MATRIX TRANSFORMATIONS OF BOUNDED VARIATION VECTOR-VALUED SEQUENCE SPACE INTO MADDOX SEQUENCE SPACE

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หัวข้อวิทยานิพนธ์ เมทริกซ์การแปลงจากปริภูมิลำดับค่าเวกเตอร์ความแปรปรวนมีขอบเขตไปยังปริภูมิลำดับแมดดอกซ์
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บทคัดย่อ

งานวิจัยนี้เป็นการกล่าวถึงการศึกษาโครงสร้างปริภูมิลำดับค่าเวกเตอร์ความแปรปรวนมีขอบเขต (Bounded Variation; $bv(X,p)$) และการให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับค่าเวกเตอร์ $bv(X,p)$ ไปยังปริภูมิลำดับค่าสเกลาร์ของแมดดอกซ์ โดยการประยุกต์ผลที่ได้นี้ทำให้ได้เงื่อนไขที่จำเป็นและเพียงพอสำหรับเมทริกซ์อนันต์ที่ส่งจาก $bv(X,p)$ ไปยัง $l_\infty(q)$, $M_\infty(q)$, $E_r(q)$, $F_r(q)$ โดยที่ $p=(p_k)$ และ $q=(q_k)$ เป็นลำดับของจำนวนจริงบวกที่มีขอบเขต

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ABSTRACT

In this research, we will discuss the structure of bounded variation vector-valued sequence spaces and to characterize an infinite matrix mapping bounded variation vector-valued sequence spaces into Maddox sequence spaces. By applying the results, we also obtain necessary and sufficient conditions for infinite matrices mapping from $bv(X,p)$ into $l_\infty(q)$, $M_\infty(q)$, $E_r(q)$, $F_r(q)$, when $p=(p_k), q=(q_k)$ are bounded sequence of positive real numbers.

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CHAPTER 1

INTRODUCTION

The theory of infinite matrix transformations between two certain sequence spaces has been widely studied for along time. However, in the past, most of problems concerned about matrix transformations between scalar-valued sequence spaces, furthermore elements of infinite matrices were scalar. In first period, domain and co-domain of transformations were classical sequence space. In the next period, mathematicians define and studied for new scalar-valued sequence space. For examples, in [5-7] Maddox and [9] Simon introduced and studied scalar-valued sequence space of Maddox. Here are the Maddox sequence spaces:

$$c_0(p)=\left\{ x=(x_k):\lim_{k\to\infty}|x_k|^{p_k}=0\right\},$$

$$c(p)=\left\{x=(x_k):\lim_{k\to\infty}|x_k-a|^{p_k}=0 \;\; for \;\; some \;\; a\right\},$$

$$l_\infty(p)=\left\{x=(x_k):\sup_k|x_k|^{p_k}<\infty\right\} \text{ and}$$

$$l(p)=\left\{x=(x_k):\sum_{k=1}^{\infty}|x_k|^{p_k}<\infty\right\},$$

where $p=(p_k)$ is a bounded sequence of positive real numbers. Grosse and Erdmann, [4-5] investigated and gave characterization for infinite matrices to transform between sequence spaces of Maddox. In 1992, S.M.Srirajudeen [10] gave characterization of infinite matrices, which transform bounded variation scalar-valued sequence space (bv) into Maddox sequence space, when bv is

$$bv=\left\{x=(x_k):\sum_{k=1}^{\infty}|x_k-x_{k-1}|<\infty; x_0=0\right\}.$$

A sequence space is a linear subspace of $W = C^N$ of all sequences. A matrix space is a linear subspace of the space $W(N^2) = C^{N^2}$ of all (complex) matrices. For a matrix $A = (a_{nk})$ and a sequence $x = (x_k)$, we put $Ax = \left(\sum_{k=1}^{\infty} a_{nk} x_k \right)_{n=1}^{\infty}$ if the series $\sum_{k=1}^{\infty} a_{nk} x_k$ converges for all $n \in N$. If E and F are sequence spaces, we say that A **maps** E into F if for each $x \in E$, Ax exists and $Ax \in F$. The matrix space (E,F) is defined as

$$(E,F) = \{ A \in W(N^2) : A \ \ maps \ \ E \ into \ \ F \ \}.$$

Let $(X, \|\cdot\|)$ be a Banach space with a scalar field K, the space of all sequences in X is denoted by $W(X)$ and let $\Phi(X)$ denote the space of all finite sequences in X. When $X = R$ or C, the corresponding spaces are written as W and Φ. Let N be the set of all natural numbers, we write $x = (x_k)$ with $x_k \in X$ for all $k \in N$. A sequence space in X is a linear subspace of $W(X)$. Let $p = (p_k)$ be a bounded sequence of positive real numbers, the X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $l_{\infty}(X,p)$, $l(X,p)$, $bv(X,p)$, $M_0(X,p)$, $l_{\underline{\infty}}(X,p)$, $E_r(X,p)$, $F_r(X,p)$ and $M_{\infty}(X,p)$ are define by

$$c_0(X,p) = \left\{ \ x = (x_k) : \lim_{k \to \infty} \|x_k\|^{p_k} = 0 \ \right\},$$

$$c(X,p) = \left\{ \ x = (x_k) : \lim_{k \to \infty} \|x_k - a\|^{p_k} = 0 \ \text{ for some } a \in X \ \right\},$$

$$l_{\infty}(X,p) = \left\{ \ x = (x_k) : \sup_{k} \|x_k\|^{p_k} < \infty \ \right\},$$

$$l(X,p) = \left\{ \ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty \ \right\},$$

$$bv(X,p) = \left\{ x=(x_k): \sum_{k=1}^{\infty} \|x_k - x_{k+1}\|^{p_k} < \infty \right\},$$

$$M_0(X,p) = \left\{ x=(x_k): \sum_{k=1}^{\infty} n^{\frac{-1}{p_k}} \|x_k\| < \infty \text{ for some } n \in N \right\},$$

$$M_\infty(X,p) = \left\{ x=(x_k): \sum_{k=1}^{\infty} n^{\frac{1}{p_k}} \|x_k\|^{p_k} < \infty \text{ for all } n \in N \right\},$$

$$\underline{l_\infty}(X,p) = \left\{ x=(x_k): \lim_{k\to\infty} \|\delta_k x_k\|^{p_k} = 0 \text{ for each } (\delta_k) \in c_0 \right\}$$

$$E_r(X,p) = \left\{ x=(x_k): \sup_k k^{-r} \|x_k\|^{p_k} < \infty \right\},$$

$$F_r(X,p) = \left\{ x=(x_k): \sum_{k=1}^{\infty} k^{r} \|x_k\|^{p_k} < \infty \right\}.$$

When $p_k = 1, \forall k \in N$; $bv(X,p)$ becomes $bv(X)$ and when $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $l_\infty(p)$, $l(p)$, $bv(p), M_0(p)$, $M_\infty(p)$ $\underline{l_\infty}(p)$, $E_r(p)$ and $F_r(p)$ respectively. This approach to matrix transformations between vector-valued sequence spaces.

The problem of matrix transformations between vector-valued sequence spaces are new problem of matrix transformations which are more general than the former problem, and elements of infinite matrix become operator or functional. These problems are studied widely by many mathematicians. For examples, in [16], Wu and Liu gave the matrix characterizations from X- valued sequence spaces $c_0(X,p)$, $l_\infty(X,p)$ and $l(X,p)$ into scalar valued sequence spaces $c_0(q)$ and $l_\infty(q)$. In Thailand there are some mathematicians study in this field. Recently, Suthep Suantai [11-12] gave characterization of infinite matrices mapping Nakano vector-valued sequence $(l(X,p))$ into any $BK-$space, l_∞ and $l_\infty(q)$. In [1] Chanun Sudsukh characterized an infinite matrix that transform Maddox vector-valued sequence space into Nakano sequence space and Nakano vector-valued sequence space into Maddox sequence space. And Orawan Tripak [8] gave characterization for infinite matrices mapping vector-values sequence spaces into scalar-valued sequence spaces or mapping between vector-valued sequence spaces.

There are many open problems about matrix transformations from vector-valued sequence spaces in to scalar-valued sequence spaces or between vector-valued sequence spaces. Since we have motivation from S.M.Sirajudeen, so we interested in the transformations of vector-valued sequence space of bounded variation $(bv(X,p))$ into scalar-valued sequence of Maddox $(l(p), l_\infty(p), c(p), c_0(p))$.

The main objective of this thesis is

(1) to discuss about structure of vector-valued sequence spaces of bounded variation $[bv(X,p)]$ in chapter 3 ,

(2) to characterize infinite matrices such elements of the matrix are bounded linear functional on the vector space which mapping vector- valued sequence space of bounded variation into Maddox scalar- valued sequence spaces , $\underline{l}_\infty(q)$, $M_\infty(q)$, $E_r(q)$ and $F_r(q)$ in chapter 4 .

CHAPTER 2

PRELIMINARIES

In this chapter, we give some general concepts, definitions and also literature reviews that contain some theorems, which will be used in the later chapters.

2.1 General Concepts and Definitions

In this part, we give some general concepts about metric spaces, fre′chet spaces, norm spaces and some related definitions. Then we conclusion some definitions of sequence space especially bounded variation vector-valued sequence space and Maddox sequence space which we shall investigate some properties and related in next chapter. Finally in this part some notations and concepts of matrix transformations which transform between sequence spaces will be given.

2.1.1 Metric Spaces, Fre′chet Spaces and Norm Spaces

Definition 2.1.1.1 (*Vector space* (*linear space*))

A vector space over a field K is a nonempty set X with an operator $+$ on $X \times X$ into X and an operator $\bullet$ on $X \times X$ into X such that for all scalars α, β and elements (vectors) $x, y, z \in X$ we have

1) $x + y = y + x$, 2) $(x + y) + z = x + (y + z)$, 3) there exists $\theta \in X$ such that $x + \theta = x$, 4) there exists $-x \in X$ such that $x + (-x) = \theta$, 5) $1 \cdot x = x$, 6) $\alpha(x + y) = \alpha x + \alpha y$, 7) $(\alpha + \beta)x = \alpha x + \beta x$, 8) $\alpha(\beta x) = (\alpha\beta)x$.

Definition 2.1.1.2 (*Metric space (semimetric)*)

A metric (semimetric) space is a pair (X, d), where X is a set and d is a metric (semimetric) on X, that is, a function define on $X \times X$ such that for all $x, y, z \in X$ we have:

Metric

(M1) d is real-valued, finite and nonnegative.

(M2) $d(x,y) = 0$ if and only if $x = y$.

(M3) $d(x,y) = d(y,x)$

(M4) $d(x,y) \leq d(x,z) + d(z,y)$

Semimetric

(M1) d is real-valued, finite and nonnegative.

(M2) $d(x,x) = 0$

(M3) $d(x,y) = d(y,x)$

(M4) $d(x,y) \leq d(x,z) + d(z,y)$

The only different between semimetric and metric is the semimetric space distinct elements can still be zero distance apart.

Definition 2.1.1.3 (*Normed space*)

A normed space X is a vector space with a norm defines on it. A norm is a real-valued function on X whose at $x \in X$ is denote by $\|x\|$ and has the properties

(N1) $\|x\| \geq 0$

(N2) $\|x\| = 0$ if and only if $x = 0$

(N3) $\|\alpha x\| = \alpha\|x\|$

(N4) $\|x + y\| \leq \|x\| + \|y\|$.

A norm on X defines a metric d on X which is given by $d(x,y) = \|x - y\|$ for all $x, y \in X$, and is call *the metric induce by the norm*. The norm space X is denoted by $(X, \|\cdot\|)$.

Definition 2.1.1.4 (*Paranorm space, Seminorm space*)

A paranorm (X,g) is a linear space together with a paranorm g on it. A paranorm $g: X \to R$, satisfies (1) $g(\theta)=0$, (2) $g(x)=g(-x)$, (3) $g(x+y) \le g(x)+g(y)$ and (4) $\lambda \to \lambda_0$, $x \to x_0$ imply $\lambda x \to \lambda_0 x_0$.

A seminorm p on a linear space X , is a function $p: X \to R$ such that

1) $p(\lambda x)=|\lambda| p(x)$, 2) $p(x+y) \le p(x)+p(y)$

By 1) and 2) we also have $p(\theta)=0$ and $p(x) \ge 0$.

Paranorm becomes seminorm.

Definition 2.1.1.5 (*Translation invariance*)

A metric d induce by a norm on a norm space X satisfies

a) $d(x+a,y+a)=d(x,y)$ b) $d(\alpha x,\alpha y)=|\alpha| d(x,y)$.

Definition 2.1.1.6 (*Convergent sequence*)

A sequence (x_n) in a metric space $X=(X,d)$ is said to converge or to be convergent if there is an $x \in X$ such that $\lim\limits_{n\to\infty} d(x_n,x)=0$, we say that (x_n) converges to x, simply $x_n \to x$. If (x_n) is not convergent, it is said to be divergent.

Definition 2.1.1.7 (*Cauchy sequence*)

A sequence (x_n) in a metric space is said to be Cauchy if for every $\varepsilon > 0$ there is an $N_\varepsilon \in N$ such that

$$d(x_m,x_n) < \varepsilon \quad \text{for all } m,n \ge N_\varepsilon .$$

Definition 2.1.1.8 (complete)

A metric space X is said to be complete if every Cauchy sequence in X converges.

Definition 2.1.1.9 (*Fre'chet space*)

A vector space X is said to be a linear metric space if X is a metric space such that both of mapping $(x,y)\to x+y$ and $(\alpha,x)\to \alpha x$ are continuous. A linear metric space is call an *Fre'chet space* (***F-space***) if the metric d on X is complete and translation invariance, i.e. $d(x+a,y+a)=d(x,y)$ for all $x,y,a\in X$.

Definition 2.1.1.10 ($K-space$)

The X - valued sequence space E is call a ***K-space*** if for each $n\in N$ the n^{th} coordinate mapping $p_n : E\to X$, define by $p_n(x)=x_n$ is continuous on E.

Definition 2.1.1.11 ($FK-space$)

If the X - valued sequence space E is an *Fre'chet* and $K-space$ then E is called ***FK-space***.

2.1.2 Sequence Spaces

Definition 2.1.2.1

A sequence space is a vector space whose elements are sequences.

Definition 2.1.2.2

Scalar sequence spaces of Maddox are following:

$$c_0(p)=\left\{\ x=(x_k):\lim_{k\to\infty}|x_k|^{p_k}\ =0\right\},$$

$$c(p)=\left\{x=(x_k):\lim_{k\to\infty}|x_k-a|^{p_k}=0\ \ for\ \ some\ a\right\},$$

$$l_\infty(p)=\left\{x=(x_k):\sup_k|x_k|^{p_k}<\infty\right\}\text{ and}$$

$$l(p)=\left\{x=(x_k):\sum_{k=1}^{\infty}|x_k|^{p_k}<\infty\right\}.$$

When $p=(p_k)$ is a bounded sequence of positive real numbers.

Definition 2.1.2.3

Let $(X,\|\cdot\|)$ be a Banach space with a scalar field K and $p = (p_k)$ be a bounded sequence of positive real numbers. The bounded variation vector-valued sequence space is

$$bv(X,p) = \left\{ \; x=(x_k): \sum_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k} < \infty \; \right\}.$$

Definition 2.1.2.4

Let E be any X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k stand for the k^{th} term of x. For $k \in N$ denote e_k by the sequence $(0,0,0\ldots,0,1,0,\ldots)$ with 1 in the k^{th} position and e by the sequence $(1,1,1,\ldots)$. For $z \in X$ and $k \in N$, let $e^k(z)$ be the sequence $(0,0,0,\ldots,z,0,\ldots)$ with z in the k^{th} position and let $e(z)$ be the sequence $(z,z,z,\ldots)$.

