.8227206	0.0000088
31	423.1894644	347.7931216	343.5112871	172.8227870	0.0000070
32	423.1895235	347.7932317	343.5113889	172.8228400	0.0000056
33	423.1895706	347.7933195	343.5114701	172.8228822	0.0000044
34	423.1896082	347.7933895	343.5115348	172.8229159	0.0000035
35	423.1896382	347.7934453	343.5115865	172.8229427	0.0000028
36	423.1896621	347.7934898	343.5116276	172.8229641	0.0000023
37	423.1896812	347.7935253	343.5116605	172.8229812	0.0000018
38	423.1896964	347.7935537	343.5116866	172.8229948	0.0000014
39	423.1897085	347.7935762	343.5117075	172.8230057	0.0000011
40	423.1897182	347.7935942	343.5117241	172.8230144	0.0000009
41	423.1897259	347.7936086	343.5117374	172.8230213	0.0000007
42	423.1897320	347.7936200	343.5117480	172.8230268	0.0000006
43	423.1897370	347.7936292	343.5117564	172.8230312	0.0000005

Estimates values are $p_1 = 423.189737$, $p_2 = 347.7936292$, $p_3 = 343.5117564$, $p_4 = 172.8230312$ after 43 iterations.

The result from the L_2-method

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	422.7880807	345.2696775	342.0277249	172.1044414	0.1573815
2	422.0982483	346.3581577	342.2703056	172.1936177	0.0372149
3	422.5037048	346.6849390	342.4984302	172.3296382	0.0182220
4	422.6509560	346.8956533	342.6892067	172.4240434	0.0140902
5	422.7530578	347.0651764	342.8440624	172.4993715	0.0113698
6	422.8354254	347.2025066	342.9696326	172.5603625	0.0091897
7	422.9021713	347.3138006	343.0714986	172.6097986	0.0074350
8	422.9562799	347.4040424	343.1541651	172.6498898	0.0060206
9	423.0001647	347.4772479	343.2212707	172.6824163	0.0048788
10	423.0357723	347.5366553	343.2757580	172.7088149	0.0039558
11	423.0646733	347.5848803	343.3200086	172.7302460	0.0032089
12	423.0881374	347.6240373	343.3559515	172.7476486	0.0026041
13	423.1071916	347.6558379	343.3851504	172.7617826	0.0021139
14	423.1226675	347.6816685	343.4088733	172.7732636	0.0017164
15	423.1352390	347.7026526	343.4281489	172.7825908	0.0013939
16	423.1454524	347.7197014	343.4438120	172.7901690	0.0011322
17	423.1537508	347.7335541	343.4565405	172.7963268	0.0009198
18	423.1604938	347.7448108	343.4668846	172.8013306	0.0007473
19	423.1659732	347.7539585	343.4752914	172.8053970	0.0006072
20	423.1704262	347.7613926	343.4821240	172.8087017	0.0004934
21	423.1740452	347.7674344	343.4876771	172.8113876	0.0004010
22	423.1769864	347.7723448	343.4921906	172.8135704	0.0003259
23	423.1793768	347.7763357	343.4958591	172.8153446	0.0002648
24	423.1813197	347.7795795	343.4988409	172.8167866	0.0002152
25	423.1828989	347.7822160	343.5012645	172.8179587	0.0001749
26	423.1841824	347.7843589	343.5032344	172.8189113	0.0001422
27	423.1852256	347.7861008	343.5048356	172.8196857	0.0001156
28	423.1860736	347.7875165	343.5061372	172.8203151	0.0000939
29	423.1867629	347.7886673	343.5071951	172.8208266	0.0000763
30	423.1873231	347.7896027	343.5080550	172.8212425	0.0000621
31	423.1877785	347.7903630	343.5087539	172.8215805	0.0000504
32	423.1881486	347.7909810	343.5093221	172.8218552	0.0000410
33	423.1884495	347.7914834	343.5097839	172.8220785	0.0000333
34	423.1886941	347.7918917	343.5101593	172.8222601	0.0000271
35	423.1888929	347.7922236	343.5104644	172.8224076	0.0000220
36	423.1890545	347.7924934	343.5107124	172.8225275	0.0000179
37	423.1891858	347.7927127	343.5109141	172.8226250	0.0000145
38	423.1892926	347.7928909	343.5110779	172.8227043	0.0000118
39	423.1893793	347.7930358	343.5112111	172.8227687	0.0000096
40	423.1894499	347.7931536	343.5113194	172.8228210	0.0000078
41	423.1895072	347.7932493	343.5114074	172.8228636	0.0000064
42	423.1895538	347.7933271	343.5114790	172.8228982	0.0000052
43	423.1895917	347.7933904	343.5115371	172.8229263	0.0000042
44	423.1896225	347.7934418	343.5115844	172.8229492	0.0000034
45	423.1896475	347.7934836	343.5116228	172.8229677	0.0000028
46	423.1896679	347.7935175	343.5116540	172.8229828	0.0000023
47	423.1896844	347.7935452	343.5116794	172.8229951	0.0000018
48	423.1896979	347.7935676	343.5117000	172.8230051	0.0000015
49	423.1897088	347.7935859	343.5117168	172.8230132	0.0000012
50	423.1897177	347.7936007	343.5117305	172.8230198	0.0000010
51	423.1897249	347.7936127	343.5117415	172.8230252	0.0000008

The result from the L_2-method (cont.)

52	423.1897308	347.7936225	343.5117505	172.8230295	0.0000007
53	423.1897355	347.7936305	343.5117579	172.8230331	0.0000005
54	423.1897394	347.7936370	343.5117638	172.8230359	0.0000004

Estimates values are $p_1 = 423.1897394$, $p_2 = 347.793637$, $p_3 = 343.5117638$, $p_4 = 172.8230359$ after 54 iterations.

The result from the L_3-method

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	422.7880807	345.2696775	341.8158919	172.3930695	0.1386174
2	422.0634434	346.2277392	342.2197141	172.2807536	0.0478780
3	422.4515679	346.6393438	342.4952191	172.3886176	0.0250521
4	422.6350959	346.8929524	342.7090200	172.4797649	0.0178248
5	422.7553929	347.0836108	342.8772976	172.5518222	0.0137745
6	422.8470660	347.2328078	343.0100831	172.6086644	0.0108044
7	422.9189843	347.3503581	343.1149565	172.6535297	0.0085034
8	422.9756895	347.4431273	343.1978319	172.6889646	0.0067024
9	423.0204587	347.5163974	343.2633515	172.7169665	0.0052881
10	423.0558284	347.5742992	343.3151677	172.7391042	0.0041755
11	423.0837858	347.6200761	343.3561578	172.7566120	0.0032990
12	423.1058928	347.6562795	343.3885906	172.7704618	0.0026077
13	423.1233790	347.6849193	343.4142571	172.7814204	0.0020621
14	423.1372136	347.7075805	343.4345717	172.7900928	0.0016311
15	423.1481612	347.7255143	343.4506521	172.7969569	0.0012905
16	423.1568255	347.7397088	343.4633820	172.8023903	0.0010212
17	423.1636837	347.7509448	343.4734601	172.8066916	0.0008083
18	423.1691127	347.7598397	343.4814393	172.8100970	0.0006398
19	423.1734107	347.7668818	343.4877569	172.8127931	0.0005065
20	423.1768135	347.7724573	343.4927593	172.8149278	0.0004010
21	423.1795077	347.7768719	343.4967202	172.8166181	0.0003174
22	423.1816410	347.7803673	343.4998566	172.8179565	0.0002513
23	423.1833301	347.7831350	343.5023401	172.8190163	0.0001990
24	423.1846676	347.7853266	343.5043068	172.8198554	0.0001576
25	423.1857267	347.7870620	343.5058641	172.8205200	0.0001248
26	423.1865653	347.7884363	343.5070972	172.8210462	0.0000988
27	423.1872294	347.7895245	343.5080738	172.8214629	0.0000782
28	423.1877553	347.7903862	343.5088471	172.8217928	0.0000620
29	423.1881717	347.7910685	343.5094595	172.8220541	0.0000491
30	423.1885014	347.7916089	343.5099444	172.8222611	0.0000388
31	423.1887626	347.7920368	343.5103284	172.8224249	0.0000308
32	423.1889694	347.7923757	343.5106325	172.8225547	0.0000244
33	423.1891331	347.7926440	343.5108733	172.8226574	0.0000193
34	423.1892628	347.7928565	343.5110640	172.8227388	0.0000153
35	423.1893655	347.7930248	343.5112150	172.8228032	0.0000121
36	423.1894468	347.7931580	343.5113346	172.8228543	0.0000096
37	423.1895112	347.7932635	343.5114293	172.8228947	0.0000076
38	423.1895622	347.7933471	343.5115043	172.8229267	0.0000060
39	423.1896026	347.7934133	343.5115637	172.8229520	0.0000048

The result from the L_3-method (cont.)

40	423.1896346	347.7934657	343.5116107	172.8229721	0.0000038
41	423.1896599	347.7935072	343.5116480	172.8229880	0.0000030
42	423.1896799	347.7935400	343.5116774	172.8230006	0.0000024
43	423.1896958	347.7935661	343.5117008	172.8230105	0.0000019
44	423.1897084	347.7935867	343.5117193	172.8230184	0.0000015
45	423.1897184	347.7936030	343.5117339	172.8230247	0.0000012
46	423.1897262	347.7936159	343.5117455	172.8230296	0.0000009
47	423.1897325	347.7936262	343.5117547	172.8230335	0.0000007
48	423.1897374	347.7936343	343.5117620	172.8230366	0.0000006
49	423.1897413	347.7936407	343.5117678	172.8230391	0.0000005

Estimates values are $p_1 = 423.1897413$, $p_2 = 347.7936407$, $p_3 = 343.5117678$, $p_4 = 172.8230391$ after 49 iterations.

