# NUMERICAL METHODS FOR SOLVING ITERATIVE ORDINARY DIFFERENTIAL EQUATIONS

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## บทคัดย่อ


เนื้อหาของการทำวิทยานิพนธ์นี้จะกล่าวถึงวิธีการหาผลเฉลยเชิงตัวเลขของสมการเชิงอนุพันธ์ซ้ำที่อยู่ในรูป

$$y'(x) = f(x, y(x), y^2(x), \ldots, y^m(x)) \quad , \quad x \in [a,b] \qquad (1)$$

กับเงื่อนไขค่าเริ่มต้น

$$y(a) = c \qquad (2)$$

เมื่อ $m$ เป็นจำนวนเต็มบวกที่มากกว่า 1 และ

$$y^2(x) = y(y(x))$$
$$y^3(x) = y(y^2(x)) = y(y(y(x)))$$
$$\vdots$$
$$y^m(x) = y(y^{m-1}(x))$$

โดยที่ ในวิทยานิพนธ์นี้จะประกอบด้วยระเบียบวิธีเชิงตัวเลขแบบ 4 วิธี คือ

1. ระเบียบวิธีที่ 1 ใช้ระเบียบวิธีรุงเงคุตตาอันดับที่ 4 กับ วิธีผลหารผลต่างสืบเนื่องของนิวตัน
2. ระเบียบวิธีที่ 2 ใช้ระเบียบวิธีรุงเงคุตตาอันดับที่ 4 กับ การประมาณค่าในช่วงด้วยพหุนามดีกรีสามของแอร์มิต
3. ระเบียบวิธีที่ 3 ใช้สูตรของซิมป์สัน กับ วิธีผลหารผลต่างสืบเนื่องของนิวตัน
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และจากการศึกษาพบว่าในการแก้ปัญหาสมการเชิงอนุพันธ์ในรูปแบบนี้ ระเบียบวิธีที่ 2 เหมาะสมที่สุด

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# ABSTRACT


The content of this thesis is to discuss the numerical methods for solving ordinary differential equations of form

$$y'(x) = f(x, y(x), y^2(x), \ldots, y^m(x)) \quad , \quad x \in [a,b] \qquad (1)$$

with the initial condition

$$y(a) = c \qquad (2)$$

where $m$ is a positive integer greater than 1 and

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y^2(x)) = y(y(y(x))) \\ &\vdots \\ y^m(x) &= y(y^{m-1}(x)). \end{aligned}$$

By this thesis will content the four methods as

1. The first method is the Forth-order Runge-Kutta Method and the Four Points Newton Divided Difference Method.
2. The second method is the Forth-order Runge-Kutta Method and the Cubic Hermite Interpolation.
3. The third method is the Simpson's Rule and the Four Points Newton Divided Difference Method.
4. The forth method is the Simpson's Rule and the Cubic Hermite Interpolation.

And finally, we choose the second method for solving the differential equations in this form.

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# CHAPTER 1

# INTRODUCTION

***The functional differential equations*** is one kind of the differential equations. There are many mathematicians who gave their best efforts to study these problem. We may separate the study of this field into three categories.

The first type of the functional differential equations is called ***"the differential equation with deviating argument"*** which has the classical form of

$$y'(x) = F(x, y(x), y(q(x))). \tag{1.1}$$

W. B. Fite is one of many mathematicians who studied the differential equations of this type. He published in [1] his work, in 1972, which included the proof of the existence theorem of the solution of the ordinary differential equation of the form

$$y^{(n)}(x) + \sum_{i=1}^{n} p_i(x) y^{(n-i)}(x) + r(x) y(q(x)) = 0. \tag{1.2}$$

Later, in [2], D. R. Anderson proved the existence theorem for a solution of the equation

$$f'(x) = F(x, f(g(x))). \tag{1.3}$$

The second type of the functional differential equations is called ***"iterative ordinary differential equation"*** which has the classical form of

$$y'(x) = G(x, y(x), y^2(x), \ldots, y^m(x)) \tag{1.4}$$

where

$$y^2(x) = y(y(x))$$
$$y^3(x) = y(y^2(x))$$
$$\vdots$$
$$y^m(x) = y(y^{m-1}(x)).$$

For this type of differential equations, G. Barba [3] found the solution of the ordinary differential equation of the form

$$f(x) f'(x) = f(f(x)). \tag{1.5}$$

In [4], [5] and [6], A. Pelczar proved the existence and uniqueness of the solution of the equation

$$\frac{dy}{dx} = f(x, y(x), y(y(x))) \tag{1.6}$$

on the interval $[0,a]$ with the initial condition

$$y(0) = c. \tag{1.7}$$

Furtherance [10], M. Podisuk proved the existence and uniqueness of the solution of the differential equation

$$\frac{dy}{dx} = f(x, y(x), y^2(x), \ldots, y^m(x)) \tag{1.8}$$

on the interval $[0,a]$ with the initial condition

$$y(0) = c \tag{1.9}$$

where $y^k(x)$ denotes the $k^{th}$ iteration of the (unknown) function $y(x)$ which satisfies (1.8) - (1.9). This work is the main interest of the author.

The third type of the functional differential equations is the mixed type of the first and the second type. The following equation is an example of this type:

$$y'(x) = H(x, y(x), y(g(x)), y(q(y(x)))). \tag{1.10}$$

Such, equation have been studied extensively by V. P. Skripnik [7] and [8].

In the fact these equations is difficult to find the exact solutions. Many mathematicians accordingly research in this field. One of many methods that they use, are numerical methods.

In this thesis, we shall study the numerical methods for solving an iterative ordinary differential equation of the form

$$y' = f(x, y(x), y^2(x), \ldots, y^m(x)) \tag{1.11}$$

on the interval $[a,b]$ with the initial condition

$$y(a) = c \tag{1.12}$$

where $y^k(x)$ denotes the $k^{th}$ iteration of the (unknown) function $y(x)$ and satisfies (1.11) - (1.12) and find the best numerical method for solving it when we can not find the exact solution.

# CHAPTER 2

# ON ITERATIVE ORDINARY DIFFERENTIAL EQUATIONS

In this chapter we shall discuss the uniqueness and existence of the solutions of the iterative ordinary differential equations

$$y'(x) = f(x, y(x), y^2(x), \ldots, y^m(x)) \tag{2.1}$$

on the interval $[a,b]$ with the initial condition

$$y(a) = c \tag{2.2}$$

where $0 \le a \le b$ and $c$ are positive real numbers, $m$ is a positive integer greater than 1 and

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y^2(x)) = y(y(y(x))) \\ &\vdots \\ y^m(x) &= y(y^{m-1}(x)). \end{aligned} \tag{2.3}$$

By a solution of the initial value problem (2.1) - (2.2) , we mean a function $y(x)$ of the class $C^1[a,b]$ satisfying (2.1) and (2.2). Thus, the initial value problem (2.1) - (2.2) is equivalent to the integral equation

$$y(x) = c + \int_a^x f(t, y(t), y^2(t), \ldots, y^m(t))dt. \tag{2.4}$$

## 2.1 Uniqueness of solution

Let $0 \le a \le b$ and $y(x)$ be a function in class $C^1$ map $[a,b]$ to $[a,b]$, that is

$$|y(x)| \le b \tag{2.5}$$

for all $x \in [a,b]$

and let

$$|y(x) - y(\bar{x})| \le K|x - \bar{x}| \tag{2.6}$$

for all $x, \bar{x} \in [a,b]$ and for some positive constant $K$.

We obtain the following lemma.

**Lemma 2.1.1** Let $u(x)$ and $v(x)$ be any functions of the class $C^1$ which maps $[a,b]$ to $[a,b]$ and satisfies the conditions (2.5) and (2.6). We have

$$\left|u^m(x) - v^m(x)\right| \le K^{m-1}w(x) + K^{m-2}w(v(x)) + \dots + Kw(v^{m-2}(x)) + w(v^{m-1}(x))$$

where $$w(x) = \left|u(x) - v(x)\right|,$$

$m$ is a positive integer greater than 1 and

$$\begin{aligned} u^2(x) &= u(u(x)) & v^2(x) &= v(v(x)) \\ u^3(x) &= u(u^2(x)) = u(u(u(x))) \quad , & v^3(x) &= v(v^2(x)) = v(v(v(x))) \\ &\vdots & &\vdots \\ u^m(x) &= u(u^{m-1}(x)) & v^m(x) &= v(v^{m-1}(x)) \quad [14]. \end{aligned}$$

**Proof.** Let $u(x)$ and $v(x)$ be any functions of the class $C^1$ which maps $[a,b]$ to $[a,b]$ the conditions (2.5) and (2.6) and let

$$w(x) = \left|u(x) - v(x)\right|.$$

We shall prove the assertion by induction.

$$\begin{aligned} \left|u^2(x) - v^2(x)\right| &= \left|u(u(x)) - v(v(x))\right| \\ &= \left|u(u(x)) - u(v(x)) + u(v(x)) - v(v(x))\right| \\ &\quad \text{(From triangular inequality)} \\ &\le \left|u(u(x)) - u(v(x))\right| + \left|u(v(x)) - v(v(x))\right| \\ &\le K\left|u(x) - v(x)\right| + w(v(x)) \\ &\le Kw(x) + w(v(x)). \end{aligned}$$

Assuming that

$$\left|u^n(x) - v^n(x)\right| \le K^{n-1}w(x) + K^{n-2}w(v(x)) + \dots + Kw(v^{n-2}(x)) + w(v^{n-1}(x)).$$

We get

$$\begin{aligned} \left|u^{n+1}(x) - v^{n+1}(x)\right| &= \left|u(u^n(x)) - v(v^n(x))\right| \\ &= \left|u(u^n(x)) - u(v^n(x)) + u(v^n(x)) - v(v^n(x))\right| \\ &\quad \text{(From triangular inequality)} \\ &\le \left|u(u^n(x)) - u(v^n(x))\right| + \left|u(v^n(x)) - v(v^n(x))\right| \\ &\le K\left|u^n(x) - v^n(x)\right| + w(v^n(x)) \\ &\le K^n w(x) + K^{n-1}w(v(x)) + \dots + Kw(v^{n-1}(x)) + w(v^n(x)). \end{aligned}$$

This completes the proof of Lemma 2.1.1.

**Remark 2.1.2** If $u(x)$ and $v(x)$ are functions which satisfy the conditions of Lemma 2.1.1. Then $w(v^{m-1}(x)) \leq b$ where $m$ is a positive integer greater than 1 [14].

Now, let $f(x,z_1,z_2,\dots,z_m)$ be defined and continuous in the domain $D=[a,b]^{m+1}$ and let

$$|f(x,z_1,z_2,\dots,z_m)-f(x,\bar{z}_1,\bar{z}_2,\dots,\bar{z}_m)|\leq M_1|z_1-\bar{z}_1|-M_2|z_2-\bar{z}_2|-\dots-M_m|z_m-\bar{z}_m| \quad (2.7)$$

for all $(x,z_1,z_2,\dots,z_m)$ , $(x,\bar{z}_1,\bar{z}_2,\dots,\bar{z}_m)$ in $D$ and $M_1, M_2,\dots,M_m$ in $R^+$ and let

$$P_m = M_1 + KM_2 + K^2M_3 + \dots + K^{m-1}M_m \quad (2.8)$$

$$S_m = M_2 + (K+1)M_3 + (K^2+K+1)M_4 + \dots + (K^{m-2} + \dots + K + 1)M_m \quad (2.9)$$

$$T_m = P_m + S_m \quad (2.10)$$

then we have the following theorem.

**Theorem 2.1.3** If $bS_m < e^{-bP_m}$ and f satisfies the above conditions then there exists at most one solution to the problem (2.1)-(2.2).

**Proof.** Let $u(x)$ and $v(x)$ be two solutions of the problem (2.1)-(2.2). Hence

$$u(x) = c + \int_a^x f(t, u(t), u^2(t),\dots, u^m(t))dt$$

and $$v(x) = c + \int_a^x f(t, v(t), v^2(t),\dots, v^m(t))dt \quad \text{for } x\in[a,b].$$

Let $w(x)=|u(x)-v(x)|$ then we have

$$w(x) = \left|\int_a^x f(t, u(t), u^2(t),..., u^m(t))dt - \int_a^x f(t, v(t), v^2(t),..., v^m(t))dt\right|$$

$$\leq \int_a^x \left|f(t, u(t), u^2(t),..., u^m(t)) - f(t, v(t), v^2(t),..., v^m(t))\right|dt$$

$$\leq \int_a^x \left[M_1|u(t) - v(t)| + M_2\left|u^2(t) - v^2(t)\right| + ... + M_m\left|u^m(t) - v^m(t)\right|\right]dt$$

$$\leq M_1 \int_a^x w(t)dt + M_2\left[K\int_a^x w(t)dt + \int_a^x w(v(t))dt\right]$$

$$+ M_3\left[K^2\int_a^x w(t)dt + K\int_a^x w(v(t))dt + \int_a^x w(v^2(t))dt\right] + ...$$

$$+ M_m\left[K^{m-1}\int_a^x w(t)dt + K^{m-2}\int_a^x w(v(t))dt + ... + \int_a^x w(v^{m-1}(t))dt\right].$$

$$w(x) \leq \left[M_1 + KM_2 + K^2M_3 + ... + K^{m-1}M_m\right]\int_a^x w(t)dt$$

$$+ \left[M_2 + KM_3 + K^2M_4 + ... + K^{m-2}M_m\right]\int_a^x w(v(t))dt$$

$$+ \left[M_3 + KM_4 + K^2M_5 + ... + K^{m-3}M_m\right]\int_a^x w(v^2(t))dt + ... \tag{2.11}$$

$$+ \left[M_{m-1} + KM_m\right]\int_a^x w(v^{m-2}(t))dt + M_m\int_a^x w(v^{m-1}(t))dt$$

$$\leq P_m\int_a^x w(t)dt + \left[M_2 + KM_3 + K^2M_4 + ... + K^{m-2}M_m\right]\int_a^x (b)dt$$

$$+ \left[M_3 + KM_4 + K^2M_5 + ... + K^{m-3}M_m\right]\int_a^x (b)dt + ...$$

$$+ \left[M_{m-1} + KM_m\right]\int_a^x w(v^{m-2}(t))dt + M_m\int_a^x (b)dt.$$

$$w(x) \leq P_m\int_a^x w(t)dt + b^2S_m. \tag{2.12}$$

From (2.12) and Gronwall inequality [13], it follows that

$$w(x) \le b^2 S_m e^{xP_m} \le b^2 S_m e^{bP_m}. \tag{2.13}$$

From (2.11) and (2.13) it follows that

$$w(x) \le P_m \int_a^x w(t)dt + (b^2 S_m e^{bP_m}) b S_m$$

$$\le b\left(bS_m e^{bP_m}\right)^2.$$

By induction on $n$, we get

$$w(x) \le b\left(bS_m e^{bP_m}\right)^n \quad , \quad n = 1,2,3,\dots .$$

Since $bS_m < e^{-bP_m}$, so $bS_m e^{bP_m} < 1$ and $w(x)$ is bounded by the sequence which tends to zero as $n \to \infty$. Since $w(x) \ge 0$, it must be equal to zero. This completes the proof of Theorem 2.1.3.

**Theorem 2.1.4** If $bT_m < 1$ and $y(x)$ satisfies the conditions of Theorem 2.1.3, then there exists at most one solution of the initial value problem (2.1)-(2.2).

**Proof.** Assume that $u(x)$ and $v(x)$ are two solutions of the initial value problem (2.1)-(2.2) and let

$$w(x) = |u(x) - v(x)|$$

and

$$P = \max_{x \in [a,b]} |u(x) - v(x)|.$$

It follows from the proof of Theorem 2.1.3 that

$$w(x) \le M_1 \int_a^x w(t)dt + M_2\left[K\int_a^x w(t)dt + \int_a^x w(v(t))dt\right]$$

$$+ M_3\left[K^2\int_a^x w(t)dt + K\int_a^x w(v(t))dt + \int_a^x w(v^2(t))dt\right] + \dots$$

$$+ M_m\left[K^{m-1}\int_a^x w(t)dt + K^{m-2}\int_a^x w(v(t))dt + \dots + \int_a^x w(v^{m-1}(t))dt\right].$$

$$\dot{w}(x) \le PM_1b + PM_2b(K+1) + PM_3b(K^2+K+1) + \ldots$$
$$+ PM_mb(K^{m-1} + K^{m-2} + \ldots + K + 1) = PbT_m.$$

By induction on $n$, it can be shown that

$$w(x) \le P(bT_m)^n \quad , \qquad n = 1,2,3,\ldots .$$

Since $bT_m < 1$, $w(x)$ must be zero. This completes the proof of Theorem 2.1.4.