Definition 2.1.2.5

Let $(X,\|\cdot\|)$ be a Banach space with a scalar field K, E be any X-valued sequence space. For a fix scalar sequence $u=(u_k)$ the sequence space E_u is define as

$$E_u = \left\{ \; x=(x_k) \in W(x) : (u_k x_k) \in E \; \right\}.$$

Definition 2.1.2.6 *(AK-property)*

Suppose that E contains $\Phi(X)$, the space of all finite sequences in X, then E is said to have **property *AB*** if the set $\left\{\sum_{k=1}^{n} e^k(x_k) : n \in N\right\}$ is bounded in E for every $x=(x_k) \in E$. It is said to have ***AK*- property** if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x=(x_k) \in E$.

2.1.3 Matrix and Transformations

Definition 2.1.3.1

Let $A = (f_k^n)$ with $f_k^n \in X'$, the topological dual of X. Suppose that E is a space of X -valued sequence space and F a space of scalar-valued sequences. Then A is said to **map** E into F, written by $A : E \to F$, If for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$.

Definition 2.1.3.2

Let (E,F) denote for the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$ {}_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}. $$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\frac{1}{u^k})$.

Definition 2.1.3.3 (weak* convergence of a sequence of functional)

Let (f_n) be a sequence of bounded linear functional on a norm space X. Then weak* convergence of (f_n) means that there is an $f \in X'$ such that $f_n(x) \to f(x)$, $\forall x \in X$. This written $f_n \xrightarrow{w^*} f$.

2.2 Literature Reviews

In this part we would like to say some literature reviews of matrices transformations and some results which necessary for this thesis.

2.2.1 Transformations between Scalar-Valued Sequence Spaces

There are many mathematicians have been studied about matrix transformation between classical sequence space for a long time. However in this part, we discuss only about some literature reviews which motivated to this research and some results will be useful for this research. In 1992, S.M.Sirajudeen [10] gave characterization of $(bv,l(q))$, $(bv,l_\infty(q))$, $(bv,c_0(q))$ and $(bv,c(q))$, these are transformations between scalar-valued sequence spaces. These results are follow:

Theorem 2.2.1.1 $A \in (bv,l(q))$ *if and only if* $\sup_j \sum_{n=1}^{\infty} \left| \sum_{k=j}^{\infty} a_{nk} \right|^{q_n} < \infty$,

Theorem 2.2.1.2 $A \in (bv,l_\infty(q)$ *if and only if* $\sup_n \left(\sup_j \left| \sum_{k=j}^{\infty} a_{nk} \right| M^{-1} \right)^{q_n} < \infty$

for some $M>1$,

Theorem 2.2.1.3 $A \in (bv,c_0(q))$ *if and only if*

$$\left| \sum_{k=j}^{\infty} a_{nk} \right|^{q_n} \to 0 ; (n \to \infty) \text{ for every } j \quad \text{and}$$

$$\lim_M \limsup_n \left(\sup_j \left| \sum_{k=j}^{\infty} a_{nk} \right| M^{-1} \right)^{q_n} = 0 \quad \text{and}$$

Theorem 2.2.1.4 $A \in (bv,c(q))$ *if and only if*

$$\sup_{n,j} \left| \sum_{k=j}^{\infty} a_{nk} \right| < \infty \quad \text{and there exist } \alpha_1, \alpha_2, \alpha_3, \ldots$$

such that

$$\left| \sum_{k=j}^{\infty} a_{nk} - \alpha_j \right|^{q_n} \to 0 ; \ (n \to \infty) \text{ for each } j \text{ and}$$

$$\lim_M \limsup_n \left(\sup_j \left| \sum_{k=j}^{\infty} a_{nk} - \alpha_j \right| M^{-1} \right)^{q_n} = 0 .$$

2.2.2 Transformations between Vector-Valued sequence spaces

When the problem of matrix transformations becomes transformation between vector-valued sequence space, which are new and more general than the former problem. In [16], Wu and Liu gave the matrix characterizations from X- valued sequence spaces $c_0(X,p)$, $l_\infty(X,p)$ and $l(X,p)$ into scalar valued sequence spaces $c_0(q)$ and $l_\infty(q)$. The important and useful results are as follow:

Theorem 2.2.2.1 *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequences of positive real numbers with* $p_k \le 1$ *for all* $k \in N$ *and* $A=(f_k^n)$ *an infinite matrix. Then* $A: l(X,p) \to l_\infty(q)$ *if and only if there exists* $M \in N$ *such that* $\|f_k^n\| \le M^{\frac{1}{p_k}+\frac{1}{q_n}}$ *for all* $n,k \in N$.

Theorem 2.2.2.2 *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequences of positive real numbers with* $p_k > 1$ *and* $\frac{1}{p_k}+\frac{1}{t_k}=1$ *for all* $k \in N$ *and* $A=(f_k^n)$ *an infinite matrix. Then* $A: l(X,p) \to l_\infty(q)$ *if and only if there exists* $M \in N$ *such that*

$$\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M^{\frac{-t_k}{q_n}} \right)^{q_n} < \infty .$$

We can notice that the condition $\frac{1}{p_k}+\frac{1}{t_k}=1$ was added for useful in the proof. The sequence (t_k) is also a sequence of positive real numbers, $t_k > 1$ and $t_k = \frac{p_k}{p_k - 1}$.

Theorem 2.2.2.3 *Let* $p=(p_k)$ *be bounded sequences of positive real numbers with* $p_k \le 1$ *for all* $k \in N$ *and* $A=(f_k^n)$ *an infinite matrix. Then* $A: l(X,p) \to c_0$ *if and only if*

(1) for each $k \in N$, $f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *and*

(2) there exists $M \in N$ *such that* $\|f_k^n\|^{p_k} \le M$ *for all* $n,k \in N$.

Suthep Suantai and Chanun Sudsukh [5] gave necessary and sufficient condition for infinite matrices mapping Nakano vector-valued $l(X,p)$ and $M_0(X,p)$ into sequence space E_r $(r \geq 0)$. Some main results are as follow:

Theorem 2.2.2.4 *Let* $r \geq 0$ *and let* $p=(p_k)$ *be bounded sequences of positive real numbers with* $p_k \leq 1$ *and* $A=(f_k^n)$ *an infinite matrix. Then* $A \in (l(X,p),E_r)$ *if and only if there is* $m_0 \in N$ *such that* $\sup_{n,k} m_0^{\frac{-1}{p_k}} n^{-r} \left\| f_k^n \right\| < \infty$.

Theorem 2.2.2.5 *Let* $r \geq 0$ *and let* $p=(p_k)$ *be bounded sequences of positive real numbers and let* $A=(f_k^n)$ *an infinite matrix. Then* $A \in (M_0(X,p),E_r)$ *if and only if for each* $s \in N$, $\sup_{n,k} n^{-r} s^{\frac{1}{p_k}} \left\| f_k^n \right\| < \infty$.

Chanun Sudsukh [1] characterized an infinite matrix transformation between Maddox vector-valued sequence space and Nakano sequence spaces. These crucial theorems which are useful for this research are as follow:

Theorem 2.2.2.6 *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequences of positive real numbers with* $p_k > 1$ *and* $\frac{1}{p_k}+\frac{1}{t_k}=1$ *and* $q_k > 1$ *for all* $k \in N$*, and* $A=(f_k^n)$ *an infinite matrix. Then* $A : l(X,p) \rightarrow l(q)$ *if the following two conditions hold;*

(1) for each $n \in N$ *there exist* $M_n \in N$ *such that* $\sum_{k=1}^{\infty} \left\| f_k^n \right\|^{t_k} M_n^{-(t_k-1)} < \infty$,

(2) there exists $M_0 \in N$ *such that*

$$\sup_K \sum_{n=1}^{\infty} \left(\sum_{k \in K} \left\| f_k^n \right\| M_0^{\frac{-1}{p_k}} \right)^{q_n} < \infty ,$$

where supremum is taken over all finite subsets K *of* N.

Theorem 2.2.2.7 *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers with* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : l(X, p) \to c_0(q)$ *if and on if*

(1) *for all* $m, k \in N$, $m^{\frac{1}{q_n}} f_k^n \xrightarrow{w^*} 0$ *as* $n \to \infty$ *and*

(2) *for each* $m \in N$, $\left(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} \left\| f_k^n \right\|^{t_k} r^{-(t_k - 1)} \right) \to 0$ *as* $r \to \infty$ *uniformly for* $n \geq 1$.

Theorem 2.2.2.8 *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers with* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$ *, and* $A = (f_k^n)$ *an infinite matrix. Then* $A : l(X, p) \to c(q)$ *if and on if there is a sequence* (f_k) *with* $f_k \in X'$ *for all* $k \in N$ *such that*

(1) *there exists* $M \in N$ *such that* $\sum_{k=1}^{\infty} \left\| f_k \right\|^{t_k} M_n^{-(t_k - 1)} < \infty$,

(2 *for all* $m, k \in N$, $m^{\frac{1}{q_n}} (f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *and*

(3) *for each* $m \in N$, $\left(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} \left\| f_k^n - f_k \right\|^{t_k} r^{-(t_k - 1)} \right) \to 0$ *as* $r \to \infty$ *uniformly for* $n \geq 1$.

Theorem 2.2.2.9 *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequences of positive real numbers with* $p_k \leq 1$ *for all* $k \in N$*, and* $A = (f_k^n)$ *an infinite matrix. Then* $A : l(X, p) \to c(q)$ *if and on if there is a sequence* (f_k) *with* $f_k \in X'$ *for all* $k \in N$ *such that*

(1) *there exists* $M \in N$ *such that* $\sup_k \left\| f_k \right\| M^{\frac{-1}{p_k}} < \infty$,

(2 *for all* $m, k \in N$, $m^{\frac{1}{q_n}} (f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *and*

(3) *for each* $m \in N$, $\sup_{m,k} m^{\frac{p_k}{q_n}} \left\| f_k^n - f_k \right\| < \infty$.

Furthermore, he gave the important lemma.

Lemma 2.2.2.10 Let $E \subseteq W(X)$ be an *FK*-space with AK property and F an *FK*-space of scalar sequences .Then for an infinite matrix $A = (f_k^n)$, $A : E \to F$ if and only if

(1) for each $n \in N$, $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for every $x = (x_k) \in E$

(2) for each $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in F$ for every $x \in X$ and

(3) $A : \Phi(X) \to F$ is continuous when $\Phi(X)$ is considered as a subspace of E .

Orawan Tripak[8] gave characterization of infinite matrices mapping between sequence space of Maddox and the sequence space $c_0(X,p)$,$M_0(X,p)$ $M_\infty(X,p)$ $F_r(X,p)$ and $E_r(X,p)$. In this research, the elements of infinite matrices are scalar-valued. We can see details in [8].

As the above mention, the main objective of this thesis is to characterize an infinite matrix to transform $bv(X,p)$ into scalar-valued sequence space of Maddox. Although now, we know about $bv(X,p)$ and scalar-valued sequence space of Maddox, some useful theorems and general concepts of transformations, but before we can characterize the infinite matrix, we should know some preliminaries structure of these sequence spaces. Hence next step, our work is shall investigate structure of these sequence spaces and some preliminaries, which will be used for characterization, in chapter 3.

CHAPTER 3

STRUCTURE OF $bv(X,p)$

The purpose of this chapter is to offer and prove some preliminary structures and useful properties of bounded variation vector-valued sequence space; $bv(X,p)$, when $p=(p_k)$ be a bounded sequence of positive real numbers, which will be used for characterization infinite matrices in the later chapter.

3.1 *FK* and *AK* Properties

We first give general property about $FK-space$ which contains $\Phi(x)$. In this way we will show that $bv(X,p)$ is an $Fre'chet$-space, K-space and contains $\Phi(x)$ property.

Proposition 3.1.1 $bv(X,p)$ *contains* $\Phi(x)$.

Proof.

Let $x=(x_k)\in\Phi(x)$, this means there exists $N_0\in N$ such that $x_k=0,\forall k\ge N_0$. So we have $\sum_{k=1}^{\infty}\|x_k-x_{k-1}\|^{p_k}=\sum_{k=1}^{N_0}\|x_k-x_{k-1}\|^{p_k}<\infty$. Hence $bv(X,p)$ contains $\Phi(x)$. #

Proposition 3.1.2 $\left(\sum_{j=1}^{\infty}\|\zeta_j+\eta_j\|^{p_j}\right)^{1/M}\le\left(\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}\right)^{1/M}+\left(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\right)^{1/M}$,

when $M=\max\{1,\sup p_j\}$.

Proof.

For the proof of this inequality we divide (p_j) into two cases.

Case $p_j\le 1$; In this case we can observe that $M=1$, that is we have to show that

$$\sum_{j=1}^{\infty}\|\zeta_j+\eta_j\|^{p_j}\le\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}+\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}.$$

Since $$\begin{aligned}\|\zeta_j+\eta_j\|^{p_j} &\le \left(\|\zeta_j\|+\|\eta_j\|\right)^{p_j}\\ &\le \|\zeta_j\|^{p_j}+\|\eta_j\|^{p_j} \qquad ;\ p_j\le 1\end{aligned}$$

$$\therefore\ \sum_{j=1}^{\infty}\|\zeta_j+\eta_j\|^{p_j} \le \sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}+\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\ .$$

Case $p_j>1$; In this case we have to show that

$$\left(\sum_{j=1}^{\infty}\|\zeta_j+\eta_j\|^{p_j}\right)^{1/M} \le \left(\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}\right)^{1/M}+\left(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\right)^{1/M}.$$

Let $$\begin{aligned}\|\zeta_j+\eta_j\|^{p_j} &= \|w_j\|^{p_j}\\ \|w_j\|^{p_j} &= \|\zeta_j+\eta_j\|\,\|w_j\|^{p_j-1}\\ &\le \|\zeta_j\|\,\|w_j\|^{p_j-1}+\|\eta_j\|\,\|w_j\|^{p_j-1}\end{aligned}$$

$$\therefore\quad \sum_{j=1}^{\infty}\|w_j\|^{p_j} \le \sum_{j=1}^{\infty}\|\zeta_j\|\,\|w_j\|^{p_j-1}+\sum_{j=1}^{\infty}\|\eta_j\|\,\|w_j\|^{p_j-1}.$$

Considers,

$$\sum_{j=1}^{\infty}\|\zeta_j\|\,\|w_j\|^{p_j-1} \le \left(\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}\right)^{1/M}\left(\sum_{j=1}^{\infty}\left(\|w_j\|^{p_j-1}\right)^{q_j}\right)^{1/Q}\text{; by Holder inequality,}$$

when $\dfrac{1}{p_j}+\dfrac{1}{q_j}=1;\forall j\in N$ and $\dfrac{1}{M}+\dfrac{1}{Q}=1$

and $$\sum_{j=1}^{\infty}\|\eta_j\|\,\|w_j\|^{p_j-1} \le \left(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\right)^{1/M}\left(\sum_{j=1}^{\infty}\left(\|w_j\|^{p_j-1}\right)^{q_j}\right)^{1/Q}. \qquad \text{; by Holder inequality,}$$

Thus $$\sum_{j=1}^{\infty}\|w_j\|^{p_j} \le \left[\left(\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}\right)^{1/M}+\left(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\right)^{1/M}\right]\left(\sum_{j=1}^{\infty}\|w_j\|^{p_j}\right)^{1/Q}; (p_j-1)q_j=p_j\,.$$

Dividing by the last factor on the right and noting that $1-\frac{1}{Q}=\frac{1}{M}$, we obtain that

$$\left(\sum_{j=1}^{\infty}\|w_j\|^{p_j}\right)^{1-1/Q} \le \left[\left(\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}\right)^{1/M}+\left(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\right)^{1/M}\right],$$

or $$\left(\sum_{j=1}^{\infty}\|w_j\|^{p_j}\right)^{1/M} \le \left(\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}\right)^{1/M}+\left(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\right)^{1/M}.$$

Hence $$\left(\sum_{j=1}^{\infty}\|\zeta_j+\eta_j\|^{p_j}\right)^{1/M} \le \left(\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j}\right)^{1/M}+\left(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j}\right)^{1/M}.$$ #

Proposition 3.1.3 *$bv(X,p)$ is an Fre'chet space, where (p_j) is bounded sequence of positive real numbers and $p_j \le 1$, for all $j \in N$.*

Proof.