The result from the L_4-method

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	422.7880807	345.2696775	341.8158919	172.3930695	0.1386174
2	422.0634434	346.2277392	342.2197141	172.2807536	0.0478780
3	422.4515679	346.6393438	342.4952191	172.3886176	0.0250521
4	422.6350959	346.8929524	342.7090200	172.4797649	0.0178248
5	422.7553929	347.0836108	342.8772976	172.5518222	0.0137745
6	422.8470660	347.2328078	343.0100831	172.6086644	0.0108044
7	422.9189843	347.3503581	343.1149565	172.6535297	0.0085034
8	422.9756895	347.4431273	343.1978319	172.6889646	0.0067024
9	423.0204587	347.5163974	343.2633515	172.7169665	0.0052881
10	423.0558284	347.5742992	343.3151677	172.7391042	0.0041755
11	423.0837858	347.6200761	343.3561578	172.7566120	0.0032990
12	423.1058928	347.6562795	343.3885906	172.7704618	0.0026077
13	423.1233790	347.6849193	343.4142571	172.7814204	0.0020621
14	423.1372136	347.7075805	343.4345717	172.7900928	0.0016311
15	423.1481612	347.7255143	343.4506521	172.7969569	0.0012905
16	423.1568255	347.7397088	343.4633820	172.8023903	0.0010212
17	423.1636837	347.7509448	343.4734601	172.8066916	0.0008083
18	423.1691127	347.7598397	343.4814393	172.8100970	0.0006398
19	423.1734107	347.7668818	343.4877569	172.8127931	0.0005065
20	423.1768135	347.7724573	343.4927593	172.8149278	0.0004010
21	423.1795077	347.7768719	343.4967202	172.8166181	0.0003174
22	423.1816410	347.7803673	343.4998566	172.8179565	0.0002513
23	423.1833301	347.7831350	343.5023401	172.8190163	0.0001990
24	423.1846676	347.7853266	343.5043068	172.8198554	0.0001576
25	423.1857267	347.7870620	343.5058641	172.8205200	0.0001248
26	423.1865653	347.7884363	343.5070972	172.8210462	0.0000988
27	423.1872294	347.7895245	343.5080738	172.8214629	0.0000782
28	423.1877553	347.7903862	343.5088471	172.8217928	0.0000620
29	423.1881717	347.7910685	343.5094595	172.8220541	0.0000491
30	423.1885014	347.7916089	343.5099444	172.8222611	0.0000388
31	423.1887626	347.7920368	343.5103284	172.8224249	0.0000308
32	423.1889694	347.7923757	343.5106325	172.8225547	0.0000244

The result from the L_4-method (cont.)

33	423.1891331	347.7926440	343.5108733	172.8226574	0.0000193
34	423.1892628	347.7928565	343.5110640	172.8227388	0.0000153
35	423.1893655	347.7930248	343.5112150	172.8228032	0.0000121
36	423.1894468	347.7931580	343.5113346	172.8228543	0.0000096
37	423.1895112	347.7932635	343.5114293	172.8228947	0.0000076
38	423.1895622	347.7933471	343.5115043	172.8229267	0.0000060
39	423.1896026	347.7934133	343.5115637	172.8229520	0.0000048
40	423.1896346	347.7934657	343.5116107	172.8229721	0.0000038
41	423.1896599	347.7935072	343.5116480	172.8229880	0.0000030
42	423.1896799	347.7935400	343.5116774	172.8230006	0.0000024
43	423.1896958	347.7935661	343.5117008	172.8230105	0.0000019
44	423.1897084	347.7935867	343.5117193	172.8230184	0.0000015
45	423.1897184	347.7936030	343.5117339	172.8230247	0.0000012
46	423.1897262	347.7936159	343.5117455	172.8230296	0.0000009
47	423.1897325	347.7936262	343.5117547	172.8230335	0.0000007
48	423.1897374	347.7936343	343.5117620	172.8230366	0.0000006
49	423.1897413	347.7936407	343.5117678	172.8230391	0.0000005

Estimates values are $p_1 = 423.1897413$, $p_2 = 347.7936407$, $p_3 = 343.5117678$, $p_4 = 172.8230391$ after 49 iterations.

The iterations of table 4.9

The result from D-method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1153780	0.9181512	0.6444943	0.2393024	1.8665805	31.6022833
2	1.0808444	0.9294546	0.6769488	0.2617288	2.0704433	13.8231705
3	1.0365805	0.8790661	0.6269273	0.2369522	1.9897429	19.8643115
4	1.0538639	0.8985646	0.6459732	0.2461915	2.1208144	8.5704631
5	1.0260083	0.8672648	0.6155828	0.2315296	2.0688263	12.5972339
6	1.0369521	0.8794931	0.6273407	0.2371357	2.1532113	5.3938081
7	1.0192711	0.8597848	0.6084610	0.2281673	2.1197013	8.0418127
8	1.0262483	0.8675342	0.6158398	0.2316479	2.1740911	3.4269437
9	1.0149562	0.8550115	0.6039441	0.2260504	2.1524767	5.1557808
10	1.0194252	0.8599561	0.6086233	0.2282428	2.1875624	2.1904902
11	1.0121841	0.851952	0.6010603	0.2247050	2.1736130	3.3146496
12	1.0150553	0.8551211	0.6040475	0.2260985	2.1962589	1.4055709
13	1.0103994	0.8499854	0.5992113	0.2238447	2.1872527	2.1348013
14	1.0122478	0.8520224	0.6011265	0.2247357	2.2018749	0.9041495
15	1.0092490	0.8487190	0.5980224	0.2232926	2.1960586	1.3765083
16	1.0104405	0.8500307	0.5992538	0.2238644	2.2055024	0.582532
17	1.0085068	0.8479024	0.5972567	0.2229374	2.2017455	0.8882274
18	1.0092755	0.8487481	0.5980498	0.2233053	2.2078458	0.3757034
19	1.0080277	0.8473756	0.596763	0.2227085	2.2054188	0.5734278
20	1.0085239	0.8479212	0.5972744	0.2229455	2.2093598	0.2424699
21	1.0077184	0.8470354	0.5964444	0.2225609	2.2077918	0.3703127
22	1.0080388	0.8473877	0.5967744	0.2227138	2.2103379	0.1565511
23	1.0075185	0.8468158	0.5962387	0.2224656	2.2093249	0.2391915

The result from D-method (cont.)

24	1.0077255	0.8470433	0.5964518	0.2225643	2.2109699	0.1011054
25	1.0073895	0.8466739	0.5961059	0.2224041	2.2103154	0.154518
26	1.0075231	0.8468208	0.5962435	0.2224678	2.2113783	0.0653085
27	1.0073061	0.8465823	0.5960201	0.2223644	2.2109554	0.0998273
28	1.0073924	0.8466772	0.596109	0.2224055	2.2116422	0.0421905
29	1.0072522	0.8465231	0.5959647	0.2223387	2.2113689	0.0644975
30	1.0073080	0.8465844	0.5960221	0.2223653	2.2118126	0.0272579
31	1.0072174	0.8464848	0.5959289	0.2223221	2.2116361	0.0416727
32	1.0072534	0.8465244	0.5959660	0.2223393	2.2119228	0.0176113
33	1.0071949	0.8464601	0.5959057	0.2223114	2.2118087	0.0269259
34	1.0072182	0.8464857	0.5959297	0.2223225	2.2119940	0.0113790
35	1.0071804	0.8464441	0.5958908	0.2223045	2.2119203	0.0173978
36	1.0071954	0.8464607	0.5959063	0.2223117	2.212040	0.0073523
37	1.0071710	0.8464338	0.5958811	0.2223000	2.2119923	0.0112415
38	1.0071807	0.8464445	0.5958911	0.2223046	2.2120697	0.0047506
39	1.0071649	0.8464271	0.5958749	0.2222971	2.2120389	0.0072637
40	1.0071712	0.8464340	0.5958814	0.2223001	2.2120889	0.0030696
41	1.0071610	0.8464228	0.5958709	0.2222953	2.2120690	0.0046934
42	1.0071650	0.8464273	0.5958750	0.2222972	2.2121013	0.0019834
43	1.0071584	0.8464200	0.5958683	0.2222941	2.2120885	0.0030327
44	1.0071611	0.8464229	0.5958710	0.2222953	2.2121093	0.0012816
45	1.0071568	0.8464182	0.5958666	0.2222933	2.2121010	0.0019596
46	1.0071585	0.8464201	0.5958683	0.2222941	2.2121145	0.0008281
47	1.0071558	0.8464171	0.5958655	0.2222928	2.2121091	0.0012662
48	1.0071568	0.8464183	0.5958666	0.2222933	2.2121179	0.0005351
49	1.0071551	0.8464163	0.5958648	0.2222924	2.2121144	0.0008181
50	1.0071558	0.8464171	0.5958655	0.2222928	2.2121200	0.0003457
51	1.0071546	0.8464158	0.5958643	0.2222922	2.2121178	0.0005286
52	1.0071551	0.8464163	0.5958648	0.2222925	2.2121214	0.0002234
53	1.0071543	0.8464155	0.5958640	0.2222921	2.2121200	0.0003416
54	1.0071546	0.8464159	0.5958643	0.2222922	2.2121223	0.0001444
55	1.0071542	0.8464153	0.5958638	0.2222920	2.2121214	0.0002207
56	1.0071544	0.8464155	0.595864	0.2222921	2.2121229	0.0000933
57	1.0071540	0.8464152	0.5958637	0.2222920	2.2121223	0.0001426
58	1.0071542	0.8464153	0.5958638	0.2222920	2.2121233	0.0000603
59	1.0071540	0.8464151	0.5958636	0.2222919	2.2121229	0.0000922
60	1.0071540	0.8464152	0.5958637	0.2222920	2.2121235	0.0000389
61	1.0071539	0.8464151	0.5958636	0.2222919	2.2121233	0.0000595
62	1.0071540	0.8464151	0.5958636	0.2222919	2.2121237	0.0000252
63	1.0071539	0.8464150	0.5958636	0.2222919	2.2121235	0.0000385
64	1.0071539	0.8464151	0.5958636	0.2222919	2.2121238	0.0000163
65	1.0071539	0.8464150	0.5958635	0.2222919	2.2121237	0.0000249
66	1.0071539	0.8464150	0.5958636	0.2222919	2.2121238	0.0000105
67	1.0071538	0.8464150	0.5958635	0.2222919	2.2121238	0.0000161
68	1.0071539	0.8464150	0.5958635	0.2222919	2.2121239	0.0000068
69	1.0071538	0.8464150	0.5958635	0.2222919	2.2121238	0.0000104
70	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000044
71	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000067
72	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000028
73	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000043
74	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000018
75	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000028

The result from D-method (cont.)

