MODIFIED NEWTON METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

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เนื้อหาวิทยานิพนธ์ฉบับนี้ กล่าวถึงการปรับปรุงทฤษฎีบทของวิธีนิวตันในการหาผลเฉลยของระบบสมการไม่เชิงเส้น

$$f(x) = 0$$

หรือ

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

เมื่อ $x \in R^n$

f เป็นฟังก์ชันจากเซตย่อยของ R^n ไปยังเซตย่อยของ R^n

พร้อมทั้งแสดงตัวอย่างการหาผลเฉลยของปัญหาระบบสมการไม่เชิงเส้นที่ประยุกต์ใช้ในสาขาวิทยาศาสตร์และเทคโนโลยี

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ABSTRACT

In this thesis we will discuss the Modified Newton's method for solving systems of nonlinear equations

$$f(x) = 0$$

or

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

where $x \in R^n$

f is a function from a subset of R^n to a subset of R^n

and giving examples for finding solution about application problems in science and technology.

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CHAPTER 1

INTRODUCTION

Systems of nonlinear equations occur widely in the mathematical modeling of real world problems, often giving more realistic model than that is obtained by using systems of linear equations. A frequent problem in applied mathematics is the approximated solution of systems of nonlinear equations

$$f_i(x_1, x_2, \ldots, x_n) = 0 \quad , \quad i = 1,2,\ldots,n \tag{1.1}$$

which we usually write in the vector form

$$f(x) = 0 \tag{1.2}$$

or

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned} \tag{1.3}$$

where $x \in R^n$ and f are real vector functions of vector x or $f : R^n \to R^n$ is continuous nonlinear functions whose components are f_i; that is $f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T$ or

$$f(x_1, x_2, \ldots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}. \tag{1.4}$$

For this purpose we use iterative methods that produce the desired solution as the limit of a sequence of approximations. We shall formulate the Newton's method, which is the theoretical basis for the study of the convergence properties and the order of convergence. The convergence behavior of the sequence of approximated values towards the solution is importance for the usefulness and the efficiency of a method. With this point of view, some methods for determining a solution of a nonlinear equation in one unknown are developed and analyzed. The considerations will emphasize on the improvement of the nonlinear system. Especially, the determination of the zeros of polynomials will be a special application of the methods.

To help develop some intuition for the solvability of nonlinear systems and to illustrate some of the main ideas used in their numerical solution, we begin the section with the first example, an examination of system of two nonlinear equations in real world problems, the positioning a robot arm [1]. Figure 1.1 shows a two-link robot arm. One end of the arm is fixed at the origin of the coordinate system, and the other end is at coordinate (x, y). There are positioning actuators control the link angles θ and ϕ, as shown. In terms of θ and ϕ, the coordinates of the joint between links are (a, b), where

$$\begin{aligned} a &= 0.5\cos\theta \\ b &= 0.5\sin\theta. \end{aligned}$$

The coordinates (x, y) are given by

$$\begin{aligned} x - a &= 0.35\cos(\phi + \theta) \\ y - b &= 0.35\sin(\phi + \theta) \end{aligned}$$

or

$$\begin{aligned} x &= 0.5\cos\theta + 0.35\cos(\phi + \theta) = 0 \\ y &= 0.5\sin\theta + 0.35\sin(\phi + \theta) = 0. \end{aligned} \tag{1.5}$$

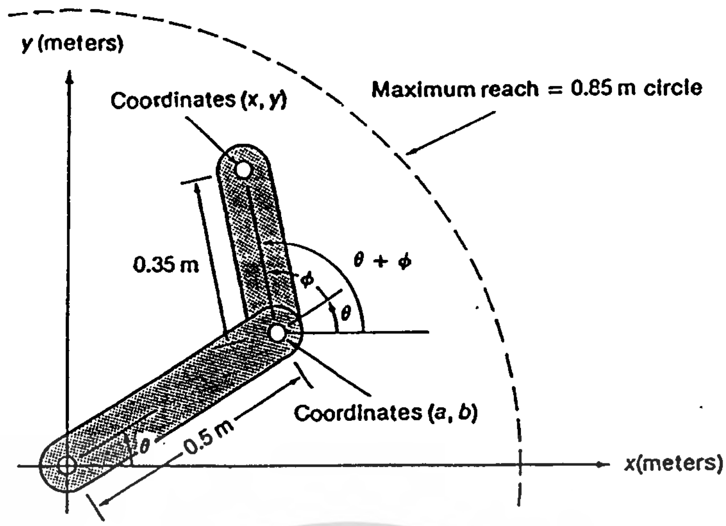


Figure 1.1 Positioning a robot arm.

The second example [2], we frequently know in advance that the absolute maximum (or minimum) value of $f(x,y)$ on R^2 occurs at an interior point of R^2 where both partial derivatives of f exist, like the surface

$$z=\frac{8}{3}x^3+4y^3-x^4-y^4. \tag{1.6}$$

In this important case we can locate every possible point at which the maximum might occur by simultaneously solving the two equations

$$f_x(x,y)=0 \quad \text{and} \quad f_y(x,y)=0. \tag{1.7}$$

Fortunately, these equations have only one simultaneous solution (x,y) interior to R^2. If so, then that solution must be the location of the desired maximum. If we find that the equations in (1.7) have several simultaneous solutions interior to R^2, then we simply evaluate f at each solution to determine which yields the largest value of $f(x,y)$ and is therefore the desired maximum point, as in Figure 1.2.

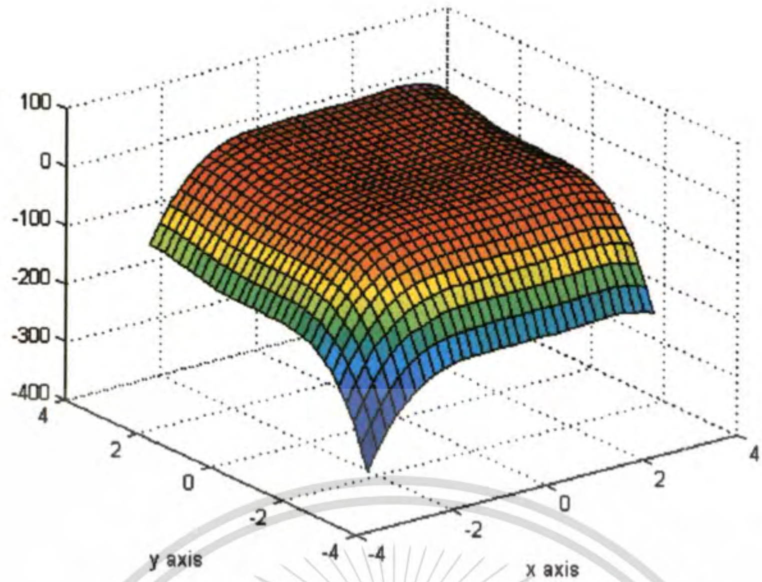


Figure 1.2 The surface $z = \frac{8}{3}x^3 + 4y^3 - x^4 - y^4$.

The development of some geometric insight concerning these equations and their numerical solution will explain in Chapter 3. There are several methods for solving nonlinear equations, which satisfied (1.2), one of the basic iteration procedures for approximating a solution of (1.2) is **Newton's method**:

$$x^{k+1} = x^k - \left[f'(x^k)\right]^{-1} f(x^k) \quad , \quad k = 0,1,2,... \tag{1.8}$$

where $f'(x)$ is the Jacobian matrix of $f(x)$ denoted by $J(x^k)$

$$J(x^k) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{1.9}$$

or
$$J(x^k) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \tag{1.10}$$

and where we have used f_{ij} to denote the partial derivative of $f_i(x)$ with respect to the j^{th} variable and evaluated at x. Then new form of an approximation solution of (1.2) by using Newton's method is

$$x^{k+1} = x^k - \left[J(x^k)\right]^{-1} f(x^k) \quad , \quad k = 0,1,2,.... \tag{1.11}$$

There are many mathematicians who gave their best efforts for studying the systems of nonlinear equations based on Newton's method. The new process is created by letting matrix $A(x)$ be an invertible operator to replace the Jacobian matrix in Newton's method, this new method called **Newton-Like method**. Then form of an approximate solution of (1.2) by using Newton-Like method is

$$x^{k+1} = x^k - \left[A(x^k)\right]^{-1} f(x^k) \quad , \quad k = 0,1,2,.... \tag{1.12}$$

M. Podisuk is one of many mathematicians who studied the systems of nonlinear equations. One of his papers [3], which is a special kind of the Newton-Like method called **Modified Newton method**. This method modified from Newton's method by letting the matrix $H(x)$ to represent the Jacobian matrix in Newton's method. Then the approximated solution of (1.2) by this Modified Newton method is

$$x^{k+1} = x^k - \left[H(x^k)\right]^{-1} f(x^k) \quad , \quad k = 0,1,2,... \tag{1.13}$$

where $H(x)$ denotes the diagonal matrix

$$H(x)=\begin{bmatrix} f_{11} & & \cdots & 0 \\ & f_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & f_{nn} \end{bmatrix}$$

and f_{ii} is the partial derivative of $f_i(x)$ with respect to the i^{th} variable.

In this thesis, we shall study the theorem that supports the research of M. Podisuk [3], to ensure the convergence properties and the order of convergence of the Modified Newton method for solving systems of nonlinear equations.

CHAPTER 2

THEOREM ON A MODIFIED NEWTON METHOD

This chapter begins with our consideration of multivariable problems by discussing the system of nonlinear equations in real variables. Let us consider the system expressed as

$$f : X \to X$$
$$\text{find } x \in X \text{ for which } f(x) = 0 \tag{2.1}$$

where $X \equiv R^n$ is n-dimensional real vector space, and f is assumed to be continuously differentiable.

Of course, (2.1) is just the standard way of denoting a system of n-nonlinear equations in n-unknowns, with the convention that right-hand side of each equation is zero, and throughout this study,

$D \subset X$ denotes the domain of $f(x)$,

D_0 denotes closed convex subset of D,

$N(x,t)$ denotes the open neighborhood of radius t around x,i.e.

$$N(x,t) = \{y \in X : \|x - y\| < t\},$$

$L(X,Y)$ denotes a real Banach space,

$f \in Lip_k(D)$ if $\|f(x) - f(y)\| \le K\|x - y\|$ for some constant K,

$\{x_k\}$ denotes sequence of vectors in R^n,

$B(x) = f'(x) - A(x)$,

and $M(x) = f'(x) - H(x)$.

In the analysis of Newton's method, it will be necessary to assume that the Jacobian matrix,

$$J(x^k) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \tag{2.2}$$

where f_{ij} denotes the partial derivative of $f_i(x^k)$ with respect to the j^{th} variable and evaluated at x, is at least continuous at the solution x^*; that is

$$\|f'(x^* + h) - f'(x^*)\| \to 0 \quad \text{as} \quad h \to 0.$$

On occasion, it will also be useful to assume, instead of continuity of f', that the derivative satisfies the following property.

Definition 2.1 The mapping $f : R^n \to R^n$ is (Totally or Frechet) differentiable at x if the Jacobian matrix (2.2) exists at x and

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|} = 0. \tag{2.3}$$

Note that if $n = 1$, Definition 2.1 reduces to the usual definition of differentiability. Note also that if f is differentiable at x, then f is continuous at x; this follows from the inequality

$$\|f(x+h) - f(x)\| \le \|f(x+h) - f(x) - f'(x)h\| + \|f'(x)h\|. \tag{2.4}$$

Finally, we note that it is possible to say that if the Jacobian matrix is continuous at x then f is differentiable at x.