Let $(x_m)_{m=1}^{\infty}$ be any Cauchy sequence in the space $bv(X,p)$ where the $x_m=(\varsigma_1^{(m)},\varsigma_2^{(m)},\varsigma_3^{(m)}\ldots)$ is a sequence in the Banach space X. Since the paranorm on $bv(X,p)$, when $p_j \le 1$, is given by

$$g(x)=\sum_{j=1}^{\infty}\|\varsigma_j-\varsigma_{j-1}\|^{p_j},$$

then the semimetric, which induce by the paranorm $g(x)$, is

$$d(x,y) = g(x-y) = \sum_{j=1}^{\infty}\|(\varsigma_j-\eta_j)-(\varsigma_{j-1}-\eta_{j-1})\|^{p_j},$$

when $x=(\varsigma_j)$ and $y=(\eta_j)$. Since (x_m) is Cauchy sequence, this means that for any $\varepsilon_1>0$ there is an $N(\varepsilon_1)\in N$ such that for all $m,n>N(\varepsilon_1)$,

$$d(x_m, x_n) = \sum_{j=1}^{\infty} \left\| (\varsigma_j^{(m)} - \varsigma_j^{(n)}) - (\varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)}) \right\|^{p_j} < \varepsilon_1, \tag{3.1}$$

It follow that for every fixed $j = 1,2,3,\ldots$, we have

$$\left\| (\varsigma_j^{(m)} - \varsigma_j^{(n)}) - (\varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)}) \right\| < \varepsilon_1^{\frac{1}{p_j}} \qquad ,\forall m,n > N(\varepsilon_1).$$

Now we want to show that $(\varsigma_j^{(m)})$ is Cauchy sequence in the Banach space X, that is we want to show that for any $\varepsilon > 0$ there is an $N(\varepsilon) \in N$ such that

$$\left\| \varsigma_j^{(m)} - \varsigma_j^{(n)} \right\| < \varepsilon \qquad , \forall m,n > N(\varepsilon).$$

Thus for any $\varepsilon > 0$, we choose $\varepsilon_1 < \left(\frac{\varepsilon}{j}\right)^{1/l}$ where $l = \min\limits_{1 \le k \le j} \left\{ \frac{1}{p_k} \right\}$; j fixed.

Since we have

$$\begin{aligned} \left\| \varsigma_j^{(m)} - \varsigma_j^{(n)} \right\| &= \left\| (\varsigma_j^{(m)} - \varsigma_j^{(n)}) - (\varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)}) + (\varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)}) \right\| \\ &\le \left\| (\varsigma_j^{(m)} - \varsigma_j^{(n)}) - (\varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)}) \right\| + \left\| \varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)} \right\| \\ &< \varepsilon_1^{\frac{1}{p_j}} + \left\| \varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)} \right\|, \end{aligned}$$

$$\begin{aligned} \left\| \varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)} \right\| &= \left\| (\varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)}) - (\varsigma_{j-2}^{(m)} - \varsigma_{j-2}^{(n)}) + (\varsigma_{j-2}^{(m)} - \varsigma_{j-2}^{(n)}) \right\| \\ &\le \left\| (\varsigma_{j-1}^{(m)} - \varsigma_{j-1}^{(n)}) - (\varsigma_{j-2}^{(m)} - \varsigma_{j-2}^{(n)}) \right\| + \left\| \varsigma_{j-2}^{(m)} - \varsigma_{j-2}^{(n)} \right\| \\ &< \varepsilon_1^{\frac{1}{p_{j-1}}} + \left\| \varsigma_{j-2}^{(m)} - \varsigma_{j-2}^{(n)} \right\|, \end{aligned}$$

$$\therefore \left\| \varsigma_j^{(m)} - \varsigma_j^{(n)} \right\| < \varepsilon_1^{\frac{1}{p_j}} + \varepsilon_1^{\frac{1}{p_{j-1}}} + \left\| \varsigma_{j-2}^{(m)} - \varsigma_{j-2}^{(n)} \right\| \quad ,$$

$$\vdots$$

$$\left\|\varsigma_j^{(m)} - \varsigma_j^{(n)}\right\| < \sum_{k=2}^{j} \varepsilon_1^{\frac{1}{p_k}} + \left\|\varsigma_1^{(m)} - \varsigma_1^{(n)}\right\|$$

$$\left\|\varsigma_1^{(m)} - \varsigma_1^{(n)}\right\| \le \left\|(\varsigma_1^{(m)} - \varsigma_1^{(n)}) - (\varsigma_0^{(m)} - \varsigma_0^{(n)})\right\| + \left\|\varsigma_0^{(m)} - \varsigma_0^{(n)}\right\|$$

$$< \varepsilon_1^{\frac{1}{p_1}} + 0 \ , \qquad ; \varsigma_0^{(m)} \text{ and } \varsigma_0^{(n)} \text{ are zero}$$

$$\therefore \left\|\varsigma_j^{(m)} - \varsigma_j^{(n)}\right\| < \sum_{k=1}^{j} \varepsilon_1^{\frac{1}{p_k}} \ .$$

Since (p_j) is bounded sequence of positive real numbers then $\left(\frac{1}{p_k}\right)$ is also bounded sequence, that is $1 \le l \le \frac{1}{p_k} \le m$, when $l = \min\limits_{1\le k\le j}\left\{\frac{1}{p_k}\right\}$ and $m = \max\limits_{1\le k\le j}\left\{\frac{1}{p_k}\right\}$.

Thus we have

$$\varepsilon_1^{\,m} \le \varepsilon_1^{1/p_k} \le \varepsilon_1^{\,l} \qquad ; \text{for all } \varepsilon_1 \le 1 .$$

Considers,

$$\sum_{k=1}^{j} \varepsilon_1^{\frac{1}{p_k}} \le \sum_{k=1}^{j} \varepsilon_1^{\,l} = j\varepsilon_1^{\,l} < j\left[\left(\frac{\varepsilon}{j}\right)^{1/l}\right]^{l} = \varepsilon .$$

Thus we can have

$$\left\|\varsigma_j^{(m)} - \varsigma_j^{(n)}\right\| < \sum_{k=1}^{j} \varepsilon_1^{\frac{1}{p_k}} \le \varepsilon \qquad , \forall m,n > N(\varepsilon) = N(\varepsilon_1) .$$

Hence for every fixed j , the sequence $(\varsigma_j^{(1)}, \varsigma_j^{(2)}, \varsigma_j^{(3)} \ldots)$ is a Cauchy sequence of X -valued [$\varsigma_j^{(m)} \in X$]. Since X is any Banach space thus it converges, say $\varsigma_j^{(m)} \to \varsigma_j$ as $m \to \infty$. Using these infinitely many limits $\varsigma_1, \varsigma_2, \varsigma_3, \ldots$, we define $x = (\varsigma_1, \varsigma_2, \varsigma_3, \ldots)$ and want to show that $x \in bv(X,p)$ and $x_m \to x$. From (3.1) with $n \to \infty$ we have

$$d(x_m,x) = \sum_{j=1}^{\infty}\left\|(\varsigma_j^{(m)} - \varsigma_j) - (\varsigma_{j-1}^{(m)} - \varsigma_{j-1})\right\|^{p_j} < \varepsilon_1 \quad ;\forall m > N(\varepsilon_1) \tag{3.2}$$

Since $x_m = (\varsigma_j{}^{(m)}) \in bv(X,p)$, there is a real number k_m such that

$$\sum_{j=1}^{\infty}\left\|\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)}\right\|^{p_j} \leq k_m$$

Now we want to that $x = (\varsigma_j) \in bv(x,p)$, that is, we need to show that

$$\sum_{j=1}^{\infty}\left\|\varsigma_j - \varsigma_{j-1}\right\|^{p_j} < \infty .$$

Since

$$\begin{aligned}\sum_{j=1}^{\infty}\left\|\varsigma_j - \varsigma_{j-1}\right\|^{p_j} &= \sum_{j=1}^{\infty}\left\|(\varsigma_j - \varsigma_{j-1}) - (\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)}) + (\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)})\right\|^{p_j} \\ &\leq \sum_{j=1}^{\infty}\left\|(\varsigma_j - \varsigma_{j-1}) - (\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)})\right\|^{p_j} + \sum_{j=1}^{\infty}\left\|\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)}\right\|^{p_j} \text{ ; by prop.3.2.1} \\ &< \varepsilon_1 + k_m ,\end{aligned}$$

hence $$\sum_{j=1}^{\infty}\left\|\varsigma_j - \varsigma_{j-1}\right\|^{p_j} < \infty .$$

That is, we can show that $x = (\varsigma_j) \in bv(X,p)$, and furthermore, the series (3.2) represents $d(x_m,x)$. Since $d(x_m,x) \to 0$ when $m \to \infty$ $(\varsigma_j^{(m)} \to \varsigma_j, \forall j \in N)$ hence $x_m \to x$. Since (x_m) was any Cauchy sequence in $bv(X,p)$, this prove completeness of $bv(X,p)$. It is not difficult to show that the semimetric d is translation invariance because the semimetric d induced by the paranorm $g(x)$. Hence we have $bv(X,p)$, when $p_j \leq 1$, is a *Fre'chet* space.

Proposition 3.1.4 *$bv(X,p)$ is an Fre'chet space, where (p_j) is bounded sequence of positive real numbers and $p_j > 1$, for all $j \in N$.*

Proof.

Let $(x_m)_{m=1}^{\infty}$ be any Cauchy sequence in the space $bv(X,p)$ where the $x_m=(\varsigma_1^{(m)},\varsigma_2^{(m)},\varsigma_3^{(m)}\ldots)$. Since the paranorm on $bv(X,p)$ when $p_j>1$ is given by

$$g(x)=\left(\sum_{j=1}^{\infty}\left\|\varsigma_j-\varsigma_{j-1}\right\|^{p_j}\right)^{\frac{1}{M}} \quad \text{when } M=\sup_j p_j,$$

where (p_j) is a bounded sequence of positive real numbers. Then the semimetric, which induce by the paranorm $g(x)$, is

$$d(x,y)=g(x-y)=\left(\sum_{j=1}^{\infty}\left\|(\varsigma_j-\eta_j)-(\varsigma_{j-1}-\eta_{j-1})\right\|^{p_j}\right)^{\frac{1}{M}},$$

when $x=(\varsigma_j)$ and $y=(\eta_j)$. Since (x_m) is Cauchy sequence, this means that for any $\varepsilon_1>0$ there is an $N(\varepsilon_1)\in N$ such that for all $m,n>N(\varepsilon_1)$,

$$d(x_m,x_n)=\left(\sum_{j=1}^{\infty}\left\|(\varsigma_j^{(m)}-\varsigma_j^{(n)})-(\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)})\right\|^{p_j}\right)^{\frac{1}{M}}<\varepsilon_1. \tag{3.3}$$

It follow that for every fixed $j=1,2,3,\ldots$, we have

$$\left\|(\varsigma_j^{(m)}-\varsigma_j^{(n)})-(\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)})\right\|<\varepsilon_1^{\frac{M}{p_j}} \quad ,\forall m,n> N(\varepsilon_1).$$

Now we want to show that $(\varsigma_j^{(m)})$ is Cauchy sequence in the Banach space X, that is we want to show that for any $\varepsilon>0$ there is an $N(\varepsilon)\in N$ such that

$$\left\|\varsigma_j^{(m)}-\varsigma_j^{(n)}\right\|<\varepsilon \qquad ,\forall m,n>N(\varepsilon).$$

Thus for any $\varepsilon>0$, we choose $\varepsilon_1<\left(\frac{\varepsilon}{j}\right)^{1/l}$ where $l=\min\limits_{1\le k\le j}\left\{\frac{M}{p_k}\right\}$; j fixed.

Since we have

$$
\begin{aligned}
\left\|\varsigma_j^{(m)}-\varsigma_j^{(n)}\right\| &= \left\|(\varsigma_j^{(m)}-\varsigma_j^{(n)})-(\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)})+(\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)})\right\| \\
&\le \left\|(\varsigma_j^{(m)}-\varsigma_j^{(n)})-(\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)})\right\| + \left\|\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)}\right\| \\
&< \varepsilon_1^{\frac{M}{p_j}} + \left\|\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)}\right\|,
\end{aligned}
$$

$$
\begin{aligned}
\left\|\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)}\right\| &= \left\|(\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)})-(\varsigma_{j-2}^{(m)}-\varsigma_{j-2}^{(n)})+(\varsigma_{j-2}^{(m)}-\varsigma_{j-2}^{(n)})\right\| \\
&\le \left\|(\varsigma_{j-1}^{(m)}-\varsigma_{j-1}^{(n)})-(\varsigma_{j-2}^{(m)}-\varsigma_{j-2}^{(n)})\right\| + \left\|\varsigma_{j-2}^{(m)}-\varsigma_{j-2}^{(n)}\right\| \\
&< \varepsilon_1^{\frac{M}{p_{j-1}}} + \left\|\varsigma_{j-2}^{(m)}-\varsigma_{j-2}^{(n)}\right\|,
\end{aligned}
$$

$$
\therefore \left\|\varsigma_j^{(m)}-\varsigma_j^{(n)}\right\| < \varepsilon_1^{\frac{M}{p_j}} + \varepsilon_1^{\frac{M}{p_{j-1}}} + \left\|\varsigma_{j-2}^{(m)}-\varsigma_{j-2}^{(n)}\right\| ,
$$

$$\vdots$$

$$
\left\|\varsigma_j^{(m)}-\varsigma_j^{(n)}\right\| < \sum_{k=2}^{j} \varepsilon_1^{\frac{M}{p_k}} + \left\|\varsigma_1^{(m)}-\varsigma_1^{(n)}\right\|
$$

$$
\begin{aligned}
\left\|\varsigma_1^{(m)}-\varsigma_1^{(n)}\right\| &\le \left\|(\varsigma_1^{(m)}-\varsigma_1^{(n)})-(\varsigma_0^{(m)}-\varsigma_0^{(n)})\right\| + \left\|\varsigma_0^{(m)}-\varsigma_0^{(n)}\right\| \\
&< \varepsilon_1^{\frac{M}{p_1}} + 0 , \qquad ; \varsigma_0^{(m)} \text{ and } \varsigma_0^{(n)} \text{ are zero}
\end{aligned}
$$

$$
\therefore \left\|\varsigma_j^{(m)}-\varsigma_j^{(n)}\right\| < \sum_{k=1}^{j} \varepsilon_1^{\frac{M}{p_k}} .
$$