76	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000012
77	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000018
78	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000008
79	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000006
80	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000005

Estimates values are $\alpha = 1.0071538$, $\beta = 0.846415$, $\gamma = 0.5958635$, $\delta = 0.2222919$ and $x = 2.212124$ after 80 iterations.

The result from U_2-method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1153780	0.9181512	0.6444943	0.2713669	1.8665805	19.3922139
2	1.0808444	0.9294546	0.6769488	0.2452388	1.9854240	7.8943157
3	1.0547894	0.8996118	0.6470068	0.2410889	2.0327866	7.6142897
4	1.0446166	0.8881091	0.6357137	0.2356021	2.0816924	4.8146811
5	1.0342315	0.8764450	0.6243963	0.2323635	2.1125124	3.8774688
6	1.0277522	0.8692092	0.6174423	0.2297150	2.1379019	2.7468176
7	1.0224522	0.8633137	0.6118140	0.2278744	2.1559367	2.1081742
8	1.0187082	0.8591618	0.6078701	0.2264622	2.1699151	1.5529981
9	1.0158181	0.8559641	0.6048437	0.2254295	2.1802419	1.1761965
10	1.0136896	0.8536130	0.6026248	0.2246489	2.1881002	0.8791118
11	1.0120737	0.8518304	0.6009458	0.2240675	2.1939857	0.6636294
12	1.0108656	0.8504989	0.5996937	0.2236292	2.1984393	0.4988768
13	1.0099526	0.8494934	0.5987492	0.2233003	2.2017917	0.3764133
14	1.0092661	0.8487377	0.5980401	0.2230523	2.2043243	0.2836600
15	1.0087478	0.8481675	0.5975052	0.2228657	2.2062344	0.2140686
16	1.0083571	0.8477378	0.5971025	0.2227249	2.2076769	0.1614994
17	1.0080623	0.8474135	0.5967986	0.2226187	2.2087657	0.1219109
18	1.0078397	0.8471689	0.5965694	0.2225386	2.2095879	0.0920227
19	1.0076718	0.8469842	0.5963964	0.2224782	2.2102087	0.0694796
20	1.0075449	0.8468448	0.5962659	0.2224325	2.2106775	0.0524600
21	1.0074492	0.8467396	0.5961674	0.2223981	2.2110315	0.0396142
22	1.0073769	0.8466601	0.5960930	0.2223721	2.2112989	0.0299147
23	1.0073223	0.8466001	0.5960368	0.2223524	2.2115008	0.0225915
24	1.0072811	0.8465548	0.5959944	0.2223376	2.2116533	0.0170613
25	1.0072499	0.8465206	0.5959623	0.2223264	2.2117685	0.0128853
26	1.0072264	0.8464947	0.5959381	0.2223180	2.2118555	0.0097315
27	1.0072086	0.8464752	0.5959199	0.2223116	2.2119212	0.0073498
28	1.0071952	0.8464604	0.5959061	0.2223067	2.2119708	0.0055510
29	1.0071851	0.8464493	0.5958957	0.2223031	2.2120083	0.0041925
30	1.0071774	0.8464409	0.5958878	0.2223003	2.2120366	0.0031665
31	1.0071717	0.8464346	0.5958818	0.2222983	2.2120580	0.0023916
32	1.0071673	0.8464298	0.5958774	0.2222967	2.2120741	0.0018063
33	1.0071640	0.8464261	0.595874	0.2222955	2.2120863	0.0013643
34	1.0071615	0.8464234	0.5958714	0.2222946	2.2120955	0.0010304
35	1.0071596	0.8464213	0.5958695	0.2222939	2.2121025	0.0007783
36	1.0071582	0.8464198	0.5958680	0.2222934	2.2121078	0.0005878
37	1.0071571	0.8464186	0.5958669	0.2222930	2.2121117	0.0004440

The result from U_2 -method (cont.)

38	1.0071563	0.8464177	0.5958661	0.2222927	2.2121147	0.0003353
39	1.0071557	0.8464170	0.5958654	0.2222925	2.2121170	0.0002533
40	1.0071552	0.8464165	0.595865	0.2222924	2.2121187	0.0001913
41	1.0071549	0.8464161	0.5958646	0.2222922	2.21212	0.0001445
42	1.0071546	0.8464158	0.5958643	0.2222921	2.212121	0.0001091
43	1.0071544	0.8464156	0.5958641	0.2222921	2.2121217	0.0000824
44	1.0071543	0.8464155	0.5958640	0.2222920	2.2121223	0.0000622
45	1.0071542	0.8464153	0.5958639	0.2222920	2.2121227	0.0000470
46	1.0071541	0.8464152	0.5958638	0.2222919	2.212123	0.0000355
47	1.0071540	0.8464152	0.5958637	0.2222919	2.2121232	0.0000268
48	1.0071540	0.8464151	0.5958637	0.2222919	2.2121234	0.0000203
49	1.0071539	0.8464151	0.5958636	0.2222919	2.2121235	0.0000153
50	1.0071539	0.8464150	0.5958636	0.2222919	2.2121237	0.0000116
51	1.0071539	0.8464150	0.5958636	0.2222919	2.2121237	0.0000087
52	1.0071539	0.8464150	0.5958635	0.2222919	2.2121238	0.0000066
53	1.0071539	0.8464150	0.5958635	0.2222919	2.2121238	0.0000050
54	1.0071539	0.8464150	0.5958635	0.2222919	2.2121239	0.0000038
55	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000028
56	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000021
57	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000016
58	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000012
59	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000009
60	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000007
61	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000005
62	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000004

Estimates values are $\alpha = 1.0071538$, $\beta = 0.846415$, $\gamma = 0.5958635$, $\delta = 0.2222919$ and $x = 2.212124$ after 62 iterations.

The result from U_3 -method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1153780	0.9181512	0.6444943	0.2713669	1.8665805	19.3922139
2	1.0808444	0.9294546	0.6769488	0.2452388	1.9854240	7.8943157
3	1.0547894	0.8996118	0.6470068	0.2410889	2.0327866	7.6142897
4	1.0446166	0.8881091	0.6357137	0.2356021	2.0816924	4.8146811
5	1.0342315	0.8764450	0.6243963	0.2323635	2.1125124	3.8774688
6	1.0277522	0.8692092	0.6174423	0.2297150	2.1379019	2.7468176
7	1.0224522	0.8633137	0.6118140	0.2278744	2.1559367	2.1081742
8	1.0187082	0.8591618	0.6078701	0.2264622	2.1699151	1.5529981
9	1.0158181	0.8559641	0.6048437	0.2254295	2.1802419	1.1761965
10	1.0136896	0.8536130	0.6026248	0.2246489	2.1881002	0.8791118
11	1.0120737	0.8518304	0.6009458	0.2240675	2.1939857	0.6636294
12	1.0108656	0.8504989	0.5996937	0.2236292	2.1984393	0.4988768
13	1.0099526	0.8494934	0.5987492	0.2233003	2.2017917	0.3764133
14	1.0092661	0.8487377	0.5980401	0.2230523	2.2043243	0.2836600
15	1.0087478	0.8481675	0.5975052	0.2228657	2.2062344	0.2140686
16	1.0083571	0.8477378	0.5971025	0.2227249	2.2076769	0.1614994
17	1.0080623	0.8474135	0.5967986	0.2226187	2.2087657	0.1219109

The result from U_3-method (cont.)

18	1.0078397	0.8471689	0.5965694	0.2225386	2.2095879	0.0920227
19	1.0076718	0.8469842	0.5963964	0.2224782	2.2102087	0.0694796
20	1.0075449	0.8468448	0.5962659	0.2224325	2.2106775	0.0524600
21	1.0074492	0.8467396	0.5961674	0.2223981	2.2110315	0.0396142
22	1.0073769	0.8466601	0.5960930	0.2223721	2.2112989	0.0299147
23	1.0073223	0.8466001	0.5960368	0.2223524	2.2115008	0.0225915
24	1.0072811	0.8465548	0.5959944	0.2223376	2.2116533	0.0170613
25	1.0072499	0.8465206	0.5959623	0.2223264	2.2117685	0.0128853
26	1.0072264	0.8464947	0.5959381	0.222318	2.2118555	0.0097315
27	1.0072086	0.8464752	0.5959199	0.2223116	2.2119212	0.0073498
28	1.0071952	0.8464604	0.5959061	0.2223067	2.2119708	0.0055510
29	1.0071851	0.8464493	0.5958957	0.2223031	2.2120083	0.0041925
30	1.0071774	0.8464409	0.5958878	0.2223003	2.2120366	0.0031665
31	1.0071717	0.8464346	0.5958818	0.2222983	2.212058	0.0023916
32	1.0071673	0.8464298	0.5958774	0.2222967	2.2120741	0.0018063
33	1.0071640	0.8464261	0.5958740	0.2222955	2.2120863	0.0013643
34	1.0071615	0.8464234	0.5958714	0.2222946	2.2120955	0.0010304
35	1.0071596	0.8464213	0.5958695	0.2222939	2.2121025	0.0007783
36	1.0071582	0.8464198	0.5958680	0.2222934	2.2121078	0.0005878
37	1.0071571	0.8464186	0.5958669	0.2222930	2.2121117	0.0004440
38	1.0071563	0.8464177	0.5958661	0.2222927	2.2121147	0.0003353
39	1.0071557	0.8464170	0.5958654	0.2222925	2.2121170	0.0002533
40	1.0071552	0.8464165	0.5958650	0.2222924	2.2121187	0.0001913
41	1.0071549	0.8464161	0.5958646	0.2222922	2.2121200	0.0001445
42	1.0071546	0.8464158	0.5958643	0.2222921	2.2121210	0.0001091
43	1.0071544	0.8464156	0.5958641	0.2222921	2.2121217	0.0000824
44	1.0071543	0.8464155	0.5958640	0.2222920	2.2121223	0.0000622
45	1.0071542	0.8464153	0.5958639	0.2222920	2.2121227	0.0000470
46	1.0071541	0.8464152	0.5958638	0.2222919	2.212123	0.0000355
47	1.0071540	0.8464152	0.5958637	0.2222919	2.2121232	0.0000268
48	1.0071540	0.8464151	0.5958637	0.2222919	2.2121234	0.0000203
49	1.0071539	0.8464151	0.5958636	0.2222919	2.2121235	0.0000153
50	1.0071539	0.8464151	0.5958636	0.2222919	2.2121237	0.0000116
51	1.0071539	0.8464151	0.5958636	0.2222919	2.2121237	0.0000087
52	1.0071539	0.8464151	0.5958635	0.2222919	2.2121238	0.0000066
53	1.0071539	0.8464151	0.5958635	0.2222919	2.2121238	0.0000050
54	1.0071539	0.8464151	0.5958635	0.2222919	2.2121239	0.0000038
55	1.0071538	0.8464151	0.5958635	0.2222919	2.2121239	0.0000028
56	1.0071538	0.8464151	0.5958635	0.2222919	2.2121239	0.0000021
57	1.0071538	0.8464151	0.5958635	0.2222919	2.2121239	0.0000016
58	1.0071538	0.8464151	0.5958635	0.2222919	2.2121239	0.0000012
59	1.0071538	0.8464151	0.5958635	0.2222919	2.2121239	0.0000009
60	1.0071538	0.8464151	0.5958635	0.2222919	2.2121239	0.0000007
61	1.0071538	0.8464151	0.5958635	0.2222919	2.2121240	0.0000005
62	1.0071538	0.8464151	0.5958635	0.2222919	2.2121240	0.0000004