One of the basic tools of nonlinear analysis is the mean value theorem. If f is a differentiable function from R^1 to R^1, this states that

$$f(x)-f(y)=f'(z)(x-y) \tag{2.5}$$

for some point z between x and y. Unfortunately, this result does not extend verbatim to mappings from R^n to R^n. However, we are able to prove some results, which are often just as useful. For the first, we define the integral $\int_a^b G(t)dt$ of a mapping $G:[a,b]\subset R^1 \to R^n$ as the vector with components $\int_a^b g_i(t)dt, i=1,2,\dots,n$ where $g_1, g_2, \dots, g_n$ are the components of G. Thus, for example, if $f: R^n \to R^n$ the relation

$$f(y)-f(x)=\int_0^1 f'(x+t(y-x))(y-x)dt \tag{2.6}$$

is equivalent to

$$f_i(y)-f_i(x)=\int_0^1 \sum_{j=1}^{n} f_{ij}(x+t(y-x))(y_j-x_j)dt \tag{2.7}$$

for $i=1,2,\dots,n$.

For $n=1$, (2.6) is simply the fundamental theorem of the integral calculus. Hence the next result is a natural extension of that theorem to n-dimensions.

Lemma 2.1 Assume that $f : R^n \to R^n$ is continuously differentiable on a convex set $D \subset R^n$. Then for any $x, y \in D$, (2.6) holds.

Proof: For fixed $x, y \in D$ define the functions $g_i : [0,1] \subset R^1 \to R^1$ by

$$g_i(t) = f_i(x + t(y - x)) \quad , \quad t \in [0,1], \; i = 1,2,\ldots,n. \tag{2.8}$$

By the convexity of D, it follows that g_i is continuously differentiable on $[0,1]$ and thus the fundamental theorem of the integral calculus implies that

$$g_i(1) - g_i(0) = \int_0^1 g_i'(t)dt. \tag{2.9}$$

But a simple calculation shown that

$$g_i'(t) = \sum_{j=1}^{n} f_{ij}(x + t(y - x))(y_j - x_j) \tag{2.10}$$

so that (2.9) is equivalent to (2.7).

For the next result, we first need a lemma on integration.

Lemma 2.2 Assume that $G : [a,b] \subset R^1 \to R^n$ is continuous. Then

$$\left\| \int_a^b G(t)dt \right\| \le \int_a^b \|G(t)\| dt. \tag{2.11}$$

proof: Since any norm is a continuous function on R^n, both integrals of (2.11) exist and therefore any $\varepsilon > 0$ there is a partition $a < t_0 < \ldots < t_p < b$ of $[a,b]$ such that

$$\left\| \int_a^b G(t)dt - \sum_{i=1}^{p} G(t_i)(t_i - t_{i-1}) \right\| < \varepsilon \tag{2.12}$$

and

$$\left| \int_a^b \|G(t)\| dt - \sum_{i=1}^{p} \|G(t_i)\| (t_i - t_{i-1}) \right| < \varepsilon .$$

Hence

$$\left\| \int_a^b G(t)dt \right\| \le \left\| \sum_{i=1}^{p} G(t_i)(t_i - t_{i-1}) \right\| + \varepsilon$$

$$\le \sum_{i=1}^{p} \|G(t_i)\| (t_i - t_{i-1}) + \varepsilon \le \int_a^b \|G(t)\| dt + 2\varepsilon$$

and, since ε was arbitrary, (2.11) must be valid.

By means of lemma 2.1 and 2.2 we can prove the following useful alternative of the mean value theorem.

Theorem 2.3 Assume that $f : R^n \to R^n$ is continuously differentiable on the convex set $D \subset R^n$. Then for any $x, y \in D$,

$$\|f(x) - f(y)\| \le \sup_{0 \le t \le 1} \|f'(x + t(y - x))\| \|y - x\| . \tag{2.13}$$

proof : By lemma 2.1 and 2.2 we have

$$\|f(x) - f(y)\| = \left\| \int_0^1 f'(x + t(y - x))(y - x) dt \right\|$$

$$\le \int_0^1 \|f'(x + t(y - x))\| \, \|y - x\| dt$$

$$\leq \sup_{0\leq t\leq 1}\|f'(x+t(y-x))\|\int_0^1\|y-x\|dt$$

which is (2.13).

Lemma 2.4 Let $f' \in Lip_k(D_0)$ and let $x, y \in D_0$, then

$$\|f(y)-f(x)-f'(x)(y-x)\| \leq \frac{1}{2}K\|x-y\|^2. \tag{2.14}$$

proof: From $f: R^n \to R^n$ is differentiable and satisfied $f' \in Lip_k(D_0)$, that means $\|f'(x)-f'(y)\| \leq K\|x-y\|$ for all x, y in closed convex subset D_0 of D.

Since $f'(x)(y-x)$ is constant with respect to the integration, we have

$$f(y)-f(x)-f'(x)(y-x) = \int_0^1 [(f'(x+t(y-x))(y-x)) - f'(x)(y-x)]dt$$

$$= \int_0^1 [f'(x+t(y-x)) - f'(x)](y-x)dt.$$

Hence, the result follows by taking norms of both side,

$$\|f(y)-f(x)-f'(x)(y-x)\| = \left\|\int_0^1 [f'(x+t(y-x)) - f'(x)](y-x)dt\right\|$$

$$\leq \int_0^1 \|f'(x+t(y-x)) - f'(x)\|\|y-x\|dt$$

$$\leq \int_0^1 K\|x+t(y-x)-x\|\|y-x\|dt$$

$$= \int_0^1 K\|t(y-x)\|\|y-x\|dt$$

$$\leq \int_0^1 K\|t\|\|y-x\|^2 dt$$

$$= K\|y-x\|^2 \int_0^1 t dt$$

$$= \frac{1}{2} K\|x-y\|^2 .$$

Definition 2.2 Let t_0 and t' be non-negative real numbers, g be a continuously differentiable real function on $[t_0, t_0 + t']$ and G a continuously differentiable operator on $\overline{N}(x_0, t) \subset X$ into X. Then the equation $t = g(t)$ will be said to majorize the equation $x = G(x)$, or g majorizes G, on $N(x_0, t')$ if

$$\|G(x_0) - x_0\| \leq g(t_0) - t_0 \tag{2.15}$$

and

$$\|G'(x)\| \leq g'(t) \text{ where } \|x - x_0\| \leq t - t_0 < t' . \tag{2.16}$$

Definition 2.3 Let t_0 and t' be non-negative real numbers, g be a real function on $[t_0, t_0 + t']$, and G an operator sending $N(x_0, t')$ into X. Then the equation $t = g(t)$ will be said to weakly majorize the equation $x = G(x)$, or g weakly majorizes G, if (2.15) holds and in addition

$$\|G(G(x)) - G(x)\| \leq g(g(t)) - g(t) \tag{2.17}$$

when $\|x - x_0\| \leq t - t_0 < t'$

and $\|G(x) - x\| \leq g(t) - t$.

We remark that Lemma 2.4 and Lemma 2.5 are know results in mathematical analysis and Lemma 2.6 is given by Ortega [4].

Lemma 2.5 (The Banach Lemma) Let $M \in L(X,X)$ and $\|I-J\| \le \delta < 1$, then J^{-1} exists in $L(X,X)$ and $J^{-1} \le (1-\delta)^{-1}$.

Lemma 2.6 Let $\{x_k\}$ be a sequence in X and $\{t_k\}$ a sequence of non-negative real numbers such that

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k \quad , \quad k = 0,1,2,\dots , \tag{2.18}$$

and $t_k \to t^* < \infty$. By these conditions, there exists a point $s \in X$ such that $x_k \to s$ and

$$\|s - x_k\| \le t^* - t_k \quad , \quad k = 0,1,2,\dots . \tag{2.19}$$

proof: The proof is immediate from

$$\begin{aligned}
\|x_{k+p} - x_k\| &\le \sum_{i=1}^{p} \|x_{k+i} - x_{k+i-1}\| \\
&\le t_{k+p} - t_k \\
&\le t^* - t_k ,
\end{aligned}$$

which show that $\{x_k\}$ is a Cauchy Sequence.

We shall say that $\{t_k\}$ majorizes $\{x_k\}$ if $\|x_{k+1} - x_k\| \le t_{k+1} - t_k$, $k = 0,1,2,\dots$ holds.

The following theorem is The Kantorovich Theorem. It is one of the fundamental theorems in numerical mathematics. The idea for proof this theorem in [5].

Theorem 2.7 (The Kantorovich Theorem) Let x_0 be in D_0 and let $\Gamma_0 = [f'(x_0)]^{-1}$ exist with $\Gamma_0 \in Lip_k(D_0)$,

$$\|\Gamma_0 f(x_0)\| \le \eta \quad \text{and} \quad h = K\eta \le \frac{1}{2}. \tag{2.20}$$

Define
$$r_0(h) = \frac{1}{h}(1-\sqrt{1-2h})\eta \tag{2.21}$$

$$r_1(h) = \frac{1}{h}(1+\sqrt{1-2h})\eta. \tag{2.22}$$

Then if $N(x_0, r_0(h)) \subset D_0$, the sequence of iterates defined by Newton's method exists, remains in $N(x_0, r_0(h))$ and converges to s in $N(x_0, r_0(h))$ such that $f(s) = 0$. If $h < \frac{1}{2}$, s is the only solution in $N(x_0, r_1(h)) \cap D_0$, and if $h = \frac{1}{2}$, s is unique in $N(x_0, r_1(h)) \cap D_0$. Furthermore, the sequence of the iterates satisfy the error bounds

$$\|s - x_m\| \le \frac{1}{h}\frac{1}{2^m}(1-\sqrt{1-2h})^{2m}\eta. \tag{2.23}$$

This theorem together with Lemma 2.6 and Definition 2.2 give the convergence of the sequence $\{x_k\}$ in X when the sequence of $\{t_k\}$ converges.

Theorem 2.8 If g majorizes G on $\overline{N}(x_0, t')$ and g has a fixed point in $[t_0, t_0 + t']$, then G has a fixed point s in $\overline{N}(x_0, t')$. Furthermore, $x_{k+1} = G(x_k)$ and $t_{k+1} = g(t_k)$, $k = 0,1,2,\ldots$, converges to s and t^* respectively with the real sequence majorizing the vector sequence.

Next lemma uses the weakly majorizing property and it is given by Dennis [6].

Lemma 2.9 If $g(t) \in (t, t_0 + t')$ when $t \in (t_0, t_0 + t')$, and g weakly majorizes G on $N(x_0, t')$, then there are elements $t^* \in [t_0, t_0 + t']$, $s \in \overline{N}(x_0, t')$ such that

$$x_{k+1} = G(x_k) \tag{2.24}$$

and

$$t_{k+1} = g(t_k) \quad , \quad k = 0,1,2,\ldots \tag{2.25}$$

converge to s and t^* respectively with the t sequence majorizing the x sequence.

Theorem 2.10 Let A be a function on $N(x_0, r)$ such that $A(x) \in L(X, Y)$ for each x and $A(x)$ is invertible for each x in $N(x_0, r)$ and that there is a real, nonvanishing, nonincreasing function $a(t)$ on $[0, r)$ such that

$$\left\|[A(x)]^{-1}\right\| \le a(\|x - x_0\|)^{-1}. \tag{2.26}$$

If $f' \in Lip_k(\overline{N}(x_0, r))$ then if $\sigma \ge 1$ and $\delta > 0$ are real numbers such that

$$a(t) + \sigma\, kt \tag{2.27}$$

is isotonic on $(0, r)$, and

$$\|B(x)\| \le a(\|x - x_0\|) + \sigma\, K\|x - x_0\|\delta, \tag{2.28}$$

for every $x \in N(x_0, r)$

then

$$g(t) = t + (a(t))^{-1}\left(\frac{1}{2}\sigma\, Kt^2 - \delta\, t + a(0)\left\|[A(x_0)]^{-1} - f(x_0)\right\|\right) \tag{2.29}$$

weakly majorizes $G(x) = x - [A(x)]^{-1} f(x)$ on $N(x_0, r)$.