## 2.2 Existence of solution

Let us assume that

$$c + bK \le b \tag{2.14}$$

and

$$bT_m < 1. \tag{2.15}$$

We consider the following sequences

$$y_{1,n+1}(x) = c + \int_a^x f(t, y_{1,n}(t), y_{1,n}^2(t), \ldots, y_{1,n}^m(t))dt \tag{2.16.1}$$

$$y_{2,n+1}(x) = c + \int_a^x f(t, y_{2,n}(t), \ldots, y_{2,n}^{m-1}(t), y_{2,n}^{m-1}(y_{2,n+1}(t)))dt \tag{2.16.2}$$

$$\vdots$$

$$y_{m,n+1}(x) = c + \int_a^x f(t, y_{m,n+1}(t), y_{m,n}(y_{m,n+1}(t)), \ldots, y_{m,n}^2(y_{m,n+1}(t)), \ldots, y_{m,n}^{m-1}(y_{m,n+1}(t))dt \tag{2.16.m}$$

$n = 0,1,2,\ldots$

where $y_{1,0}(x), y_{2,0}(x), \ldots, y_{m+1,0}(x)$ are fixed functions of the class $C^1$ map $[a,b]$ to $[a,b]$, such that

$$|y'_{1,0(x)}|, |y'_{2,0(x)}|, \ldots, |y'_{m+1,0(x)}| \le K.$$

We have the following theorem.

**Theorem 2.2.1** Let the assumption of Theorem 2.1.3, the conditions (2.14) and (2.15) are satisfied. Then the sequences (2.16.1)-(2.16.*m*) converge uniformly to the (unique) solution $y = y(x)$ of initial value problem (2.1)-(2.2).

**Proof.** We put

$$Y_{1,n} = \max_{x\in[a,b]} \left| y_{1,n}(x) - y_{1,n-1}(x) \right|$$

$$Y_{2,n} = \max_{x\in[a,b]} \left| y_{2,n}(x) - y_{2,n-1}(x) \right|$$

$$\vdots$$

$$Y_{m,n} = \max_{x\in[a,b]} \left| y_{m,n}(x) - y_{m,n-1}(x) \right|$$

where $n = 1,2,3,\ldots$ .

It can be shown by induction $n$ on that

$$Y_{1,n} \le W_1^{n-1} Y_{1,0}$$

$$Y_{2,n} \le W_2^{n-1} Y_{2,0}$$

$$\vdots$$

$$Y_{m,n} \le W_m^{n-1} Y_{m,0}$$

where

$$W_1 = \frac{bU_1}{V_1}, W_2 = \frac{bU_2}{V_2}, \ldots, W_m = \frac{bU_m}{V_m}$$

$$U_1 = M_1 + (K+1)M_2 + (K^2+K+1)M_3 + \ldots + (K^{m-1} + K^{m-2} + \ldots + 1)M_m$$

$$U_2 = M_1 + (K+1)M_2 + \ldots + (K^{m-3} + \ldots + K + 1)M_{m-2} + (K^{m-2} + \ldots + K + 1)(M_{m-1} + M_m)$$

$$U_3 = M_1 + (K+1)M_2 + \ldots + (K^{m-4} + \ldots + K + 1)M_{m-3} + (K^{m-3} + \ldots + K + 1)(M_{m-2} + M_{m-1}) + (K^{m-2} + \ldots + K + 1)M_m$$

$$\vdots$$

$$U_m = (M_1 + M_2) + (K+1)M_3 + (K^2+K+1)M_4 + \ldots + (K^{m-2} + \ldots + K + 1)M_m$$

$$V_1 = 1$$

$$V_2 = 1 - bK^{m-1}M_m$$

$$V_3 = 1 - bK^{m-2}M_{m-1} - bK^{m-1}M_m$$

$$\vdots$$

$$V_m = 1 - bKM_2 - bK^2M_3 - \ldots - bK^{m-2}M_{m-1} - bK^{m-1}M_m.$$

Since $W_1 = bT_m < 1$, so $W_j < 1$ for $j = 2,3,\ldots,m$ , $W_j^{n-1}$ tends to zero as $n \to \infty$. Hence $Y_{j,n}$ tends to zero as $n \to \infty$ for $j = 2,3,\ldots,m$. This implies that if $\{ y_{i,n_j} \}$ is a subsequence of $\{ y_{i,n} \}$ tending uniformly to some $\bar{y}_i(z)$ for $i = 1,2,\ldots,m$ , then $\bar{y}_i(z)$ is a solution of the initial value problem (2.1)-(2.2). Since the family $\{ y_{i,n} \}$ is the Arzela-Ascoli family [13], there exists a subsequence $\{ y_{i,m_j} \}$ uniformly convergent for every subsequence $\{ y_{i,n_j} \}$ of $\{ y_{i,n} \}$ The limit must be a solution for the initial value problem (2.1)-(2.2). Thus, the sequence $\{ y_{i,n} \}$ converge uniformly to the (unique) solution $y = y(x)$ of the initial value problem (2.1)-(2.2).

# CHAPTER 3

# NUMERICAL METHODS FOR SOLVING ITERATIVE ORDINARY DIFFERENTIAL EQUATIONS

In this chapter we shall look for a numerical method for solving the iterative differential equation of the form

$$y'(x) = f(x, y(x), y^2(x), ..., y^m(x)) \quad (3.1)$$

on the interval $[a,b]$ with the initial condition

$$y(a) = c \quad (3.2)$$

by using the iterative integral equation

$$y_{k+1}(x) = c + \int_a^x f(t, y_k(t), y_k^2(t), ..., y_k^m(t))dt. \quad (3.3)$$

The formula (3.3) shall insure that the following methods shall converge because of the theorem (2.2.1).

The regular numerical method for the solving the initial value problem

$$y'(x) = g(x, y(x)) \quad (3.4)$$

on the interval $[a,b]$ with the initial condition

$$y(a) = c \quad (3.5)$$

such as the Taylor Method and the Runge-Kutta Method can not be used to solve the problem (3.1)-(3.2). To find the value of $y_{k+1}$ at the point $x_{k+1}$ by using the previous value $y_k$ at the point $x_k$ can not be accomplished because the right hand side of the problem (3.1)-(3.2) is not $g(x, y(x))$. We have $y^m(x)$ instead of $g(x, y(x))$ where we know only the value $y_k$ but not the values of $y_k(y_k), y_k(y_k^2), ..., y_k(y_k^{m-1})$. The author has devised the four methods to solve this kind of problems.

These four method use the same idea which include the iteration technique, the combination of numerical method for solving ordinary differential equation and interpolation. For each iteration, we shall solve for the value of $y(x)$ for the whole interval $[a,b]$. After that we partition the closed interval $[a,b]$ into $n$ equally subintervals by the points $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

## 3.1 First method

This method uses the Forth-order Runge-Kutta Method and the Four Points Newton Divided Difference Method to solve the problem (3.1)-(3.2).

First let $\quad y_0(x_k) = c \quad , k = 1,2,\ldots, n$

then we use the Forth-order Runge-Kutta Method to find $y_j(x_k)$ at $x_1, x_2, \ldots, x_n$ where $y_j(x_0) = c$ and

$$y_j(x_{k+1}) = y_j(x_k) + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \tag{3.6}$$

where

$$h = \frac{x_n - x_0}{n}$$

$$\begin{aligned}
k_1 &= f(x_k, y_j(x_k), y_j(y_j(x_k)), \ldots, y_j^{m-1}(y_j(x_k))) \\
k_2 &= f(x_k + \frac{h}{2}, y_j(x_k) + \frac{k_1}{2}, y_j(y_j(x_k) + \frac{k_1}{2}), \ldots, y_j^{m-1}(y_j(x_k) + \frac{k_1}{2})) \\
k_3 &= f(x_k + \frac{h}{2}, y_j(x_k) + \frac{k_2}{2}, y_j(y_j(x_k) + \frac{k_2}{2}), \ldots, y_j^{m-1}(y_j(x_k) + \frac{k_2}{2})) \\
k_4 &= f(x_k + h, y_j(x_k) + k_3, y_j(y_j(x_k) + k_3), \ldots, y_j^{m-1}(y_j(x_k) + k_3)).
\end{aligned} \tag{3.7}$$

By the Four Points Newton Divided Difference Method we find the value $y_j(x), x \neq x_k$ for $k = 0,1,2,\ldots, n$, from the interpolation polynomial of degree 3 , $P_{j,l}(x)$, that

$$\begin{aligned}
P_{j,l}(x) = {} & y_{j-1}(x_{l-2}) + (x - x_{l-2}) f[x_{l-1}, x_{l-2}] \\
& + (x - x_{l-2})(x - x_{l-1}) f[x_l, x_{l-1}, x_{l-2}] \\
& + (x - x_{l-2})(x - x_{l-1})(x - x_l) f[x_{l+1}, x_l, x_{l-1}, x_{l-2}]
\end{aligned} \tag{3.8}$$

where $x_{l-1} < x < x_l$ and

$$\begin{aligned}
f[x_{l-1}, x_{l-2}] &= \frac{y_{j-1}(x_{l-1}) - y_{j-1}(x_{l-2})}{x_{l-1} - x_{l-2}} \\
f[x_l, x_{l-1}, x_{l-2}] &= \frac{f[x_l, x_{l-1}] - f[x_{l-1}, x_{l-2}]}{x_l - x_{l-2}} \\
f[x_{l+1}, x_l, x_{l-1}, x_{l-2}] &= \frac{f[x_{l+1}, x_l, x_{l-1}] - f[x_l, x_{l-1}, x_{l-2}]}{x_{l+1} - x_{l-2}}.
\end{aligned} \tag{3.9}$$

Repeat until $$\sum_{k=1}^{n} \left| y_{j-1}(x_k) - y_j(x_k) \right| < E \quad , \ j = 1,2,3,\ldots$$

for a sufficiently small $E$.

### 3.2 Second method

This method uses the Forth-order Runge-Kutta Method and the Cubic Hermite Interpolation to solve the problem (3.1)-(3.2).

First let $y_0(x_k) = c \quad , k = 1,2,\ldots, n$

then we use the Forth-order Runge-Kutta Method to find $y_j(x_k)$ at $x_1, x_2, \ldots, x_n$ like the first method. By the Cubic Hermite Interpolation, we find the value $y_j(x)$, $x \neq x_k$ for $k = 0,1,2,\ldots, n$, from the interpolation polynomial of degree 3 , $P_{j,l}(x)$ , that

$$P_{j,l} = c_1 + c_2(x - x_l) + c_3(x - x_l)^2 + c_4(x - x_l)^3 \tag{3.10}$$

where $x_l < x < x_{l+1}$ and

$$\begin{aligned} c_1 &= y_{j-1}(x_l) \\ c_2 &= s_l \\ c_3 &= \frac{y_{j-1}(x_{l+1}) - y_{j-1}(x_l)}{h^2} - \frac{s_l}{h} - c_4 h \\ c_4 &= \frac{s_{l+1}}{h^2} + \frac{s_l}{h^2} - 2\,\frac{y_{j-1}(x_{l+1}) - y_{j-1}(x_l)}{h^3} \end{aligned} \tag{3.11}$$

$$s_l = f(x_l, y_{j-1}(x_l), y_{j-1}(y_{j-1}(x_l)), \ldots, y_{j-1}^{m-1}(y_{j-1}(x_l))).$$

Repeat until $$\sum_{k=1}^{n} \left| y_{j-1}(x_k) - y_j(x_k) \right| < E \quad , \ j = 1,2,3,\ldots$$

for a sufficiently small $E$.

### 3.3 Third method

This method uses the Simpson's Rule and the Four Points Newton Divided Difference Method to solve the problem (3.1)-(3.2).

First let $y_0(x_k) = c \quad , k = 1,2,\ldots, n$

then we use the Simpson's Rule to find $y_j(x_k)$ at $x_1, x_2, \ldots, x_n$ where $y_j(x_0) = c$ and

$$y_j(x_{k+1}) = y_j(x_k) + \frac{1}{6} h(f_1 + 4f_2 + f_3) \qquad (3.12)$$

where

$$h = \frac{x_n - x_0}{n}$$

$$f_1 = f(x_k, y_{j-1}(x_k), y_{j-1}(y_{j-1}(x_k)), \ldots, y_{j-1}^{m-1}(y_{j-1}(x_k)))$$

$$f_2 = f(x_k + \frac{h}{2}, y_{j-1}\left((x_k) + \frac{h}{2}\right), y_{j-1}(y_{j-1}\left((x_k) + \frac{h}{2}\right)), \ldots, y_{j-1}^{m-1}(y_{j-1}\left((x_k) + \frac{h}{2}\right))) \qquad (3.13)$$

$$f_3 = f(x_{k+1}, y_{j-1}(x_{k+1}), y_{j-1}(y_{j-1}(x_{k+1})), \ldots, y_{j-1}^{m-1}(y_j(x_{k+1}))).$$

By the Four Points Newton Divided Difference Method, we find the value $y_{j-1}(x), x \neq x_k$ for $k = 0,1,2,.., n$, like the first method.

Repeat until $$\sum_{k=1}^{n} \left| y_{j-1}(x_k) - y_j(x_k) \right| < E \quad , \; j = 1,2,3,\ldots$$

for a sufficiently small $E$.

## 3.4 Forth method

This method uses the Simpson's Rule and the Cubic Hermite Interpolation to solve the problem (3.1)-(3.2).

First let $y_0(x_k) = c \quad , k = 1,2,\ldots, n$

then we use the Simpson's Rule to find $y_j(x_k)$ at $x_1, x_2, \ldots, x_n$ like the third method. By the Cubic Hermite Interpolation, we find the value $y_{j-1}(x), x \neq x_k$ for $k = 0,1,2,.., n$, like the second method.

Repeat until $$\sum_{k=1}^{n} \left| y_{j-1}(x_k) - y_j(x_k) \right| < E \quad , \; j = 1,2,3,\ldots$$

for a sufficiently small $E$.

# CHAPTER 4

# EXAMPLES OF THE ITERATIVE ORDINARY DIFFERENTIAL EQUATIONS

In this chapter, we shall find the analytic solutions and numerical solutions by using the four methods obtained from chapter 3 to solve some iterative ordinary differential equation problems.

## 4.1 Analytic solution

From chapter 2, we can find the analytic solutions of iterative ordinary differential equations

$$y'(x) = f(x, y(x), y^2(x), ..., y^m(x)) \tag{4.1}$$

on the interval $[a,b]$ with the initial condition

$$y(a) = c \tag{4.2}$$

by letting

$$y(x) = \lim_{n\to\infty} y_{n+1}(x) \tag{4.3}$$

where

$$\begin{aligned} y_0(x) &= c \\ y_{n+1}(x) &= c + \int_a^x f(t, y_n(t), y_n^2(t), ..., y_n^m(t))dt \quad , n = 0,1,2,... \end{aligned} \tag{4.4}$$

**Example 4.1** Find the solution on the interval $\left[0, \frac{1}{2}\right]$ of the equation

$$y'(x) = y^2(x) \tag{4.5}$$

with the initial condition

$$y(0) = 0.25 . \tag{4.6}$$

Let $y_0(x) = 0.25$ then by the equation (4.4), we get

$$\begin{aligned} y_1(x) &= 0.25 + \int_0^x y_0(y_0(t))dt \\ &= 0.25 + 0.25x \end{aligned}$$

$$y_2(x) = 0.25 + \int_0^x y_1(y_1(t))dt$$

$$= 0.25 + 0.3125x + 0.03125x^2$$

$$y_3(x) = 0.25 + \int_0^x y_2(y_2(t))dt$$

$$= 0.25 + (3.30078 \times 10^{-1})x + (5.12695 \times 10^{-2})x^2 + (4.435221 \times 10^{-3})x^3 + (1.52587 \times 10^{-4})x^4 + (6.10351 \times 10^{-6})x^5$$

$$y_4(x) = 0.25 + \int_0^x y_3(y_3(t))dt$$

$$\begin{aligned} = {} & 0.25 + (3.36086 \times 10^{-1})x + (5.92627 \times 10^{-2})x^2 + (8.369 \times 10^{-3})x^3 \\ & + (1.03805 \times 10^{-3})x^4 + (1.65254 \times 10^{-4})x^5 + (5.13064 \times 10^{-5})x^6 \\ & + (2.27891 \times 10^{-5})x^7 + (1.11981 \times 10^{-5})x^8 + (5.64832 \times 10^{-6})x^9 \\ & + (2.88698 \times 10^{-6})x^{10} + (1.49034 \times 10^{-6})x^{11} + (6.62681 \times 10^{-7})x^{12} \\ & + (1.75158 \times 10^{-7})x^{13} + (6.74152 \times 10^{-8})x^{14} + (3.35821 \times 10^{-8})x^{15} \\ & + (1.77152 \times 10^{-8})x^{16} + (8.02786 \times 10^{-9})x^{17} + (2.12115 \times 10^{-9})x^{18} \\ & + (8.02153 \times 10^{-10})x^{19} + (3.93583 \times 10^{-10})x^{20} + (2.04969 \times 10^{-10})x^{21} \\ & + (8.5901 \times 10^{-11})x^{22} + (1.12052 \times 10^{-11})x^{23} + (8.48685 \times 10^{-13})x^{24} \\ & + (2.81777 \times 10^{-14})x^{25} + (1.01015 \times 10^{-15})x^{26}. \end{aligned}$$

In this problem, we may use $y_4(x)$ as the approximation to the solution of the initial value problem (4.5)-(4.6).