Estimates values are $\alpha = 1.0071538$, $\beta = 0.8464151$, $\gamma = 0.5958635$, $\delta = 0.2222919$ and $x = 2.212124$ after 62 iterations.

The result from U_4-method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1153780	0.9398068	0.6444943	0.2713669	1.8665805	18.3699618
2	1.0808444	0.9004528	0.6769488	0.2461807	1.9786337	9.8304445
3	1.0562593	0.8861962	0.6486567	0.2401852	2.0400394	7.3504071
4	1.0430681	0.8749618	0.6340125	0.2349757	2.0875397	5.1122860
5	1.0329983	0.8671753	0.6230665	0.2314560	2.1210235	3.7091426
6	1.0259717	0.8615043	0.6155455	0.2289187	2.1456266	2.6685001
7	1.0208465	0.8574224	0.6101193	0.2271086	2.1634681	1.9396265
8	1.0171498	0.8544522	0.6062362	0.2257995	2.1765175	1.4102514
9	1.0144566	0.8522916	0.6034234	0.2248517	2.1860466	1.0287332
10	1.0124957	0.8507149	0.6013838	0.2241622	2.1930203	0.7511970
11	1.0110636	0.8495633	0.5998988	0.2236599	2.1981245	0.5492718
12	1.0100171	0.8487210	0.5988159	0.2232931	2.2018631	0.4019109
13	1.0092514	0.8481046	0.5980249	0.2230251	2.2046021	0.2942718
14	1.0086910	0.8476533	0.5974466	0.2228290	2.2066094	0.2155494
15	1.0082805	0.8473227	0.5970234	0.2226855	2.2080807	0.1579378
16	1.0079797	0.8470804	0.5967136	0.2225804	2.2091593	0.1157508
17	1.0077593	0.8469028	0.5964866	0.2225033	2.2099501	0.0848470
18	1.0075978	0.8467727	0.5963203	0.2224469	2.2105299	0.0622017
19	1.0074793	0.8466773	0.5961984	0.2224055	2.2109550	0.0456045
20	1.0073925	0.8466073	0.5961090	0.2223752	2.2112667	0.0334382
21	1.0073289	0.8465560	0.5960435	0.222353	2.2114953	0.0245187
22	1.0072822	0.8465184	0.5959955	0.2223367	2.211663	0.0179792
23	1.0072479	0.8464908	0.5959603	0.2223247	2.2117859	0.0131841
24	1.0072228	0.8464706	0.5959345	0.222316	2.2118760	0.0096681
25	1.0072044	0.8464557	0.5959156	0.2223095	2.2119422	0.0070899
26	1.0071909	0.8464449	0.5959017	0.2223048	2.2119906	0.0051993
27	1.0071810	0.8464369	0.5958915	0.2223014	2.2120262	0.0038128
28	1.0071738	0.8464310	0.5958840	0.2222988	2.2120523	0.0027961
29	1.0071685	0.8464268	0.5958786	0.2222970	2.2120714	0.0020505
30	1.0071646	0.8464236	0.5958745	0.2222956	2.2120854	0.0015037
31	1.0071617	0.8464213	0.5958716	0.2222946	2.2120957	0.0011028
32	1.0071596	0.8464196	0.5958694	0.2222939	2.2121032	0.0008087
33	1.0071581	0.8464184	0.5958679	0.2222933	2.2121088	0.0005931
34	1.0071569	0.8464175	0.5958667	0.2222929	2.2121128	0.0004349
35	1.0071561	0.8464168	0.5958658	0.2222926	2.2121158	0.0003190
36	1.0071555	0.8464163	0.5958652	0.2222924	2.212118	0.0002339
37	1.007155	0.8464159	0.5958648	0.2222923	2.2121196	0.0001715
38	1.0071547	0.8464157	0.5958644	0.2222922	2.2121207	0.0001258
39	1.0071545	0.8464155	0.5958642	0.2222921	2.2121216	0.0000923
40	1.0071543	0.8464153	0.5958640	0.2222920	2.2121222	0.0000677
41	1.0071542	0.8464152	0.5958639	0.2222920	2.2121227	0.0000496
42	1.0071541	0.8464152	0.5958638	0.2222919	2.2121230	0.0000364
43	1.0071540	0.8464151	0.5958637	0.2222919	2.2121233	0.0000267
44	1.0071540	0.8464151	0.5958636	0.2222919	2.2121235	0.0000196
45	1.0071539	0.8464150	0.5958636	0.2222919	2.2121236	0.0000144
46	1.0071539	0.8464150	0.5958636	0.2222919	2.2121237	0.0000105
47	1.0071539	0.8464150	0.5958636	0.2222919	2.2121238	0.0000077
48	1.0071539	0.8464150	0.5958635	0.2222919	2.2121238	0.0000057

The result from U_3 -method (cont.)

49	1.0071539	0.8464150	0.5958635	0.2222919	2.2121239	0.0000042
50	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000003
51	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000022
52	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000016
53	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000012
54	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000009
55	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000006
56	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000005

Estimates values are $\alpha = 1.0071538$, $\beta = 0.8464150$, $\gamma = 0.5958635$, $\delta = 0.2222919$ and $x = 2.2121240$ after 56 iterations.

The result from U_5 -method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1340095	0.8964956	0.6444943	0.2713669	1.8665805	20.2703681
2	1.0527067	0.9602390	0.6769488	0.2446839	1.9894238	6.8119827
3	1.0452429	0.9089067	0.6460382	0.2414744	2.0296854	7.7973926
4	1.0345229	0.9011812	0.6364435	0.2357788	2.0799969	4.5115102
5	1.0285229	0.8837711	0.6247827	0.2327712	2.1087314	3.9715703
6	1.0230777	0.8762570	0.6182883	0.2300341	2.1348110	2.7043414
7	1.0194053	0.8681701	0.6124944	0.2282229	2.1525330	2.1790510
8	1.0164126	0.8632918	0.6086110	0.2267549	2.1670090	1.5840959
9	1.0142431	0.8590464	0.6054707	0.2256997	2.1775397	1.2351004
10	1.0125435	0.8561150	0.6032040	0.2248767	2.1858049	0.9230651
11	1.0112743	0.8537570	0.6014354	0.2242646	2.1919888	0.7106761
12	1.0102949	0.8520333	0.6001180	0.2237938	2.1967659	0.5375989
13	1.0095541	0.8506890	0.5991038	0.2234384	2.2003829	0.4119085
14	1.0089860	0.8496818	0.5983379	0.2231664	2.2031592	0.3132922
15	1.0085538	0.8489060	0.5977512	0.2229597	2.2052722	0.2396155
16	1.0082231	0.8483184	0.5973053	0.2228017	2.2068898	0.1827054
17	1.0079709	0.8478680	0.5969644	0.2226813	2.2081238	0.1396526
18	1.0077781	0.8475252	0.5967045	0.2225893	2.2090674	0.1066101
19	1.0076309	0.8472630	0.5965059	0.2225191	2.2097880	0.0814738
20	1.0075183	0.8470630	0.5963544	0.2224655	2.2103389	0.0622322
21	1.0074324	0.8469101	0.5962385	0.2224245	2.2107597	0.0475580
22	1.0073667	0.8467933	0.5961501	0.2223932	2.2110814	0.0363364
23	1.0073165	0.8467041	0.5960825	0.2223693	2.2113272	0.0277689
24	1.0072782	0.8466359	0.5960309	0.2223511	2.2115150	0.0212197
25	1.0072488	0.8465838	0.5959914	0.2223371	2.2116586	0.0162168
26	1.0072264	0.8465440	0.5959612	0.2223264	2.2117683	0.0123931
27	1.0072093	0.8465136	0.5959382	0.2223183	2.2118521	0.0094714
28	1.0071962	0.8464903	0.5959206	0.2223120	2.2119162	0.0072384
29	1.0071862	0.8464726	0.5959071	0.2223073	2.2119652	0.0055320
30	1.0071786	0.8464590	0.5958968	0.2223036	2.2120026	0.0042279
31	1.0071728	0.8464486	0.5958890	0.2223009	2.2120312	0.0032313
32	1.0071683	0.8464407	0.5958830	0.2222987	2.2120531	0.0024695
33	1.0071649	0.8464346	0.5958784	0.2222971	2.2120698	0.0018874

The result from U_3-method (cont.)