Theorem 2.11 Let $f' \in Lip_k(\overline{N}(x_0,r))$ and $[A(x_0)]^{-1}$ exist and be bounded in the norm by $[a(0)]^{-1}$. If $\|B(x_0)\| < a(0)$,

$$h' = \frac{K}{a(0)}\left\|[A(x_0)]^{-1} f(x_0)\right\|a(0) - (\|B(x_0)\|)^2 \le \frac{1}{2} \tag{2.30}$$

and

$$r_0' = \frac{1}{K}\left(1-\sqrt{1-2h'}\right)\left[a(0) - \|B(x_0)\|\right] \le r . \tag{2.31}$$

Then if f has a unique zero $s \in \overline{N}(x_0, r_0')$, and

$$x'_{m+1} = x'_m - [A(x_0)]^{-1} f(x'_m) \quad , \quad m = 0,1,2,\ldots$$

converges to s from any $x'_0 \in \overline{N}(x_0, r)$ such that

$$\|x'_0 - x_0\| < r_1' = \frac{1}{K}\left(1-\sqrt{1-2h}\right)\left[a(0) - \|B(x_0)\|\right].$$

If, in addition σ, δ and a satisfy the conditions of Theorem 2.10 and

$$h = \frac{1}{\delta^2}\sigma K\left\|[A(x_0)]^{-1} f(x_0)\right\|a(0) \le \frac{1}{2} \tag{2.32}$$

and

$$r_0 = \frac{1}{\sigma . K}\left(1-\sqrt{1-2h}\right)\delta < r, \tag{2.33}$$

then

$$x_{m+1} = x'_m - [A(x_m)]^{-1} f(x'_m) \quad , \quad m = 0,1,2,\ldots \tag{2.34}$$

converges to s.

In the following theorem, we impose one more condition on $A(x)$ and one condition on $B(x)$ instead of $B(x_0)$.

Theorem 2.12 Let $f' \in Lip_k(\overline{D})$ where $x_0 \in D$ and D is an open convex subset of X. Assume that

$$\left\|[A(x_0)]^{-1} f(x_0)\right\| \le \alpha \tag{2.35}$$

$$\left\|[A(x_0)]^{-1}\right\| \le \beta \tag{2.36}$$

$$\|A(x) - A(x_0)\| \le \eta_0 + \eta_1 \|x - x_0\|, \ \forall x \in D \tag{2.37}$$

$$\|B(x)\| \le \delta_0 + \delta_1 \|x - x_0\|, \ \ \forall x \in D. \tag{2.38}$$

Then

$$\beta.\delta_0 < 1, \ h' = \frac{\beta.K.\alpha}{(1-\beta.\delta_0)^2} \le \frac{1}{2}$$

and $N(x_0, r_0') \subset D$ where

$$r_0' = \frac{1-\sqrt{1-2h'}}{\beta.K}(1-\beta.\delta_0)$$

implied that f has a solution $r \in \overline{N}(x_0, r_0')$ which is unique in $D \cap N(x_0, r_0')$ where

$$r_1' = \frac{1-\sqrt{1+2h'}}{\beta.K}(1-\beta.\delta_0).$$

Furthermore $$x_{m+1}' = x_m' - [A(x_0)]^{-1}(f(x_m'))$$

converges to s from any $x_0' \in \overline{D} \cap N(x_0, r_1')$.

If, in addition $\beta(\delta_0 + \eta_0) < 1$ and $h = \dfrac{\sigma.\beta.K.\alpha}{(1-\beta.\eta_0-\beta.\delta_0)^2} \le \dfrac{1}{2}$ where $\sigma = \max\left(1, \dfrac{\delta_1+\eta_1}{K}\right)$, and $N(x_0, r_0) \subset D$,

$$r_0' = \frac{1-\sqrt{1-2h}}{\sigma.\beta.K}(1-\beta.\eta_0-\beta.\delta_0)$$

then

$$x_{m+1} = x_m - [A(x_m)]^{-1} f(x_m)$$

converges to s.

The next theorems will ensure that the convergence of the Modified Newton method for solving systems of nonlinear equations which is a special kind of the Newton-Like method.

Theorem 2.13 Let D be an open convex subset of the space X and $f' \in Lip_k(\overline{D})$. Assuming that $f(x)$ and $H(x)$ satisfy all the conditions of the previous theorems, then there exists a unique zero s in D so that for any point x_0 in D the sequence $\{x_m\}$ where

$$x_{m+1} = x_m - [H(x_m)]^{-1} f(x_m)$$

converges to s.

Proof: By the results of the previous theorems, the existence and uniqueness of s in D is ensured and the sequence $\{x_m\}$ of points in D, where

$$x_{m+1} = x_m - [H(x_m)]^{-1} f(x_m)$$

for x_0 in D, converges to this unique point s.

The last theorem of this section, Theorem 2.14, shows that the order of the convergence of the Modified Newton method for solving systems of nonlinear equations which is of second order.

Theorem 2.14 Let the conditions of Theorem 2.12 be satisfied and $\delta_0 = 0$, that is

$$\|M(x)\| \le \delta_1 \|x - x_0\| \quad , \quad \forall x \in D. \tag{2.39}$$

Then the order of the convergence of the method is equal to 2.

Proof: Let $\qquad Q = \sup\left(a(\|x - x_0\|)\right)^{-1} \quad , \quad x \in N(x_0, r_0)$

$$P = Q\left(\frac{1}{2}K + \delta_1\right)$$

and $\qquad e_k = \|s - x_k\|$

then

$$\begin{aligned}
e_{k+1} &= \|s - x_{k+1}\| \\
&= \left\|s - x_k + [H(x_k)]^{-1} f(x_k)\right\| \\
&= \left\|[H(x_k)]^{-1} f(s) + [H(x_k)]^{-1} f(x_k) + s - x_k\right\| \\
&= \left\|[H(x_k)]^{-1} f(s) + [H(x_k)]^{-1} f(x_k) + [H(x_k)]^{-1} H(x_k)(s - x_k)\right\| \\
&\le \left\|[H(x_k)]^{-1}\right\| \cdot \left\|f(s) + f(x_k) + H(x_k)(s - x_k)\right\| \\
&= \left\|[H(x_k)]^{-1}\right\| \cdot \left\|f(s) + f(x_k) + f'(x_k)(s - x_k) - f'(x_k)(s - x_k) + H(x_k)(s - x_k)\right\| \\
&= \left\|[H(x_k)]^{-1}\right\| \cdot \left\|f(s) + f(x_k) + f'(x_k)(s - x_k) - f'(x_k) - H(x_k)(s - x_k)\right\| \\
&\le \left(a(\|x_0 - x_k\|)\right)^{-1} \left\|f(s) + f(x) + f'(x_k)(s - x_k) - M(x_k)(s - x_k)\right\| \\
&\le \left(a(\|x_0 - x_k\|)\right)^{-1} \left(\frac{1}{2}K\|s - x_k\|^2 + \|M(x_k)(s - x_k\|\right) \\
&\le \left(a(\|x_0 - x_k\|)\right)^{-1} \left(\frac{1}{2}Ke_k^2 + \|M(x_k)\|e_k\right) \\
&\le Q\left(\frac{1}{2}K + \delta_1\right)e_k^2 \\
&= Pe_k^2.
\end{aligned}$$

CHAPTER 3

EXAMPLES OF SYSTEMS OF NONLINEAR EQUATIONS

In many practical problems, the approximate location of a desired root is well known. The solution of linear systems is so straightforward and deal with the intersections of lines, planes and hyperplanes. Nonlinear systems deal with the intersections of surfaces and hypersurfaces, which are not plane. Even if these surfaces, defined by $F(x_1, x_2,...)$, $G(x_1, x_2,...)$, $H(x_1, x_2,...)$,..., are perfectly smooth, the problem of finding the initial approximating value is difficult.

Some systems of nonlinear equation often have an infinite number of real roots; the three dimensional graphs, although possible, would not be practical. Figure 3.1 is an impression of portions of three surfaces, F, G and H, meeting in space. If we go to four or more dimensions, the graphical method is clearly impossible.

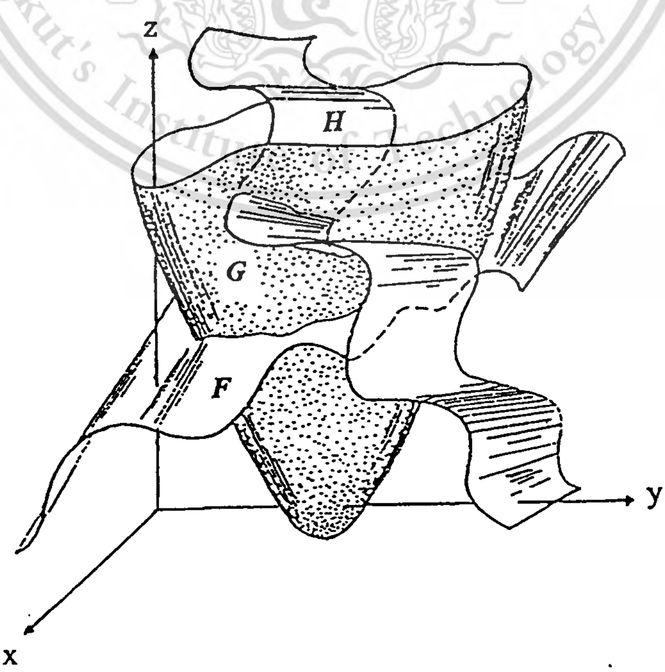


Figure 3.1 Three intersecting surfaces.

In actual problems of science and technology, we are interested in a solution within a specific domain D. The limitation of this study that is to requested the region using numerical technique. In this chapter, we shall find the solutions of some equations of the systems of nonlinear equations.