**Example 4.2** Find the solution on the interval $\left[0, \frac{1}{2}\right]$ of the equation

$$y'(x) = y^3(x) \tag{4.7}$$

with the initial condition

$$y(0) = 0.2\,. \tag{4.8}$$

Let $y_0(x) = 0.2$ then by the equation (4.4), we get

$$y_1(x) = 0.2 + \int_0^x y_0(y_0(y_0(t)))dt$$

$$= 0.2 + 0.2x$$

$$y_2(x) = 0.2 + \int_0^x y_1(y_1(y_1(t)))dt$$

$$= 0.2 + 0.248x + 0.004x^2$$

$$y_3(x) = 0.2 + \int_0^x y_2(y_2(y_2(t)))dt$$

$$= 0.2 + (2.6219 \times 10^{-1})x + (7.73754 \times 10^{-3})x^2 + (1.08809 \times 10^{-4})x^3$$
$$+ (6.50056 \times 10^{-7})x^4 + (5.22482 \times 10^{-9})x^5 + (1.84497 \times 10^{-11})x^6$$
$$+ (1.27008 \times 10^{-13})x^7 + (5.07904 \times 10^{-16})x^8 + (1.82044 \times 10^{-18})x^9.$$

In this problem, we may use $y_3(x)$ as the approximation to the solution of the initial value problem (4.7)-(4.8).

**Example 4.3** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = xy^2(x) + 1 - x^2 \qquad (4.9)$$

with the initial condition

$$y(0) = 0\,. \qquad (4.10)$$

Let $y_0(x) = 0$ then by the equation (4.4), we get

$$y_1(x) = x - \frac{x^3}{3}$$

$$y_2(x) = x - \frac{2x^5}{3 \times 5} + \frac{x^7}{21} - \frac{x^8}{81} + \frac{x^{11}}{891}$$

$$y_3(x) = x - \frac{2^2 x^7}{3 \times 5 \times 7} + \text{(higher power terms)}$$

$$y_4(x) = x - \frac{2^3 x^9}{3 \times 5 \times 7 \times 9} + \text{(higher power terms)}$$

$$y_5(x) = x - \frac{2^4 x^{11}}{3 \times 5 \times 7 \times 9 \times 11} + \text{(higher power terms)}$$

$\vdots$

$$y_n(x) = x - \frac{2^{n-1} x^{2n+1}}{3 \times 5 \times 7 \times \ldots \times (2n+1)} + \text{(higher power terms).}$$

Hence we have

$y_0(x) = R_0$

$y_1(x) = x - R_1 x^3$

$y_2(x) = x - R_2 x^5$ + (higher power terms)

$y_3(x) = x - R_3 x^7$ + (higher power terms)

$\vdots$

$y_n(x) = x - R_n x^{2n+1}$ + (higher power terms)

where

$$R_0 = 0, \quad R_1 = \frac{1}{3}, \quad R_2 = \frac{2^2}{3 \times 5}, \quad R_3 = \frac{2^3}{3 \times 5 \times 7}, \ldots,$$

$$R_n = \frac{2^{n-1}}{3 \times 5 \times 7 \times \ldots \times (2n+1)}.$$

Thus we have $R_n$ and the coefficients of the higher power terms tend to zero as $n$ tend to infinity. Hence the exact solution of the given equation (4.9)-(4.10) is $y(x) = x$.

## 4.2 Solution from numerical methods

Now we shall find the solution of the iterative ordinary differential equation using the four numerical methods in chapter 3. Then we shall compare them with the exact solution.

**Example 4.4** Find the solution on the interval $[0,0.5]$ of the equation

$$y'(x) = y^2(x) \tag{4.11}$$

with the initial condition

$$y(0) = 0.25 . \tag{4.12}$$

We use the analytic solution from $y_4(x)$ in example 4.1, we then divide the interval $[0,0.5]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.1-4.4.

Table 4.1 The result of example 4.4 from the first method.

| i | X | Approximate value | $y_4(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2714254693 | 0.2712389291 | 0.0001865402 |
| 2 | 0.12500 | 0.2933598062 | 0.2929533341 | 0.0004064721 |
| 3 | 0.18750 | 0.3158195747 | 0.3151560698 | 0.0006635049 |
| 4 | 0.25000 | 0.3388221084 | 0.3378604148 | 0.0009616936 |
| 5 | 0.31250 | 0.3623855533 | 0.3610800974 | 0.0013054559 |
| 6 | 0.37500 | 0.3865289105 | 0.3848293264 | 0.0016995841 |
| 7 | 0.43750 | 0.4112720923 | 0.4091228263 | 0.0021492660 |
| 8 | 0.50000 | 0.4366359604 | 0.4339758805 | 0.0026600799 |

The arithmetic mean of absolute error is 0.0011147330.

Table 4.2 The result of example 4.4 from the second method.

| i | X | Approximate value | $y_4(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2714254661 | 0.2712389291 | 0.0001865370 |
| 2 | 0.12500 | 0.2933597942 | 0.2929533341 | 0.0004064601 |
| 3 | 0.18750 | 0.3158195510 | 0.3151560698 | 0.0006634812 |
| 4 | 0.25000 | 0.3388220715 | 0.3378604148 | 0.0009616567 |
| 5 | 0.31250 | 0.3623854908 | 0.3610800974 | 0.0013053934 |
| 6 | 0.37500 | 0.3865288169 | 0.3848293264 | 0.0016994905 |
| 7 | 0.43750 | 0.4112719526 | 0.4091228263 | 0.0021491263 |
| 8 | 0.50000 | 0.4366357479 | 0.4339758805 | 0.0026598674 |

The arithmetic mean of absolute error is 0.0011146681.

Table 4.3 The result of example 4.4 from the third method.

| i | X | Approximate value | $y_4(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2714942459 | 0.2712389291 | 0.0002553168 |
| 2 | 0.12500 | 0.2935139464 | 0.2929533341 | 0.0005606124 |
| 3 | 0.18750 | 0.3160797215 | 0.3151560698 | 0.0009236517 |
| 4 | 0.25000 | 0.3391818314 | 0.3378604148 | 0.0013214166 |
| 5 | 0.31250 | 0.3628642722 | 0.3610800974 | 0.0017841748 |
| 6 | 0.37500 | 0.3871146587 | 0.3848293264 | 0.0022853323 |
| 7 | 0.43750 | 0.4119794902 | 0.4091228263 | 0.0028566639 |
| 8 | 0.50000 | 0.4374918480 | 0.4339758805 | 0.0035159675 |

The arithmetic mean of absolute error is 0.0015003485.

Table 4.4 The result of example 4.4 from the forth method.

| i | X | Approximate value | $y_4(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2714254722 | 0.2712389291 | 0.0001865431 |
| 2 | 0.12500 | 0.2933598107 | 0.2929533341 | 0.0004064766 |
| 3 | 0.18750 | 0.3158195842 | 0.3151560698 | 0.0006635144 |
| 4 | 0.25000 | 0.3388221302 | 0.3378604148 | 0.0009617154 |
| 5 | 0.31250 | 0.3623855890 | 0.3610800974 | 0.0013054916 |
| 6 | 0.37500 | 0.3865289742 | 0.3848293264 | 0.0016996479 |
| 7 | 0.43750 | 0.4112721982 | 0.4091228263 | 0.0021493719 |
| 8 | 0.50000 | 0.4366361521 | 0.4339758805 | 0.0026602716 |

The arithmetic mean of absolute error is 0.0011147814.

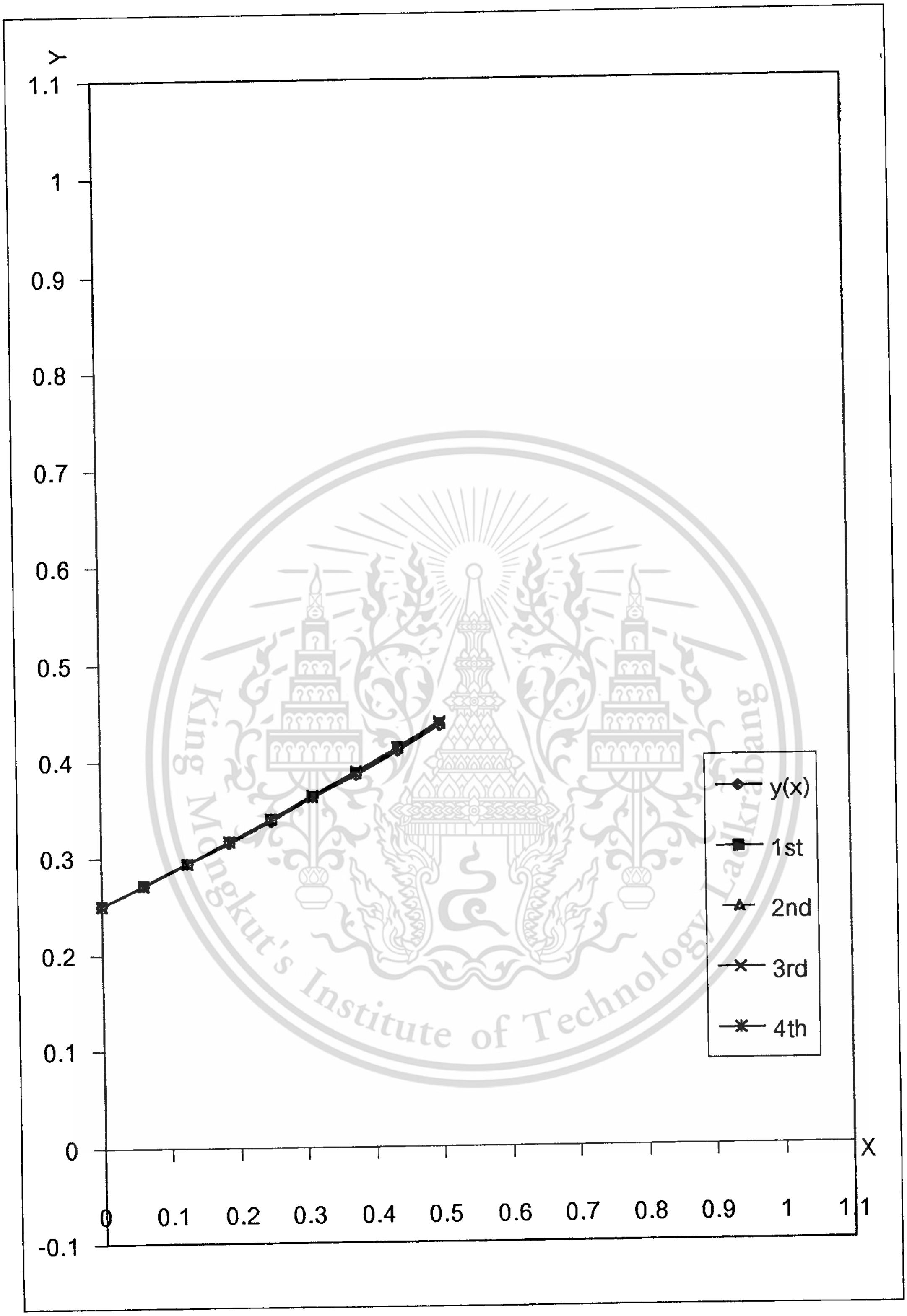


**Figure 4.1** Graph of the result from the four numerical methods of example 4.4 compare with analytic solution.

**Example 4.5** Find the solution on the interval $[0,0.5]$ of the equation

$$y'(x) = y^3(x) \tag{4.13}$$

with the initial condition

$$y(0) = 0.2\,. \tag{4.14}$$

We use the analytic solution from $y_3(x)$ in example 4.2, we then divide the interval $[0,0.5]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.5-4.8.

Table 4.5 The result of example 4.5 from the first method.

| i | X | Approximate value | $y_3(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2168553220 | 0.2164171263 | 0.0004381957 |
| 2 | 0.12500 | 0.2337898263 | 0.2328948617 | 0.0008949646 |
| 3 | 0.18750 | 0.2508039954 | 0.2494333659 | 0.0013706295 |
| 4 | 0.25000 | 0.2678983063 | 0.2660327989 | 0.0018655074 |
| 5 | 0.31250 | 0.2850732259 | 0.2826933209 | 0.0023799050 |
| 6 | 0.37500 | 0.3023292038 | 0.2994150924 | 0.0029141114 |
| 7 | 0.43750 | 0.3196666612 | 0.3161982741 | 0.0034683871 |
| 8 | 0.50000 | 0.3370859741 | 0.3330430269 | 0.0040429472 |

The arithmetic mean of absolute error is 0.0019305164.

Table 4.6 The result of example 4.5 from the second method.

| i | X | Approximate value | $y_3(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2168553294 | 0.2164171263 | 0.0004382031 |
| 2 | 0.12500 | 0.2337898444 | 0.2328948617 | 0.0008949827 |
| 3 | 0.18750 | 0.2508040288 | 0.2494333659 | 0.0013706628 |
| 4 | 0.25000 | 0.2678983617 | 0.2660327989 | 0.0018655627 |
| 5 | 0.31250 | 0.2850733113 | 0.2826933209 | 0.0023799904 |
| 6 | 0.37500 | 0.3023293300 | 0.2994150924 | 0.0029142376 |
| 7 | 0.43750 | 0.3196668442 | 0.3161982741 | 0.0034685701 |
| 8 | 0.50000 | 0.3370862358 | 0.3330430269 | 0.0040432089 |

The arithmetic mean of absolute error is 0.0019306020.

Table 4.7 The result of example 4.5 from the third method.

| i | X | Approximate value | $y_3(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2170606569 | 0.2164171263 | 0.0006435306 |
| 2 | 0.12500 | 0.2341633601 | 0.2328948617 | 0.0012684984 |
| 3 | 0.18750 | 0.2513081099 | 0.2494333659 | 0.0018747440 |
| 4 | 0.25000 | 0.2684948521 | 0.2660327989 | 0.0024620532 |
| 5 | 0.31250 | 0.2857239597 | 0.2826933209 | 0.0030306388 |
| 6 | 0.37500 | 0.3029955062 | 0.2994150924 | 0.0035804138 |
| 7 | 0.43750 | 0.3203094231 | 0.3161982741 | 0.0041111490 |
| 8 | 0.50000 | 0.3376656624 | 0.3330430269 | 0.0046226355 |

The arithmetic mean of absolute error is 0.0023992959.

Table 4.8 The result of example 4.5 from the forth method.

| i | X | Approximate value | $y_3(x)$ | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 1 | 0.06250 | 0.2170532168 | 0.2164171263 | 0.0006360905 |
| 2 | 0.12500 | 0.2341481269 | 0.2328948617 | 0.0012532652 |
| 3 | 0.18750 | 0.2512848509 | 0.2494333659 | 0.0018514850 |
| 4 | 0.25000 | 0.2684634622 | 0.2660327989 | 0.0024306633 |
| 5 | 0.31250 | 0.2856840668 | 0.2826933209 | 0.0029907459 |
| 6 | 0.37500 | 0.3029467813 | 0.2994150924 | 0.0035316889 |
| 7 | 0.43750 | 0.3202517024 | 0.3161982741 | 0.0040534283 |
| 8 | 0.50000 | 0.3375989627 | 0.3330430269 | 0.0045559358 |

The arithmetic mean of absolute error is 0.0023670336.