Since (p_j) is bounded sequence of positive real numbers then $\left(\dfrac{M}{p_k}\right)$ is also bounded sequence, that is $l \le \dfrac{M}{p_k} \le m$, when $l = \min\limits_{1\le k\le j}\left\{\dfrac{M}{p_k}\right\}$ and $m = \max\limits_{1\le k\le j}\left\{\dfrac{M}{p_k}\right\}$. Thus we have

$$
\varepsilon_1^{\,m} \le \varepsilon_1^{M/p_k} \le \varepsilon_1^{\,l} \qquad ; \text{for all } \varepsilon_1 \le 1 .
$$

Considers,

$$\sum_{k=1}^{j} \varepsilon_1^{\frac{M}{p_k}} \le \sum_{k=1}^{j} \varepsilon_1^{\,l} = j\varepsilon_1^{\,l} < j\left[\left(\frac{\varepsilon}{j}\right)^{1/l}\right]^{l} = \varepsilon .$$

Thus we can have

$$\left\|\varsigma_j^{(m)} - \varsigma_j^{(n)}\right\| < \sum_{k=1}^{j} \varepsilon_1^{\frac{M}{p_k}} \le \varepsilon \qquad , \forall m,n > N(\varepsilon) = N(\varepsilon_1) .$$

Hence for every fixed j, the sequence $(\varsigma_j^{(1)}, \varsigma_j^{(2)}, \varsigma_j^{(3)} \ldots)$ is a Cauchy sequence of X-valued [$\varsigma_j^{(m)} \in X$]. Since X is any Banach space thus it converges, say $\varsigma_j^{(m)} \to \varsigma_j$ as $m \to \infty$. Using these infinitely many limits $\varsigma_1, \varsigma_2, \varsigma_3, \ldots,$ we define $x = (\varsigma_1, \varsigma_2, \varsigma_3, \ldots)$ and want to show that $x \in bv(X,p)$ and $x_m \to x$. From (3.3) with $n \to \infty$ we have

$$d(x_m, x) = \left(\sum_{j=1}^{\infty} \left\|(\varsigma_j^{(m)} - \varsigma_j) - (\varsigma_{j-1}^{(m)} - \varsigma_{j-1})\right\|^{p_j}\right)^{\frac{1}{M}} \le \varepsilon_1 \quad ; \forall m > N(\varepsilon_1) \qquad (3.4)$$

Since $x_m = (\varsigma_j^{(m)}) \in bv(X,p)$, there is a real number k_m such that

$$\sum_{j=1}^{\infty} \left\|\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)}\right\|^{p_j} \le (k_m)^M \quad \text{or} \quad \left(\sum_{j=1}^{\infty} \left\|\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)}\right\|^{p_j}\right)^{\frac{1}{M}} \le (k_m) .$$

Now we want to that $x = (\varsigma_j) \in bv(x,p)$, that is we need to show that $\sum_{j=1}^{\infty} \left\|\varsigma_j - \varsigma_{j-1}\right\|^{p_j} < \infty$. Since

$$\left(\sum_{j=1}^{\infty} \left\|\varsigma_j - \varsigma_{j-1}\right\|^{p_j}\right)^{\frac{1}{M}}$$

$$= \left(\sum_{j=1}^{\infty} \left\|(\varsigma_j - \varsigma_{j-1}) - (\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)}) + (\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)})\right\|^{p_j}\right)^{\frac{1}{M}}$$

$$\le \left(\sum_{j=1}^{\infty} \left\|(\varsigma_j - \varsigma_{j-1}) - (\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)})\right\|^{p_j}\right)^{\frac{1}{M}} + \left(\sum_{j=1}^{\infty} \left\|\varsigma_j^{(m)} - \varsigma_{j-1}^{(m)}\right\|^{p_j}\right)^{\frac{1}{M}} \text{;by prop.3.1.2}$$

$$< \varepsilon + (k_m),$$

$$\therefore \quad \sum_{j=1}^{\infty}\left\|\varsigma_j - \varsigma_{j-1}\right\|^{p_j} < \infty .$$

That is, we can show that $x=(\varsigma_j)\in bv(X,p)$, and furthermore, the series (3.4) represents $d(x_m,x)$. Since $d(x_m,x)\to 0$ when $m\to\infty$ $(\varsigma_j^{(m)}\to\varsigma_j, \forall j\in N)$ hence $x_m\to x$. Since (x_m) was any Cauchy sequence in $bv(X,p)$, this prove completeness of $bv(X,p)$. It is not difficult to show that the semimetric d is translation invariance because the semimetric d induced by the paranorm $g(x)$. Hence we have $bv(X,p)$, when $p_j>1$, is a *Fre'chet* space. #

Next we want to show that $bv(X,p)$ is a K space when $p_k\le 1$. In several situations it is immaterial whether we work with a norm or paranorm. Now we define norm on $bv(X,p)$ when $p_k\le 1$ by $\|x\| = \sum_{k=1}^{\infty}\left\|x_k - x_{k+1}\right\|^{p_k} + \|\lim x\|$.

We can show that the $p_n : bv(X,p)\to X$ is linear mapping. Since

$$\begin{aligned} p_n(\alpha x+\beta y) &= (\alpha x+\beta y)_n \\ &= \alpha x_n + \beta y_n \\ &= \alpha p_n(x) + \beta p_n(y). \end{aligned}$$

Since $bv(X,p)$ and X are normed space and $p_n : bv(X,p)\to X$ is a linear operator. Where $D(p_n)\subset bv(X)$, p_n is said to be ***bounded*** if there is a real number c such that for all $x\in D(p_n)$, $\|p_n(x)\|\le c\|x\|$, and p_n is continuous if and only if p_n is bounded. Hence we will also have $bv(X,p)$ is $K-space$ when $p_k\le 1$, if we can show that p_n is bounded linear mapping.

Proposition 3.1.5 Let $x=(x_k)$ be any sequence and $p=(p_k)$ be a bounded sequence of positive real numbers then $\lim_{k\to\infty}\|x_k\|^{p_k}=0$ implies $\lim_{k\to\infty}\|x_k\|=0$.

Proof.

Since we have $p=(p_k)$ be a bounded sequence of positive real numbers.

For show this we also divide into two case, considers any $k \in N$;

if $\|x_k\| \le 1$; we have $\|x_k\|^{\sup p_k} \le \|x_k\|^{p_k} \le \|x_k\|^{\inf p_k}$. If $\lim\limits_{k\to\infty}\|x_k\|^{p_k} = 0$ since $\lim\limits_{k\to\infty}\|x_k\|^{\sup p_k} \le \lim\limits_{k\to\infty}\|x_k\|^{p_k}$, let $\sup p_k = m$ thus $\lim\limits_{k\to\infty}\|x_k\|^{m} = 0$. This implies $\lim\limits_{k\to\infty}\|x_k\| = 0$.

if $\|x_k\| > 1$; we have $\|x_k\|^{\inf p_k} \le \|x_k\|^{p_k} \le \|x_k\|^{\sup p_k}$. If $\lim\limits_{k\to\infty}\|x_k\|^{p_k} = 0$, since $\lim\limits_{k\to\infty}\|x_k\|^{\inf p_k} \le \lim\limits_{k\to\infty}\|x_k\|^{p_k}$, let $\inf p_k = l$ thus $\lim\limits_{k\to\infty}\|x_k\|^{l} = 0$. This implies $\lim\limits_{k\to\infty}\|x_k\| = 0$.

It is true that $\lim\limits_{k\to\infty}\|x_k\|^{p_k} = 0$ implies $\lim\limits_{k\to\infty}\|x_k\| = 0$ both $p_k \le 1$ and $p_k > 1$. #

Proposition 3.1.6 $bv(X,p)$ is $K-space$ when $p=(p_k)$ be a bounded sequence of positive real numbers and $p_k \le 1$.

Proof.

In the proof $bv(X,p)$ is $K-$ space. This means that we want to show that for each $n \in N$ the n^{th} coordinate mapping $p_n : bv(X,p) \to X$, define by $p_n(x) = x_n$, is continuous on $bv(X,p)$. Since we know that p_n is continuous if and only if it is bounded on $bv(X,p)$. Now we need to show that p_n is bounded on $bv(X,p)$. For each $(x_k) \in bv(X,p)$ it means $\sum\limits_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k} < \infty$, then $\lim\limits_{k\to\infty}\|x_k - x_{k+1}\|^{p_k} = 0$, this implies $\lim\limits_{k\to\infty}\|x_k - x_{k+1}\| = 0$, by proposition 3.1.5. Then there exists $k_0 \in N$ such that $\|x_k - x_{k+1}\| < 1, \forall k > k_0$ then we have $\|x_k - x_{k+1}\| \le \|x_k - x_{k+1}\|^{p_k}$, $\forall k > k_0$ because $p_k \le 1$. For each $1 \le k \le k_0$, if $\|x_k - x_{k+1}\| \le 1$ implies $\|x_k - x_{k+1}\| \le \|x_k - x_{k+1}\|^{p_k}$, if $\|x_k - x_{k+1}\| > 1$, we have $\|x_k - x_{k+1}\| = \dfrac{\|x_k - x_{k+1}\|}{\|x_k - x_{k+1}\|^{p_k}}\|x_k - x_{k+1}\|^{p_k}$, let $L_k = \dfrac{\|x_k - x_{k+1}\|}{\|x_k - x_{k+1}\|^{p_k}}$ ($\max\limits_{1\le k\le k_0} L_k < \infty$ and $L_k > 1$ since $\|x_k - x_{k+1}\|^{p_k} \le \|x_k - x_{k+1}\|$ where $p_k < 1$).

That is $\|x_k - x_{k+1}\| \le \max\limits_{1\le k\le k_0} L_k \|x_k - x_{k+1}\|^{p_k}$ for $1 \le k \le k_0$. Thus $\|x_k - x_{k+1}\| \le L\|x_k - x_{k+1}\|^{p_k}$, $\forall k \in N$ when $L = \max(1, \max\limits_{1\le k\le k_0}\sup L_k)$. Hence we obtain

$$\sum_{k=1}^{\infty}\|x_k - x_{k+1}\| \le L\sum_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k} \text{ , } \forall k \in N \text{ and } \|\lim x\| \le L\|\lim x\|.$$

Since $$x_n = \sum_{k=n}^{m}(x_k - x_{k+1}) + x_{m+1} \text{ ,}$$

thus $$\|x_n\| = \left\|\sum_{k=n}^{m}(x_k - x_{k+1}) + x_{m+1}\right\| . \quad \text{Let } m \to \infty ,$$

we have $$\|x_n\| = \left\|\sum_{k=n}^{\infty}(x_k - x_{k+1}) + \lim_{m\to\infty} x_{m+1}\right\|$$

$$= \left\|\sum_{k=n}^{\infty}(x_k - x_{k+1}) + \lim x\right\|$$

$$\le \left\|\sum_{k=n}^{\infty}(x_k - x_{k+1})\right\| + \|\lim x\|$$

$$\le \sum_{k=1}^{\infty}\|x_k - x_{k+1}\| + \|\lim x\|$$

$$\le L\sum_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k} + L\|\lim x\| \qquad \text{; from above}$$

$$= L\left(\sum_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k} + \|\lim x\|\right)$$

$$= L\|x\| \text{ ,}$$

$$\therefore \|p_n(x)\| = \|x_n\| \le L\|x\| \qquad , \forall x \in D(p_n) .$$

Thus p_n is a bounded linear operator. Hence p_n is a continuous linear operator. So we obtain $p_n(x) = x_n$ is continuous coordinate. Hence we yield that $bv(X,p)$ *is* $K-space$. #

Corollary 3.1.7 $bv(X,p)$ is $FK-space$ when $p_k \leq 1$.

Proof. From proposition 3.1.3 and 3.1.6 we obtain corollary 3.1.7. #

Remark 3.1.1 $bv(X)$ is a Banach space and in addition it is ***BK****-space*.

Since $bv(X)$ is $bv(X,p)$ when $p_k = 1, \forall k \in N$ and it is a normed space if we define

$$\|x\|_{bv(X)} = \|\lim x\| + \sum_{k=1}^{\infty}\|x_k - x_{k-1}\| \quad ,$$

we can check that $\|x\|_{bv(X)}$ satisfied norm definition as follow:

1. $\|x\|_{bv(X)} \geq 0$ obviously that $\|\lim x\| + \sum_{k=1}^{\infty}\|x_k - x_{k-1}\| \geq 0$

2. $\|x\|_{bv(X)} = 0$ if and only if $x = 0 = (0,0,0\ldots)$

($\Leftarrow$) If $x = 0$ then $\|x\|_{bv(X)} = \|\lim x\| + \sum_{k=1}^{\infty}\|x_k - x_{k-1}\| = \|0\| + \sum_{k=1}^{\infty}\|0\| = 0$.

($\Rightarrow$) If $\|\lim x\| + \sum_{k=1}^{\infty}\|x_k - x_{k-1}\| = 0$ then $\|\lim x\| = 0$ and $\sum_{k=1}^{\infty}\|x_k - x_{k-1}\| = 0$.

We have $\lim x = 0$ and $\|x_k - x_{k-1}\| = 0$, thus $x_k = x_{k-1}, \forall k \in N$. Since $\lim x = \lim_{k\to\infty} x_k = \lim_{k\to\infty} x_{k-1} = 0$, hence $x_k = x_{k-1} = 0, \forall k \in N$. That is $x = 0$

3. $\|\alpha x\|_{bv(X)} = |\alpha|\ \|x\|_{bv(X)}$.

Since $\|\lim \alpha x\| + \sum_{k=1}^{\infty}\|\alpha x_k - \alpha x_{k-1}\| = |\alpha|\ \|\lim x\| + |\alpha|\sum_{k=1}^{\infty}\|x_k - x_{k-1}\|$

$$= |\alpha| \, (\|\lim x\| + \sum_{k=1}^{\infty} \|x_k - x_{k-1}\|)$$

$$= |\alpha| \; \|x\|_{bv(X)} .$$

4. $\|x + y\|_{bv(X)} \leq \|x\|_{bv(X)} + \|y\|_{bv(X)}$.

Since $\|\lim(x + y)\| + \sum_{k=1}^{\infty} \|(x + y)_k - (x + y)_{k-1}\|$

$$\begin{aligned}
&= \|\lim(x + y)\| + \sum_{k=1}^{\infty} \|(x_k - x_{k-1}) + (y_{k-1} - y_{k-1})\| \\
&\leq \|\lim x\| + \|\lim y\| + \sum_{k=1}^{\infty} (\|x_k - x_{k-1}\| + \|y_k - y_{k-1}\|) \\
&\leq \|\lim x\| + \|\lim y\| + (\sum_{k=1}^{\infty} \|x_k - x_{k-1}\| + \sum_{k=1}^{\infty} \|y_k - y_{k-1}\|) \\
&= \|x\|_{bv(X)} + \|y\|_{bv(X)} .
\end{aligned}$$

When we induce a metric d by the norm. We can say that $bv(X)$ is Banach space. Furthermore, we know that $bv(X,p)$ is $K-space$ when $p_k \leq 1$, thus $bv(X)$ is also ***BK***-space.

Proposition 3.1.8 $bv(X,p)$ has property AK.