For the system of nonlinear equation in the form

$$f(x) = 0 \tag{3.1}$$

the approximate solution by using the Newton's method with the Jacobian matrix

$$J(x^k) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \tag{3.2}$$

where we have used f_{ij} to denote the partial derivative of $f_i(x)$ with respect to the j^{th} variable and evaluated at x, could be written as follows:

$$x^{k+1} = x^k - \left[J(x_1^k, x_2^k, \ldots, x_n^k)\right]^{-1} \begin{bmatrix} f_1(x_1^k, x_2^k, \ldots, x_n^k) \\ f_2(x_1^k, x_2^k, \ldots, x_n^k) \\ \vdots \\ f_n(x_1^k, x_2^k, \ldots, x_n^k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots. \tag{3.3}$$

Then using the idea of Modified Newton method with the diagonal matrix

$$H(x^k) = \begin{bmatrix} f_{11} & & \cdots & 0 \\ \vdots & f_{22} & & \vdots \\ & & \ddots & \\ 0 & \cdots & & f_{nn} \end{bmatrix} \tag{3.4}$$

therefore the Modified Newton method algorithm could be written as follows:

$$x_1^{k+1} = x_1^k - \frac{f_1(x_1^k, x_2^k, \ldots, x_n^k)}{f_{11}(x_1^k, x_2^k, \ldots, x_n^k)} \quad ; \quad k = 0,1,2,\ldots$$

$$x_2^{k+1} = x_2^k - \frac{f_2(x_1^k, x_2^k, \ldots, x_n^k)}{f_{22}(x_1^k, x_2^k, \ldots, x_n^k)} \quad ; \quad k = 0,1,2,\ldots$$

$$\vdots \tag{3.5}$$

$$x_n^{k+1} = x_n^k - \frac{f_n(x_1^k, x_2^k, \ldots, x_n^k)}{f_{nn}(x_1^k, x_2^k, \ldots, x_n^k)} \quad ; \quad k = 0,1,2,\ldots.$$

The formula of Modified Newton method can be improved as follows:

$$x_1^{k+1} = x_1^k - \frac{f_1(x_1^k, x_2^k, \ldots, x_n^k)}{f_{11}(x_1^k, x_2^k, \ldots, x_n^k)} \quad ; \quad k = 0,1,2,\ldots$$

$$x_2^{k+1} = x_2^k - \frac{f_2(x_1^{k+1}, x_2^k, \ldots, x_n^k)}{f_{22}(x_1^{k+1}, x_2^k, \ldots, x_n^k)} \quad ; \quad k = 0,1,2,\ldots$$

$$\vdots \tag{3.6}$$

$$x_n^{k+1} = x_n^k - \frac{f_n(x_1^{k+1}, x_2^{k+1}, \ldots, x_{n-1}^{k+1}, x_n^k)}{f_{nn}(x_1^{k+1}, x_2^{k+1}, \ldots, x_{n-1}^{k+1}, x_n^k)} \quad ; \quad k = 0,1,2,\ldots.$$

3.1 Some Examples of Systems of Nonlinear Equations

Example 3.1 Find the solutions of system

$$f(x,y) = e^x \cos y - x^2 + y^2 = 0$$
$$g(x,y) = e^x \sin y - 2xy = 0 \tag{3.7}$$
$$\varepsilon < 0.0000001$$

The following matrix is the Jacobian matrix for the system (3.7)

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} e^x \cos y - 2x & -e^x \sin y + 2y \\ e^x \sin y - 2y & e^x \cos y - 2x \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} e^x \cos y - 2x & 0 \\ 0 & e^x \cos y - 2x \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

We obtain the results from Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.1.

Table 3.1 The solution of the example 3.1

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0,0)	4	x=-0.70346747 y=0.00000000	4	x=-0.70346747 y=0.00000000	4	x=-0.70346747 y=0.00000000
(1,0)	5	x=-0.70346742 y=0.00000000	5	x=-0.70346742 y=0.00000000	5	x=-0.70346742 y=0.00000000
(0,1)	6	x=-0.70346742 y=0.00000000	6	x=-0.70346742 y=0.00000000	6	x=-0.70346742 y=0.00000000
(-1,1)	5	x=-0.70346742 y=0.00000000	5	x=-0.70346742 y=0.00000000	5	x=-0.70346742 y=0.00000000
(-1,-1)	5	x=-0.70346742 y=0.00000000	5	x=-0.70346742 y=0.00000000	5	x=-0.70346742 y=0.00000000

Example 3.2 Find the solutions of system

$$\begin{aligned} f(x,y) &= e^x + xy - 1 = 0 \\ g(x,y) &= \sin xy + x + y - 1 = 0 \\ \varepsilon &< 0.0000001 \end{aligned} \tag{3.8}$$

The following matrix is the Jacobian matrix for the system (3.8)

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} e^x + y & x \\ y\cos xy + 1 & x\cos xy + 1 \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} e^x + y & 0 \\ 0 & x\cos xy + 1 \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results from Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.2.

Table 3.2 The solution of the example 3.2

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(1,1)	5	x=-0.00000000 y=1.00000000	6	x=-0.00000000 y=1.00000000	4	x=0.00000000 y=0.99999999
(1,-1)	5	x=0.00000000 y=1.00000000	6	x=-0.00000000 y=1.00000000	5	x=0.00000000 y=1.00000000
(0,0)	1	x=0.00000000 y=1.00000000	1	x=0.00000000 y=1.00000000	1	x=0.00000000 y=1.00000000
(2,0)	6	x=0.00000000 y=1.00000000	7	x=-0.00000000 y=1.00000000	6	x=0.00000000 y=1.00000000
(-1,0)	6	x=0.00000000 y=1.00000000	3	x=-0.00000000 y=1.00000000	5	x=0.00000000 y=1.00000000
(-1,1)	4	x=0.00000001 y=1.00000000	5	x=-0.00000000 y=1.00000000	4	x=0.00000000 y=1.00000000

Example 3.3 Find the solutions of system

$$f(x,y) = x - \cos y - 3.9578 = 0$$
$$g(x,y) = e^{-x} + y + 2.4204 = 0 \tag{3.9}$$
$$\varepsilon < 0.0000001$$

The following matrix is the Jacobian matrix for the system (3.9)

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1 & \sin y \\ -e^{-x} & 1 \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results from Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.3.

Table 3.3 The solution of the example 3.3

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(1,1)	5	x=3.17997969 y=-2.46198650	11	x=3.17997970 y=-2.46198650	6	x=3.17997970 y=-2.46198650
(1,-1)	4	x=3.17997969 y=-2.46198650	11	x=3.17997970 y=-2.46198650	6	x=3.17997970 y=-2.46198650
(2,-1)	4	x=3.17997969 y=-2.46198650	10	x=3.17997967 y=-2.46198649	6	x=3.17997970 y=-2.46198650
(2,-2)	3	x=3.17997973 y=-2.46198644	10	x=3.17997967 y=-2.46198649	6	x=3.17997970 y=-2.46198650
(3,-2)	3	x=3.17997973 y=-2.46198644	10	x=3.17997969 y=-2.46198649	6	x=3.17997970 y=-2.46198650
(3,-3)	3	x=3.17997962 y=-2.46198650	10	x=3.17997969 y=-2.46198650	6	x=3.17997970 y=-2.46198650

Example 3.4 Find the solutions of system

$$
\begin{aligned}
&f(x,y) = \frac{1}{2}\sin xy + \frac{y}{4\pi} - \frac{x}{2} = 0 \\
&g(x,y) = \left(1 - \frac{1}{4\pi}\right)\left(e^{2x} - e\right) + \frac{ey}{\pi} - 2ex = 0 \\
&\varepsilon < 0.0000001
\end{aligned}
\tag{3.10}
$$

The following matrix is the Jacobian matrix for the system (3.10)

$$
J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -0.5 + 0.5y\cos xy & -\dfrac{0.25}{\pi} + 0.5x\cos xy \\ -2e + \left(2 - \dfrac{0.5}{\pi}\right)e^{2x} & \dfrac{e}{\pi} \end{bmatrix}.
$$

Therefore the Newton's method algorithm could be written as follows:

$$
\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots
$$

The following matrix is the matrix H for Modified Newton method

$$
H(x,y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} -0.5 + 0.5y\cos xy & 0 \\ 0 & \dfrac{e}{\pi} \end{bmatrix}.
$$

Therefore the Modified Newton method algorithm could be written as follows:

$$
\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots
$$

or $$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results from Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.4.

Table 3.4 The solution of the example 3.4

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0.5,3)	3	x=0.50000001 y=3.14159267	11	x=0.49999995 y=3.14159270	6	x=0.50000005 y=3.14159268
(0.3,3)	3	x=0.29944869 y=2.83692777	13	x=0.29944868 y=2.83692778	6	x=0.29944876 y=2.83692794
(0.2,2)	4	x=0.29944868 y=2.83692776	15	x=0.29944868 y=2.83692781	7	x=0.29944876 y=2.83692799
(0.2,3)	4	x=0.29944869 y=2.83692777	15	x=0.29944870 y=2.83692781	6	x=0.29944811 y=2.83692805
(0,3)	5	x=0.29944869 y=2.83692777	18	x=0.29944870 y=2.83692781	6	x=0.29944854 y=2.83692741

Example 3.5 Find the solution of the system

$$
\begin{aligned}
f(x,y,z) &= e^x + yz - 3 = 0 \\
g(x,y,z) &= \frac{x}{y} + z^2 - \log y = 0 \\
h(x,y,z) &= \frac{x}{x+y+z} - \sin z = 0 \\
\varepsilon &< 0.0000001
\end{aligned}
\tag{3.11}
$$

The following matrix is the Jacobian matrix for the system (3.11)

$$
J(x,y,z) = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} e^x & z & y \\ \frac{1}{y} & -\frac{x}{y^2} - \frac{1}{y\ln 10} & 2z \\ \frac{y+z}{(x+y+z)^2} & \frac{-x}{(x+y+z)^2} & \frac{-x}{(x+y+z)^2} - \cos z \end{bmatrix}.
$$

Therefore the Newton's method algorithm could be written as follows:

$$
\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [J(x_k,y_k,z_k)]^{-1} \begin{bmatrix} f(x_k,y_k,z_k) \\ g(x_k,y_k,z_k) \\ h(x_k,y_k,z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.
$$

The following matrix is the matrix H for Modified Newton method

$$
H(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ 0 & g_y & 0 \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} e^x & 0 & 0 \\ 0 & -\frac{x}{y^2} - \frac{1}{y\ln 10} & 0 \\ 0 & 0 & \frac{-x}{(x+y+z)^2} - \cos z \end{bmatrix}.
$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - \left[H(x_k, y_k, z_k)\right]^{-1} \begin{bmatrix} f(x_k, y_k, z_k) \\ g(x_k, y_k, z_k) \\ h(x_k, y_k, z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k, z_k)}{f_x(x_k, y_k, z_k)} \\ \dfrac{g(x_k, y_k, z_k)}{g_y(x_k, y_k, z_k)} \\ \dfrac{h(x_k, y_k, z_k)}{h_z(x_k, y_k, z_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.5.

Table 3.5 The solution of the example 3.5

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0,1,0)	40	x=0.87097685 y=2.52834967 z=0.24156325	45	x=0.87097685 y=2.52834975 z=0.24156334	13	x=0.87097677 y=2.52834968 z=0.24156333
(1,1,0)	38	x=0.87097684 y=2.52834967 z=0.24156325	42	x=0.87097677 y=2.52834861 z=0.24156331	11	x=0.87097678 y=2.52834968 z=0.24156333
(0,1,1)	44	x=0.87097669 y=2.52834969 z=0.24156339	43	x=0.87097677 y=2.52834858 z=0.24156331	12	x=0.87097675 y=2.52834969 z=0.24156332
(1,1,1)	39	x=0.87097682 y=2.52834967 z=0.24156327	38	x=0.87097673 y=2.52834983 z=0.24156334	13	x=0.87097677 y=2.52834967 z=0.24156333

Example 3.6 Find the solution of the system

$$\begin{aligned} f(x,y,z) &= x^3 - 10x + y - z + 3 = 0 \\ g(x,y,z) &= y^3 + 10y - 2x - 2z - 5 = 0 \\ h(x,y,z) &= x + y - 10z + 2\sin z + 5 = 0 \\ \varepsilon &< 0.0000001 \end{aligned} \tag{3.12}$$

The following matrix is the Jacobian matrix for the system (3.12)

$$J(x,y,z) = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} 3x^2 - 10 & 1 & -1 \\ -2 & 3y^2 + 10 & -2 \\ 1 & 1 & -10 + 2\cos z \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [J(x_k, y_k, z_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k) \\ g(x_k, y_k, z_k) \\ h(x_k, y_k, z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ 0 & g_y & 0 \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} 3x^2 - 10 & 0 & 0 \\ 0 & 3y^2 + 10 & 0 \\ 0 & 0 & -10 + 2\cos z \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [H(x_k, y_k, z_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k) \\ g(x_k, y_k, z_k) \\ h(x_k, y_k, z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or
$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k, z_k)}{f_x(x_k, y_k, z_k)} \\ \dfrac{g(x_k, y_k, z_k)}{g_y(x_k, y_k, z_k)} \\ \dfrac{h(x_k, y_k, z_k)}{h_z(x_k, y_k, z_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.6.