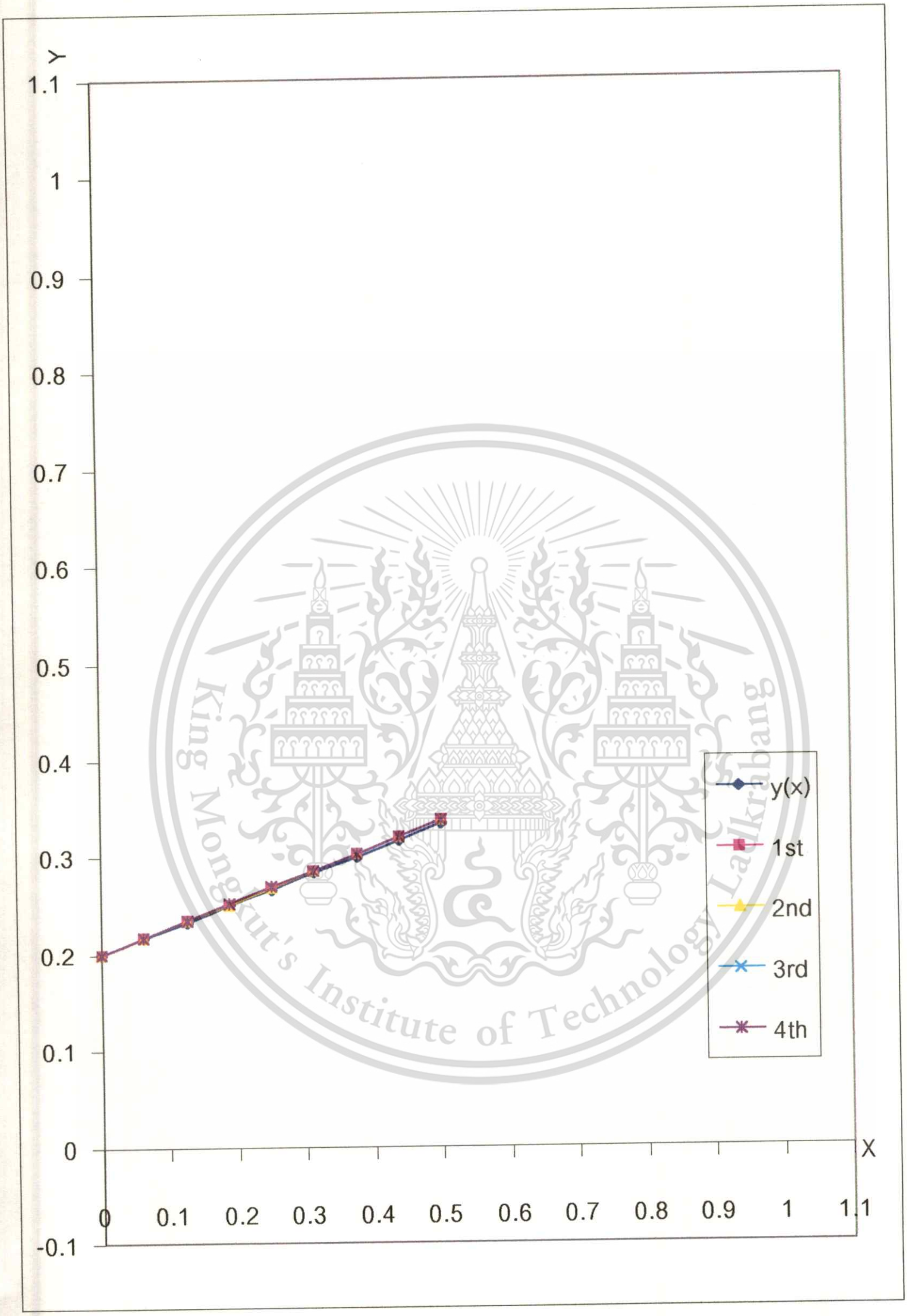


Figure 4.2 Graph of the result from the four numerical methods of example 4.5 compare with analytic solution.

**Example 4.6** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = \frac{1}{1+2x} - \frac{1}{1+x} + \frac{1}{(1+x)^2} + 2y^2(x) - y \qquad (4.15)$$

with the initial condition

$$y(0) = 0 . \qquad (4.16)$$

The exact solution is $y = \dfrac{x}{1+x}$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.9-4.12.

Table 4.9 The result of example 4.6 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1110894270 | 0.1111111111 | 0.0000216841 |
| 2 | 0.25000 | 0.1999815166 | 0.2000000000 | 0.0000184834 |
| 3 | 0.37500 | 0.2727075940 | 0.2727272727 | 0.0000196787 |
| 4 | 0.50000 | 0.3332896388 | 0.3333333333 | 0.0000436945 |
| 5 | 0.62500 | 0.3845534598 | 0.3846153846 | 0.0000619248 |
| 6 | 0.75000 | 0.4284957725 | 0.4285714286 | 0.0000756561 |
| 7 | 0.87500 | 0.4665703864 | 0.4666666667 | 0.0000962803 |
| 8 | 1.00000 | 0.4998876741 | 0.5000000000 | 0.0001123259 |

The arithmetic mean of absolute error is 0.0000499698.

Table 4.10 The result of example 4.6 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1111166877 | 0.1111111111 | 0.0000055766 |
| 2 | 0.25000 | 0.2000096192 | 0.2000000000 | 0.0000096192 |
| 3 | 0.37500 | 0.2727404636 | 0.2727272727 | 0.0000131909 |
| 4 | 0.50000 | 0.3333506783 | 0.3333333333 | 0.0000173450 |
| 5 | 0.62500 | 0.3846364627 | 0.3846153846 | 0.0000210781 |
| 6 | 0.75000 | 0.4285964830 | 0.4285714286 | 0.0000250544 |
| 7 | 0.87500 | 0.4666960276 | 0.4666666667 | 0.0000293609 |
| 8 | 1.00000 | 0.5000333949 | 0.5000000000 | 0.0000333949 |

The arithmetic mean of absolute error is 0.0000171800.

Table 4.11 The result of example 4.6 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1125129076 | 0.1111111111 | 0.0014017968 |
| 2 | 0.25000 | 0.2002883146 | 0.2000000000 | 0.0002883146 |
| 3 | 0.37500 | 0.2681330711 | 0.2727272727 | 0.0045942016 |
| 4 | 0.50000 | 0.3203010014 | 0.3333333333 | 0.0130323319 |
| 5 | 0.62500 | 0.3602928473 | 0.3846153846 | 0.0243225373 |
| 6 | 0.75000 | 0.3922372966 | 0.4285714286 | 0.0363341320 |
| 7 | 0.87500 | 0.4228271950 | 0.4666666667 | 0.0438394717 |
| 8 | 1.00000 | 0.4473178894 | 0.5000000000 | 0.0526821106 |

The arithmetic mean of absolute error is 0.0196105441.

Table 4.12 The result of example 4.6 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1111153254 | 0.1111111111 | 0.0000042143 |
| 2 | 0.25000 | 0.2000074919 | 0.2000000000 | 0.0000074919 |
| 3 | 0.37500 | 0.2727376663 | 0.2727272727 | 0.0000103936 |
| 4 | 0.50000 | 0.3333472019 | 0.3333333333 | 0.0000138686 |
| 5 | 0.62500 | 0.3846322731 | 0.3846153846 | 0.0000168885 |
| 6 | 0.75000 | 0.4285915533 | 0.4285714286 | 0.0000201247 |
| 7 | 0.87500 | 0.4666903396 | 0.4666666667 | 0.0000236729 |
| 8 | 1.00000 | 0.5000267960 | 0.5000000000 | 0.0000267960 |

The arithmetic mean of absolute error is 0.0000137167.

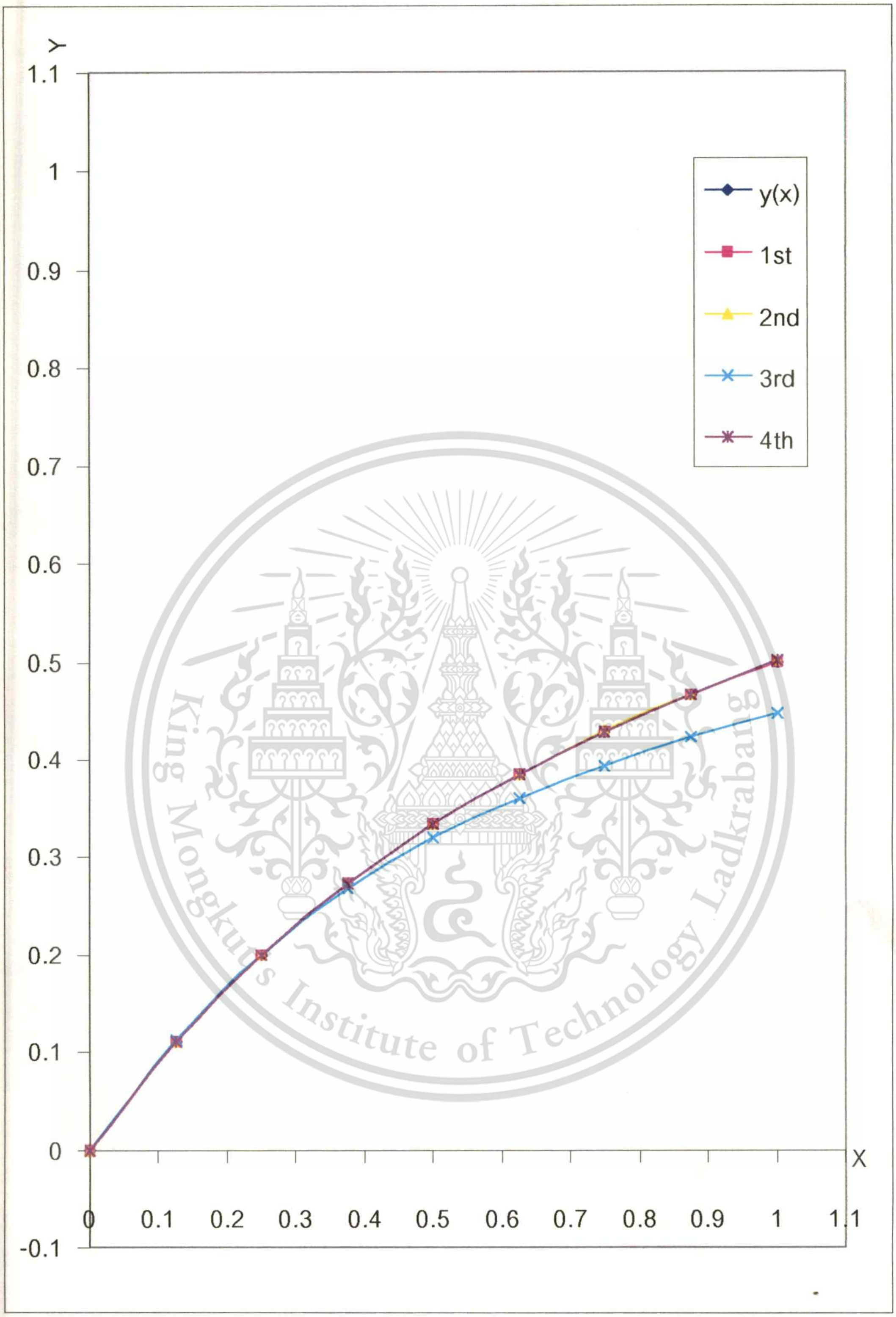


**Figure 4.3 Graph of the result from the four numerical methods of example 4.6 compare with exact solution.**

**Example 4.7** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = y^2(x) - \frac{x}{4} - \frac{3}{4} \tag{4.17}$$

with the initial condition

$$y(0) = 0.5\ . \tag{4.18}$$

The exact solution is $y = \frac{1-x}{2}$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.13-4.16.

Table 4.13 The result of example 4.7 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4374998840 | 0.4375000000 | 0.0000001160 |
| 2 | 0.25000 | 0.3749997394 | 0.3750000000 | 0.0000002606 |
| 3 | 0.37500 | 0.3124995502 | 0.3125000000 | 0.0000004498 |
| 4 | 0.50000 | 0.2499992936 | 0.2500000000 | 0.0000007064 |
| 5 | 0.62500 | 0.1874989370 | 0.1875000000 | 0.0000010630 |
| 6 | 0.75000 | 0.1249984342 | 0.1250000000 | 0.0000015658 |
| 7 | 0.87500 | 0.0624977181 | 0.0625000000 | 0.0000022819 |
| 8 | 1.00000 | -0.0000033078 | 0.0000000000 | 0.0000033078 |

The arithmetic mean of absolute error is 0.0000010835.

Table 4.14 The result of example 4.7 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4374999015 | 0.4375000000 | 0.0000000985 |
| 2 | 0.25000 | 0.3749997833 | 0.3750000000 | 0.0000002167 |
| 3 | 0.37500 | 0.3124996267 | 0.3125000000 | 0.0000003733 |
| 4 | 0.50000 | 0.2499994184 | 0.2500000000 | 0.0000005816 |
| 5 | 0.62500 | 0.1874991303 | 0.1875000000 | 0.0000008697 |
| 6 | 0.75000 | 0.1249987278 | 0.1250000000 | 0.0000012722 |
| 7 | 0.87500 | 0.0624981609 | 0.0625000000 | 0.0000018391 |
| 8 | 1.00000 | -0.0000026480 | 0.0000000000 | 0.0000026480 |

The arithmetic mean of absolute error is 0.0000008777.

Table 4.15 The result of example 4.7 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4374998565 | 0.4375000000 | 0.0000001435 |
| 2 | 0.25000 | 0.3749996904 | 0.3750000000 | 0.0000003096 |
| 3 | 0.37500 | 0.3125002240 | 0.3125000000 | 0.0000002240 |
| 4 | 0.50000 | 0.2500002180 | 0.2500000000 | 0.0000002180 |
| 5 | 0.62500 | 0.1874994704 | 0.1875000000 | 0.0000005296 |
| 6 | 0.75000 | 0.1249997057 | 0.1250000000 | 0.0000002943 |
| 7 | 0.87500 | 0.0625004700 | 0.0625000000 | 0.0000004700 |
| 8 | 1.00000 | -0.0000003613 | 0.0000000000 | 0.0000003613 |

The arithmetic mean of absolute error is 0.0000002834.

Table 4.16 The result of example 4.7 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4374999077 | 0.4375000000 | 0.0000000923 |
| 2 | 0.25000 | 0.3749997977 | 0.3750000000 | 0.0000002023 |
| 3 | 0.37500 | 0.3124996537 | 0.3125000000 | 0.0000003463 |
| 4 | 0.50000 | 0.2499994631 | 0.2500000000 | 0.0000005369 |
| 5 | 0.62500 | 0.1874992034 | 0.1875000000 | 0.0000007967 |
| 6 | 0.75000 | 0.1249988416 | 0.1250000000 | 0.0000011584 |
| 7 | 0.87500 | 0.0624983408 | 0.0625000000 | 0.0000016592 |
| 8 | 1.00000 | -0.0000024444 | 0.0000000000 | 0.0000024444 |

The arithmetic mean of absolute error is 0.0000008040.

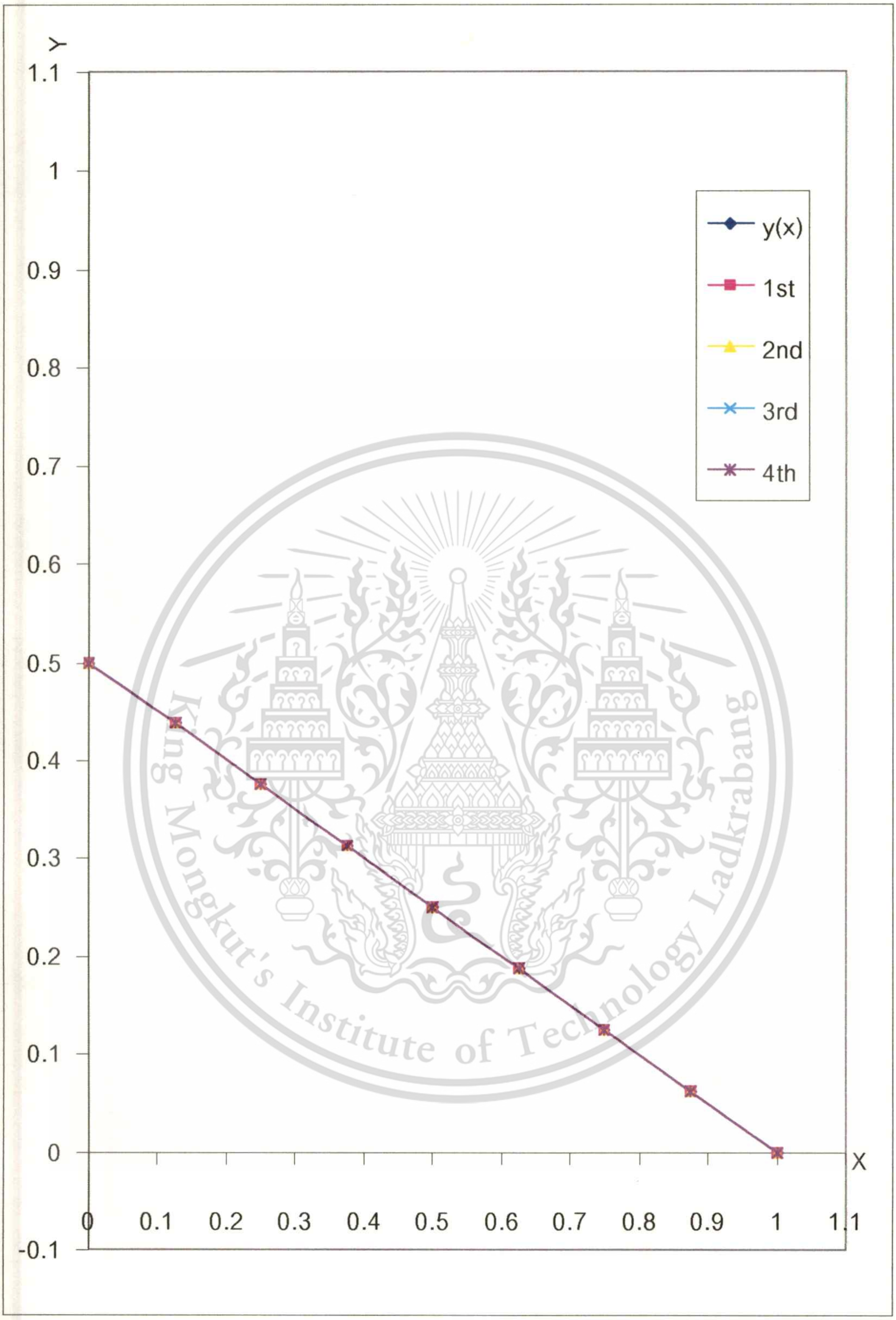


**Figure 4.4** Graph of the result from the four numerical methods of example 4.7 compare with exact solution.

**Example 4.8** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = y^3(x) + \frac{x}{8} - \frac{7}{8} \tag{4.19}$$

with the initial condition

$$y(0) = 0.5 . \tag{4.20}$$

The exact solution is $y = \dfrac{1-x}{2}$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.17-4.20.