34	1.0071623	0.8464300	0.5958749	0.2222959	2.2120826	0.0014425
35	1.0071603	0.8464264	0.5958722	0.2222949	2.2120923	0.0011025
36	1.0071588	0.8464237	0.5958701	0.2222942	2.2120998	0.0008426
37	1.0071576	0.8464217	0.5958686	0.2222936	2.2121055	0.0006440
38	1.0071567	0.8464201	0.5958674	0.2222932	2.2121098	0.0004922
39	1.007156	0.8464189	0.5958665	0.2222929	2.2121132	0.0003762
40	1.0071555	0.8464179	0.5958658	0.2222927	2.2121157	0.0002875
41	1.0071551	0.8464172	0.5958652	0.2222925	2.2121177	0.0002197
42	1.0071548	0.8464167	0.5958648	0.2222923	2.2121191	0.0001679
43	1.0071546	0.8464163	0.5958645	0.2222922	2.2121203	0.0001283
44	1.0071544	0.8464160	0.5958643	0.2222921	2.2121212	0.0000981
45	1.0071543	0.8464157	0.5958641	0.2222921	2.2121218	0.0000750
46	1.0071542	0.8464156	0.5958639	0.2222920	2.2121223	0.0000573
47	1.0071541	0.8464154	0.5958638	0.2222920	2.2121227	0.0000438
48	1.0071540	0.8464153	0.5958638	0.2222919	2.2121230	0.0000335
49	1.0071540	0.8464152	0.5958637	0.2222919	2.2121232	0.0000256
50	1.0071539	0.8464152	0.5958637	0.2222919	2.2121234	0.0000195
51	1.0071539	0.8464151	0.5958636	0.2222919	2.2121235	0.0000149
52	1.0071539	0.8464151	0.5958636	0.2222919	2.2121236	0.0000114
53	1.0071539	0.8464150	0.5958636	0.2222919	2.2121237	0.0000087
54	1.0071539	0.8464150	0.5958636	0.2222919	2.2121238	0.0000067
55	1.0071539	0.8464150	0.5958635	0.2222919	2.2121238	0.0000051
56	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000039
57	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000003
58	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000023
59	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000017
60	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000013
61	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000010
62	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000008
63	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000006
64	1.0071538	0.8464150	0.5958635	0.2222919	2.2121240	0.0000005

Estimates values are $\alpha = 1.0071538$, $\beta = 0.846415$, $\gamma = 0.5958635$, $\delta = 0.2222919$ and $x = 2.212124$ after 64 iterations.

The result from L_2-method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1153780	0.9181512	0.6444943	0.2393024	2.4359713	58.4547319
2	0.9615696	0.7977931	0.5516698	0.2024285	2.1908938	23.475503
3	1.0115416	0.8512488	0.6003894	0.2243782	2.2841659	12.7452526
4	0.9925800	0.8304789	0.5810648	0.2154990	2.2273117	4.6468931
5	1.0040597	0.8430186	0.5926898	0.2208251	2.2412753	2.8074714
6	1.0012247	0.8399127	0.5897971	0.2194931	2.2262532	0.7365879
7	1.0042745	0.8432540	0.5929096	0.2209266	2.2263351	0.7207509
8	1.0042578	0.8432357	0.5928925	0.2209188	2.2212807	0.0278637
9	1.0052866	0.8443644	0.5939462	0.2214052	2.2197471	0.2495180
10	1.0055990	0.8447073	0.5942666	0.2215532	2.2175674	0.0838234
11	1.0060433	0.8451951	0.5947224	0.2217639	2.2163765	0.1101400

The result from L_2-method (cont.)

12	1.0062861	0.8454617	0.5949717	0.2218792	2.2152733	0.0623289
13	1.0065111	0.8457088	0.5952028	0.2219861	2.2145308	0.0564176
14	1.0066626	0.8458752	0.5953585	0.2220581	2.2139294	0.0384600
15	1.0067853	0.8460101	0.5954846	0.2221165	2.2134932	0.0309107
16	1.0068743	0.8461078	0.5955761	0.2221588	2.2131557	0.0225212
17	1.0069432	0.8461835	0.5956469	0.2221916	2.2129043	0.0173814
18	1.0069945	0.8462399	0.5956997	0.2222160	2.2127129	0.0129641
19	1.0070336	0.8462828	0.5957399	0.2222346	2.2125690	0.0098657
20	1.0070630	0.8463151	0.5957701	0.2222486	2.2124600	0.0074190
21	1.0070852	0.8463396	0.5957929	0.2222592	2.2123778	0.0056183
22	1.0071020	0.8463580	0.5958102	0.2222672	2.2123157	0.0042370
23	1.0071147	0.8463719	0.5958232	0.2222732	2.2122688	0.0032031
24	1.0071243	0.8463825	0.5958331	0.2222778	2.2122333	0.0024180
25	1.0071315	0.8463904	0.5958405	0.2222812	2.2122066	0.0018269
26	1.0071370	0.8463964	0.5958462	0.2222838	2.2121864	0.0013796
27	1.0071411	0.8464010	0.5958504	0.2222858	2.2121711	0.0010421
28	1.0071442	0.8464044	0.5958536	0.2222873	2.2121596	0.000787
29	1.0071466	0.846407	0.5958560	0.2222884	2.2121508	0.0005945
30	1.0071483	0.8464089	0.5958579	0.2222892	2.2121443	0.000449
31	1.0071497	0.8464104	0.5958592	0.2222899	2.2121393	0.0003391
32	1.0071507	0.8464115	0.5958603	0.2222904	2.2121356	0.0002561
33	1.0071515	0.8464124	0.5958611	0.2222907	2.2121327	0.0001935
34	1.0071520	0.8464130	0.5958617	0.222291	2.2121306	0.0001461
35	1.0071525	0.8464135	0.5958621	0.2222912	2.212129	0.0001104
36	1.0071528	0.8464138	0.5958625	0.2222914	2.2121277	0.0000834
37	1.0071531	0.8464141	0.5958627	0.2222915	2.2121268	0.000063
38	1.0071532	0.8464143	0.5958629	0.2222916	2.2121261	0.0000475
39	1.0071534	0.8464145	0.595863	0.2222916	2.2121256	0.0000359
40	1.0071535	0.8464146	0.5958632	0.2222917	2.2121252	0.0000271
41	1.0071536	0.8464147	0.5958632	0.2222917	2.2121249	0.0000205
42	1.0071536	0.8464147	0.5958633	0.2222918	2.2121247	0.0000155
43	1.0071537	0.8464148	0.5958634	0.2222918	2.2121245	0.0000117
44	1.0071537	0.8464148	0.5958634	0.2222918	2.2121244	0.0000088
45	1.0071537	0.8464149	0.5958634	0.2222918	2.2121243	0.0000067
46	1.0071538	0.8464149	0.5958634	0.2222918	2.2121242	0.000005
47	1.0071538	0.8464149	0.5958635	0.2222918	2.2121241	0.0000038
48	1.0071538	0.8464149	0.5958635	0.2222918	2.2121241	0.0000029
49	1.0071538	0.8464149	0.5958635	0.2222918	2.2121241	0.0000022
50	1.0071538	0.8464149	0.5958635	0.2222918	2.212124	0.0000016
51	1.0071538	0.8464149	0.5958635	0.2222918	2.212124	0.0000012
52	1.0071538	0.8464149	0.5958635	0.2222918	2.212124	0.0000009
53	1.0071538	0.8464149	0.5958635	0.2222918	2.212124	0.0000007
54	1.0071538	0.8464149	0.5958635	0.2222918	2.212124	0.0000005
55	1.0071538	0.8464149	0.5958635	0.2222918	2.212124	0.0000004

Estimates values are $\alpha = 1.0071538$, $\beta = 0.8464149$, $\gamma = 0.5958635$, $\delta = 0.2222918$ and $x = 2.212124$ after 55 iterations.

The result from L_3 -method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.115378	0.9181512	0.6444943	0.2393024	2.3307744	41.91756
2	0.9825267	0.8201022	0.5717086	0.211311	2.2203271	10.0284037
3	1.0054849	0.8445825	0.5941491	0.2214976	2.2343594	2.4733806
4	1.0026273	0.8414485	0.5912264	0.2201508	2.2244235	0.3401767
5	1.0046468	0.8436623	0.5932906	0.2211025	2.2215432	0.2222984
6	1.0052331	0.8443057	0.5938914	0.2213799	2.2186369	0.0742587
7	1.0058253	0.8449557	0.5944987	0.2216605	2.2167625	0.0699744
8	1.0062074	0.8453753	0.5948909	0.2218418	2.2153985	0.0458865
9	1.0064856	0.8456808	0.5951766	0.221974	2.2144416	0.0332328
10	1.0066808	0.8458952	0.5953772	0.2220668	2.2137630	0.0233549
11	1.0068193	0.8460474	0.5955195	0.2221326	2.2132834	0.0165566
12	1.0069172	0.8461549	0.5956201	0.2221792	2.2129441	0.0117058
13	1.0069864	0.846231	0.5956913	0.2222122	2.212704	0.0082822
14	1.0070354	0.8462848	0.5957417	0.2222355	2.2125343	0.0058584
15	1.0070701	0.8463229	0.5957774	0.222252	2.2124142	0.0041441
16	1.0070946	0.8463498	0.5958026	0.2222636	2.2123293	0.0029313
17	1.0071119	0.8463689	0.5958204	0.2222719	2.2122692	0.0020735
18	1.0071242	0.8463824	0.595833	0.2222777	2.2122267	0.0014666
19	1.0071329	0.8463919	0.5958419	0.2222819	2.2121966	0.0010374
20	1.007139	0.8463987	0.5958482	0.2222848	2.2121754	0.0007338
21	1.0071433	0.8464034	0.5958527	0.2222869	2.2121603	0.000519
22	1.0071464	0.8464068	0.5958559	0.2222883	2.2121497	0.0003671
23	1.0071486	0.8464092	0.5958581	0.2222894	2.2121422	0.0002597
24	1.0071501	0.8464109	0.5958597	0.2222901	2.2121368	0.0001837
25	1.0071512	0.8464121	0.5958608	0.2222906	2.2121331	0.0001299
26	1.007152	0.8464129	0.5958616	0.222291	2.2121304	0.0000919
27	1.0071525	0.8464135	0.5958621	0.2222912	2.2121285	0.000065
28	1.0071529	0.8464139	0.5958625	0.2222914	2.2121272	0.000046
29	1.0071532	0.8464142	0.5958628	0.2222915	2.2121262	0.0000325
30	1.0071534	0.8464144	0.595863	0.2222916	2.2121256	0.000023
31	1.0071535	0.8464146	0.5958632	0.2222917	2.2121251	0.0000163
32	1.0071536	0.8464147	0.5958633	0.2222917	2.2121248	0.0000115
33	1.0071537	0.8464148	0.5958633	0.2222918	2.2121245	0.0000081
34	1.0071537	0.8464148	0.5958634	0.2222918	2.2121244	0.0000058
35	1.0071537	0.8464149	0.5958634	0.2222918	2.2121243	0.0000041
36	1.0071538	0.8464149	0.5958634	0.2222918	2.2121242	0.0000029
37	1.0071538	0.8464149	0.5958635	0.2222918	2.2121241	0.0000020
38	1.0071538	0.8464149	0.5958635	0.2222918	2.2121241	0.0000014
39	1.0071538	0.8464149	0.5958635	0.2222918	2.2121240	0.0000010
40	1.0071538	0.8464149	0.5958635	0.2222918	2.2121240	0.0000007
41	1.0071538	0.8464149	0.5958635	0.2222918	2.2121240	0.0000005
42	1.0071538	0.8464149	0.5958635	0.2222918	2.2121240	0.0000004

Estimates values are $\alpha = 1.0071538$, $\beta = 0.8464149$, $\gamma = 0.5958635$, $\delta = 0.2222918$ and $x = 2.212124$ after 80 iterations.