Table 3.6 The solution of the example 3.6

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0,0,0)	4	x=0.29703611 y=0.67480979 z=0.73065630	11	x=0.29703611 y=0.67480978 z=0.73065630	7	x=0.29703611 y=0.67480979 z=0.73065630
(1,0,0)	4	x=0.29703611 y=0.67480979 z=0.73065630	11	x=0.29703611 y=0.67480979 z=0.73065630	7	x=0.29703611 y=0.67480978 z=0.73065630
(0,1,0)	3	x=0.29703611 y=0.67480979 z=0.73065630	11	x=0.29703611 y=0.67480979y z=0.73065630	7	x=0.29703611 y=0.67480979 z=0.73065630
(0,0,1)	3	x=0.29703611 y=0.67480979 z=0.73065630	11	x=0.29703611 y=0.67480979 z=0.73065630	7	x=0.29703611 y=0.67480979 z=0.73065630
(0,1,1)	3	x=0.29703611 y=0.67480979 z=0.73065630	10	x=0.29703611 y=0.67480979 z=0.73065630	7	x=0.29703612 y=0.67480979 z=0.73065630
(1,1,1)	4	x=0.29703611 y=0.67480979 z=0.73065630	10	x=0.29703611 y=0.67480979 z=0.73065630	7	x=0.29703611 y=0.67480979 z=0.73065630

Example 3.7 Find the solution of the system

$$\begin{aligned} f(x,y,z) &= 7744x^3y + 7744x^3z - 97.336 = 0 \\ g(x,y,z) &= 0.5\log x + 8.75\log y + 4.4 = 0 \\ h(x,y,z) &= 0.5\log x + 8.75\log z + 6.2 = 0 \\ \varepsilon &< 0.0000001 \end{aligned} \tag{3.13}$$

The following matrix is the Jacobian matrix for the system (3.13)

$$J(x,y,z) = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} 23232x^2y + 23232x^2z & 7744x^3 & 7744x^3 \\ \dfrac{0.5}{x\ln 10} & \dfrac{8.75}{y\ln 10} & 0 \\ \dfrac{0.5}{x\ln 10} & 0 & \dfrac{8.75}{z\ln 10} \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [J(x_k,y_k,z_k)]^{-1} \begin{bmatrix} f(x_k,y_k,z_k) \\ g(x_k,y_k,z_k) \\ h(x_k,y_k,z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ 0 & g_y & 0 \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} 23232x^2y + 23232x^2z & 0 & 0 \\ 0 & \dfrac{8.75}{y\ln 10} & 0 \\ 0 & 0 & \dfrac{8.75}{z\ln 10} \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [H(x_k,y_k,z_k)]^{-1} \begin{bmatrix} f(x_k,y_k,z_k) \\ g(x_k,y_k,z_k) \\ h(x_k,y_k,z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

$$\text{or} \qquad \begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k,y_k,z_k)}{f_x(x_k,y_k,z_k)} \\ \dfrac{g(x_k,y_k,z_k)}{g_y(x_k,y_k,z_k)} \\ \dfrac{h(x_k,y_k,z_k)}{h_z(x_k,y_k,z_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.7.

Table 3.7 The solution of the example 3.7

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0.1, 0.1, 0.1)	7	x=0.28416236 y=0.33757294 z=0.21020998	17	x=0.28416236 y=0.33757294 z=0.21020998	12	x=0.28416236 y=0.33757294 z=0.21020998
(0.2, 0.2, 0.2)	5	x=0.28416236 y=0.33757294 z=0.21020998	12	x=0.28416236 y=0.33757294 z=0.21020998	8	x=0.28416236 y=0.33757294 z=0.21020998
(0.2, 0.3, 0.2)	5	x=0.28416236 y=0.33757294 z=0.21020998	12	x=0.28416236 y=0.33757294 z=0.21020998	7	x=0.28416236 y=0.33757294 z=0.21020998
(0.25, 0.3, 0.2)	4	x=0.28416236 y=0.33757294 z=0.21020998	10	x=0.28416236 y=0.33757294 z=0.21020998	7	x=0.28416236 y=0.33757294 z=0.21020998
(0.25, 0.35, 0.2)	4	x=0.28416236 y=0.33757294 z=0.21020998	10	x=0.28416236 y=0.33757294 z=0.21020998	6	x=0.28416236 y=0.33757294 z=0.21020998

Example 3.8 Find the solution of the system

$$\begin{aligned} f(x,y,z) &= \ln x - \cos^2 y - \sin 2z + 3.0309 = 0 \\ g(x,y,z) &= e^x + \ln y + z - 0.5875 = 0 \\ h(x,y,z) &= \sin 3x - e^{-y} + \ln z + 0.8458 = 0 \\ \varepsilon &< 0.0000001 \end{aligned} \tag{3.14}$$

The following matrix is the Jacobian matrix for the system (3.14)

$$J(x,y,z) = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & 2\cos y \sin y & -2\cos 2z \\ e^x & \frac{1}{y} & 1 \\ 3\cos 3x & e^{-y} & \frac{1}{z} \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [J(x_k, y_k, z_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k) \\ g(x_k, y_k, z_k) \\ h(x_k, y_k, z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\dots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y,z) = \begin{bmatrix} f_x & 0 & 0 \\ 0 & g_y & 0 \\ 0 & 0 & h_z \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [H(x_k, y_k, z_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k) \\ g(x_k, y_k, z_k) \\ h(x_k, y_k, z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\dots$$

or
$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k, z_k)}{f_x(x_k, y_k, z_k)} \\ \dfrac{g(x_k, y_k, z_k)}{g_y(x_k, y_k, z_k)} \\ \dfrac{h(x_k, y_k, z_k)}{h_z(x_k, y_k, z_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.8.

Table 3.8 The solution of the example 3.8

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0.1, 0.2, 0.3)	4	x=0.25799717 y=0.31700946 z=0.44198851	19	x=0.25799719 y=0.31700945 z=0.44198851	11	x=0.25799718 y=0.31700947 z=0.44198851
(0.2, 0.3, 0.4)	3	x=0.25799717 y=0.31700946 z=0.44198851	16	x=0.25799717 y=0.31700945 z=0.44198850	11	x=0.25799718 y=0.31700946 z=0.44198851
(0.3, 0.4, 0.5)	3	x=0.25799718 y=0.31700944 z=0.44198851	18	x=0.25799718 y=0.31700946 z=0.44198851	11	x=0.25799718 y=0.31700945 z=0.44198852
(0.3, 0.3, 0.6)	4	x=0.25799718 y=0.31700945 z=0.44198851	18	x=0.25799717 y=0.31700945 z=0.44198850	11	x=0.25799719 y=0.31700945 z=0.44198850
(0.25, 0.31, 0.44)	3	x=0.25799718 y=0.31700945 z=0.44198851	15	x=0.25799718 y=0.31700945 z=0.44198850	8	x=0.25799718 y=0.31700947 z=0.44198851
(0.26, 0.32, 0.44)	2	x=0.25799718 y=0.31700945 z=0.44198851	14	x=0.25799717 y=0.31700947 z=0.44198852	9	x=0.25799718 y=0.31700945 z=0.44198851

Example 3.9 Find the solution of the system

$$
\begin{aligned}
f(a,b,c,d) &= 137a + 55b + 20c - 76 = 0 \\
g(a,b,c,d) &= 76b - 21bd - 19 = 0 \\
h(a,b,c,d) &= a + b + c - 1 = 0 \\
k(a,b,c,d) &= 76a + 61ad - 38 = 0 \\
\varepsilon &< 0.0000001
\end{aligned}
\tag{3.15}
$$

The following matrix is the Jacobian matrix for the system (3.15)

$$
J(a,b,c,d) = \begin{bmatrix} f_a & f_b & f_c & f_d \\ g_a & g_b & g_c & g_d \\ h_a & h_b & h_c & h_d \\ k_a & k_b & k_c & k_d \end{bmatrix} = \begin{bmatrix} 137 & 55 & 20 & 0 \\ 0 & 76-21d & 0 & 21b \\ 1 & 1 & 1 & 0 \\ 76-61d & 0 & 0 & 61a \end{bmatrix}.
$$

Therefore the Newton's method algorithm could be written as follows:

$$
\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix} - [J(a_k,b_k,c_k,d_k)]^{-1} \begin{bmatrix} f(a_k,b_k,c_k,d_k) \\ g(a_k,b_k,c_k,d_k) \\ h(a_k,b_k,c_k,d_k) \\ k(a_k,b_k,c_k,d_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.
$$

The following matrix is the matrix H for Modified Newton method

$$
H(a,b,c,d) = \begin{bmatrix} f_a & 0 & 0 & 0 \\ 0 & g_b & 0 & 0 \\ 0 & 0 & h_c & 0 \\ 0 & 0 & 0 & k_d \end{bmatrix} = \begin{bmatrix} 137 & 0 & 0 & 0 \\ 0 & 76-21d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 61a \end{bmatrix}.
$$

Therefore the Modified Newton method algorithm could be written as follows:

$$
\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix} - [H(a_k,b_k,c_k,d_k)]^{-1} \begin{bmatrix} f(a_k,b_k,c_k,d_k) \\ g(a_k,b_k,c_k,d_k) \\ h(a_k,b_k,c_k,d_k) \\ k(a_k,b_k,c_k,d_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots
$$

or

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(a_k,b_k,c_k,d_k)}{f_a(a_k,b_k,c_k,d_k)} \\ \dfrac{g(a_k,b_k,c_k,d_k)}{g_b(a_k,b_k,c_k,d_k)} \\ \dfrac{h(a_k,b_k,c_k,d_k)}{h_c(a_k,b_k,c_k,d_k)} \\ \dfrac{k(a_k,b_k,c_k,d_k)}{k_d(a_k,b_k,c_k,d_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.9.