Table 4.17 The result of example 4.8 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4375000083 | 0.4375000000 | 0.0000000083 |
| 2 | 0.25000 | 0.3750000335 | 0.3750000000 | 0.0000000335 |
| 3 | 0.37500 | 0.3125000726 | 0.3125000000 | 0.0000000726 |
| 4 | 0.50000 | 0.2500000871 | 0.2500000000 | 0.0000000871 |
| 5 | 0.62500 | 0.1874999747 | 0.1875000000 | 0.0000000253 |
| 6 | 0.75000 | 0.1249995629 | 0.1250000000 | 0.0000004371 |
| 7 | 0.87500 | 0.0624987258 | 0.0625000000 | 0.0000012742 |
| 8 | 1.00000 | -0.0000022655 | 0.0000000000 | 0.0000022655 |

The arithmetic mean of absolute error is 0.0000004971.

Table 4.18 The result of example 4.8 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4375000194 | 0.4375000000 | 0.0000000194 |
| 2 | 0.25000 | 0.3750000554 | 0.3750000000 | 0.0000000554 |
| 3 | 0.37500 | 0.3125000881 | 0.3125000000 | 0.0000000881 |
| 4 | 0.50000 | 0.2500000498 | 0.2500000000 | 0.0000000498 |
| 5 | 0.62500 | 0.1874998107 | 0.1875000000 | 0.0000001893 |
| 6 | 0.75000 | 0.1249991823 | 0.1250000000 | 0.0000008177 |
| 7 | 0.87500 | 0.0624982145 | 0.0625000000 | 0.0000017855 |
| 8 | 1.00000 | -0.0000025306 | 0.0000000000 | 0.0000025306 |

The arithmetic mean of absolute error is 0.0000006151.

Table 4.19 The result of example 4.8 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4329121310 | 0.4375000000 | 0.0045878690 |
| 2 | 0.25000 | 0.3677303233 | 0.3750000000 | 0.0072696767 |
| 3 | 0.37500 | 0.3044856181 | 0.3125000000 | 0.0080143819 |
| 4 | 0.50000 | 0.2431794202 | 0.2500000000 | 0.0068205798 |
| 5 | 0.62500 | 0.1838525868 | 0.1875000000 | 0.0036474132 |
| 6 | 0.75000 | 0.1264276029 | 0.1250000000 | 0.0014276029 |
| 7 | 0.87500 | 0.0709473886 | 0.0625000000 | 0.0084473886 |
| 8 | 1.00000 | 0.0173992783 | 0.0000000000 | 0.0173992783 |

The arithmetic mean of absolute error is 0.0064015767.

Table 4.20 The result of example 4.8 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4325262590 | 0.4375000000 | 0.0049737410 |
| 2 | 0.25000 | 0.3670084330 | 0.3750000000 | 0.0079915670 |
| 3 | 0.37500 | 0.3034606656 | 0.3125000000 | 0.0090393344 |
| 4 | 0.50000 | 0.2418796364 | 0.2500000000 | 0.0081203636 |
| 5 | 0.62500 | 0.1822628408 | 0.1875000000 | 0.0052371592 |
| 6 | 0.75000 | 0.1246161447 | 0.1250000000 | 0.0003838553 |
| 7 | 0.87500 | 0.0689190819 | 0.0625000000 | 0.0064190819 |
| 8 | 1.00000 | 0.0153338411 | 0.0000000000 | 0.0153338411 |

The arithmetic mean of absolute error is 0.0063887715.

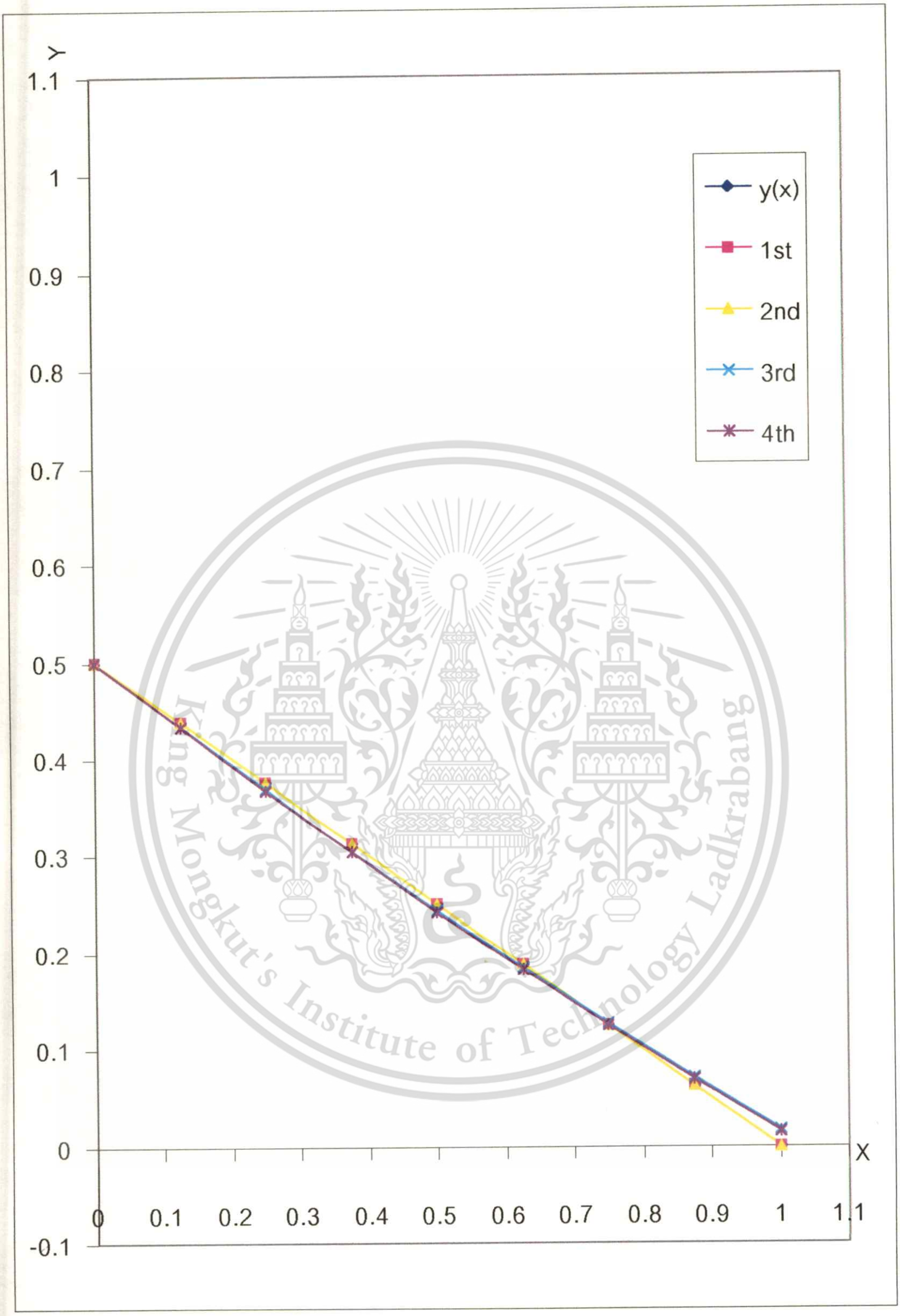


Figure 4.5 Graph of the result from the four numerical methods of example 4.8 compare with exact solution.

**Example 4.9** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = y^4(x) - \frac{x}{16} - \frac{13}{16} \tag{4.21}$$

with the initial condition

$$y(0) = 0.5 . \tag{4.22}$$

The exact solution is $y = \frac{1-x}{2}$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.21-4.24.

Table 4.21 The result of example 4.9 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4374999537 | 0.4375000000 | 0.0000000463 |
| 2 | 0.25000 | 0.3749998870 | 0.3750000000 | 0.0000001130 |
| 3 | 0.37500 | 0.3124997863 | 0.3125000000 | 0.0000002137 |
| 4 | 0.50000 | 0.2499996292 | 0.2500000000 | 0.0000003708 |
| 5 | 0.62500 | 0.1874993801 | 0.1875000000 | 0.0000006199 |
| 6 | 0.75000 | 0.1249989784 | 0.1250000000 | 0.0000010216 |
| 7 | 0.87500 | 0.0624983270 | 0.0625000000 | 0.0000016730 |
| 8 | 1.00000 | -0.0000027280 | 0.0000000000 | 0.0000027280 |

The arithmetic mean of absolute error is 0.0000007504.

Table 4.22 The result of example 4.9 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4374999566 | 0.4375000000 | 0.0000000434 |
| 2 | 0.25000 | 0.3749998951 | 0.3750000000 | 0.0000001049 |
| 3 | 0.37500 | 0.3124998038 | 0.3125000000 | 0.0000001962 |
| 4 | 0.50000 | 0.2499996646 | 0.2500000000 | 0.0000003354 |
| 5 | 0.62500 | 0.1874994469 | 0.1875000000 | 0.0000005531 |
| 6 | 0.75000 | 0.1249991047 | 0.1250000000 | 0.0000008953 |
| 7 | 0.87500 | 0.0624985618 | 0.0625000000 | 0.0000014382 |
| 8 | 1.00000 | -0.0000025306 | 0.0000000000 | 0.0000025306 |

The arithmetic mean of absolute error is 0.0000006525.

Table 4.23 The result of example 4.9 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4400924607 | 0.4375000000 | 0.0025924607 |
| 2 | 0.25000 | 0.3790959835 | 0.3750000000 | 0.0040959835 |
| 3 | 0.37500 | 0.3169996112 | 0.3125000000 | 0.0044996112 |
| 4 | 0.50000 | 0.2538082429 | 0.2500000000 | 0.0038082429 |
| 5 | 0.62500 | 0.1895141887 | 0.1875000000 | 0.0020141887 |
| 6 | 0.75000 | 0.1241125408 | 0.1250000000 | 0.0008874592 |
| 7 | 0.87500 | 0.0576039057 | 0.0625000000 | 0.0048960943 |
| 8 | 1.00000 | -0.0099844082 | 0.0000000000 | 0.0099844082 |

The arithmetic mean of absolute error is 0.0036420499.

Table 4.24 The result of example 4.9 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4387583399 | 0.4375000000 | 0.0012583399 |
| 2 | 0.25000 | 0.3770174955 | 0.3750000000 | 0.0020174955 |
| 3 | 0.37500 | 0.3147784427 | 0.3125000000 | 0.0022774427 |
| 4 | 0.50000 | 0.2520431610 | 0.2500000000 | 0.0020431610 |
| 5 | 0.62500 | 0.1888153152 | 0.1875000000 | 0.0013153152 |
| 6 | 0.75000 | 0.1250984049 | 0.1250000000 | 0.0000984049 |
| 7 | 0.87500 | 0.0608966550 | 0.0625000000 | 0.0016033450 |
| 8 | 1.00000 | -0.0037984898 | 0.0000000000 | 0.0037984898 |

The arithmetic mean of absolute error is 0.0016014438.

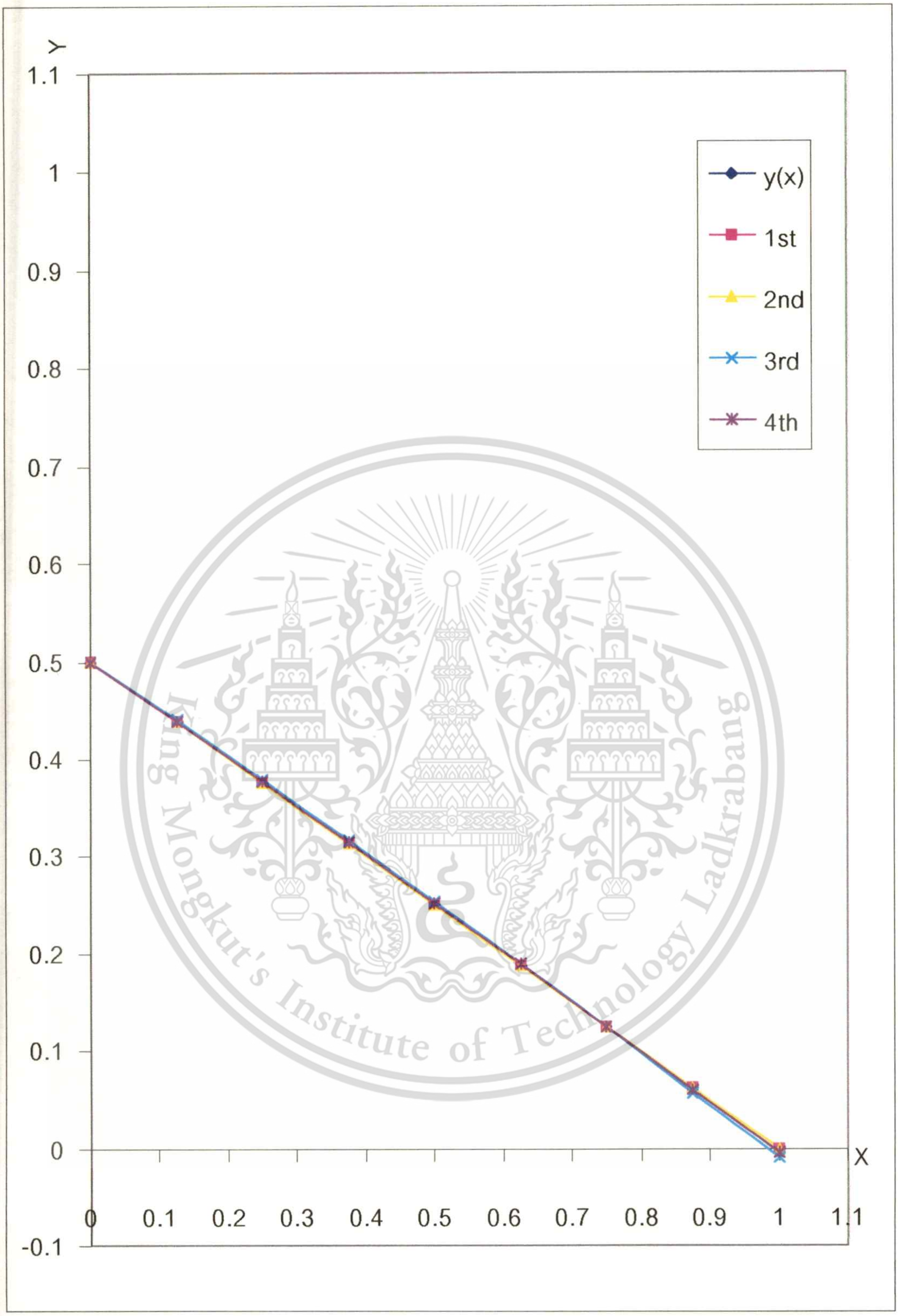


**Figure 4.6 Graph of the result from the four numerical methods of example 4.9 compare with exact solution.**

**Example 4.10** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = y^5(x) + \frac{x}{32} - \frac{27}{32} \tag{4.23}$$

with the initial condition

$$y(0) = 0.5 . \tag{4.24}$$

The exact solution is $y = \dfrac{1-x}{2}$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.25-4.28.

Table 4.25 The result of example 4.10 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4374999999 | 0.4375000000 | 0.0000000001 |
| 2 | 0.25000 | 0.3750000019 | 0.3750000000 | 0.0000000019 |
| 3 | 0.37500 | 0.3125000101 | 0.3125000000 | 0.0000000101 |
| 4 | 0.50000 | 0.2500000319 | 0.2500000000 | 0.0000000319 |
| 5 | 0.62500 | 0.1875000719 | 0.1875000000 | 0.0000000719 |
| 6 | 0.75000 | 0.1250001022 | 0.1250000000 | 0.0000001022 |
| 7 | 0.87500 | 0.0624999470 | 0.0625000000 | 0.0000000530 |
| 8 | 1.00000 | -0.0000009500 | 0.0000000000 | 0.0000009500 |

The arithmetic mean of absolute error is 0.0000001357.

Table 4.26 The result of example 4.10 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4375000004 | 0.4375000000 | 0.0000000004 |
| 2 | 0.25000 | 0.3750000025 | 0.3750000000 | 0.0000000025 |
| 3 | 0.37500 | 0.3125000076 | 0.3125000000 | 0.0000000076 |
| 4 | 0.50000 | 0.2500000168 | 0.2500000000 | 0.0000000168 |
| 5 | 0.62500 | 0.1875000193 | 0.1875000000 | 0.0000000193 |
| 6 | 0.75000 | 0.1249999666 | 0.1250000000 | 0.0000000334 |
| 7 | 0.87500 | 0.0624996937 | 0.0625000000 | 0.0000000363 |
| 8 | 1.00000 | -0.0000012819 | 0.0000000000 | 0.0000012819 |

The arithmetic mean of absolute error is 0.0000001853.