The result from L_4-method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1153780	0.9181512	0.6444943	0.2393024	2.1481172	13.2034673
2	1.0200422	0.8608960	0.6095605	0.2286896	2.1492340	3.1482388
3	1.0200977	0.8607015	0.6093307	0.2285753	2.1715277	0.1083822
4	1.0154853	0.8555963	0.6044962	0.2263083	2.1847091	0.2357891
5	1.0127706	0.8525989	0.6016693	0.2249886	2.1937211	0.1428766
6	1.0109199	0.8505587	0.5997499	0.2240951	2.1997662	0.0960739
7	1.0096808	0.8491942	0.5984684	0.2234996	2.2038278	0.0640066
8	1.0088494	0.8482792	0.597610	0.2231012	2.2065552	0.0427869
9	1.0082916	0.8476657	0.5970349	0.2228345	2.2083862	0.0286352
10	1.0079173	0.8472541	0.5966493	0.2226558	2.2096154	0.0191815
11	1.0076661	0.8469780	0.5963907	0.2225360	2.2104404	0.0128565
12	1.0074976	0.8467928	0.5962172	0.2224556	2.2109941	0.0086206
13	1.0073845	0.8466685	0.5961008	0.2224017	2.2113657	0.0057819
14	1.0073086	0.8465851	0.5960228	0.2223656	2.2116151	0.0038786
15	1.0072577	0.8465291	0.5959704	0.2223413	2.2117825	0.0026022
16	1.0072235	0.8464916	0.5959352	0.2223251	2.2118948	0.0017460
17	1.0072006	0.8464664	0.5959116	0.2223141	2.2119702	0.0011715
18	1.0071852	0.8464495	0.5958958	0.2223068	2.2120208	0.0007861
19	1.0071749	0.8464381	0.5958852	0.2223019	2.2120547	0.0005275
20	1.0071680	0.8464305	0.595878	0.2222986	2.2120775	0.0003540
21	1.0071633	0.8464254	0.5958733	0.2222964	2.2120928	0.0002375
22	1.0071602	0.8464220	0.5958700	0.2222949	2.2121030	0.0001594
23	1.0071581	0.8464196	0.5958679	0.2222939	2.2121099	0.0001070
24	1.0071567	0.8464181	0.5958664	0.2222932	2.2121145	0.0000718
25	1.0071557	0.8464171	0.5958655	0.2222928	2.2121176	0.0000482
26	1.0071551	0.8464164	0.5958648	0.2222925	2.2121197	0.0000323
27	1.0071547	0.8464159	0.5958644	0.2222923	2.2121211	0.0000217
28	1.0071544	0.8464156	0.5958641	0.2222921	2.2121221	0.0000146
29	1.0071542	0.8464154	0.5958639	0.2222920	2.2121227	0.0000098
30	1.0071541	0.8464152	0.5958638	0.2222920	2.2121231	0.0000066
31	1.0071540	0.8464151	0.5958637	0.2222919	2.2121234	0.0000044
32	1.0071539	0.8464151	0.5958636	0.2222919	2.2121236	0.0000030
33	1.0071539	0.8464150	0.5958636	0.2222919	2.2121237	0.0000020
34	1.0071539	0.8464150	0.5958636	0.2222919	2.2121238	0.0000013
35	1.0071539	0.8464150	0.5958635	0.2222919	2.2121239	0.0000009
36	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000006
37	1.0071538	0.8464150	0.5958635	0.2222919	2.2121239	0.0000004

Estimates values are $\alpha = 1.0071538$, $\beta = 0.846415$, $\gamma = 0.5958635$, $\delta = 0.2222919$ and $x = 2.2121239$ after 37 iterations.

The result from L_5 -method

k	α^k	β^k	γ^k	δ^k	x^k	ER
1	1.1153780	0.9181512	0.6444943	0.2393024	2.3401243	43.3873917
2	0.9806452	0.8180855	0.5698783	0.2104907	2.2369899	8.5189410
3	1.0020968	0.8408678	0.5906851	0.2199005	2.2392093	1.5683084
4	1.0016434	0.8403710	0.5902234	0.2196892	2.2294057	0.0782401
5	1.0036335	0.8425511	0.5922540	0.2206243	2.2242877	0.1651231
6	1.0046744	0.8436926	0.5933189	0.2211156	2.2205043	0.0916258
7	1.0054447	0.8445380	0.5941083	0.2214801	2.2179238	0.0668110
8	1.0059706	0.8451153	0.5946478	0.2217295	2.2161340	0.0457575
9	1.0063356	0.8455160	0.5950225	0.2219027	2.2148970	0.0317378
10	1.0065879	0.8457932	0.5952817	0.2220226	2.2140416	0.0219477
11	1.0067624	0.8459849	0.5954611	0.2221056	2.2134500	0.0151841
12	1.0068831	0.8461175	0.5955852	0.2221630	2.2130409	0.0105023
13	1.0069666	0.8462093	0.5956710	0.2222027	2.2127581	0.0072637
14	1.0070244	0.8462727	0.5957304	0.2222302	2.2125625	0.0050235
15	1.0070643	0.8463166	0.5957714	0.2222492	2.2124272	0.0034741
16	1.0070919	0.8463469	0.5957998	0.2222624	2.2123336	0.0024025
17	1.0071110	0.8463679	0.5958195	0.2222715	2.2122690	0.0016614
18	1.0071242	0.8463824	0.5958331	0.2222778	2.2122242	0.0011489
19	1.0071334	0.8463925	0.5958424	0.2222821	2.2121933	0.0007945
20	1.0071397	0.8463994	0.5958489	0.2222851	2.2121719	0.0005494
21	1.0071440	0.8464042	0.5958534	0.2222872	2.2121571	0.0003799
22	1.0071471	0.8464075	0.5958565	0.2222886	2.2121469	0.0002627
23	1.0071491	0.8464098	0.5958587	0.2222896	2.2121398	0.0001817
24	1.0071506	0.8464114	0.5958602	0.2222903	2.2121349	0.0001256
25	1.0071516	0.8464125	0.5958612	0.2222908	2.2121316	0.0000869
26	1.0071523	0.8464133	0.5958619	0.2222911	2.2121292	0.0000601
27	1.0071528	0.8464138	0.5958624	0.2222913	2.2121276	0.0000415
28	1.0071531	0.8464141	0.5958627	0.2222915	2.2121265	0.0000287
29	1.0071533	0.8464144	0.5958630	0.2222916	2.2121257	0.0000199
30	1.0071535	0.8464146	0.5958631	0.2222917	2.2121252	0.0000137
31	1.0071536	0.8464147	0.5958632	0.2222917	2.2121248	0.0000095
32	1.0071537	0.8464148	0.5958633	0.2222918	2.2121245	0.0000066
33	1.0071537	0.8464148	0.5958634	0.2222918	2.2121244	0.0000045
34	1.0071537	0.8464149	0.5958634	0.2222918	2.2121242	0.0000031
35	1.0071538	0.8464149	0.5958634	0.2222918	2.2121242	0.0000022
36	1.0071538	0.8464149	0.5958635	0.2222918	2.2121241	0.0000015
37	1.0071538	0.8464149	0.5958635	0.2222918	2.2121241	0.0000010
38	1.0071538	0.8464149	0.5958635	0.2222918	2.2121240	0.0000007
39	1.0071538	0.8464149	0.5958635	0.2222918	2.2121240	0.0000005

Estimates values are $\alpha = 1.0071538$, $\beta = 0.8464150$, $\gamma = 0.5958635$, $\delta = 0.2222919$ and $x = 2.2121240$ after 39 iterations.