Proof. From definition, $bv(X,p)$ has property AK if $\sum_{k=1}^{n} e^k(x_k) \to x$, $\forall x \in bv(X,p)$ as $n \to \infty$. Let $x = (x_k)$ be any sequence in $bv(X,p)$. Since $x = \sum_{k=1}^{\infty} e^k(x_k)$, thus $\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} e^k(x_k) \right\| = \left\| x - \sum_{k=1}^{\infty} e^k(x_k) \right\|$=0. That is $\sum_{k=1}^{n} e^k(x_k) \to x$, $\forall x \in bv(X,p)$ as $n \to \infty$. #

In addition, we already know from [1], the space $l(q)$ is a $FK-space$ with property AK under the paranorm $g(x)=\left(\sum_{k=1}^{\infty}\|x_k\|^{p_k}\right)^{1/M}$, where $M=\max\{1,\sup p_k\}$.

3.2 $Köthe-toeplizt$ dual of $bv(X,p)$

In the later chapter we shall characterize infinite matrices which transform bounded variation vector-valued sequence space into Maddox sequence space . For this, we need to use [1,lemma3.2.1]. We can see that the condition (1) in this lemma is equivalent to $Köthe\_toeplizt$ dual or β - dual .So in this part before we find out the β - dual, we now show that $l(X,p)\subset bv(X,p)$.

Proposition 3.2.1 Let $x=(x_k)$ be any sequences , If $\lim\limits_{k\to\infty}\|x_k\|=0$ then there exist L such that $\|x_{k+1}\|\le L\|x_k\|, \forall k\in N$.

Proof.

For each $k\in N$, if $\|x_{k+1}\|\le\|x_k\|$ nothing to do, if $\|x_{k+1}\|>\|x_k\|$, we have $\|x_{k+1}\|=\frac{\|x_{k+1}\|}{\|x_k\|}\|x_k\|$, let $L_k=\frac{\|x_{k+1}\|}{\|x_k\|}$ ($L_k>1$), so $\|x_{k+1}\|\le L_k\|x_k\|$. Since $\lim\limits_{k\to\infty}\|x_k\|=0$ that means $\sup L_k<\infty$; $L_k>1$ occurred finite time if not $\lim\limits_{k\to\infty}\|x_k\|\ne 0$ and let $L=\sup L_k$, Thus $\|x_{k+1}\|\le L\|x_k\|, \forall k\in N$. #

Proposition 3.2.2 $l(X,p)\subset bv(X,p)$, when (p_k) is a bounded sequence of positive real numbers.

Proof

Let $x=(x_k)\in l(X,p)$ this means $\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty$. Next we have to show that $\sum_{k=1}^{\infty}\|x_k-x_{k+1}\|^{p_k}<\infty$. Since $\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty$ that means $\lim\limits_{k\to\infty}\|x_k\|^{p_k}=0$, this implies $\lim\limits_{k\to\infty}\|x_k\|=0$, by proposition 3.1.5 then there exists a positive real number $L_1>1$ such that $\|x_{k+1}\|\le L_1\|x_k\|, \forall k\in N$, by proposition 3.2.1. Thus $\|x_{k+1}\|^{p_k}\le(L_1\|x_k\|)^{p_k}\le$

$L\|x_k\|^{p_k}$, $\forall k \in N$ when $L = \sup L_1^{p_k}$ ($\sup L_1^{p_k} < \infty$ because (p_k) bounded sequence). Hence $\sum_{k=1}^{\infty}\|x_{k+1}\|^{p_k} \leq L\sum_{k=1}^{\infty}\|x_k\|^{p_k}$. For show that $\sum_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k} < \infty$, we divide (p_k) in to two cases :

Case $p_k \leq 1$; we know that

$$\begin{aligned}\sum_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k} &\leq \sum_{k=1}^{\infty}\|x_k\|^{p_k} + \sum_{k=1}^{\infty}\|x_{k+1}\|^{p_k} \text{ ; by proposition 3.1.2} \\ &\leq \sum_{k=1}^{\infty}\|x_k\|^{p_k} + L\sum_{k=1}^{\infty}\|x_k\|^{p_k} \text{ ; by above} \\ &< \infty.\end{aligned}$$

$\therefore$ $x = (x_k) \in bv(X,p)$, thus $l(X,p) \subset bv(X,p)$ for $p_k \leq 1$.

Case $p_k > 1$; for $P = \sup p_k$, we know that

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\|x_k - x_{k+1}\|^{p_k}\right)^{1/P} &\leq \left(\sum_{k=1}^{\infty}\|x_k\|^{p_k}\right)^{1/P} + \left(\sum_{k=1}^{\infty}\|x_{k+1}\|^{p_k}\right)^{1/P} \text{ ; by proposition 3.1.2} \\ &\leq \left(\sum_{k=1}^{\infty}\|x_k\|^{p_k}\right)^{1/P} + \left(\sum_{k=1}^{\infty}(L_1\|x_k\|)^{p_k}\right)^{1/P} \text{ ; by above} \\ &\leq \left(\sum_{k=1}^{\infty}\|x_k\|^{p_k}\right)^{1/P} + \left(L\sum_{k=1}^{\infty}\|x_k\|^{p_k}\right)^{1/P} \text{ ; by above} \\ &< \infty.\end{aligned}$$

$\therefore$ $x = (x_k) \in bv(X,p)$, thus $l(X,p) \subset bv(X,p)$ for $p_k > 1$.

Thus we can conclude that $l(X,p) \subset bv(X,p)$ both $p_k > 1$ and $p_k \leq 1$. #

The following propositions are β - dual of $bv(X,p)$ when $p_k \leq 1$ and $p_k > 1$.

Proposition 3.2.3 *Let* (f_k) *be a sequence of continuous linear functional on X and* $p = (p_k)$ *a bounded sequence of positive real numbers with* $p_k \leq 1$. *Then* $\sum_{k=1}^{\infty} f_k(x_k)$ *converges for all* $x = (x_k) \in bv(X,p)$ *if and only if there exists* $M \in N$ *such that*

$\sup_j \|g_j\| M^{-\frac{1}{p_j}} < \infty$, *where* g_j *is the bounded linear functional on* X *define by* $g_j(x) = \sum_{k=j}^{\infty} f_k(x)$ *for all* $x \in X$.

Proof.

Suppose that there exist $M \in N$ such that $\sup_j \|g_j\| M^{-\frac{1}{p_j}} < \infty$, then there exist a $K > 0$ such that $\|g_j\| \le KM^{\frac{1}{p_j}}$ for all $j \in N$, and for each $x = (x_k) \in bv(X,p)$, we know that $(z_j) = (x_j - x_{j-1}) \in l(X,p)$ then there exist $j_0 \in N$ such that $M^{\frac{1}{p_j}} \|z_j\| \le 1$,$\forall j \ge j_0$. By $p_j \le 1$ for all $j \ge j_0$,

we have
$$M^{\frac{1}{p_j}} \|z_j\| \le \left(M^{\frac{1}{p_j}} \|z_j\| \right)^{p_j} = M \|z_j\|^{p_j}.$$

Consider,

$$\begin{aligned}
\sum_{k=1}^{\infty} f_k(x_k) &= \lim_{J\to\infty} \sum_{k=1}^{J} f_k(x_k) \\
&= \lim_{J\to\infty} \left[\sum_{j=1}^{J} g_j(x_j - x_{j-1}) - g_{J+1}(x_J) \right] \\
&= \sum_{j=1}^{\infty} g_j(x_j - x_{j-1}) && ;\ g_{J+1}(x_J) \to 0, J \to \infty \\
&= \sum_{j=1}^{\infty} g_j(z_j) && ;\ z_j = x_j - x_{j-1}
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{\infty} |g_j(z_j)| &\le \sum_{j=1}^{\infty} \|g_j\| \|z_j\| \\
&= \sum_{j=1}^{j_0} \|g_j\| \|z_j\| + \sum_{j_0+1}^{\infty} \|g_j\| \|z_j\| \\
&\le \sum_{j=1}^{j_0} \|g_j\| \|z_j\| + \sum_{j_0+1}^{\infty} KM^{\frac{1}{p_j}} \|z_j\| && ;\quad \|g_j\| \le KM^{\frac{1}{p_j}} \\
&\le \sum_{j=1}^{j_0} \|g_j\| \|z_j\| + K \sum_{j_0+1}^{\infty} M \|z_j\|^{p_j} && ;\quad \text{above} \\
&= \sum_{j=1}^{j_0} \|g_j\| \|z_j\| + KM \sum_{j_0+1}^{\infty} \|z_j\|^{p_j} \\
&< \infty.
\end{aligned}$$

This implies $\sum_{j=1}^{\infty} g_j(z_j)$ converges, since we have $\sum_{k=1}^{\infty} f_k(x_k) = \sum_{j=1}^{\infty} g_j(z_j)$, so we obtain that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $(x_k) \in bv(X,p)$.

Conversely, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $(x_k) \in bv(X,p)$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $(x_k) \in l(X,p)$ because $l(X,p) \subset bv(X,p)$. For each $x=(x_k) \in l(X,p)$, choose scalar sequence (t_k) with $|t_k|=1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in l(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \qquad \text{for all } x=(x_k) \in l(X,p). \tag{3.5}$$

Now suppose that $\sup_j \|g_j\| m^{-\frac{1}{p_j}} = \infty$ for all $m \in N$. For each $i \in N$, choose sequence (m_i) and (j_i) of positive integers with $m_1 < m_2 < m_3 \ldots$ and $j_1 < j_2 < j_3 \ldots$ such that $m_i > 2^i$ and $\|g_{j_i}\| m_i^{-\frac{1}{p_{j_i}}} > 1$. Choose $x_{j_i} \in X$ with $\|x_{j_i}\|=1$ such that

$$|g_{j_i}(x_{j_i})| m_i^{-\frac{1}{p_{j_i}}} = \left|\sum_{k=j_i}^{\infty} f_k(x_{j_i})\right| m_i^{-\frac{1}{p_{j_i}}} > 1$$

with also

$$|f_{j_i}(x_{j_i})| m_i^{-\frac{1}{p_{j_i}}} > 1 \quad . \tag{3.6}$$

Let $y=(y_k)$, $y_k = m_i^{-\frac{1}{p_{j_i}}} x_k$ if $k=j_i$ for some i, and 0 otherwise. Then $\sum_{k=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} \frac{1}{m_i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$, so that $(y_k) \in l(X,p)$ and

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(y_k)| &= \sum_{i=1}^{\infty} \left| f_{j_i}\left(m_i^{-\frac{1}{p_{j_i}}} x_{j_i}\right)\right| \\ &= \sum_{i=1}^{\infty} m_i^{-\frac{1}{p_{j_i}}} |f_{j_i}(x_{j_i})| = \infty \qquad \text{; by (3.6)}\end{aligned}$$

and this contradictory with (3.5). Therefore, there exist $M \in N$ such that $\sup_j \|g_j\| M^{-\frac{1}{p_j}} < \infty$. The proof is complete. #

Proposition 3.2.4 *Let* (f_k) *be a sequence of continuous linear functional on* X *and* (p_k) *be a bounded sequence of positive real numbers with* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$. *Then* $\sum_{k=1}^{\infty} f_k(x_k)$ *converges for all* $x = (x_k) \in bv(X, p)$ *if and only if there exists* $M \in N$ *such that* $\sum_{j=1}^{\infty} \|g_j\|^{t_j} M^{-(t_j-1)} < \infty$, *where* g_j *is the bounded linear functional on* X *define by* $g_j(x) = \sum_{k=j}^{\infty} f_k(x)$ *for all* $x \in X$.

Proof.

Suppose that $\sum_{j=1}^{\infty} \|g_j\|^{t_j} M^{-(t_j-1)} < \infty$ for some $M \in N$. For each $x = (x_k) \in bv(X, p)$. Considers,

$$\begin{aligned}
\sum_{k=1}^{\infty} f_k(x_k) &= \lim_{J\to\infty} \sum_{k=1}^{J} f_k(x_k) \\
&= \lim_{J\to\infty} \left[\sum_{j=1}^{J} g_j(x_j - x_{j-1}) - g_{J+1}(x_J) \right] \\
&= \sum_{j=1}^{\infty} g_j(x_j - x_{j-1}) && ;\ g_{J+1}(x_J) \to 0, J \to \infty \\
&= \sum_{j=1}^{\infty} g_j(z_j) && ;\ z_j = x_j - x_{j-1}
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{\infty} |g_j(z_j)| &\le \sum_{j=1}^{\infty} \|g_j\| \|z_j\| \\
&= \sum_{j=1}^{\infty} \|g_j\| M^{-\frac{1}{p_j}} M^{\frac{1}{p_j}} \|z_j\| \\
&\le \sum_{j=1}^{\infty} \left(\|g_j\|^{t_j} M^{-\frac{t_j}{p_j}} + M \|z_j\|^{p_j} \right)
\end{aligned}$$

$$= \sum_{j=1}^{\infty} \|g_j\|^{t_j} M^{-(t_j-1)} + M \sum_{j=1}^{\infty} \|z_j\|^{p_j}$$

$$< \infty .$$

Which implies $\sum_{j=1}^{\infty} g_j(z_j)$ converges, thus $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in bv(X,p)$.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in bv(X,p)$ then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $(x_k) \in l(X,p)$ because $l(X,p) \subset bv(X,p)$. For each $x = (x_k) \in l(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in l(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \qquad \text{for all } x = (x_k) \in l(X,p). \tag{3.7}$$

We want to show that there exists $M \in N$ such that $\sum_{j=1}^{\infty} \|g_j\|^{t_j} M^{-(t_j-1)} < \infty$. Suppose that $\sum_{j=1}^{\infty} \|g_j\|^{t_j} m^{-(t_j-1)} = \infty, \forall m \in N$. For each $i \in N$, choose (m_i) and (j_i) of positive integers with $m_1 < m_2 < m_3 \ldots$ and $j_1 < j_2 < j_3 \ldots$ such that $m_i > 2^i$ and $\sum_{j_{i-1} < j \le j_i} \|g_j\|^{t_j} m_i^{-(t_j-1)} > 1$. Choose $x_j \in X$ with $\|x_j\| = 1$ such that

$$\sum_{j_{i-1} < j \le j_i} |g_j(x_j)|^{t_j} m_i^{-(t_j-1)} = \sum_{j_{i-1} < j \le j_i} \left| \sum_{k=j}^{\infty} f_k(x_j) \right|^{t_j} m_i^{-(t_j-1)} > 1 \text{ with also}$$

$$\sum_{j_{i-1} < j \le j_i} |f_j(x_j)|^{t_j} m_i^{-(t_j-1)} > 1 \qquad , \forall i \in N.$$

Let $\qquad a_i = \sum_{j_{i-1} < j \le j_i} |f_j(x_j)|^{t_j} m_i^{-(t_j-1)} \qquad$ and

put $y = (y_j)$, $y_j = a_i^{-1} m_i^{-(t_j-1)} |f_j(x_j)|^{t_j-1} x_j \qquad$ for $j_{i-1} < j < j_i$.