Table 3.9 The solution of the example 3.9

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(1,0, 0,0)	17	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	33	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	20	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000
(1,0, 1,0)	17	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	34	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	19	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000
(1,1, 1,1)	18	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	38	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	21	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000
(0.3, 0.2, 0.3, 0.3)	17	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	35	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000	19	a=0.39645780 b=0.27469823 c=0.32884398 d=0.32539000

3.2 Some Applied Examples of Systems of Nonlinear Equations

Example 3.10 Figure 3.2 (a) shows a two-link robot arm. One end of the arm is fixed at the origin of the coordinate system, and the other end is at coordinate (x, y). There are positioning actuators control the link angles θ and ϕ, as shown. In terms of θ and ϕ, the coordinates of the joint between links are (a, b), where

$$a = 0.5\cos\theta$$
$$b = 0.5\sin\theta.$$

The coordinates (x, y) are given by

$$x - a = 0.35\cos(\phi + \theta)$$
$$y - b = 0.35\sin(\phi + \theta)$$

or

$$x = 0.5\cos\theta + 0.35\cos(\phi + \theta) = 0$$
$$y = 0.5\sin\theta + 0.35\sin(\phi + \theta) = 0.$$

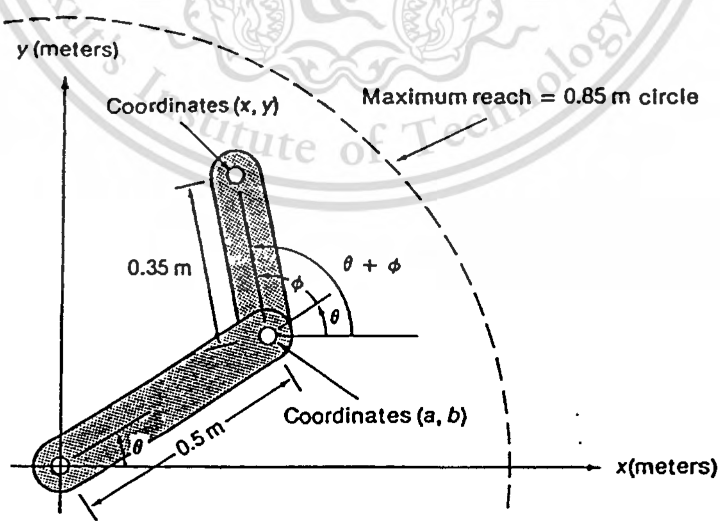
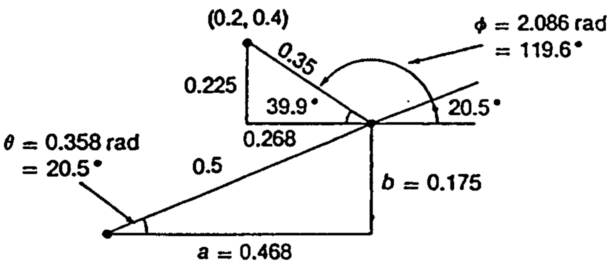
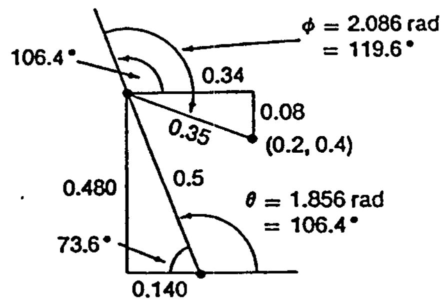


Figure 3.2 Positioning a robot arm.

(a) Robot arm geometry.



(b) First solution



(c) Second solution

The angles θ and ϕ that will position the end of the arm at coordinates $(0.2,0.4)$ are given by simultaneous solution of

$$\begin{aligned} f(\theta,\phi) &= 0.5\cos\theta + 0.35\cos(\phi+\theta) - 0.2 = 0 \\ g(\theta,\phi) &= 0.5\sin\theta + 0.35\sin(\phi+\theta) - 0.4 = 0 \end{aligned} \tag{3.16}$$

or

$$\begin{aligned} f(x,y) &= 0.5\cos x + 0.35\cos(x+y) - 0.2 = 0 \\ g(x,y) &= 0.5\sin x + 0.35\sin(x+y) - 0.4 = 0. \end{aligned}$$

The following matrix is the Jacobian matrix for the system (3.16)

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -0.5\sin x - 0.35\sin(x+y) & -0.35\sin(x+y) \\ 0.5\cos x + 0.35\cos(x+y) & 0.35\cos(x+y) \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k,y_k)]^{-1} \begin{bmatrix} f(x_k,y_k) \\ g(x_k,y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} -0.5\sin x - 0.35\sin(x+y) & 0 \\ 0 & 0.35\cos(x+y) \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results from Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.10.

Table 3.10 The solution of the example 3.10

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(1,-1)	6	x=1.85623372 y=-2.08616669	18	x=1.85623364 y=-2.08616668	11	x=1.85623376 y=-8.36935203
(1.5, -2.1)	4	x=1.85623372 y=-2.08616669	14	x=1.85623375 y=-2.08616679	7	x=1.85623385 y=-2.08616677
(3,-2)	6	x=1.85623372 y=-2.08616669	15	x=1.85623371 y=-2.08616675	8	x=1.85623381 y=-2.08616675
(1,2)	5	x=0.35806372 y=2.08616669	32	x=0.35806386 y=2.08616661	16	x=0.35806360 y=2.08616661
(0,1)	>1000	-	34	x=0.35806384 y=2.08616647	20	x=0.35806364 y=2.08616664

There are two sets of link angles that will position the arm as desired, $\theta \approx 0.358$, $\phi \approx 2.086$ and $\theta \approx 1.856$, $\phi \approx -2.086$. The geometry of these solutions is shown in Figure 3.2 (b) and (c) respectively. The existence of multiple solutions, as in this case, is one of the complications of robotics.

Example 3.11 To maximize the cross-sectional area of a rectangular beam cut from a circular log with $r = \sqrt{2}$ as the given radius of the log.

With the coordinate system indicated in Figure 3.3, we want to maximize the area $A = f(x,y) = 4xy$ of the beam's rectangular cross section subject to the constraint $g(x,y) = x^2 + y^2 - 2 = 0$ that describes the circular log, then we have system

$$\begin{aligned} f(x,y) &= 4xy = 0 \\ g(x,y) &= x^2 + y^2 - 2 = 0 \end{aligned} \tag{3.17}$$

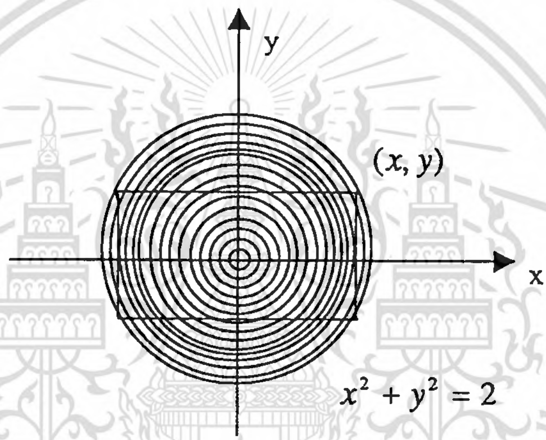


Figure 3.3 Cutting a rectangular beam from a circular log.

The following matrix is the Jacobian matrix for system (3.17)

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 4y & 4x \\ 2x & 2y \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,....$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} 4y & 0 \\ 0 & 2y \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.11.

Table 3.11 The solution of the example 3.11

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0,1)	4	x=0.00000000 y=1.41421356	4	x=0.00000000 y=1.41421356	4	x=0.00000000 y=1.41421356
(0.5,1)	5	x=-0.00000000 y=1.41421356	4	x=0.00000000 y=1.41421356	4	x=0.00000000 y=1.41421356
(0.5,-1)	5	x=-0.00000000 y=-1.41421356	4	x=0.00000000 y=-1.41421356	4	x=0.00000000 y=-1.41421356
(-0.5, -1)	5	x=0.00000000 y=-1.41421356	4	x=0.00000000 y=-1.41421356	4	x=0.00000000 y=-1.41421356
(-1.2,2)	5	x=-0.00000000 y=1.41421356	5	x=0.00000000 y=1.41421356	4	x=0.00000000 y=1.41421356

Example 3.12 Find the highest point on the surface

$$z = \frac{8}{3}x^3 + 4y^3 - x^4 - y^4. \tag{3.18}$$

Because the partial derivative of z with respect to x and y exist everywhere, then we can solve the equations $z_x = 0$ and $z_y = 0$, that is

$$z_x(x,y) = 8x^2 - 4x^3 = 0$$

and

$$z_y(x,y) = 12y^2 - 4y^3 = 0.$$

Let $f(x,y) = z_x(x,y)$ and $g(x,y) = z_y(x,y)$, then we have system of nonlinear equations as

$$\begin{aligned} f(x,y) &= 8x^2 - 4x^3 = 0 \\ g(x,y) &= 12y^2 - 4y^3 = 0. \end{aligned} \tag{3.19}$$

The following matrix is the Jacobian matrix for system (3.19)

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 16x - 12x^2 & 0 \\ 0 & 24y - 12y^2 \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots .$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} 16x - 12x^2 & 0 \\ 0 & 24y - 12y^2 \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.12.

Table 3.12 The solution of the example 3.12

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(9,6)	23	x=2.00000000 y=3.00000000	23	x=2.00000000 y=3.00000000	23	x=2.00000000 y=3.00000000
(3,4)	19	x=2.00000000 y=3.00000000	19	x=2.00000000 y=3.00000000	19	x=2.00000000 y=3.00000000
(0.2, -0.1)	11	x=0.00008268 y=-0.00005044	11	x=0.00008268 y=-0.00005044	11	x=0.00008268 y=-0.00005044
(0.2,1)	13	x=0.00002067 y=-0.00007195	13	x=0.00002067 y=-0.00007195	13	x=0.00002067 y=-0.00007195
(3,1)	19	x=2.00000000 y=0.00000112	19	x=2.00000000 y=0.00000112	19	x=2.00000000 y=0.00000112
(2,0.1)	11	x=2.00000000 y=0.00004718	11	x=2.00000000 y=0.00004718	11	x=2.00000000 y=0.00004718
(0.2,4)	11	x=0.00008268 y=3.00000000	11	x=0.00008268 y=3.00000000	11	x=0.00008268 y=3.00000000
(0.2, 2.5)	11	x=0.00008268 y=3.00000000	11	x=0.00008268 y=3.00000000	11	x=0.00008268 y=3.00000000

It follow that either four critical points $(0,0)$, $(0,3)$, $(2,0)$ and $(2,3)$ on the surface. Consequently we need only inspect the values

$$z(0,0) = 0,$$
$$z(0,3) = 27,$$
$$z(2,0) = \frac{16}{3} = 5.333...$$
$$z(2,3) = \frac{97}{3} = 32.333....$$

Thus the highest point on the surface is the point $(2,3,\frac{97}{3})$.

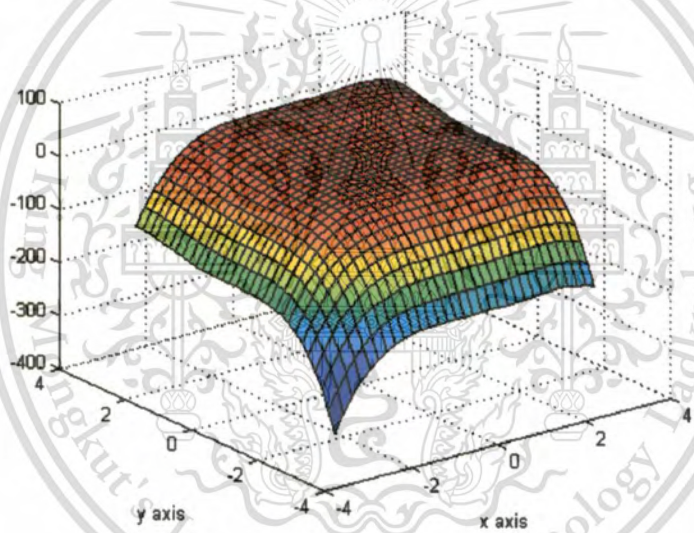


Figure 3.4 The surface $z = \frac{8}{3}x^3 + 4y^3 - x^4 - y^4$.