The iterations of table 4.10

The result from D-method

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	ER
1	1.6229508	-0.6229508	0.3750000	0.2099237	-0.1315789	0.2080292	0.2900763	-0.5000001	9.4831571
2	-0.7486339	1.7486339	0.9216553	0.1543522	0.1320775	0.5112832	0.2132867	0.5018946	2.0577042
3	1.0274951	-0.0274951	0.7135703	0.3500626	0.0724816	0.3958492	0.4837229	0.2754301	7.1213203
4	0.6721943	0.3278057	0.5774558	0.1795569	0.0316916	0.3203404	0.2481150	0.1204279	5.4694085
5	0.3012492	0.6987508	0.7887515	0.1989407	0.1135644	0.4375556	0.2748999	0.4315446	3.8854350
6	0.8221950	0.1778050	0.6874950	0.2242107	0.0756696	0.3813841	0.3098185	0.2875445	9.2483794
7	0.6125066	0.3874934	0.7001196	0.1902690	0.0812625	0.3883875	0.2629172	0.3087975	4.0499591
8	0.6419603	0.3580397	0.7284685	0.2026152	0.0917619	0.4041139	0.2799773	0.3486952	3.4808954
9	0.7043788	0.2956212	0.7056229	0.2007851	0.0831339	0.3914405	0.2774485	0.3159088	2.3654109
10	0.6544698	0.3455302	0.7160810	0.1970141	0.0871345	0.3972420	0.2722377	0.3311110	0.8665662
11	0.6777119	0.3222881	0.7158514	0.2000178	0.0869790	0.3971147	0.2763883	0.3305203	0.3675769
12	0.6772090	0.3227910	0.7130031	0.1986077	0.0859203	0.3955346	0.2744398	0.3264970	0.3452595
13	0.6709424	0.3290576	0.7154720	0.1986380	0.0868489	0.3969042	0.2744816	0.3300257	0.2419886
14	0.6763772	0.3236228	0.7145131	0.1990164	0.0864774	0.3963722	0.2750044	0.3286142	0.0828487
15	0.6742708	0.3257292	0.7145062	0.1986882	0.0864798	0.3963684	0.2745509	0.3286233	0.0377310
16	0.6742556	0.3257444	0.7148320	0.1988152	0.0866002	0.3965491	0.2747265	0.3290807	0.0365770
17	0.6749720	0.3250280	0.7145846	0.1988162	0.0865064	0.3964119	0.2747278	0.3287243	0.0244792
18	0.6744280	0.3255720	0.7146774	0.1987729	0.0865422	0.3964634	0.2746680	0.3288603	0.0086496
19	0.6746322	0.3253678	0.7146849	0.1988058	0.0865444	0.3964675	0.2747134	0.3288686	0.0042831
20	0.6746486	0.3253514	0.7146499	0.1987934	0.0865313	0.3964481	0.2746964	0.3288191	0.0039182
21	0.6745716	0.3254284	0.7146752	0.1987924	0.0865409	0.3964622	0.2746950	0.3288555	0.0024582
22	0.6746274	0.3253726	0.7146666	0.1987971	0.0865376	0.3964574	0.2747014	0.3288428	0.0008796
23	0.6746084	0.3253916	0.7146653	0.1987937	0.0865372	0.3964567	0.2746968	0.3288412	0.0004723
24	0.6746056	0.3253944	0.7146691	0.1987949	0.0865386	0.3964588	0.2746984	0.3288465	0.0004163
25	0.6746139	0.3253861	0.7146666	0.1987950	0.0865376	0.3964574	0.2746986	0.3288429	0.0002457
26	0.6746083	0.3253917	0.7146674	0.1987945	0.0865379	0.3964578	0.2746979	0.3288440	0.0000897
27	0.6746101	0.3253899	0.7146676	0.1987949	0.0865380	0.3964579	0.2746984	0.3288443	0.0000518
28	0.6746105	0.3253895	0.7146672	0.1987948	0.0865378	0.3964577	0.2746982	0.3288437	0.0000440
29	0.6746096	0.3253904	0.7146674	0.1987947	0.0865379	0.3964578	0.2746982	0.3288441	0.0000244
30	0.6746102	0.3253898	0.7146673	0.1987948	0.0865379	0.3964578	0.2746983	0.3288440	0.0000091
31	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288439	0.0000056
32	0.6746099	0.3253901	0.7146674	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000046
33	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000024
34	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000009
35	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000006
36	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000005

Estimates values are $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 = 0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746982$ and $x_8 = 0.328844$ after 36 iterations.

The result from U_2-method

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	ER
1	1.0000000	0.0000000	0.6052632	0.2631579	-0.1315789	0.3357664	0.3636364	-0.5000000	41.9187127
2	0.3905916	0.6094084	0.9582813	0.1732976	0.0791045	0.5316013	0.2394658	0.3005972	42.1121886

The result from U_2 -method (cont.)

3	1.0740633	-0.0740633	0.7025199	0.2183755	0.0602455	0.3897191	0.3017553	0.2289329	16.0576535
4	0.6474405	0.3525595	0.7632044	0.1765501	0.0811910	0.4233835	0.2439601	0.3085256	7.1637201
5	0.7745385	0.2254615	0.7186153	0.2001937	0.0875412	0.3986479	0.2766313	0.3326564	3.4248584
6	0.6832425	0.3167575	0.7194373	0.1930216	0.0854528	0.3991039	0.2667207	0.3247208	0.7150549
7	0.6850278	0.3149722	0.7163707	0.1981764	0.0879409	0.3974027	0.2738438	0.3341754	0.3572970
8	0.6783462	0.3216538	0.7138346	0.1982245	0.0865141	0.3959958	0.2739102	0.3287535	0.2773199
9	0.6727769	0.3272231	0.7149304	0.1985555	0.0868668	0.3966037	0.2743676	0.3300939	0.1196387
10	0.6751882	0.3248118	0.7142166	0.1989166	0.0865865	0.3962077	0.2748666	0.3290286	0.0743354
11	0.6736183	0.3263817	0.7146526	0.1987610	0.0865594	0.3964496	0.2746515	0.3289257	0.0342358
12	0.6745775	0.3254225	0.7145857	0.1988549	0.0865523	0.3964125	0.2747814	0.3288989	0.0097458
13	0.6744305	0.3255695	0.7146500	0.1987977	0.0865279	0.3964482	0.2747023	0.3288062	0.0060717
14	0.6745718	0.3254282	0.7146669	0.1988052	0.0865394	0.3964576	0.2747126	0.3288496	0.0025617
15	0.6746091	0.3253909	0.7146634	0.1987973	0.0865342	0.3964556	0.2747017	0.3288299	0.0008308
16	0.6746012	0.3253988	0.7146711	0.1987947	0.0865376	0.3964599	0.2746981	0.3288427	0.0007291
17	0.6746183	0.3253817	0.7146671	0.1987953	0.0865374	0.3964577	0.2746990	0.3288420	0.0003278
18	0.6746096	0.3253904	0.7146684	0.1987943	0.0865377	0.3964584	0.2746975	0.3288434	0.0001465
19	0.6746123	0.3253877	0.7146675	0.1987948	0.0865379	0.3964579	0.2746983	0.3288441	0.0000782
20	0.6746103	0.3253897	0.7146674	0.1987946	0.0865379	0.3964579	0.2746980	0.3288439	0.0000187
21	0.6746102	0.3253898	0.7146674	0.1987947	0.0865379	0.3964578	0.2746982	0.3288441	0.0000084
22	0.6746101	0.3253899	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000070
23	0.6746099	0.3253901	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000031
24	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000018
25	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000008
26	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000002

Estimates values are $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 = 0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746982$ and $x_8 = 0.328844$ after 26 iterations.

The result from U_3 -method

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	ER
1	1.0000000	0.0000000	1.1447368	-0.0131579	-0.1315789	0.6350365	-0.0181818	-0.5000000	31.5277372
2	1.2649331	-0.2649331	0.7256975	0.1734748	0.1008277	0.4025767	0.2397107	0.3831453	14.4872778
3	0.6984926	0.3015074	0.7117597	0.1941054	0.0941349	0.3948448	0.2682183	0.3577126	1.0642761
4	0.6681911	0.3318089	0.7119800	0.1993524	0.0886676	0.3949670	0.2754687	0.3369369	0.0352962
5	0.6686793	0.3313207	0.7140419	0.1992306	0.0867274	0.3961109	0.2753005	0.3295642	0.1183955
6	0.6732338	0.3267662	0.7146417	0.1988876	0.0864707	0.3964436	0.2748265	0.3285886	0.0329900
7	0.6745536	0.3254464	0.7146955	0.1987967	0.0865079	0.3964734	0.2747008	0.3287299	0.0030997
8	0.6746718	0.3253282	0.7146769	0.1987900	0.0865331	0.3964631	0.2746917	0.3288258	0.0011043
9	0.6746309	0.3253691	0.7146685	0.1987933	0.0865382	0.3964584	0.2746962	0.3288452	0.0004658
10	0.6746125	0.3253875	0.7146671	0.1987946	0.0865383	0.3964577	0.2746980	0.3288454	0.0000731
11	0.6746095	0.3253905	0.7146672	0.1987948	0.0865380	0.3964577	0.2746983	0.3288443	0.0000061
12	0.6746097	0.3253903	0.7146673	0.1987948	0.0865379	0.3964578	0.2746983	0.3288440	0.0000058
13	0.6746099	0.3253901	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000013
14	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000001

Estimates values are $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 =0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746982$ and $x_8 = 0.328844$ after 14 iterations.

The result from U_4-method

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	ER
1	1.0000000	0.0000000	1.1447368	-0.0131579	-0.1315789	0.6350365	-0.0181818	-0.5000000	31.5277372
2	1.2649331	-0.2649331	0.7256975	0.1734748	0.1008277	0.4025767	0.2397107	0.3831453	14.4872778
3	0.6984926	0.3015074	0.7117597	0.1941054	0.0941349	0.3948448	0.2682183	0.3577126	1.0642761
4	0.6681911	0.3318089	0.7119800	0.1993524	0.0886676	0.3949670	0.2754687	0.3369369	0.0352962
5	0.6686793	0.3313207	0.7140419	0.1992306	0.0867274	0.3961109	0.2753005	0.3295642	0.1183955
6	0.6732338	0.3267662	0.7146417	0.1988876	0.0864707	0.3964436	0.2748265	0.3285886	0.0329900
7	0.6745536	0.3254464	0.7146955	0.1987967	0.0865079	0.3964734	0.2747008	0.3287299	0.0030997
8	0.6746718	0.3253282	0.7146769	0.1987900	0.0865331	0.3964631	0.2746917	0.3288258	0.0011043
9	0.6746309	0.3253691	0.7146685	0.1987933	0.0865382	0.3964584	0.2746962	0.3288452	0.0004658
10	0.6746125	0.3253875	0.7146671	0.1987946	0.0865383	0.3964577	0.2746980	0.3288454	0.0000731
11	0.6746095	0.3253905	0.7146672	0.1987948	0.0865380	0.3964577	0.2746983	0.3288443	0.0000061
12	0.6746097	0.3253903	0.7146673	0.1987948	0.0865379	0.3964578	0.2746983	0.3288440	0.0000058
13	0.6746099	0.3253901	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000013
14	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000001

Estimates values are $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 = 0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746982$ and $x_8 = 0.328844$ after 14 iterations.