For each $i \in N$, we have

$$\begin{aligned}
\sum_{j_{i-1}<j<j_i} \left\|y_j\right\|^{p_j} &= \sum_{j_{i-1}<j\le j_i} \left\| a_i^{-1} m_i^{-(t_j-1)} \left|f_j(x_j)\right|^{t_j-1} x_j \right\|^{p_j} \\
&= \sum_{j_{i-1}<j<j_i} a_i^{-p_j} m_i^{-t_j} \left|f_j(x_j)\right|^{t_j} \qquad ;\ p_j(t_j-1)=t_j \\
&= \sum_{j_{i-1}<j<j_i} a_i^{-p_j} m_i^{-1} m_i^{-(t_j-1)} \left|f_j(x_j)\right|^{t_j} \\
&\le a_i^{-1} m_i^{-1} \sum_{j_{i-1}<j<j_i} m_i^{-(t_j-1)} \left|f_j(x_j)\right|^{t_j} \qquad ;\ -p_j<-1 \\
&= a_i^{-1} m_i^{-1} a_i \quad = \quad m_i^{-1} < \frac{1}{2^i}.
\end{aligned}$$

So we have that $\sum_{j=1}^{\infty} \left\|y_j\right\|^{p_j} \le \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty$. Hence, $y=(y_j) \in l(X,p)$.

For each $i \in N$, we have

$$\begin{aligned}
\sum_{j_{i-1}<j<j_i} \left|f_j(y_j)\right| &= \sum_{j_{i-1}<j<j_i} \left| f_j(a_i^{-1} m_i^{-(t_j-1)} \left|f_j(x_j)\right|^{t_j-1} x_j \right| \\
&= \sum_{j_{i-1}<j<j_i} a_i^{-1} m_i^{-(t_j-1)} \left|f_j(x_j)\right|^{t_j} \\
&= a_i^{-1} \sum_{j_{i-1}<j<j_i} m_i^{-(t_j-1)} \left|f_j(x_j)\right|^{t_j} \\
&= 1.
\end{aligned}$$

Then $\sum_{j=1}^{\infty} \left|f_j(y_j)\right| = \infty$, which contradicts with (3.7). The proof is complete. #

CHAPTER 4

MATRIX TRANSFORMATIONS ON THE BOUNDED VARIATION VECTOR-VALUED SEQUENCE SPACE

The main objective, to characterize infinite matrices which transform vector-value sequence space of $bv(X,p)$ into Maddox scalar-valued sequence space and apply the previous results to obtain necessary and sufficient conditions for infinite matrices mapping $bv(X,p)$ into $M_\infty(q)$, $\underline{l_\infty}(q)$, $E_r(q)$, $F_r(q)$, when $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers, will be shown in this chapter. We categorize this chapter into two parts.

4.1 Mapping into Maddox Sequence Spaces

In the first part we discuss some theorems that characterize infinite matrices which transform $bv(X,p)$ into Maddox scalar-valued sequence space. Furthermore, we divide this part in to two cases, in the case $p_k \le 1$ and in the case $p_k > 1$.

4.1.1 In the Case $p_k \le 1$

Theorem 4.1.1.1($A : bv(X,p) \to l(q); p_k \le 1$)

Let $A = \left(f_k^n\right)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers with* $p_k \le 1$. *Then* $A : bv(X,p) \to l(q)$ *if and only if*

(1) for each $n \in N$, *there is* $M_n \in N$ *such that* $\sup_j \left\|g_j^n\right\| M_n^{-1/p_j} < \infty$, *where* g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$, $\forall x \in X$.

(2) for every $k \in N$, $\sum_{n=1}^{\infty} \left| f_k^n(x) \right|^{q_n} < \infty$ *for all* $x \in X$ *and*

(3) for each $r \in N$, *there exists* $M_r \in N$ *such that*

$$\|\lim x\| + \sum_{k \in K} \|x_k - x_{k-1}\|^{p_k} < \frac{1}{M_r} \Rightarrow \sum_{n=1}^{\infty} \left| \sum_{k \in K} f_k^n(x_k) \right|^{q_n} < \frac{1}{r}$$

for all $x = (x_k) \in \Phi(X)$ *and for all subsets* K *of* N.

Proof.

Assume that $A : bv(X,p) \to l(q)$. Since $bv(X,p)$ and $l(q)$ are $FK-$space, by [1,Lemma 3.2.1] and proposition 3.2.3 the condition (1) and (2) are obtained. Now we have to show that (3) holds. Since $bv(X,p)$ and $l(q)$ are $FK-$space and $bv(X,p)$ has $AK-$property, we have by [1,Lemma 3.2.1] that $A : \Phi(X) \to l(q)$ is continuous when $\Phi(X)$ consider as a subspace of $bv(X,p)$. Let $\varepsilon > 0$ be given, then there exists $\delta > 0$ such that

$$\sum_{n=1}^{\infty} \left| \sum_{k \in K} f_k^n(x_k) \right|^{q_n} < \varepsilon \qquad \text{for all } x = (x_k) \in \Phi(X),$$

$$\|x\| = \|\lim x\| + \sum_{k \in K} \|x_k - x_{k-1}\|^{p_k} < \delta .$$

That is for each $\varepsilon > 0, \exists r \in N$ with $r \geq \frac{1}{\varepsilon}$, for each r there exists $M_r \in N$ with $M_r \geq \frac{1}{\delta}$ such that

$$\sum_{n=1}^{\infty} \left| \sum_{k \in K} f_k^n(x_k) \right|^{q_n} < \frac{1}{r} \qquad \text{for all } x = (x_k) \in \Phi(X)$$

when $$\|x\| = \|\lim x\| + \sum_{k \in K} \|x_k - x_{k-1}\|^{p_k} < \frac{1}{M_r}.$$

Thus the condition (3) holds.

Conversely, assume that the condition (1), (2), (3) hold .The condition (1), by proposition 3.2.3, implies that $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x=(x_k)\in bv(X,p)$. By condition (2) we have $\left(f_k^n(x)\right)_{n=1}^{\infty}\in l(q)$, for all $k\in N$ and for all $x\in X$, thus $A:\Phi(X)\to l(q)$. Now, we shall show that $A:\Phi(X)\to l(q)$ is continuous when $\Phi(X)$ is considered as subspace $bv(X,p)$. For each $r\in N$, there exist $\varepsilon>0$ such that $\varepsilon\geq\frac{1}{r}$. For each $M_r\in N$, there exist $\delta>0$ such that $\delta\geq\frac{1}{M_r}$, thus we have $\sum_{n=1}^{\infty}\left|\sum_{k\in K} f_k^n(x_k)\right|^{q_n}<\frac{1}{r}\leq\varepsilon$ when $\|\lim x\|+\sum_{k\in K}\|x_k-x_{k-1}\|^{p_k}<\frac{1}{M_r}$ for all $x=(x_k)\in\Phi(X)$ and for all subsets K of N, that is $A:\Phi(X)\to l(q)$ is continuous By [1,Lemma 3.2.1] $A:bv(X,p)\to l(q)$ is obtained. #

Theorem 4.1.1.2($A:bv(X,p)\to l(q)\,;\,p_k\leq 1$)

Let $A=\left(f_k^n\right)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers with* $p_k\leq 1$. *Then* $A:bv(X,p)\to l(q)$ *if*

$$\sup_j\sum_{n=1}^{\infty}\left\|g_j^n\right\|^{q_n}<\infty\ , \tag{4.1}$$

when g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x\in X$ *and for all* $n\in N$.

Proof.

Let $\sup_j\sum_{n=1}^{\infty}\left\|g_j^n\right\|^{q_n}<\infty$, that is for each $n\in N,\sup_j\left\|g_j^n\right\|<\infty$. And let $x=(x_k)\in bv(X,p)$, so that we have $\sum_{k=1}^{\infty}\|x_k-x_{k-1}\|^{p_k}<\infty$, say converges to L. Considers,

$$A_n(x)\ =\ \sum_{k=1}^{\infty} f_k^n(x_k)\ =\lim_{J\to\infty}\sum_{k=1}^{J} f_k^n(x_k)$$

$$= \lim_{J\to\infty}\left[\sum_{j=1}^{J} g_j^n(x_j - x_{j-1}) - g_{J+1}^n(x_J)\right]$$

$$= \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1}) \qquad ;\ g_{J+1}(x_J) \to 0, J \to \infty,$$

Let us write $H = \max\{1, \sup q_n\}$ then we observe that $|\lambda|^{q_n} \le \max(1, |\lambda|^H)$, considers

$$|A_n(x)|^{q_n} = \left|\sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1})\right|^{q_n}$$

$$\le \left(\sum_{j=1}^{\infty} \|g_j^n\| \|x_j - x_{j-1}\|\right)^{q_n}$$

$$\le \left(\sup_j \|g_j^n\| \sum_{j=1}^{\infty} \|x_j - x_{j-1}\|\right)^{q_n} \qquad ;\ \|g_j^n\| \le \sup_j \|g_j^n\|, \forall n \in N$$

$$\le \left(\sup_j \|g_j^n\| L_1 \sum_{j=1}^{\infty} \|x_j - x_{j-1}\|^{p_j}\right)^{q_n} \qquad ;\ \sum_{j=1}^{\infty} \|x_j - x_{j-1}\| \le L_1 \sum_{j=1}^{\infty} \|x_j - x_{j-1}\|^{p_j}$$

$$= \sup_j \|g_j^n\|^{q_n} L_2^{q_n} \qquad ;\ L_2 = L_1 L.$$

So that $\sum_{n=1}^{\infty} |A_n(x)|^{q_n} \le \sum_{n=1}^{\infty} \sup_j \|g_j^n\|^{q_n} L_2^{q_n}$

$$\le M \sum_{n=1}^{\infty} \sup_j \|g_j^n\|^{q_n},$$

where $M = \max(1, L_2^H)$. Hence from $\sup_j \sum_{n=1}^{\infty} \|g_j^n\|^{q_n} < \infty$ so we have $\sum_{n=1}^{\infty} |A_n(x)|^{q_n}$ $< \infty$. Thus $A : bv(X,p) \to l(q)$. #

The following from here we shall characterize infinite matrices which mapping from $bv(X,p)$ into $l_\infty(q)$, $c_0(q)$ and $c(q)$. When g_j^n is the bounded linear functional on X define by $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ for all $x \in X$ and for all $n \in N$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} f_k^n(x_k) &= \lim_{J\to\infty}\sum_{k=1}^{J} f_k^n(x_k)\\ &= \lim_{J\to\infty}\left[\sum_{j=1}^{J} g_j^n(x_j - x_{j-1}) - g_{J+1}^n(x_J)\right]\\ &= \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1}) \qquad ;\ g_{J+1}(x_J)\to 0, J\to\infty.\end{aligned}$$

Let $z_j = x_j - x_{j-1}$ so that when $(x_k)\in bv(X,p)$, we have $(z_j)\in l(X,p)$. When the infinite matrix $A=\left(f_k^n\right)$ and $B=\left(g_j^n\right)$, we have $A: bv(X,p)\to l_\infty(q)$, $A: bv(X,p)\to c_0(q)$, $A: bv(X,p)\to c(q)$ if and only if $B: l(X,p)\to l_\infty(q)$, $B: l(X,p)\to c_0(q)$, $B: l(X,p)\to c(q)$ respectively. Since we already have some theorems from [1] to characterize infinite matrix $A: bv(X,p)\to l_\infty(q)$, $A: bv(X,p)\to c_0(q)$, $A: bv(X,p)\to c(q)$.

Theorem 4.1.1.3$(A: bv(X,p)\to l_\infty(q))$

Let $A=\left(f_k^n\right)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers with* $p_k\le 1$. *Then* $A: bv(X,p)\to l_\infty(q)$ *if and only if there exists* $M\in N$ *such that*

$$\left\|g_j^n\right\|\le M^{1/p_j + 1/q_n} \qquad \text{for all } n, j\in N. \tag{4.2}$$

When g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x\in X$ *and, for all* $n\in N$.

Proof.

When g_j^n is the bounded linear functional on X define by $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$, for all $x\in X$ and for all $n\in N$, so we have $\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1})$. Let $z_j = x_j - x_{j-1}$, so that when $(x_k)\in bv(X,p)$, we have $(z_j)\in l(X,p)$. Hence if

$A=\left(f_k^n\right)$ and $B=\left(g_j^n\right)$, we have $A: bv(X,p)\to l_\infty(q)$ if and only if $B: l(X,p)\to l_\infty(q)$. Since we have theorem about $B: l(X,p)\to l_\infty(q)$. By [1,theorem1.3], thus we have $A: bv(X,p)\to l_\infty(q)$ if and only if $\left\|g_j^n\right\|\le M_1^{1/p_1+1/q_n}$, $\forall n,j\in N$ .The proof is complete. #

Theorem 4.1.1.4($A: bv(X,p)\to c_0(q)$)

Let $A=\left(f_k^n\right)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers with* $p_k\le 1$. *Then* $A: bv(X,p)\to c_0(q)$ *if and only if*

(1) for all $m,k\in N$, $m^{1/q_n} g_j^n \xrightarrow{w^*} 0$ *as* $n\to\infty$,

(2) for each $m\in N$ *there exists* $M_m\in N$ *such that* $m^{p_j/q_n}\left\|g_j^n\right\|^{p_j}\le M_m$ *for all* $n,j\in N$.

When g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x\in X$ *and for all* $n\in N$.

Proof.

When g_j^n is the bounded linear functional on X define by $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ for all $x\in X$ and, for all $n\in N$, so we have $\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{j=1}^{\infty} g_j^n(x_j-x_{j-1})$. Let $z_j=x_j-x_{j-1}$ so that when $(x_k)\in bv(X,p)$, we have $(z_j)\in l(X,p)$. Hence if $A=\left(f_k^n\right)$ and $B=\left(g_j^n\right)$, we have $A: bv(X,p)\to c_0(q)$ if and only if $B: l(X,p)\to c_0(q)$.

$$\Leftrightarrow\ B: l(X,p)\to \bigcap_{m=1}^{\infty} c_{0(m^{1/q_n})}$$

$$\Leftrightarrow\ (m^{1/q_n} g_j^n): l(X,p)\to c_0,\ \text{for all } m\in N.$$

By[1,theorem1.5 and proposition 2.3(i), we have $A: bv(X,p)\to c_0(q)$ if and only if the conditions (1)and (2) hold. #

Theorem 4.1.1.5($A : bv(X,p) \to c(q)$) *Let* $A = (f_k^n)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers with* $p \le 1$. *Then* $A : bv(X,p) \to c(q)$ *if and only if there is a sequence* (g_j) *with* $g_j \in X'$ *for all* $j \in N$ *such that*

(1) for some $M \in N$, $\sup_j \|g_j\| M^{-1/p_j} < \infty$,

(2) for all $m, j \in N$, $m^{1/q_n}(g_j^n - g_j) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *and*

(3) for each $m \in N$ $\sup_j m^{p_j/q_n} \|g_j^n - g_j\|^{p_j} < \infty$ *for all* $n, j \in N$.

Where g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$ *and for all* $n \in N$.

Proof.

When g_j^n is the bounded linear functional on X define by $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ for all $x \in X$ and, for all $n \in N$, so we have $\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1})$. Let $z_j = x_j - x_{j-1}$ so that when $(x_k) \in bv(X,p)$, we have $(z_j) \in l(X,p)$. Hence if $A = (f_k^n)$ and $B = (g_j^n)$, we have $A : bv(X,p) \to c(q)$ if and only if $B : l(X,p) \to c(q)$. By [1,theorem 3.1.8], we have $A : bv(X,p) \to c(q)$ if and only if the conditions (1), (2) and (3) hold. #

4.1.2 In the Case $p_k > 1$

This part we give characterization infinite matrix which transformations from $bv(X,p)$ into Maddox scalar-value sequence space when $p_k > 1$. Similar in the previous section we can have $A : bv(X,p) \to l(q)$, $A : bv(X,p) \to l_\infty(q)$, $A : bv(X,p) \to c_0(q)$, $A : bv(X,p) \to c(q)$ because we already have theorems that characterize the infinite matrices $A:l(X,p)\to l(q)$, $A:l(X,p)\to l_\infty(q)$, $A:l(X,p)\to c_0(q)$, $A:l(X,p)\to c(q)$ in [1].