Table 4.27 The result of example 4.10 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4366315837 | 0.4375000000 | 0.0008684163 |
| 2 | 0.25000 | 0.3736290018 | 0.3750000000 | 0.0013709982 |
| 3 | 0.37500 | 0.3109930856 | 0.3125000000 | 0.0015069144 |
| 4 | 0.50000 | 0.2487249259 | 0.2500000000 | 0.0012750741 |
| 5 | 0.62500 | 0.1868242718 | 0.1875000000 | 0.0006757282 |
| 6 | 0.75000 | 0.1252905537 | 0.1250000000 | 0.0002905537 |
| 7 | 0.87500 | 0.0641260027 | 0.0625000000 | 0.0016260027 |
| 8 | 1.00000 | 0.0033297928 | 0.0000000000 | 0.0033297928 |

The arithmetic mean of absolute error is 0.0012159423.

Table 4.28 The result of example 4.10 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.4365742662 | 0.4375000000 | 0.0009257338 |
| 2 | 0.25000 | 0.3735127873 | 0.3750000000 | 0.0014872127 |
| 3 | 0.37500 | 0.3108159301 | 0.3125000000 | 0.0016840699 |
| 4 | 0.50000 | 0.2484840578 | 0.2500000000 | 0.0015159422 |
| 5 | 0.62500 | 0.1865177242 | 0.1875000000 | 0.0009822758 |
| 6 | 0.75000 | 0.1249178651 | 0.1250000000 | 0.0000821349 |
| 7 | 0.87500 | 0.0636848375 | 0.0625000000 | 0.0011848375 |
| 8 | 1.00000 | 0.0028205269 | 0.0000000000 | 0.0028205269 |

The arithmetic mean of absolute error is 0.0011869704.

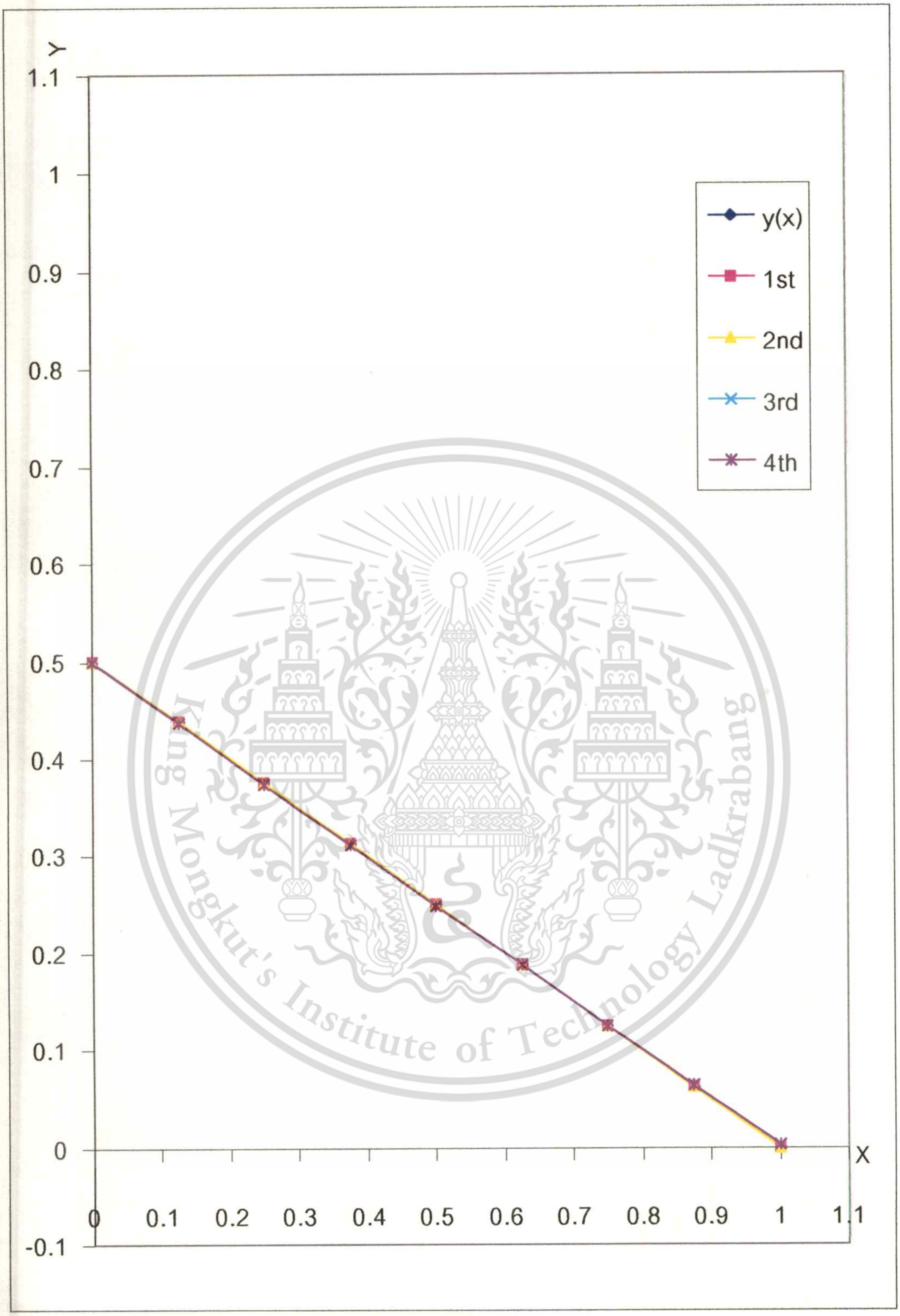


**Figure 4.7** Graph of the result from the four numerical methods of example 4.10 compare with exact solution.

**Example 4.11** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = xy^2(x) + 1 - x^2 \qquad (4.25)$$

with the initial condition

$$y(0) = 0 . \qquad (4.26)$$

The exact solution is $y = x$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.29-4.32.

Table 4.29 The result of example 4.11 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999995 | 0.7500000000 | 0.0000000005 |
| 7 | 0.87500 | 0.8749999771 | 0.8750000000 | 0.0000000229 |
| 8 | 1.00000 | 0.9999995908 | 1.0000000000 | 0.0000004092 |

The arithmetic mean of absolute error is 0.0000000481.

Table 4.30 The result of example 4.11 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7500000000 | 0.7500000000 | 0.0000000000 |
| 7 | 0.87500 | 0.8749999931 | 0.8750000000 | 0.0000000069 |
| 8 | 1.00000 | 0.9999997471 | 1.0000000000 | 0.0000002529 |

The arithmetic mean of absolute error is 0.0000000289.

Table 4.31 The result of example 4.11 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999999 | 0.7500000000 | 0.0000000001 |
| 7 | 0.87500 | 0.8749999940 | 0.8750000000 | 0.0000000060 |
| 8 | 1.00000 | 0.9999998984 | 1.0000000000 | 0.0000001016 |

The arithmetic mean of absolute error is 0.0000000120.

Table 4.32 The result of example 4.11 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999999 | 0.7500000000 | 0.0000000001 |
| 7 | 0.87500 | 0.8749999855 | 0.8750000000 | 0.0000000145 |
| 8 | 1.00000 | 0.9999995820 | 1.0000000000 | 0.0000004180 |

The arithmetic mean of absolute error is 0.0000000481.

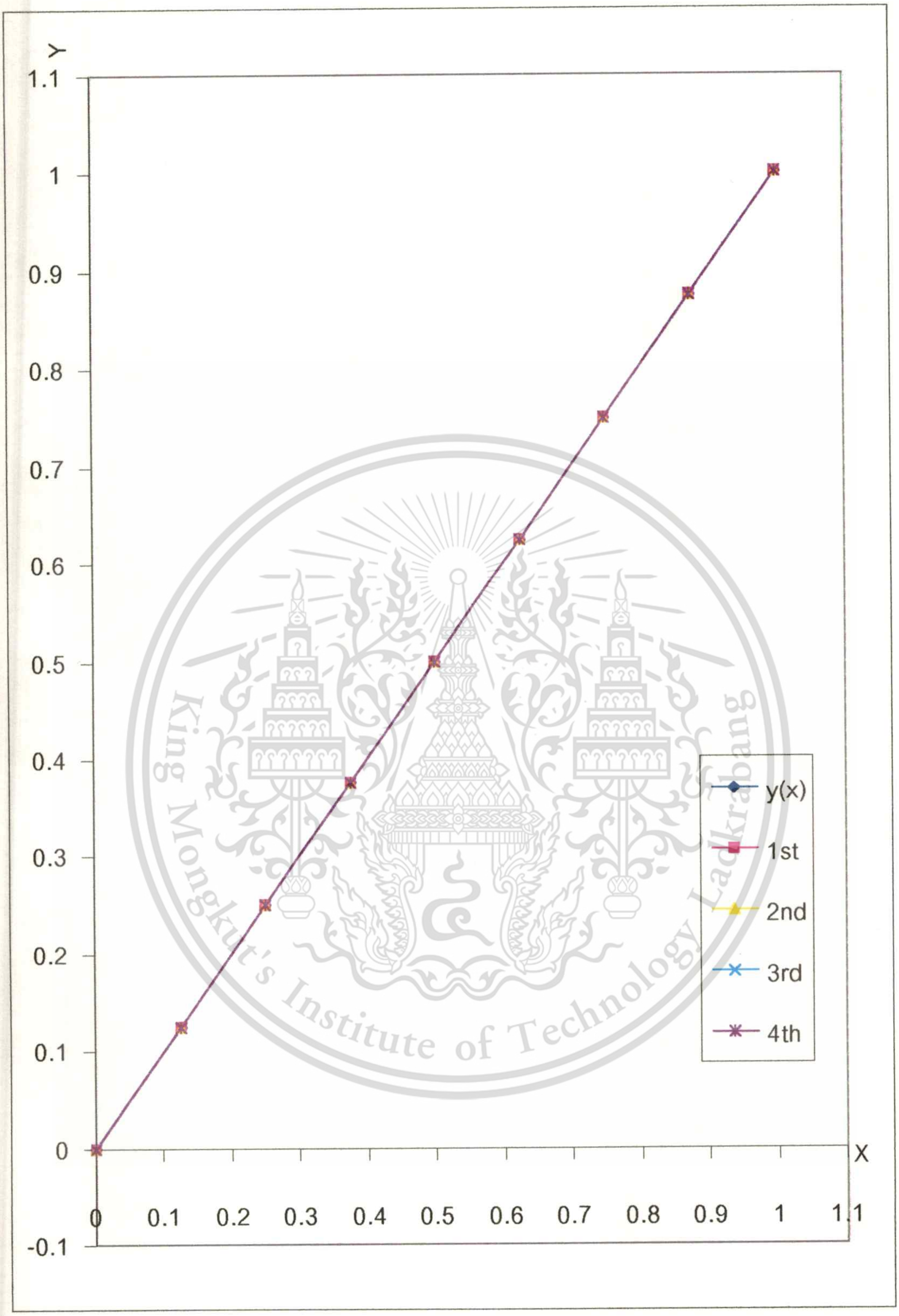


**Figure 4.8 Graph of the result from the four numerical methods of example 4.11 compare with exact solution.**

**Example 4.12** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = xy^3(x) + 1 - x^2 \tag{4.27}$$

with the initial condition

$$y(0) = 0\ . \tag{4.28}$$

The exact solution is $y = x$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.33-4.36.

Table 4.33 The result of example 4.12 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999998 | 0.7500000000 | 0.0000000002 |
| 7 | 0.87500 | 0.8749999927 | 0.8750000000 | 0.0000000073 |
| 8 | 1.00000 | 0.9999998945 | 1.0000000000 | 0.0000001055 |

The arithmetic mean of absolute error is 0.0000000126.

Table 4.34 The result of example 4.12 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7500000000 | 0.7500000000 | 0.0000000000 |
| 7 | 0.87500 | 0.8749999990 | 0.8750000000 | 0.0000000010 |
| 8 | 1.00000 | 0.9999999518 | 1.0000000000 | 0.0000000482 |

The arithmetic mean of absolute error is 0.0000000055.

Table 4.35 The result of example 4.12 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999997 | 0.7500000000 | 0.0000000003 |
| 7 | 0.87500 | 0.8749999810 | 0.8750000000 | 0.0000000190 |
| 8 | 1.00000 | 0.9999997537 | 1.0000000000 | 0.0000002463 |

The arithmetic mean of absolute error is 0.0000000295.

Table 4.36 The result of example 4.12 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999999 | 0.7500000000 | 0.0000000001 |
| 7 | 0.87500 | 0.8749999821 | 0.8750000000 | 0.0000000179 |
| 8 | 1.00000 | 0.9999995929 | 1.0000000000 | 0.0000004071 |

The arithmetic mean of absolute error is 0.0000000472.

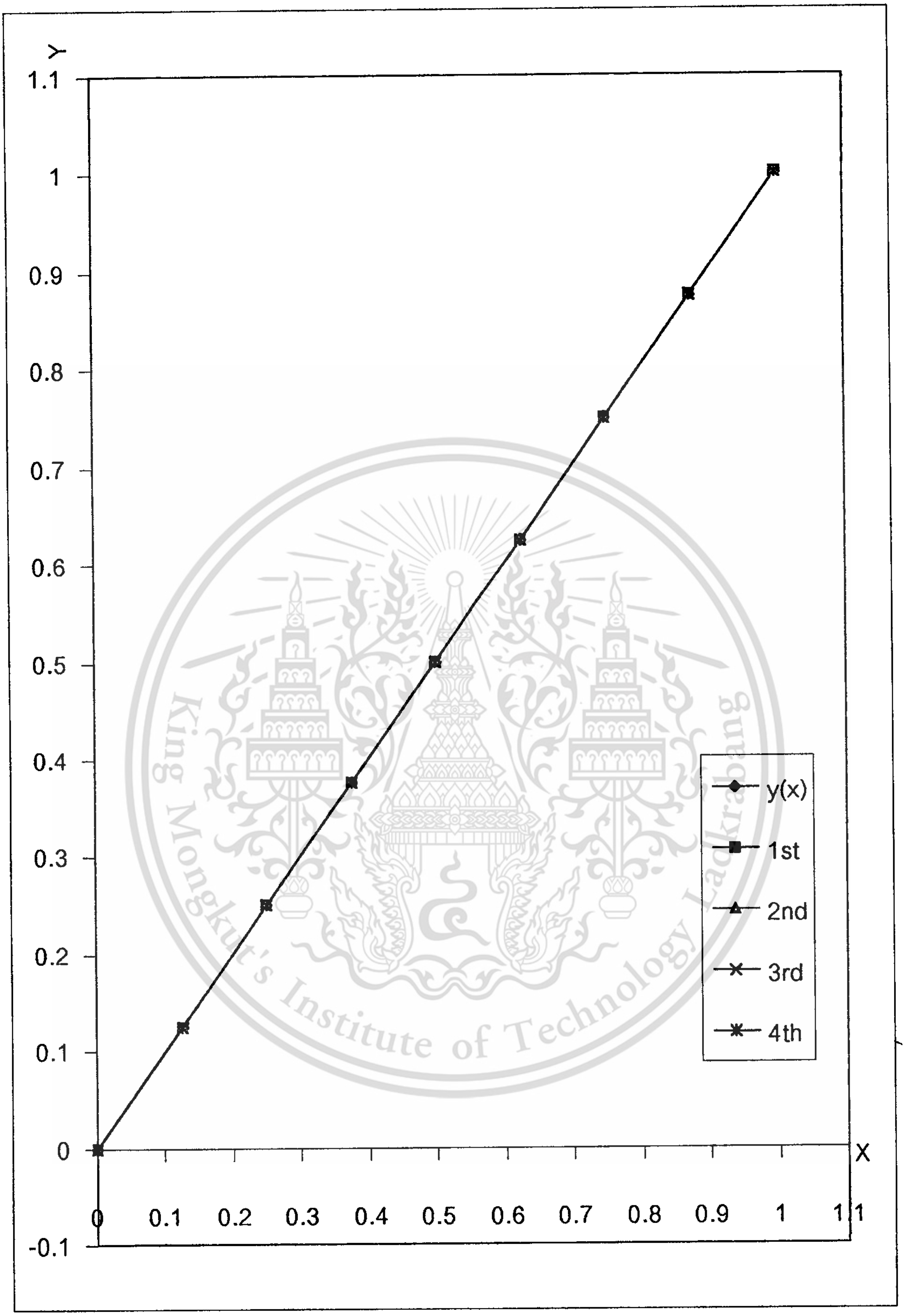


**Figure 4.9 Graph of the result from the four numerical methods of example 4.12 compare with exact solution.**

**Example 4.13** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = xy^4(x) + 1 - x^2 \tag{4.29}$$

with the initial condition

$$y(0) = 0 . \tag{4.30}$$

The exact solution is $y = x$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.37-4.40.