Example 3.13 Find the lowest point on the surface

$$z = x^4 + y^4 - x^2 y^2 . \tag{3.20}$$

Because the partial derivative of z with respect to x and y exist everywhere, then we can solve the equations $z_x = 0$ and $z_y = 0$, that is

$$z_x(x, y) = 4x^3 - 2xy^2 = 0$$

and

$$z_y(x, y) = 4y^3 - 2x^2 y = 0 .$$

Let $f(x, y) = z_x(x, y)$ and $g(x, y) = z_y(x, y)$, then we have system of nonlinear equations as

$$\begin{aligned} f(x, y) &= 4x^3 - 2xy^2 = 0 \\ g(x, y) &= 4y^3 - 2x^2 y = 0. \end{aligned} \tag{3.21}$$

The following matrix is the Jacobian matrix for system (3.21)

$$J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 12x^2 - 2y^2 & -4xy \\ -4xy & 12y^2 - 2x^2 \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [J(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x, y) = \begin{bmatrix} f_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} 12x^2 - 2y^2 & 0 \\ 0 & 12y^2 - 2x^2 \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - [H(x_k, y_k)]^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots$$

or

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} \dfrac{f(x_k, y_k)}{f_x(x_k, y_k)} \\ \dfrac{g(x_k, y_k)}{g_y(x_k, y_k)} \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.13.

Table 3.13 The solution of the example 3.13

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(0.2, 2.5)	17	x=0.00020299 y=0.00253740	17	x=0.00000000 y=0.00254011	17	x=0.00000000 y=0.00253740
(8,35)	24	x=0.00047523 y=0.00207911	24	x=0.00000000 y=0.00210114	24	x=0.00000000 y=0.00208080
(-1,-9)	20	x=-0.00030073 y=-0.00270656	20	x=0.00000000 y=-0.00271217	20	x=0.00000000 y=-0.00270657
(-1,7)	20	x=-0.00030073 y=0.00210510	20	x=0.00000000 y=0.00211242	20	x=0.00000000 y=0.00210516
(35, 125)	27	x=0.00061603 y=0.00220012	27	x=0.00000000 y=0.00225910	27	x=0.00000000 y=0.00221371
(-2,-1)	17	x=-0.00202992 y=-0.00101496	30	x=-0.00277159 y=-0.00277159	24	x=-0.00255616 y=-0.00223417

It follow that critical points is $(0,0)$ on the surface. Consequently we need only inspect the values $z(0,0)=0$. Thus the lowest point on the surface is the point $(0,0,0)$.

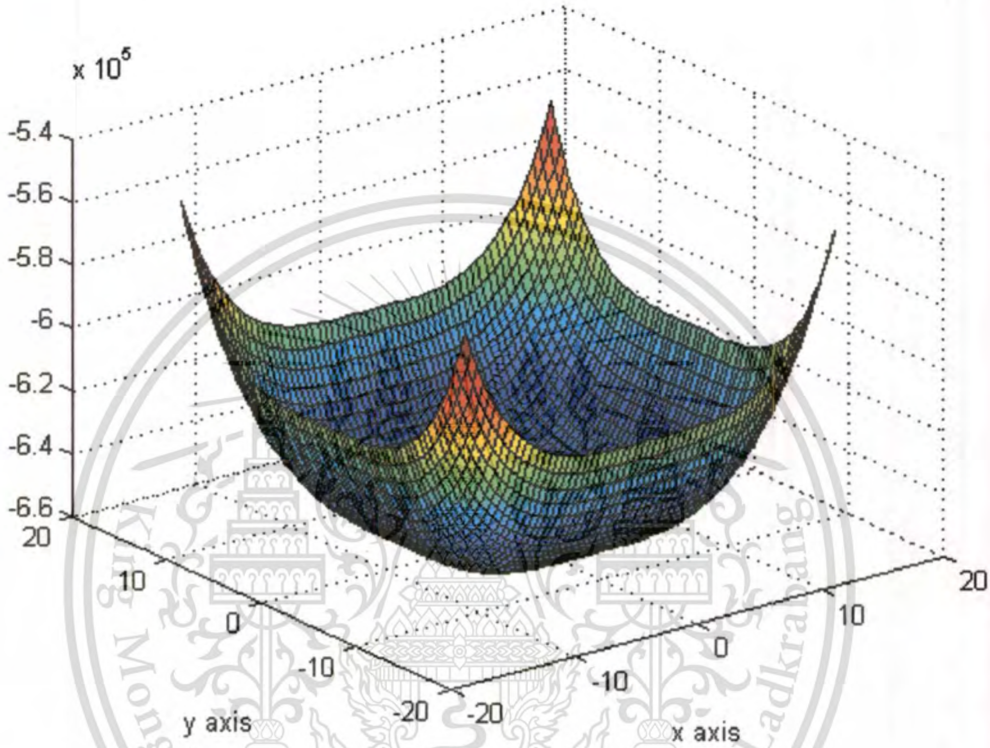


Figure 3.5 The surface $z=x^4+y^4-x^2y^2$.

Example 3.14 Flow in a pipe network.

A network consists of a number of horizontal pipes, of specified diameters and lengths that are joined at n nodes, numbered $i = 1,2,\ldots,n$. The pressure is specified at some of these nodes. There is at most a single pipe connected directly between any two nodes.

For flow of a liquid from point i to point j in a horizontal pipe, the pressure drop is given by the *Fanning* equation:

$$p_i - p_j = \frac{1}{2} f_M \rho u_m^2 \frac{L}{D}. \tag{3.22}$$

Here, f_M is the dimensionless *Moody* friction factor, ρ is the liquid density, u_m is the mean velocity, L and D are the length and diameter of the pipe, respectively. Since the volumetric flow rate is $Q = \left(\frac{\pi D^2}{4}\right) u_m$, equation (3.22) becomes

$$p_i - p_j = \frac{8 f_M \rho Q^2 L}{\pi^2 D^5}.$$

Here, all quantities are in consistent units. However, if p_i and p_j are expressed in psi ($lb\, sq\, in.$), ρ in $lb_m / cu\, ft.$, Q in *gpm* (gallons/min), L in *ft.*, and D in *inches*, we obtain

$$p_i - p_j = C \frac{LQ^2}{D^5} \tag{3.23}$$

where

$$C = \frac{8 \times 12^5}{\pi^2 \times 144 \times 32.2 \times (7.48 \times 60)^2} f_M \rho. \tag{3.24}$$

Let $c_{ij} = \frac{CL_{ij}}{D_{ij}^5}$, where the subscripts ij now emphasize that we are concerned with the pipe joining nodes i and j. The flow rate, Q_{ij} between nodes i and j is then given by

$$|p_i - p_j| = c_{ij} Q_{ij}^2, \tag{3.25}$$

in which Q_{ij} is plus or minus for flow from i to j or vice versa, respectively. In the following version, Q_{ij} will automatically have the correct sign:

$$Q_{ij} = (p_i - p_j)\sqrt{\frac{1}{c_{ij}|p_i - p_j|}}.$$

At any *free* node j, where the pressure is not specified, the sum of the flows from neighboring nodes i must be zero:

$$\sum_i Q_{ij} = \sum_i (p_i - p_j)\sqrt{\frac{1}{c_{ij}|p_i - p_j|}} = 0. \tag{3.26}$$

When applied at all the free nodes, equation (3.26) yields a system of nonlinear simultaneous equations in the unknown pressures. We shall solve this system by Newton's method, Modified Newton method and improve Modified Newton method. First, note that $(p_i - p_j)$ is more sensitive than $(|p_i - p_j|)^{1/2}$ to variations in p_j. Thus an appropriate version is

$$p_j = \frac{\sum_i a_{ij} p_i}{\sum_i a_{ij}}, \tag{3.27}$$

in which

$$a_{ij} = \left(c_{ij}|p_i - p_j|\right)^{-1/2}. \tag{3.28}$$

Equation (3.27) is applied repeatedly at all free nodes unite either each computed pressure p_j does not change by more than a small amount ε from one iteration to the next or a preassigned number of iterations, *itmax*, has been exceeded. The most recent estimated values of p_i will always be used in the right-hand side of equation (3.27).

The data used above relate to the network shown in Figure 3.6, with $f_M = 0.056$, $\rho = 50\ lb_m / cu\ ft.$, and two pressure fixed; $p_1 = 50$, $p_3 = 0$ psi.

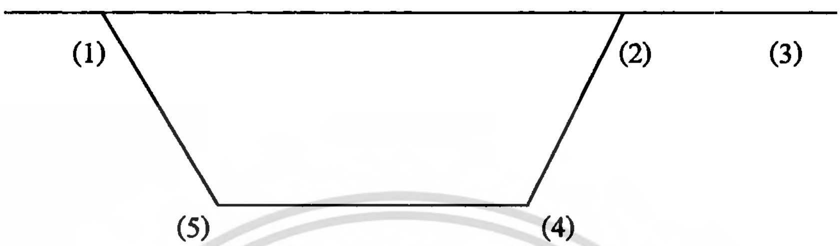


Figure 3.6 Pipe network for calculation.

Although the method is computationally straight forward, it need many iterations to give a reasonable degree of convergence. Also, referring to equation (3.28), we can see that a starting guess of $p_i = p_j$ for any two nodes that are directly connected would be unfortunate.

Note that the bulk of the pressure drop occurs in the pipe 2-3, and that the flow in the branch 1-5-4-2 is appreciably greater than that in the pipe 1-2, even though the latter is much shorter. Both these observations can be reconciled by noting that pressure drop is proportional to $\frac{Q^2}{D^5}$, and that pipe 2-3 must take the combined flow along 1-5-4-2 and 1-2.

At point 5; $$\frac{p_5 - p_4}{\sqrt{|p_5 - p_4|}} + \frac{p_5 - p_1}{\sqrt{|p_5 - p_1|}} = 0$$

$$\frac{p_5 - p_4}{\sqrt{|p_5 - p_4|}} - \frac{50 - p_5}{\sqrt{|50 - p_5|}} = 0$$

$$p_4 - 2p_5 + 50 = 0. \qquad (3.29)$$

At point 4; $$\frac{p_4 - p_5}{\sqrt{|p_4 - p_5|}} + \frac{p_4 - p_2}{\sqrt{|p_4 - p_2|}} = 0$$

$$\frac{p_4 - p_2}{\sqrt{|p_4 - p_2|}} - \frac{p_5 - p_4}{\sqrt{|p_5 - p_4|}} = 0$$

$$p_2 - 2p_4 + p_5 = 0\,. \tag{3.30}$$

At point 2; $$\frac{p_2 - p_3}{\sqrt{|p_2 - p_3|}} + \frac{p_2 - p_1}{\sqrt{|p_2 - p_1|}} + \frac{p_2 - p_4}{\sqrt{|p_2 - p_4|}} = 0$$

$$\frac{p_2}{\sqrt{|p_2|}} - \frac{50 - p_2}{\sqrt{|50 - p_2|}} - \frac{p_4 - p_2}{\sqrt{|p_4 - p_2|}} = 0$$

$$5p_2^2 - 100p_2 - 2p_2p_4 - 100p_4 + p_4^2 + 2500 = 0\,. \tag{3.31}$$

From (3.29), (3.30) and (3.31), let $x = p_5$, $y = p_4$ and $z = p_2$, then we obtain the system

$$\begin{aligned} f(x,y,z) &= 2x - y - 50 = 0 \\ g(x,y,z) &= x - 2y + z = 0 \\ h(x,y,z) &= y^2 - 100y - 2yz - 100z + 5z^2 + 2500 = 0. \end{aligned} \tag{3.32}$$

The Jacobian matrix for this system could be written as follows:

$$J(a,b,c) = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2y - 2z - 100 & -2y + 10z - 100 \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} - [J(x_k, y_k, z_k)]^{-1} \begin{bmatrix} f(x_k, y_k, z_k) \\ g(x_k, y_k, z_k) \\ h(x_k, y_k, z_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(x,y,z)=\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2y+10z-100\end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix}x_{k+1}\\ y_{k+1}\\ z_{k+1}\end{bmatrix}=\begin{bmatrix}x_k\\ y_k\\ z_k\end{bmatrix}-[H(x_k,y_k,z_k)]^{-1}\begin{bmatrix}f(x_k,y_k,z_k)\\ g(x_k,y_k,z_k)\\ h(x_k,y_k,z_k)\end{bmatrix} \quad ; \quad k=0,1,2,\ldots$$

or

$$\begin{bmatrix}x_{k+1}\\ y_{k+1}\\ z_{k+1}\end{bmatrix}=\begin{bmatrix}x_k\\ y_k\\ z_k\end{bmatrix}-\begin{bmatrix}\dfrac{f(x_k,y_k,z_k)}{f_x(x_k,y_k,z_k)}\\ \dfrac{g(x_k,y_k,z_k)}{g_y(x_k,y_k,z_k)}\\ \dfrac{h(x_k,y_k,z_k)}{h_z(x_k,y_k,z_k)}\end{bmatrix} \quad ; \quad k=0,1,2,\ldots.$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.14.