Theorem 4.1.2.1($bv(X,p)\to l(q)$)

Let $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers such that* $p_k>1$ *and* $\frac{1}{p_k}+\frac{1}{t_k}=1$ *for all* $k\in N$, *and* $A=\left(f_k^n\right)$ *an infinite matrix. Then* $A:bv(X,p)\to l(q)$ *if and only if*

(1) for each $n\in N$, *there exists* $M_n\in N$ *such that* $\sum_{j=1}^{\infty}\left\|g_j^n\right\|^{t_j}M_n^{-(t_j-1)}<\infty$,

(2) for each $j\in N$, $\sum_{n=1}^{\infty}\left|\sum_{k=j}^{\infty}f_k^n(x)\right|^{q_n}<\infty$ *for every* $x\in X$ *and*

(3) for each $r\in N$ *there exists* $M_r\in N$ *such that*

$$\sum_{k\in K}\left\|x_k-x_{k-1}\right\|^{p_k}\ \Rightarrow\ \sum_{n=1}^{\infty}\left|\sum_{j\in K}\left(\sum_{k=j}^{\infty}f_k^n(x_j-x_{j-1})\right)\right|^{q_n}<\frac{1}{r}$$

for all $x=(x_k)\in\Phi(X)$ *and all finite subsets* K *of* N. *Where* g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty}f_k^n(x)$ *for all* $x\in X$.

Proof.

When g_j^n is the bounded linear functional on X define by $g_j^n(x)=\sum_{k=j}^{\infty}f_k^n(x)$ for all $x\in X$ and, for all $n\in N$, so we have $\sum_{k=1}^{\infty}f_k^n(x_k)\ =\sum_{j=1}^{\infty}g_j^n(x_j-x_{j-1})$. Let $z_j=x_j-x_{j-1}$ so that when $(x_k)\in bv(X,p)$, we have $(z_j)\in l(X,p)$. Then if $A=\left(f_k^n\right)$ and $B=\left(g_j^n\right)$, we have $A:bv(X,p)\to l(q)$ if and only if $B:l(X,p)\to l(q)$. Since we have $B:l(X,p)\to l(q)$ in [1,theorem 4.1.1] and proposition 3.2.4, thus the condition (1), (2) and (3) are hold. #

Theorem 4.1.2.2 ($bv(X,p) \to l_\infty(q)$)

Let $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers such that* $p_k > 1$ *and* $\dfrac{1}{p_k}+\dfrac{1}{t_k}=1$ *for all* $k \in N$ *and* $A=\left(f_k^n\right)$ *an infinite matrix. Then* $A: bv(X,p) \to l_\infty(q)$ *if and only if there exists* $M \in N$ *such that*

$$\sup_n \left(\sum_{j=1}^{\infty} \left\| g_j^n \right\|^{t_j} M^{-t_j/q_n} \right) < \infty \tag{4.3}$$

Where g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$.

Proof.

When g_j^n is the bounded linear functional on X define by $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ for all $x \in X$ and, for all $n \in N$, so we have $\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1})$. Let $z_j = x_j - x_{j-1}$ so that when $(x_k) \in bv(X,p)$, we have $(z_j) \in l(X,p)$. Hence if $A=\left(f_k^n\right)$ and $B=\left(g_j^n\right)$, we have $A: bv(X,p) \to l_\infty(q)$ if and only if $B: l(X,p) \to l_\infty(q)$. By [1,theorem 1.4] thus the $\sup_n \left(\sum_{j=1}^{\infty} \left\| g_j^n \right\|^{t_j} M^{-t_j/q_n} \right) < \infty$ holds. #

Theorem 4.1.2.3 ($bv(X,p) \to c_0(q)$)

Let $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers such that* $p_k > 1$ *and* $\dfrac{1}{p_k}+\dfrac{1}{t_k}=1$ *for all* $k \in N$ *and* $A=\left(f_k^n\right)$ *an infinite matrix. Then* $A: bv(X,p) \to c_0(q)$ *if and only if*

(1) for all $m,k \in N$, $m^{1/q_n} g_j^n \xrightarrow{w^*} 0$ *as* $n \to \infty$,

(2) *for each* $m \in N$, $\sum_{j=1}^{\infty}(m^{t_j/q_n}\left\|g_j^n\right\|^{t_j} r^{-(t_j-1)})$ *as* $r \to \infty$, *uniformly for* $n \geq 1$.

Where g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$.

Proof.

When g_j^n is the bounded linear functional on X define by $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ for all $x \in X$ and, for all $n \in N$, so we have $\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1})$. Let $z_j = x_j - x_{j-1}$ so that when $(x_k) \in bv(X,p)$, we have $(z_j) \in l(X,p)$. Hence if $A = (f_k^n)$ and $B = (g_j^n)$, we have $A : bv(X,p) \to c_0(q)$ if and only if $B : l(X,p) \to c_0(q)$. Since we have [1,theorem 4.1.3], thus the conditions (1) and (2) hold. #

Theorem 4.1.2.4 ($bv(X,p) \to c(q)$)

Let $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers such that* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : bv(X,p) \to c(q)$ *if and only if there is a sequence* (g_j) *with* $g_j \in X'$ *for all* $j \in N$ *such that*

(1) there exists $M \in N$, $\sum_{j=1}^{\infty}\left\|g_j\right\|^{t_j} M^{-(t_j-1)} < \infty$,

(2) for all $m,k \in N$, $m^{1/q_n}(g_j^n - g_j) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *and*

(3) for each $m \in N$, $\sum_{j=1}^{\infty}(m^{t_j/q_n}\left\|g_j^n - g_j\right\|^{t_j} r^{-(t_j-1)})$ *as* $r \to \infty$, *uniformly for* $n \geq 1$.

Where g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$.

Proof.

When g_j^n is the bounded linear functional on X define by $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$, for all $x \in X$ and, for all $n \in N$, so we have $\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1})$. Let $z_j = x_j - x_{j-1}$ so that when $(x_k) \in bv(X,p)$, we have $(z_j) \in l(X,p)$. Hence if $A=(f_k^n)$ and $B=(g_j^n)$, we have $A: bv(X,p) \to c(q)$ if and only if $B: l(X,p) \to c(q)$. Since we have [1,theorem 4.1.4], thus the conditions (1), (2) and (3) are hold. #

4.2 Mapping into $M_\infty(q)$, $\underline{l_\infty}(q)$, $E_r(q)$ and $F_r(q)$

We now give characterization of the infinite matrix transformation $bv(X,p)$ into $M_\infty(q)$, $\underline{l_\infty}(q)$, $E_r(q)$ and $F_r(q)$ by use the previous results. Similarly in the previous part we divide in to two cases.

4.2.1 In the Case of $p_k \le 1$

Theorem 4.2.1.1$(A: bv(X,p) \to M_\infty(q); p_k \le 1)$

Let $A=(f_k^n)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers with* $p_k \le 1$. *Then* $A: bv(X,p) \to M_\infty(q)$ *if and only if*

(1) for each $m,n \in N$, *there is* $M_n \in N$ *such that* $\sup_j \left\| m^{1/q_n} g_j \right\| M_n^{-1/p_j} < \infty$, *where* g_j *is the bounded linear functional on* X *define by* $g_j(x)=\sum_{k=j} f_k^n(x)$, $\forall x \in X$.

(2) for every $m,k \in N$, $\sum_{n=1}^{\infty}\left|m^{1/q_n} f_k^n(x)\right| < \infty$ *for all* $x \in X$ *and*

(3) for each $m,r \in N$, *there exists* $M_r \in N$ *such that*

$$\|\lim x\| + \sum_{k\in K}\|x_k - x_{k-1}\|^{p_k} < \frac{1}{M_r} \Rightarrow \sum_{n=1}^{\infty}\left|\sum_{k\in K} m^{1/q_n} f_k^n(x_k)\right| < \frac{1}{r}$$

for all $x = (x_k) \in \Phi(X)$ *and for all subsets* K *of* N.

Proof.

By [1, proposition 2.3(vii)], we have $M_\infty(q) = \bigcap_{m=1}^{\infty} l_{(m^{1/q_n})}$. By [1, proposition 2.2 (ii) and (iv)], we have

$$\begin{aligned} A : bv(X,p) \to M_\infty(q) &\Leftrightarrow A : bv(X,p) \to \bigcap_{m=1}^{\infty} l_{(m^{1/q_n})} \\ &\Leftrightarrow A : bv(X,p) \to l_{(m^{1/q_n})} \quad \text{for all } m \in N \\ &\Leftrightarrow \left(m^{1/q_n} f_k^n\right)_{n,k} : bv(X,p) \to l \quad \text{for all } m \in N \end{aligned}$$

and we have the theorem 4.1.1.1 , $A : bv(X,p) \to l(q)$, taking $q_n = 1, \forall n \in N$.

Thus the conditions (1)-(3) hold. #

Theorem 4.2.1.2($A : bv(X,p) \to \underline{l_\infty}(q)$)

Let $A = \left(f_k^n\right)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers with* $p_k \le 1$. *Then* $A : bv(X,p) \to \underline{l_\infty}(q)$ *if and only if there exists* $M \in N$ *such that*

$$m^{1/q_n}\left\|g_j^n\right\| \le M^{\frac{1}{p_j}+1} \qquad \textit{for all } m,n,j \in N. \tag{4.4}$$

When g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$ *and, for all* $n \in N$.

Proof.

By [1, proposition 2.3(vi)], we have $\underline{l_\infty}(q)=\bigcap_{m=1}^{\infty} l_{\infty(m^{1/q_n})}$. By [1, proposition 2.2 (ii) and (iv)], we have

$$\begin{aligned} A: bv(X,p) \rightarrow \underline{l_\infty}(q) &\Leftrightarrow A: bv(X,p) \rightarrow \bigcap_{m=1}^{\infty} l_{\infty(m^{1/q_n})} \\ &\Leftrightarrow A: bv(X,p) \rightarrow l_{\infty(m^{1/q_n})} \quad \text{for all } m \in N \\ &\Leftrightarrow \left(m^{1/q_n} f_k^n\right)_{n,k} : bv(X,p) \rightarrow l_\infty \quad \text{for all } m \in N, \end{aligned}$$

and we have the theorem 4.1.1.3 ; $A: bv(X,p) \rightarrow l_\infty(q)$, taking $q_n = 1, \forall n \in N$. Thus the condition (4.4) hold. #

Theorem 4.2.1.3($A: bv(X,p) \rightarrow E_r(q)$**)**Let $A=(f_k^n)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers with* $p_k \le 1$, *and let* $r \ge 0$. *Then* $A: bv(X,p) \rightarrow E_r(q)$ *if and only if there exists* $M \in N$ *such that*

$$\left\| n^{-r/q_n} g_j^n \right\| \le M^{1/p_j + 1/q_n} \qquad \text{for all } n, j \in N. \tag{4.5}$$

When g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$ *and, for all* $n \in N$.

Proof.

By [1, proposition 2.3(viii)], we have $E_r(q)=l_\infty(q)_{(n^{-r/q_n})}$.

By [1, proposition 2.2(iv)], we have

$$A : bv(X,p) \rightarrow E_r(q) \;\Leftrightarrow\; A : bv(X,p) \rightarrow l_\infty(q)_{n^{-r/q_n}}$$

$$\Leftrightarrow\; \left(n^{-r/q_n} f_k^n\right)_{n,k} : bv(X,p) \rightarrow l_\infty(q) \;,$$

and we have the theorem 4.1.1.3 ; $A : bv(X,p) \rightarrow l_\infty(q)$.Thus the condition (4.5) holds. #

Theorem 4.2.1.4 ($A : bv(X,p) \rightarrow F_s(q) ; p_k \le 1$)

Let $A = (f_k^n)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers with* $p_k \le 1$ *and let* $s \ge 0$. *Then* $A : bv(X,p) \rightarrow F_s(q)$ *if and only if*

(1) for each $n \in N$, *there is* $M_n \in N$ *such that* $\sup_j \left\| n^{s/q_n} g_j \right\| M_n^{-1/p_j} < \infty$, *where* g_j *is the bounded linear functional on* X *define by* $g_j(x) = \sum_{k=j}^{\infty} f_k^n(x)$, $\forall x \in X$.

(2) for every $k \in N$, $\sum_{n=1}^{\infty} n^s \left| f_k^n(x) \right|^{q_n} < \infty$ *for all* $x \in X$ *and*

(3) for each $r \in N$, *there exists* $M_r \in N$ *such that*

$$\|\lim x\| + \sum_{k \in K} \|x_k - x_{k-1}\|^{p_k} < \frac{1}{M_r} \Rightarrow \sum_{n=1}^{\infty} n^s \left| \sum_{k \in K} f_k^n(x_k) \right|^{q_n} < \frac{1}{r}$$

for all $x = (x_k) \in \Phi(X)$ *and for all subsets* K *of* N.

Proof.

By [1, proposition 2.3(ix)], we have $F_s(q) = l(q)_{(n^{s/q_n})}$. By [1,proposition 2.2 (iv)], we have

$$A: bv(X,p) \to F_s(q) \;\Leftrightarrow\; A: bv(X,p) \to l(q)_{n^{s/q_n}}$$

$$\Leftrightarrow\; \left(n^{s/q_n} f_k^n\right)_{n,k} : bv(X,p) \to l(q)\;,$$

and we have the theorem 4.1.1.1 , $A: bv(X,p) \to l_\infty(q)$.Thus the conditions (1)-(3) hold.

#

4.2.2 In the Case of $p_k > 1$

Theorem 4.2.2.1 ($bv(X,p) \to M_\infty(q)$)

Let $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers such that* $p_k > 1$ *and* $\frac{1}{p_k}+\frac{1}{t_k}=1$ *for all* $k \in N$, *and* $A=\left(f_k^n\right)$ *an infinite matrix. Then* $A: bv(X,p) \to M_\infty(q)$ *if and only if*

(1) for each $n,m \in N$, *there exists* $M_n \in N$ *such that*

$$\sum_{j=1}^{\infty} m^{t_j/q_n} \|g_j\|^{t_j} M_n^{-(t_j-1)} < \infty,$$

(2) for each $j,m \in N$, $\sum_{n=1}^{\infty} m^{1/q_n} \left|\sum_{k=j}^{\infty} f_k^n(x)\right| < \infty$ *for every* $x \in X$ *and*

(3) for each $m,r \in N$ *there exists* $M_r \in N$ *such that*

$$\sum_{k\in K} \|x_k - x_{k-1}\|^{p_k} \;\Rightarrow\; \sum_{n=1}^{\infty} m^{1/q_n} \left|\sum_{j\in K}\left(\sum_{k=j}^{\infty} f_k^n(x_j - x_{j-1})\right)\right| < \frac{1}{r}$$

for all $x=(x_k) \in \Phi(X)$ *and all finite subsets* K *of* N. *Where* g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$.

Proof.