Table 4.37 The result of example 4.13 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999997 | 0.7500000000 | 0.0000000003 |
| 7 | 0.87500 | 0.8749999808 | 0.8750000000 | 0.0000000192 |
| 8 | 1.00000 | 0.9999997598 | 1.0000000000 | 0.0000002402 |

The arithmetic mean of absolute error is 0.0000000289.

Table 4.38 The result of example 4.13 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7500000000 | 0.7500000000 | 0.0000000000 |
| 7 | 0.87500 | 0.8749999993 | 0.8750000000 | 0.0000000007 |
| 8 | 1.00000 | 0.9999999153 | 1.0000000000 | 0.0000000847 |

The arithmetic mean of absolute error is 0.0000000095.

Table 4.39 The result of example 4.13 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999996 | 0.7500000000 | 0.0000000004 |
| 7 | 0.87500 | 0.8749999573 | 0.8750000000 | 0.0000000427 |
| 8 | 1.00000 | 0.9999995131 | 1.0000000000 | 0.0000004869 |

The arithmetic mean of absolute error is 0.0000000589.

Table 4.40 The result of example 4.13 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7500000000 | 0.7500000000 | 0.0000000000 |
| 7 | 0.87500 | 0.8749999827 | 0.8750000000 | 0.0000000173 |
| 8 | 1.00000 | 0.9999996289 | 1.0000000000 | 0.0000003711 |

The arithmetic mean of absolute error is 0.0000000432.

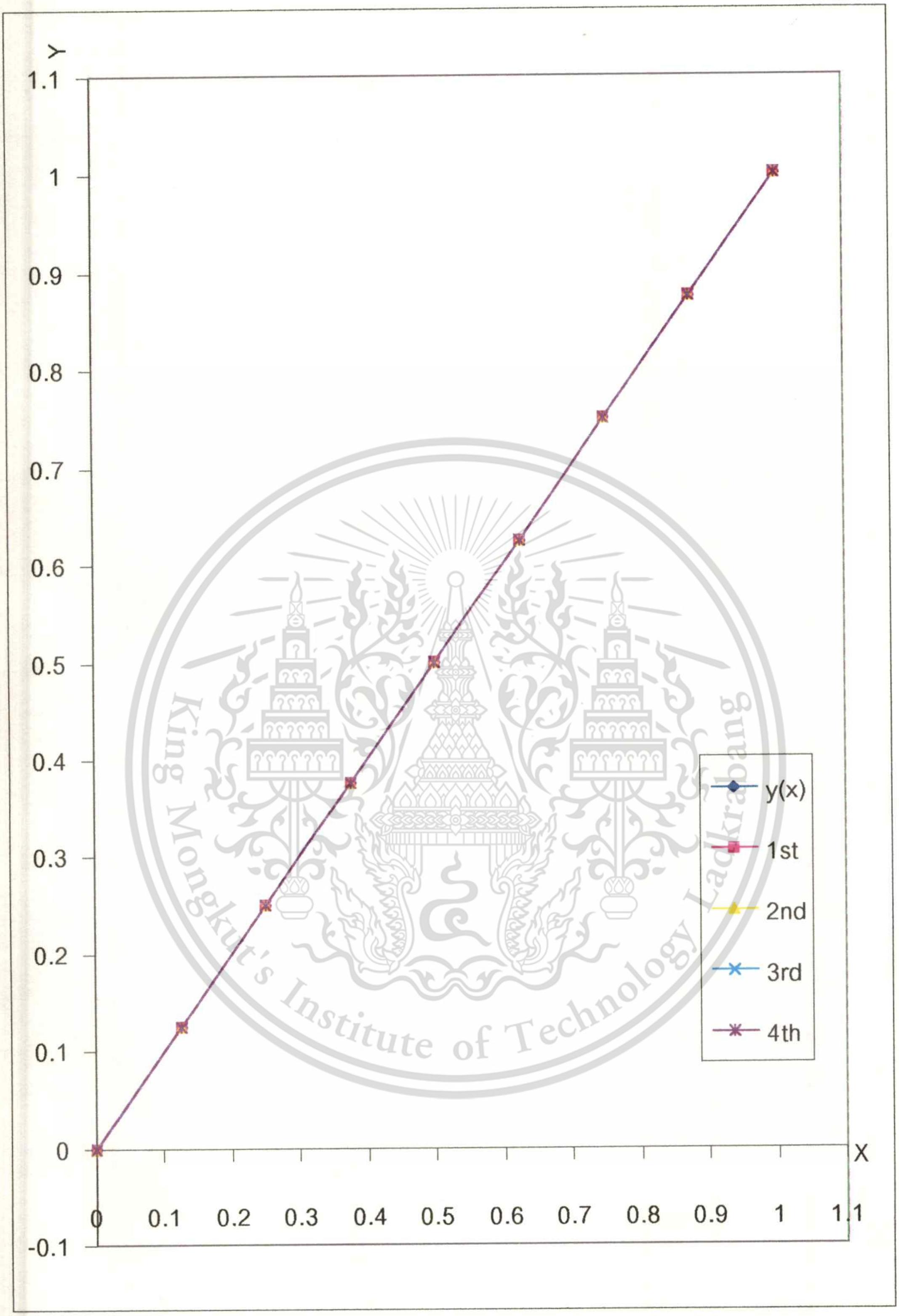


**Figure 4.10** Graph of the result from the four numerical methods of example 4.13 compare with exact solution.

**Example 4.14** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = xy^5(x) + 1 - x^2 \tag{4.31}$$

with the initial condition

$$y(0) = 0 . \tag{4.32}$$

The exact solution is $y = x$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.41-4.44.

Table 4.41 The result of example 4.14 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999994 | 0.7500000000 | 0.0000000006 |
| 7 | 0.87500 | 0.8749999599 | 0.8750000000 | 0.0000000401 |
| 8 | 1.00000 | 0.9999995432 | 1.0000000000 | 0.0000004568 |

The arithmetic mean of absolute error is 0.0000000553.

Table 4.42 The result of example 4.14 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7500000001 | 0.7500000000 | 0.0000000001 |
| 7 | 0.87500 | 0.8750000017 | 0.8750000000 | 0.0000000017 |
| 8 | 1.00000 | 0.9999999082 | 1.0000000000 | 0.0000000918 |

The arithmetic mean of absolute error is 0.0000000104.

Table 4.43 The result of example 4.14 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7499999993 | 0.7500000000 | 0.0000000007 |
| 7 | 0.87500 | 0.8749999100 | 0.8750000000 | 0.0000000900 |
| 8 | 1.00000 | 0.9999990490 | 1.0000000000 | 0.0000009510 |

The arithmetic mean of absolute error is 0.0000001158.

Table 4.44 The result of example 4.14 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.1250000000 | 0.1250000000 | 0.0000000000 |
| 2 | 0.25000 | 0.2500000000 | 0.2500000000 | 0.0000000000 |
| 3 | 0.37500 | 0.3750000000 | 0.3750000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.62500 | 0.6250000000 | 0.6250000000 | 0.0000000000 |
| 6 | 0.75000 | 0.7500000000 | 0.7500000000 | 0.0000000000 |
| 7 | 0.87500 | 0.8749999834 | 0.8750000000 | 0.0000000166 |
| 8 | 1.00000 | 0.9999996404 | 1.0000000000 | 0.0000003596 |

The arithmetic mean of absolute error is 0.0000000418.

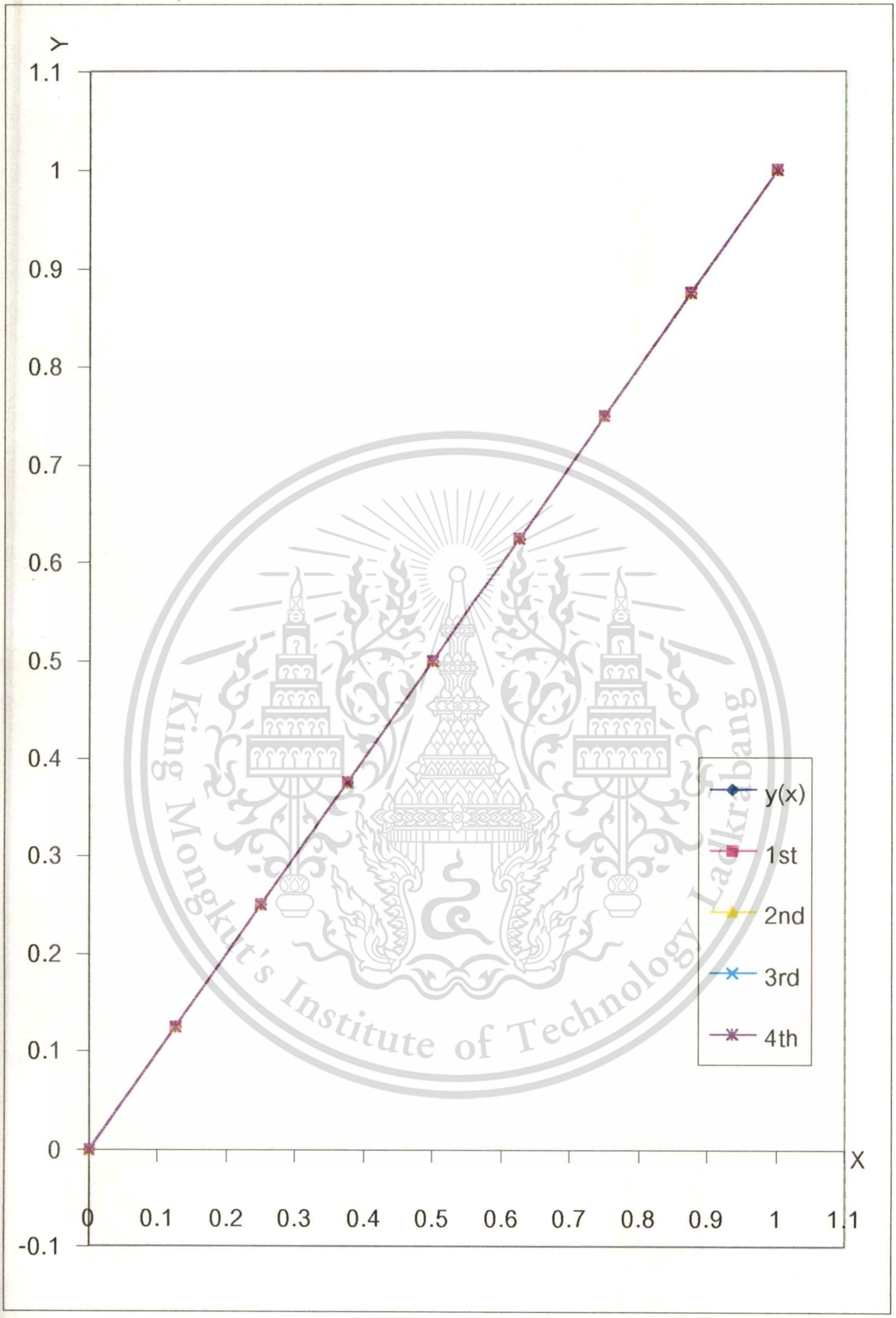


**Figure 4.11** Graph of the result from the four numerical methods of example 4.14 compare with exact solution.

**Example 4.15** Find the solution on the interval $[0,1]$ of the equation

$$y'(x) = y^3(x) - x^4y^2(x) + 2x \tag{4.29}$$

with the initial condition

$$y(0) = 0 . \tag{4.30}$$

The exact solution is $y = x^2$. We shall divide the interval $[0,1]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.45-4.48.

Table 4.45 The result of example 4.15 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.0156250001 | 0.0156250000 | 0.0000000001 |
| 2 | 0.25000 | 0.0625000080 | 0.0625000000 | 0.0000000080 |
| 3 | 0.37500 | 0.1406250688 | 0.1406250000 | 0.0000000688 |
| 4 | 0.50000 | 0.2500003048 | 0.2500000000 | 0.0000003048 |
| 5 | 0.62500 | 0.3906259814 | 0.3906250000 | 0.0000009814 |
| 6 | 0.75000 | 0.5625027467 | 0.5625000000 | 0.0000027467 |
| 7 | 0.87500 | 0.7656325833 | 0.7656250000 | 0.0000075833 |
| 8 | 1.00000 | 1.0000231188 | 1.0000000000 | 0.0000231188 |

The arithmetic mean of absolute error is 0.0000038680.

Table 4.46 The result of example 4.15 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.0156250001 | 0.0156250000 | 0.0000000001 |
| 2 | 0.25000 | 0.0625000080 | 0.0625000000 | 0.0000000080 |
| 3 | 0.37500 | 0.1406250687 | 0.1406250000 | 0.0000000687 |
| 4 | 0.50000 | 0.2500003044 | 0.2500000000 | 0.0000003044 |
| 5 | 0.62500 | 0.3906259810 | 0.3906250000 | 0.0000009810 |
| 6 | 0.75000 | 0.5625027478 | 0.5625000000 | 0.0000027478 |
| 7 | 0.87500 | 0.7656325843 | 0.7656250000 | 0.0000075843 |
| 8 | 1.00000 | 1.0000209933 | 1.0000000000 | 0.0000209933 |

The arithmetic mean of absolute error is 0.0000036320.

Table 4.47 The result of example 4.15 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.0156290773 | 0.0156250000 | 0.0000040773 |
| 2 | 0.25000 | 0.0625874829 | 0.0625000000 | 0.0000874829 |
| 3 | 0.37500 | 0.1403369205 | 0.1406250000 | 0.0002880795 |
| 4 | 0.50000 | 0.2495457834 | 0.2500000000 | 0.0004542166 |
| 5 | 0.62500 | 0.3934492682 | 0.3906250000 | 0.0028242682 |
| 6 | 0.75000 | 0.5741033197 | 0.5625000000 | 0.0116033197 |
| 7 | 0.87500 | 0.7965139554 | 0.7656250000 | 0.0308889554 |
| 8 | 1.00000 | 1.0862104930 | 1.0000000000 | 0.0862104930 |

The arithmetic mean of absolute error is 0.0147067658.

Table 4.48 The result of example 4.15 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.00000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.12500 | 0.0156250000 | 0.0156250000 | 0.0000000000 |
| 2 | 0.25000 | 0.0624999998 | 0.0625000000 | 0.0000000002 |
| 3 | 0.37500 | 0.1406249282 | 0.1406250000 | 0.0000000718 |
| 4 | 0.50000 | 0.2499959781 | 0.2500000000 | 0.0000040219 |
| 5 | 0.62500 | 0.3905455678 | 0.3906250000 | 0.0000794322 |
| 6 | 0.75000 | 0.5616944876 | 0.5625000000 | 0.0008055124 |
| 7 | 0.87500 | 0.7608380838 | 0.7656250000 | 0.0047869162 |
| 8 | 1.00000 | 0.9856693078 | 1.0000000000 | 0.0143306922 |

The arithmetic mean of absolute error is 0.0022229608.

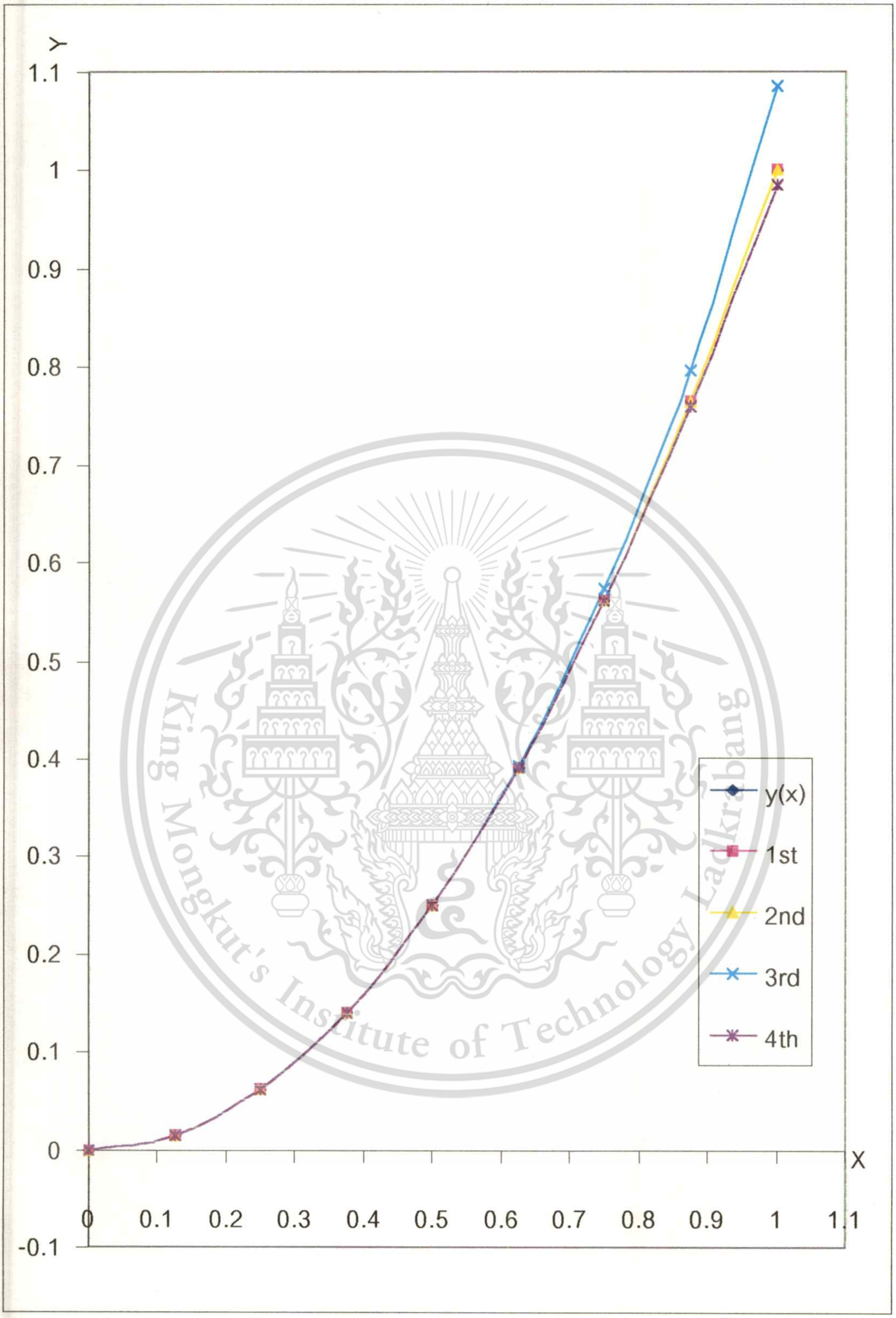


Figure 4.12 Graph of the result from the four numerical methods of example 4.15 compare with exact solution.