Table 3.14 The solution of the example 3.14

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(5,12, 25)	6	x=45.22175894 y=40.44351788 z=35.66527682	78	x=45.22175894 y=40.44351788 z=35.66527682	30	x=45.22175890 y=40.44351784 z=35.66527680
(15,45, 45)	5	x=45.22175894 y=40.44351788 z=35.66527682	71	x=45.22175894 y=40.44351788 z=35.66527682	29	x=45.22175897 y=40.44351791 z=35.66527684

Table 3.14 (Continued)

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(35,30, 45)	5	x=45.22175894 y=40.44351788 z=35.66527686	68	x=45.22175894 y=40.44351788 z=35.66527682	28	x=45.22175898 y=40.44351792 z=35.66527684
(56,25, 45)	5	x=45.22175894 y=40.44351788 z=35.66527682	71	x=45.22175894 y=40.44351788 z=35.66527682	27	x=45.22175899 y=40.44351793 z=35.66527684
(135, 122, 110)	7	x=45.22175894 y=40.44351788 z=35.66527682	69	x=45.22175894 y=40.44351788 z=35.66527682	32	x=45.22175899 y=40.44351793 z=35.66527684
(545, -340, 235)	9	x=45.22175894 y=40.44351788 z=35.66527682	75	x=45.22175894 y=40.44351788 z=35.66527682	33	x=45.22175899 y=40.44351793 z=35.66527685

Example 3.15 This application is taken from the Bohr theory of the hydrogen spectrum in atomic physics [7]. A certain constant know as the Rydberg constant, usually denoted by R_∞, plays an important part in that theory, but it cannot be measured directly. The value must be inferred from measurement of other constants. Approximate values R_H and R_{He} can be determined from measurements of the spectrum of hydrogen and helium, respectively. If we knew the exact mass M_H of the nucleus of a hydrogen atom and the exact mass m of an electron, then theoretically, we could determine R_∞ from the equation

$$R_\infty = \left(1+\frac{m}{M_H}\right)R_H .$$

Suppose, for the moment, that m and M_H are unknown (because even our best values are inferred from other measured variables). The atomic weight A_H of hydrogen can be measured. Also, $A_H = M_H + m$. Similar equations hold for the helium atom. So we have a system of four equations:

$$R_\infty = \left(1+\frac{m}{M_H}\right)R_H \qquad R_\infty = \left(1+\frac{m}{M_{He}}\right)R_{He}$$

$$A_H = M_H + m \qquad A_{He} = M_{He} + m .$$

In this system, the numbers A_H, A_{He}, R_H and R_{He} can be taken as known because their values can be determined from laboratory measurements. The number R_∞ and masses m, M_H and M_{He} will be regarded as unknown. Their values can be determined by solving the system of four equations.

The work is simplified if we first make some algebraic changes in the system. Let $a = R_\infty$, $b = m$, $c = M_H$ and $d = M_{He}$. Then our system becomes

$$a = \left(1 + \frac{b}{d}\right) R_{He}$$
$$a = \left(1 + \frac{b}{c}\right) R_H$$
$$b + c = A_H$$
$$2b + d = A_{He}$$

or

$$\begin{aligned} f(a,b,c,d) &= a - \left(1 + \frac{b}{d}\right) R_{He} = 0 \\ g(a,b,c,d) &= a - \left(1 + \frac{b}{c}\right) R_H = 0 \\ h(a,b,c,d) &= b + c - A_H = 0 \\ k(a,b,c,d) &= 2b + d - A_{He} = 0. \end{aligned} \tag{3.33}$$

Suppose laboratory measurements produced the following values: $A_H = 1.00812$, $A_{He} = 4.00388$, $R_H = 109677.68$ and $R_{He} = 109722.34$. What are the corresponding values for Rydberg constant R_∞ and mass m, M_H and M_{He}?

The following matrix is the Jacobian matrix for the system (3.33)

$$J(a,b,c,d) = \begin{bmatrix} f_a & f_b & f_c & f_d \\ g_a & g_b & g_c & g_d \\ h_a & h_b & h_c & h_d \\ k_a & k_b & k_c & k_d \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{-R_{He}}{d} & 0 & \dfrac{bR_{He}}{d^2} \\ 1 & \dfrac{-R_H}{c} & \dfrac{bR_H}{c^2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}.$$

Therefore the Newton's method algorithm could be written as follows:

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix} - \left[J(a_k, b_k, c_k, d_k)\right]^{-1} \begin{bmatrix} f(a_k, b_k, c_k, d_k) \\ g(a_k, b_k, c_k, d_k) \\ h(a_k, b_k, c_k, d_k) \\ k(a_k, b_k, c_k, d_k) \end{bmatrix} \quad ; \quad k = 0,1,2,\ldots.$$

The following matrix is the matrix H for Modified Newton method

$$H(a,b,c,d)=\begin{bmatrix} f_a & 0 & 0 & 0 \\ 0 & g_b & 0 & 0 \\ 0 & 0 & h_c & 0 \\ 0 & 0 & 0 & k_d \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-R_H}{c} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the Modified Newton method algorithm could be written as follows:

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \\ d_{k+1} \end{bmatrix}=\begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix}-\left[H(a_k,b_k,c_k,d_k)\right]^{-1}\begin{bmatrix} f(a_k,b_k,c_k,d_k) \\ g(a_k,b_k,c_k,d_k) \\ h(a_k,b_k,c_k,d_k) \\ k(a_k,b_k,c_k,d_k) \end{bmatrix} \quad ; \quad k=0,1,2,...$$

or

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \\ d_{k+1} \end{bmatrix}=\begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix}-\begin{bmatrix} \dfrac{f(a_k,b_k,c_k,d_k)}{f_a(a_k,b_k,c_k,d_k)} \\ \dfrac{g(a_k,b_k,c_k,d_k)}{g_b(a_k,b_k,c_k,d_k)} \\ \dfrac{h(a_k,b_k,c_k,d_k)}{h_c(a_k,b_k,c_k,d_k)} \\ \dfrac{k(a_k,b_k,c_k,d_k)}{k_d(a_k,b_k,c_k,d_k)} \end{bmatrix} \quad ; \quad k=0,1,2,....$$

We obtain the results by using Newton's method, Modified Newton method and Improved Modified Newton method as follows in the table 3.15.

Table 3.15 The solution of the example 3.15

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(-10, 0.2, -1.3, 0.4)	9	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327	40	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327	21	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327

Table 3.15 (Continued)

Initial vector	Newton's method		Modified Newton method		Improved Modified Newton method	
	Iterations	Solution	Iterations	Solution	Iterations	Solution
(10550 ,0, 1, 4)	5	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327	43	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327	16	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327
(-150, 3, 1.5, 6.5)	6	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327	50	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327	22	a=109737.3715 b=0.00054837 c=1.00757163 d=4.00278327

For this problem, we obtain that

$R_{\infty} = 109737.37$

electron mass: $m = 0.00054837$

mass of hydrogen nucleus: $M_H = 1.00757163$

mass of helium nucleus: $M_{He} = 4.00278327$

hydrogen mass / electron mass: $\frac{M_H}{m} = 1837.41$.

The value of one atomic mass unit is 1.6599×10^{-24} g, so the computed mass of the electron in this case is equivalent to

$$0.00054837 \times 1.6599 \times 10^{-24} = 9.103 \times 10^{-28} \text{ g}.$$

The accepted value is 9.11×10^{-28} g, so the computed value is quite satisfactory.

CHAPTER 4

CONCLUSION AND SUGGESTIONS

4.1 Conclusion

The high effective and accurate method for finding the solutions of systems of nonlinear equations, $f(x)=0$ is to find a proper iterative algorithms, Newton's method, for instance. The classical Newton's method is a well-known method, which has been widely used. Its procedure starts by estimating an initial vector, then using iterative algorithm of the method to enable the initial vector to converge to the exact value of the desired solution in a finite number of steps.

Newton's method is one of the basic iteration procedures used to refine an approximate solution to the system of nonlinear equations $f(x)=0$. However, it is time-consuming because it requires the calculation of the approximate inverse of the Jacobian matrix $J(x^k)$ recalculated for each iteration.

This research study is known as a simplified Newton-Like method i.e. Modified Newton method that needs to calculate the inverse of the diagonal matrix $H(x^k)$, which is easier and faster than calculating the inverse of the Jacobian matrix $J(x^k)$. Although this method has a greater number of iteration in converging to the solution than the classical scheme (Newton's method), this method takes a shorter computation time than the classical scheme for each iteration because there is less multiplied operator in computing the inverse of the Jacobian matrix.

An important question to be asked in using any iterative method is when will be stop. For the algorithms, an absolute tolerance or absolute error; ε is set, and iterated until either $\sum_{i,j=1}^{n}\left|f_i(x_j)-f_i(x_{j-1})\right|<\varepsilon$, or a maximum number of iterations is reached. It may happen that if the smaller the tolerance is, the closer the value is to its solution. In this study, it is assume that the absolute error is $\varepsilon < 0.0000001$ for all examples.

The order of convergence of Newton's method is 2. For the Modified Newton method which is a Newton-Like method together with the condition

$$\|M(x)\| \le \delta_1 \|x - x_0\| \quad , \quad \forall x \in D ,$$

the order of convergence is also 2. Without this condition, the order of convergence of the Modified Newton method is linear.

The Modified Newton method does not need matrix computations. Therefore, it causes the Modified Newton method to be more interesting and attractive than the Newton's method and other Newton-Like methods because the later methods need to compute the inverse of Jacobian matrix or the inverse of the matrix $\acute{A}$ respectively.

4.2 Suggestions

The inverse diagonal matrix $H(x^k)$ in each step of Modified Newton method may also be determined. We can improve this method in the new form, for example, any elements of upper diagonal or lower diagonal in the matrix $H(x^k)$ can be increased and then the inverse of these new matrices can be determined. These new matrices can be used to replace matrix $H(x^k)$ for solving systems of nonlinear equations in the form of Modified Newton method. They may converge to the solution faster than using matrix $H(x^k)$.

For the complicated function or function which is hard to find its derivative, this method may be modified by using

$$f_{ij} = \frac{f_i(x^{k+1}) - f_i(x^k)}{x_j^{k+1} - x_j^k} \quad , \; i, j = 1,2,\dots \; , \; k = 0,1,2,\dots .$$

It is clearly seen that this method need not find the derivative of functions in computing the Jacobian matrix.

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