By [1, proposition 2.3(vii)], we have $M_{\infty}(q)=\bigcap_{m=1}^{\infty} l_{(m^{1/q_n})}$. By [1, proposition 2.2 (ii) and (iv)], we have

$$\begin{aligned} A:bv(X,p)\to M_{\infty}(q) &\Leftrightarrow A:bv(X,p)\to \bigcap_{m=1}^{\infty} l_{(m^{1/q_n})} \\ &\Leftrightarrow A:bv(X,p)\to l_{(m^{1/q_n})} \quad \text{for all } m\in N \\ &\Leftrightarrow \left(m^{1/q_n} f_k^n\right)_{n,k} :bv(X,p)\to l \quad \text{for all } m\in N, \end{aligned}$$

and we have the theorem 4.1.2.1 ; $A:bv(X,p)\to l(q)$, taking $q_n=1, \forall n\in N$.

Thus the conditions (1)-(3) hold. #

Theorem 4.2.2.2 ($bv(X,p)\to \underline{l}_{\infty}(q)$)

Let $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers such that* $p_k>1$ *and* $\frac{1}{p_k}+\frac{1}{t_k}=1$ *for all* $k\in N$ *and* $A=\left(f_k^n\right)$ *an infinite matrix. Then* $A:bv(X,p)\to \underline{l}_{\infty}(q)$ *if and only if there exists* $M\in N$ *such that*

$$\sup_n\left(\sum_{j=1}^{\infty} m^{t_j/q_n}\left\|g_j^n\right\|^{t_j} M^{-t_j/q_n}\right)<\infty \qquad \textit{for all } m\in N. \tag{4.6}$$

Where g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x\in X$.

Proof.

By [1, proposition 2.3(vi)], we have $\underline{l}_{\infty}(q)=\bigcap_{m=1}^{\infty} l_{\infty(m^{1/q_n})}$. By [1, proposition 2.2 (ii) and (iv)], we have

$$A:bv(X,p)\to \underline{l}_{\infty}(q) \Leftrightarrow A:bv(X,p)\to \bigcap_{m=1}^{\infty} l_{\infty(m^{1/q_n})}$$

$$\Leftrightarrow \; A : bv(X,p) \to l_{\infty(m^{1/q_n})} \quad \text{for all } m \in N$$

$$\Leftrightarrow \; \left(m^{1/q_n} f_k^n\right)_{n,k} : bv(X,p) \to l_\infty \quad \text{for all } m \in N,$$

and we have the theorem 4.1.2.2 ; $A : bv(X,p) \to l_\infty(q)$, taking $q_n = 1, \forall n \in N$. Thus the condition (4.6) hold. #

Theorem 4.2.2.3 $(bv(X,p) \to E_r(q))$

Let $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers such that* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix and let* $r \ge 0$. *Then* $A : bv(X,p) \to E_r(q)$ *if and only if there exists* $M \in N$ *such that*

$$\sup_n \left(n^{-rt_j/q_n} \sum_{j=1}^{\infty} \|g_j^n\|^{t_j} M^{-t_j/q_n} \right) < \infty \tag{4.7}$$

Where g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$.

Proof.

By [1, proposition 2.3(viii)], we have $E_r(q) = l_\infty(q)_{(n^{-r/q_n})}$. By [1, proposition 2.2(iv)], we have

$$A : bv(X,p) \to E_r(q) \;\Leftrightarrow\; A : bv(X,p) \to l_\infty(q)_{n^{-r/q_n}}$$

$$\Leftrightarrow\; \left(n^{-r/q_n} f_k^n\right)_{n,k} : bv(X,p) \to l_\infty(q)$$

and we have the theorem 4.1.2.2 ; $A : bv(X,p) \to l_\infty(q)$.Thus the condition (4.7) holds. #

Theorem 4.2.2.4 ($bv(X,p) \to F_s(q)$)

Let $p=(p_k)$ *and* $q=(q_k)$ *be bounded sequence of positive real numbers such that* $p_k > 1$ *and* $\frac{1}{p_k}+\frac{1}{t_k}=1$ *for all* $k \in N$, *and* $A=\left(f_k^n\right)$ *an infinite matrix and let* $s \geq 0$.

Then $A: bv(X,p) \to F_s(q)$ *if and only if*

(1) for each $n \in N$, *there exists* $M_n \in N$ *such that* $\sum_{j=1}^{\infty}\left\| n^{s/q_n} g_j \right\|^{t_j} M_n^{-(t_j-1)} < \infty$,

(2) for each $j \in N$, $\sum_{n=1}^{\infty} n^s \left| \sum_{k=j}^{\infty} f_k^n(x) \right|^{q_n} < \infty$ *for every* $x \in X$ *and*

(3) for each $r \in N$ *there exists* $M_r \in N$ *such that*

$$\sum_{k\in K}\left\|x_k - x_{k-1}\right\|^{p_k} \Rightarrow \sum_{n=1}^{\infty} n^s \left| \sum_{j\in K}\left(\sum_{k=j}^{\infty} f_k^n(x_j - x_{j-1}) \right) \right|^{q_n} < \frac{1}{r}$$

for all $x=(x_k) \in \Phi(X)$ *and all finite subsets* K *of* N*.Where* g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x)=\sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$.

Proof.

By [1, proposition 2.3(ix)], we have $F_s(q)=l(q)_{(n^{s/q_n})}$. By [1,proposition 2.2 (iv)], we have

$$\begin{aligned} A: bv(X,p) \to F_s(q) &\Leftrightarrow A: bv(X,p) \to l(q)_{n^{s/q_n}} \\ &\Leftrightarrow \left(n^{s/q_n} f_k^n\right)_{n,k} : bv(X,p) \to l(q)\ , \end{aligned}$$

and we have the theorem 4.1.2.1; $A: bv(X,p) \to l_\infty(q)$.Thus the conditions (1)-(3) hold. #

CHAPTER 5

CONCLUDIONS AND SUGGESTIONS

In this chapter we shall give conclusions, suggestions and the possible way to apply the matrix transformation in daily life or real world problem.

5.1 Conclusions

In this research, when we let $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers, let $A=\left(f_k^n\right)$ be an infinite matrix of continuous functional and let $r,s\geq 0$, we obtain the main following results:

Theorem 4.1.1.1 ($A:bv(X,p)\to l(q)\,;p_k\leq 1$)

$A:bv(X,p)\to l(q)$ if and only if

(1) for each $n\in N$, there is $M_n\in N$ such that $\sup_j\left\|g_j^n\right\|M_n^{-1/p_j}<\infty$, where g_j is the bounded linear functional on X define by $g_j^n(x)=\sum_{k=j}^{\infty}f_k^n(x)\ ,\forall x\in X$.

(2) for every $k\in N$, $\sum_{n=1}^{\infty}\left|f_k^n(x)\right|^{q_n}<\infty$ for all $x\in X$ and

(3) for each $r\in N$, there exists $M_r\in N$ such that

$$\|\lim x\|+\sum_{k\in K}\|x_k-x_{k-1}\|^{p_k}<\frac{1}{M_r}\Rightarrow\sum_{n=1}^{\infty}\left|\sum_{k\in K}f_k^n(x_k)\right|^{q_n}<\frac{1}{r}$$

for all $x=(x_k)\in\Phi(X)$ and for all subsets K of N.

Theorem 4.1.1.2 ($A:bv(X,p)\to l(q)\,;p_k\leq 1$)

$A:bv(X,p)\to l(q)$ if

$$\sup_j\sum_{n=1}^{\infty}\left\|g_j^n\right\|^{q_n}<\infty \tag{4.1}$$

Theorem 4.1.1.3 ($A : bv(X,p) \to l_\infty(q)$, $p_k \le 1$)

$A : bv(X,p) \to l_\infty(q)$ if and only if there exists $M \in N$ such that

$$\left\|g_j^n\right\| \le M^{1/p_j + 1/q_n} \quad \textit{for all } n, j \in N. \tag{4.2}$$

Theorem 4.1.1.4 ($A : bv(X,p) \to c_0(q)$, $p_k \le 1$)

$A : bv(X,p) \to c_0(q)$ if and only if

(1) *for all $m, k \in N$, $m^{1/q_n} g_j^n \xrightarrow{w^*} 0$ as $n \to \infty$,*

(2) *for each $m \in N$ there exists $M_m \in N$ such that $m^{p_j/q_n} \left\|g_j^n\right\|^{p_j} \le M_m$ for all $n, j \in N$.*

Theorem 4.1.1.5 ($A : bv(X,p) \to c(q)$, $p_k \le 1$)

$A : bv(X,p) \to c(q)$ if and only if there is a sequence (g_j) with $g_j \in X'$ for all $j \in N$ such that

(1) for some $M \in N$, $\sup_j \left\|g_j\right\| M^{-1/p_j} < \infty$,

(2) for all $m, j \in N$, $m^{1/q_n}(g_j^n - g_j) \xrightarrow{w^*} 0$ as $n \to \infty$ and

(3) for each $m \in N$ $\sup_j m^{p_j/q_n} \left\|g_j^n - g_j\right\|^{p_j} < \infty$ for all $n, j \in N$.

Theorem 4.1.2.1 ($bv(X,p) \to l(q)$, $p_k > 1$)

$A : bv(X,p) \to l(q)$ if and only if

(1) for each $n \in N$, there exists $M_n \in N$ such that $\sum_{j=1}^{\infty} \left\|g_j^n\right\|^{t_j} M_n^{-(t_j - 1)} < \infty$,

(2) for each $j \in N$, $\sum_{n=1}^{\infty} \left| \sum_{k=j}^{\infty} f_k^n(x) \right|^{q_n} < \infty$ for every $x \in X$ and

(3) for each $r \in N$ there exists $M_r \in N$ such that

$$\sum_{k\in K}\left\|x_k - x_{k-1}\right\|^{p_k} \Rightarrow \sum_{n=1}^{\infty}\left|\sum_{j\in K}\left(\sum_{k=j}^{\infty} f_k^n(x_j - x_{j-1})\right)\right|^{q_n} < \frac{1}{r}$$

for all $x=(x_k)\in\Phi(X)$ *and all finite subsets* K *of* N.

Theorem 4.1.2.2 $(bv(X,p)\to l_\infty(q), p_k>1)$

$A: bv(X,p)\to l_\infty(q)$ *if and only if there exists* $M\in N$ *such that*

$$\sup_n\left(\sum_{j=1}^{\infty}\left\|g_j^n\right\|^{t_j} M^{-t_j/q_n}\right)<\infty \tag{4.3}$$

Theorem 4.1.2.3 $(bv(X,p)\to c_0(q), p_k>1)$

$A: bv(X,p)\to c_0(q)$ *if and only if*

(1) for all $m,k\in N$, $m^{1/q_n} g_j^n \xrightarrow{w^*} 0$ *as* $n\to\infty$,

(2) for each $m\in N$, $\sum_{j=1}^{\infty}\left(m^{t_j/q_n}\left\|g_j^n\right\|^{t_j} r^{-(t_j-1)}\right)$ *as* $r\to\infty$, *uniformly for* $n\geq 1$.

Theorem 4.1.2.4 $(bv(X,p)\to c(q), p_k>1)$

$A: bv(X,p)\to c(q)$ *if and only if there is a sequence* (g_j) *with* $g_j\in X'$ *for all* $j\in N$ *such that*

(1) there exists $M\in N$, $\sum_{j=1}^{\infty}\left\|g_j\right\|^{t_j} M^{-(t_j-1)}<\infty$,

(2) for all $m,k\in N$, $m^{1/q_n}(g_j^n - g_j)\xrightarrow{w^*} 0$ *as* $n\to\infty$ *and*

(3) for each $m \in N$, $\sum_{j=1}^{\infty}(m^{t_j/q_n}\|g_j^n - g_j\|^{t_j} r^{-(t_j-1)})$ *as* $r \to \infty$*, uniformly for* $n \geq 1$.

Where g_j^n is the bounded linear functional on X define by $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ for all $x \in X$.

And by applying these results, we obtain the characterization of infinite matrices which transform $bv(X,p)$ into $\underline{l_\infty}(q)$, $M_\infty(q)$, $E_r(q)$ and $F_r(q)$.

When we compare this results with theorem 2.1-2.4, we can see the results of this research are generalize and cover these theorems, if we taking any Banach space X be scalar-field (R or C) space, $p_k = 1, \forall k \in N$ and any continuous functions in an infinite matrix be any scalars, which S.M. Srirajudeen study in 1992.

For example in the theorem 4.1.1.3 say that

" *Let* $A = (f_k^n)$ *be an infinite matrix of bounded linear functional on* X. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers with* $p_k \leq 1$. *Then* $A : bv(X,p) \to l_\infty(q)$ *if and only if there exists* $M \in N$ *such that*

$$\|g_j^n\| \leq M^{1/p_j + 1/q_n} \quad \text{for all } n, j \in N. \tag{4.2}$$

When g_j^n *is the bounded linear functional on* X *define by* $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ *for all* $x \in X$ *and, for all* $n \in N$."

We take $p_k = 1$ for all $k \in N$, we obtain $\|g_j^n\| \leq M^{1+1/q_n}$ for all $n, j \in N$ that is $\|g_j^n\| \leq M^1 \cdot M^{1/q_n} \Rightarrow \|g_j^n\| M^{-1} \leq M^{1/q_n}$ for all $n, j \in N$. This means

$\sup_j \|g_j^n\| M^{-1} \le M^{1/q_n}$ for all $n \in N \Rightarrow \left(\sup_j \|g_j^n\| M^{-1} \right)^{q_n} \le M$ for all $n \in N$,that is $\sup_n \left(\sup_j |g_j^n| M^{-1} \right)^{q_n} \le \infty$ for some $M \in N$ .When we take the Banach space $X = R$, we have $A = (f_k^n) : R \to R$ and when we define $f_k^n(x_k) = a_{nk} x_k , \forall x_k \in X = R$, that is any continuous functions in an infinite matrix (f_k^n) will be any scalars in an infinite matrix (a_{nk}). Thus we obtain $\sup_n \left(\sup_j \left| \sum_{k=j}^{\infty} a_{nk} \right| M^{-1} \right)^{q_n} < \infty$, the theorem 2.2.1.2 which S.M.Srirajudeen characterized in 1992.

We should to realize that this research is the one form of characterizations. There can have difference ways to characterize an infinite matrix that mapping $bv(X,p)$ into Maddox sequence space. And furthermore there are many open problems to characterize infinite matrix transformations between vector-valued sequence space. If some one interested in to study in this field, they can do it.

In the finally we shall offer some possible way to application matrices transformation with daily life problem. Maybe, for this we need to increase more and more research.

5.2 Application of Infinite Matrix Transformations

In the mathematical, the transformations is used to transform problems in one system into another system that easy than ones. When we obtain the solution, we can transform it back to the original system. The matrix transformation is also similar to another transformation.

Generally, we use finite matrices with daily life. For example, it was used in the computer graphic (4×4) and image processing (4×4). We know that an infinite matrix contains finite matrix, so we can apply infinite matrix as well as finite matrix.

In addition, we expect that the infinite matrix transformation can apply with data communication. If we require to send a message (sequence of binary bits) to another destination and we need to secure the message. We can use infinite matrix transformation to transform and send it. The destination can encode with certain matrix, which can convert it to the real original message. We can set the length of each input data as long as you request. We can check that in each one message (sequence of binary bits) is in $bv(p)$ and $l(p)$. So we can transform the message with infinite matrices which we already characterize it. The de-multiplexed signals are synchronized by input clock where the output levels have to be digitized by logical devices.

Furthermore, infinite matrix transformation is probably to apply with Quantum Teleportation and Dense Coding [14].

All of we discuss as above it will true when in each field, that we want to apply with, when we study more and more.

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