The result from L_2-method

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	ER
1	1.6229508	-0.6229508	0.3750000	0.2099237	-0.0763359	0.2080292	0.2900763	-0.2900763	75.0747602
2	-0.7486339	1.7486339	0.8664122	0.1543522	0.1522853	0.4806374	0.2132867	0.5786841	73.8432028
3	0.9498086	0.0501914	0.6933625	0.3500626	0.0093789	0.3846390	0.4837229	0.0356397	28.9519777
4	0.6263291	0.3736709	0.6405585	0.1834655	0.0952224	0.3553463	0.2535159	0.3618450	10.7826606
5	0.4928209	0.5071791	0.7213121	0.2017522	0.0962813	0.4001440	0.2787849	0.3658688	7.0872515
6	0.6890849	0.3109151	0.7019665	0.2104080	0.0813449	0.3894121	0.2907455	0.3091105	2.7055424
7	0.6461803	0.3538197	0.7082472	0.1979249	0.0887083	0.3928962	0.2734963	0.3370916	0.9789843
8	0.6603664	0.3396336	0.7133667	0.2005256	0.0868457	0.3957363	0.2770900	0.3300138	0.4161670
9	0.6717452	0.3282548	0.7126286	0.1996582	0.0864138	0.3953268	0.2758913	0.3283724	0.2044064
10	0.6701148	0.3298852	0.7139280	0.1989678	0.0867726	0.3960477	0.2749374	0.3297358	0.1046222
11	0.6729828	0.3270172	0.7142596	0.1990665	0.0865470	0.3962316	0.2750736	0.3288787	0.0440462
12	0.6737130	0.3262870	0.7143865	0.1988930	0.0865617	0.3963020	0.2748340	0.3289344	0.0231038
13	0.6739923	0.3260077	0.7145453	0.1988489	0.0865592	0.3963901	0.2747731	0.3289249	0.0136865
14	0.6743416	0.3256584	0.7145919	0.1988321	0.0865421	0.3964159	0.2747498	0.3288602	0.0070964
15	0.6744441	0.3255559	0.7146258	0.1988110	0.0865430	0.3964347	0.2747206	0.3288635	0.0039501
16	0.6745186	0.3254814	0.7146460	0.1988048	0.0865403	0.3964460	0.2747121	0.3288532	0.0021921
17	0.6745631	0.3254369	0.7146549	0.1988003	0.0865390	0.3964509	0.2747058	0.3288482	0.0011760
18	0.6745827	0.3254173	0.7146607	0.1987976	0.0865387	0.3964541	0.2747021	0.3288470	0.0006536
19	0.6745954	0.3254046	0.7146637	0.1987964	0.0865383	0.3964558	0.2747005	0.3288454	0.0003568
20	0.6746020	0.3253980	0.7146653	0.1987956	0.0865381	0.3964567	0.2746994	0.3288448	0.0001947
21	0.6746056	0.3253944	0.7146663	0.1987952	0.0865380	0.3964572	0.2746989	0.3288444	0.0001073
22	0.6746076	0.3253924	0.7146667	0.1987950	0.0865380	0.3964575	0.2746986	0.3288442	0.0000586
23	0.6746087	0.3253913	0.7146670	0.1987949	0.0865379	0.3964576	0.2746984	0.3288441	0.0000321
24	0.6746093	0.3253907	0.7146672	0.1987948	0.0865379	0.3964577	0.2746983	0.3288441	0.0000176
25	0.6746096	0.3253904	0.7146672	0.1987948	0.0865379	0.3964577	0.2746983	0.3288440	0.0000096
26	0.6746098	0.3253902	0.7146673	0.1987948	0.0865379	0.3964578	0.2746983	0.3288440	0.0000053
27	0.6746099	0.3253901	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000029

The result from L_2 -method (cont.)

28	0.6746099	0.3253901	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000016
29	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000009
30	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000005

Estimates values are $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 = 0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746982$ and $x_8 = 0.328844$ after 30 iterations.

The result from L_3 -method

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	ER
1	1.6229508	-0.6229508	0.3750000	0.2099237	0.1320775	0.2080292	0.2900763	0.5018945	58.4433733
2	-0.7486339	1.7486339	0.6579988	0.1543522	0.1109716	0.3650213	0.2132867	0.4216920	45.1270719
3	0.5392865	0.4607135	0.7346762	0.3500626	0.0286104	0.4075576	0.4837229	0.1087195	26.1256483
4	0.7174040	0.2825960	0.6213270	0.2073124	0.0970669	0.3446777	0.2864681	0.3688542	12.5806589
5	0.4385589	0.5614411	0.6956207	0.1962450	0.0902456	0.3858918	0.2711749	0.3429333	7.4912330
6	0.6315868	0.3684132	0.7135094	0.2141420	0.0811261	0.3958154	0.2959053	0.3082793	2.8437824
7	0.6720600	0.3279400	0.7047319	0.2014259	0.0870316	0.3909462	0.2783339	0.3307199	1.2216486
8	0.6524576	0.3475424	0.7115426	0.1989488	0.0869380	0.3947243	0.2749111	0.3303646	0.6757711
9	0.6677096	0.3322904	0.7141132	0.2001409	0.0861293	0.3961504	0.2765583	0.3272913	0.2549940
10	0.6733906	0.3266094	0.7137298	0.1992121	0.0865230	0.3959377	0.2752749	0.3287873	0.0939026
11	0.6725460	0.3274540	0.7142649	0.1988684	0.0865699	0.3962345	0.2748000	0.3289655	0.0559215
12	0.6737246	0.3262754	0.7145617	0.1989194	0.0865080	0.3963992	0.2748705	0.3287303	0.0234219
13	0.6743778	0.3256222	0.7145726	0.1988482	0.0865323	0.3964052	0.2747721	0.3288227	0.0079106
14	0.6744017	0.3255983	0.7146195	0.1988088	0.0865398	0.3964313	0.2747176	0.3288512	0.0050592
15	0.6745048	0.3254952	0.7146514	0.1988073	0.0865356	0.3964490	0.2747156	0.3288354	0.0027175
16	0.6745750	0.3254250	0.7146570	0.1988011	0.0865371	0.3964521	0.2747070	0.3288409	0.0010649
17	0.6745873	0.3254127	0.7146618	0.1987969	0.0865379	0.3964547	0.2747011	0.3288441	0.0005828
18	0.6745978	0.3254022	0.7146652	0.1987961	0.0865377	0.3964566	0.2747001	0.3288433	0.0003146
19	0.6746053	0.3253947	0.7146662	0.1987955	0.0865378	0.3964571	0.2746992	0.3288436	0.0001359
20	0.6746074	0.3253926	0.7146667	0.1987951	0.0865379	0.3964574	0.2746986	0.3288439	0.0000690
21	0.6746086	0.3253914	0.7146671	0.1987949	0.0865379	0.3964576	0.2746984	0.3288439	0.0000366
22	0.6746094	0.3253906	0.7146672	0.1987949	0.0865379	0.3964577	0.2746983	0.3288439	0.0000168
23	0.6746097	0.3253903	0.7146673	0.1987948	0.0865379	0.3964578	0.2746983	0.3288440	0.0000083
24	0.6746098	0.3253902	0.7146673	0.1987948	0.0865379	0.3964578	0.2746983	0.3288440	0.0000043
25	0.6746099	0.3253901	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000020
26	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000010
27	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000005
28	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000002

Estimates values are $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 = 0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746982$ and $x_8 = 0.328844$ after 28 iterations.

The result from L_4-method

k	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	ER
1	0.2402477	0.7597523	0.3750000	0.2099237	0.1320775	0.2080292	0.2900763	0.5018945	35.6389108
2	0.7649601	0.2350399	0.6579988	0.2289942	0.0838291	0.3650213	0.3164283	0.3185505	10.7291506
3	0.6049565	0.3950435	0.6871768	0.1934871	0.0924812	0.3812076	0.2673640	0.3514284	3.0973205
4	0.6758787	0.3241213	0.7140317	0.2030897	0.0850689	0.3961052	0.2806330	0.3232618	0.6488772
5	0.6683952	0.3316048	0.7118414	0.1987182	0.0869783	0.3948901	0.2745925	0.3305174	0.2308249
6	0.6738284	0.3261716	0.7143035	0.1991706	0.0864543	0.3962560	0.2752175	0.3285265	0.0372928
7	0.6739672	0.3260328	0.7143751	0.1988420	0.0865634	0.3962957	0.2747634	0.3289409	0.0174886
8	0.6744503	0.3255497	0.7145946	0.1988336	0.0865344	0.3964175	0.2747519	0.3288307	0.0056326
9	0.6745324	0.3254676	0.7146320	0.1988044	0.0865395	0.3964382	0.2747115	0.3288502	0.0022990
10	0.6745852	0.3254148	0.7146561	0.1987995	0.0865378	0.3964515	0.2747047	0.3288438	0.0008108
11	0.6745998	0.3254002	0.7146627	0.1987963	0.0865380	0.3964552	0.2747003	0.3288445	0.0003125
12	0.6746064	0.3253936	0.7146657	0.1987954	0.0865379	0.3964569	0.2746991	0.3288440	0.0001141
13	0.6746086	0.3253914	0.7146667	0.1987950	0.0865379	0.3964574	0.2746985	0.3288440	0.0000430
14	0.6746095	0.3253905	0.7146671	0.1987949	0.0865379	0.3964577	0.2746983	0.3288440	0.0000159
15	0.6746098	0.3253902	0.7146673	0.1987948	0.0865379	0.3964577	0.2746983	0.3288440	0.0000060
16	0.6746099	0.3253901	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000022
17	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000008
18	0.6746100	0.3253900	0.7146673	0.1987948	0.0865379	0.3964578	0.2746982	0.3288440	0.0000003

Estimates values are $x_1 = 0.67461$, $x_2 = 0.32539$, $x_3 = 0.7146673$, $x_4 = 0.1987948$, $x_5 = 0.0865379$, $x_6 = 0.3964578$, $x_7 = 0.2746982$ and $x_8 = 0.328844$ after 18 iterations.

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