**Example 4.16** Find the solution on the interval $[0.1,0.9]$ of the equation

$$y'(x) = xy^2(x) + 1 - x^2 \qquad (4.31)$$

with the initial condition

$$y(0.1) = 0.1 \qquad (4.32)$$

The exact solution is $y = x$. We shall divide the interval $[0.1,0.9]$ into 8 equally subintervals and repeat the methods until the error, $E$, less than 0.000005.

We obtain the result as follows in the table 4.49-4.52.

Table 4.49 The result of example 4.16 from the first method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.10000 | 0.1000000000 | 0.1000000000 | 0.0000000000 |
| 1 | 0.20000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 2 | 0.30000 | 0.3000000000 | 0.3000000000 | 0.0000000000 |
| 3 | 0.40000 | 0.4000000000 | 0.4000000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.60000 | 0.5999999999 | 0.6000000000 | 0.0000000001 |
| 6 | 0.70000 | 0.6999999996 | 0.7000000000 | 0.0000000004 |
| 7 | 0.80000 | 0.7999999897 | 0.8000000000 | 0.0000000103 |
| 8 | 0.90000 | 0.8999997950 | 0.9000000000 | 0.0000002050 |

The arithmetic mean of absolute error is 0.0000000680.

Table 4.50 The result of example 4.16 from the second method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.10000 | 0.1000000000 | 0.1000000000 | 0.0000000000 |
| 1 | 0.20000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 2 | 0.30000 | 0.3000000000 | 0.3000000000 | 0.0000000000 |
| 3 | 0.40000 | 0.4000000000 | 0.4000000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.60000 | 0.6000000000 | 0.6000000000 | 0.0000000000 |
| 6 | 0.70000 | 0.6999999999 | 0.7000000000 | 0.0000000001 |
| 7 | 0.80000 | 0.7999999981 | 0.8000000000 | 0.0000000019 |
| 8 | 0.90000 | 0.8999999381 | 0.9000000000 | 0.0000000619 |

The arithmetic mean of absolute error is 0.0000000206.

Table 4.51 The result of example 4.16 from the third method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.10000 | 0.1000000000 | 0.1000000000 | 0.0000000000 |
| 1 | 0.20000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 2 | 0.30000 | 0.3000000000 | 0.3000000000 | 0.0000000000 |
| 3 | 0.40000 | 0.4000000000 | 0.4000000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.60000 | 0.5999999999 | 0.6000000000 | 0.0000000001 |
| 6 | 0.70000 | 0.6999999997 | 0.7000000000 | 0.0000000003 |
| 7 | 0.80000 | 0.7999999961 | 0.8000000000 | 0.0000000039 |
| 8 | 0.90000 | 0.8999999123 | 0.9000000000 | 0.0000000877 |

The arithmetic mean of absolute error is 0.0000000291.

Table 4.52 The result of example 4.16 from the forth method.

| i | X | Approximate value | Exact value | Absolute error |
|---|---|---|---|---|
| 0 | 0.10000 | 0.1000000000 | 0.1000000000 | 0.0000000000 |
| 1 | 0.20000 | 0.2000000000 | 0.2000000000 | 0.0000000000 |
| 2 | 0.30000 | 0.3000000000 | 0.3000000000 | 0.0000000000 |
| 3 | 0.40000 | 0.4000000000 | 0.4000000000 | 0.0000000000 |
| 4 | 0.50000 | 0.5000000000 | 0.5000000000 | 0.0000000000 |
| 5 | 0.60000 | 0.6000000000 | 0.6000000000 | 0.0000000000 |
| 6 | 0.70000 | 0.6999999999 | 0.7000000000 | 0.0000000001 |
| 7 | 0.80000 | 0.7999999894 | 0.8000000000 | 0.0000000106 |
| 8 | 0.90000 | 0.8999997080 | 0.9000000000 | 0.0000002920 |

The arithmetic mean of absolute error is 0.0000000969.

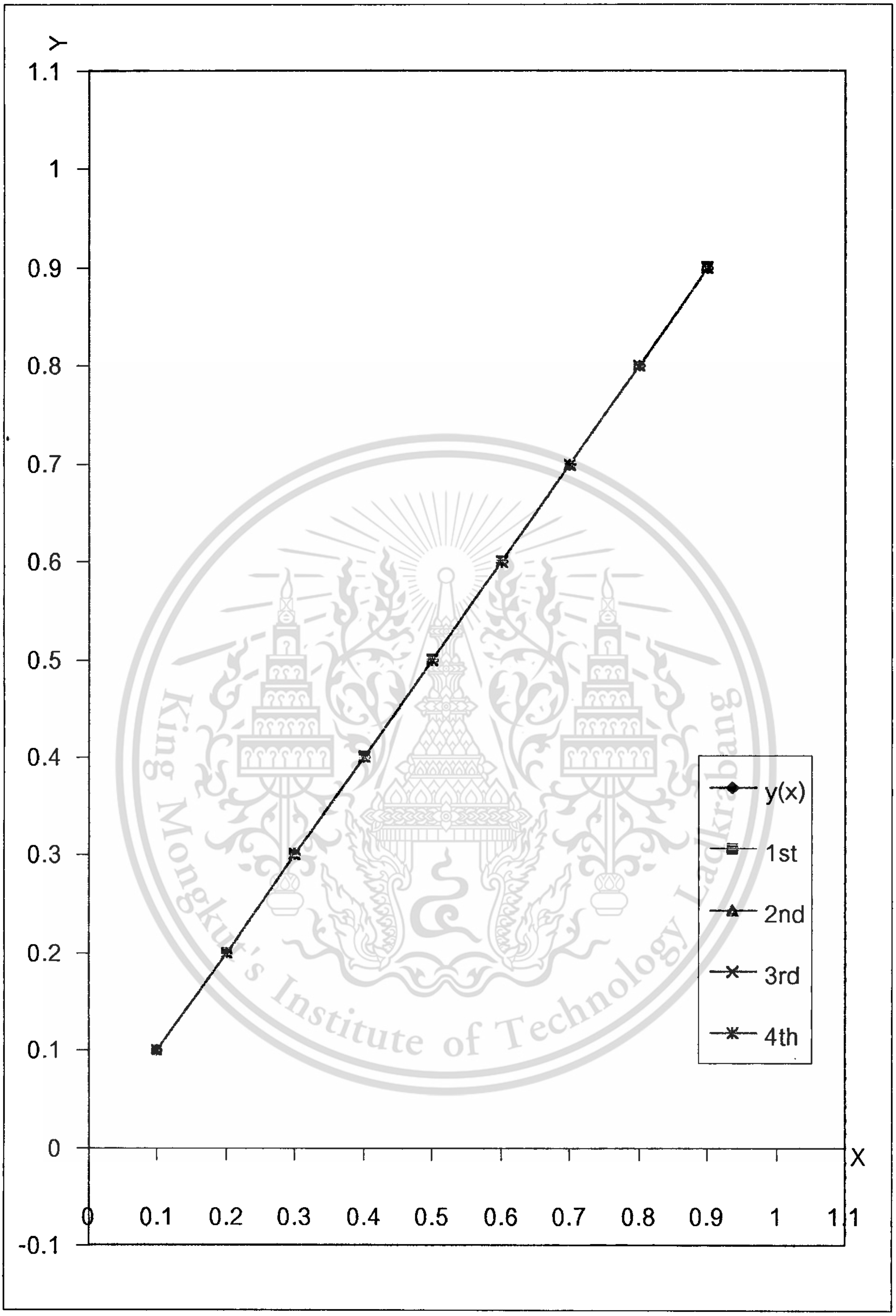


Figure 4.13 Graph of the result from the four numerical methods of example 4.16 compare with exact solution.

# CHAPTER 5
# CONCLUSION

In this chapter, we select the best of the four numerical methods by consider the arithmetic mean value of errors from the examples in chapter 4. In order to achieve the result, we consider the table 5.1-5.2.

Table 5.1 The arithmetic mean value of errors from the examples in chapter 4.

| Example | The arithmetic mean value | | | |
|---|---|---|---|---|
| | First Method | Second Method | Third Method | Forth Method |
| $y'(x) = y^2(x)$ on $[0,0.5]$ | 0.0011147330 | 0.0011146681 | 0.0015003485 | 0.0011147814 |
| $y'(x) = y^3(x)$ on $[0,0.5]$ | 0.0019305164 | 0.0019306020 | 0.0023992959 | 0.0023670336 |
| $y'(x) = \frac{1}{1+2x} - \frac{1}{1+x} + \frac{1}{(1+x)^2} + 2y^2(x) - y$ on $[0,1]$ | 0.0000499698 | 0.0000171800 | 0.0196105441 | 0.0000137167 |
| $y'(x) = y^2(x) - \frac{x}{4} - \frac{3}{4}$ on $[0,1]$ | 0.0000010835 | 0.0000008777 | 0.0000002834 | 0.0000008040 |
| $y'(x) = y^3(x) + \frac{x}{8} - \frac{7}{8}$ on $[0,1]$ | 0.0000004671 | 0.0000006151 | 0.0050653312 | 0.0063887715 |
| $y'(x) = y^4(x) - \frac{x}{16} - \frac{13}{16}$ on $[0,1]$ | 0.0000007540 | 0.0000006525 | 0.0036420499 | 0.0016014438 |
| $y'(x) = y^5(x) + \frac{x}{32} - \frac{27}{32}$ on $[0,1]$ | 0.0000001357 | 0.0000001853 | 0.0012159423 | 0.0011869704 |
| $y'(x) = xy^2(x) + 1 - x^2$ on $[0,1]$ | 0.0000000481 | 0.0000000287 | 0.0000000120 | 0.0000000480 |
| $y'(x) = xy^3(x) + 1 - x^2$ on $[0,1]$ | 0.0000000126 | 0.0000000055 | 0.0000000295 | 0.0000000473 |
| $y'(x) = xy^4(x) + 1 - x^2$ on $[0,1]$ | 0.0000000289 | 0.0000000095 | 0.0000001611 | 0.0000000432 |
| $y'(x) = xy^5(x) + 1 - x^2$ on $[0,1]$ | 0.0000000553 | 0.0000000104 | 0.0000001158 | 0.0000000418 |
| $y'(x) = y^3(x) - x^4y^2(x) + 2x$ on $[0,1]$ | 0.0000038680 | 0.0000036320 | 0.0147067658 | 0.0022229608 |
| $y'(x) = xy^2(x) + 1 - x^2$ on $[0.1,0.9]$ | 0.0000000680 | 0.0000000206 | 0.0000000291 | 0.0000000969 |

Table 5.2 The order of arithmetic mean value of errors from table 5.1.

| Example | The arithmetic mean value | | | |
|---|---|---|---|---|
| | First Method | Second Method | Third Method | Forth Method |
| $y'(x) = y^2(x)$ on $[0,0.5]$ | $2^{nd}$ | $1^{st}$ | $4^{th}$ | $3^{rd}$ |
| $y'(x) = y^3(x)$ on $[0,0.5]$ | $1^{st}$ | $2^{nd}$ | $4^{th}$ | $3^{rd}$ |
| $y'(x) = \frac{1}{1+2x} - \frac{1}{1+x} + \frac{1}{(1+x)^2} + 2y^2(x) - y$ on $[0,1]$ | $3^{rd}$ | $2^{nd}$ | $4^{th}$ | $1^{st}$ |
| $y'(x) = y^2(x) - \frac{x}{4} - \frac{3}{4}$ on $[0,1]$ | $4^{th}$ | $3^{rd}$ | $1^{st}$ | $2^{nd}$ |
| $y'(x) = y^3(x) + \frac{x}{8} - \frac{7}{8}$ on $[0,1]$ | $1^{st}$ | $2^{nd}$ | $3^{rd}$ | $4^{th}$ |
| $y'(x) = y^4(x) - \frac{x}{16} - \frac{13}{16}$ on $[0,1]$ | $2^{nd}$ | $1^{st}$ | $4^{th}$ | $3^{rd}$ |
| $y'(x) = y^5(x) + \frac{x}{32} - \frac{27}{32}$ on $[0,1]$ | $1^{st}$ | $2^{nd}$ | $4^{th}$ | $3^{rd}$ |
| $y'(x) = xy^2(x) + 1 - x^2$ on $[0,1]$ | $4^{th}$ | $2^{nd}$ | $1^{st}$ | $3^{rd}$ |
| $y'(x) = xy^3(x) + 1 - x^2$ on $[0,1]$ | $2^{nd}$ | $1^{st}$ | $3^{rd}$ | $4^{th}$ |
| $y'(x) = xy^4(x) + 1 - x^2$ on $[0,1]$ | $2^{nd}$ | $1^{st}$ | $4^{th}$ | $3^{rd}$ |
| $y'(x) = xy^5(x) + 1 - x^2$ on $[0,1]$ | $3^{rd}$ | $1^{st}$ | $4^{th}$ | $2^{nd}$ |
| $y'(x) = y^3(x) - x^4y^2(x) + 2x$ on $[0,1]$ | $2^{nd}$ | $1^{st}$ | $4^{th}$ | $3^{rd}$ |
| $y'(x) = xy^2(x) + 1 - x^2$ on $[0.1,0.9]$ | $3^{rd}$ | $1^{st}$ | $2^{nd}$ | $4^{th}$ |

Where, the first method is the Forth-order Runge-Kutta Method and the Four Points Newton Divided Difference Method. The second method is the Forth-order Runge-Kutta Method and the Cubic Hermite Interpolation. The third method is the Simpson's Rule and the Four Points Newton Divided Difference Method. And the forth method is the Simpson's Rule and the Cubic Hermite Interpolation.

In all of the above examples, we use the similarity condition by dividing to 8 subintervals and the error less than 0.000005. We observe that the second method has the minimum arithmetic mean value of error 7 times from 13 examples. And when it is not be the minimum, it has the least differentiate from the minimum value. From this reason, we shall choose the second method that uses the forth-order Runge-Kutta method and the cubic Hermite interpolation for solving the iterative ordinary differential equations. Otherwise the first method and the forth method can also solve this problem satisfactorily.

Thorough study of this thesis, we find that we will unable solve some Iterative Ordinary Differential Equations. This case will happen when the solution $y(x)$ maps from $[a, b]$ to $[c, d]$ as $[c, d] \not\subset [a, b]$ that it causes the numerical solutions tend to infinity. Therefore, it is suggested that further research should focus on this problem.

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# APPENDIX

# ALGORITHM

Algorithm is given in this Appendix for the following procedures.

**A1 Iterative Ordinary Differential Equations**

**A1.1 Forth-order Runge-Kutta Method**

**A1.2 Simpson's Rule**

**A2 Interpolations**

**A2.1 Four Points Newton's Divided Difference Method**

**A2.2 Cubic Hermite Interpolation**

## A1 Iterative Ordinary Differential Equations

We shall use these algorithms to solve the Iterative Ordinary Differential Equation $y' = f(x, y, y^2, \ldots, y^m)$ where $x \in [a, b]$ and $y(a) = c$ in $n$ equally subintervals then $x_0 = a$ and $x_i = a + ih$ for $i = 1,2,\ldots,n$ where $h = \dfrac{b-a}{n}$.

### A1.1 Forth-order Runge-Kutta method

The algorithm is

for $i = 0$ to $n$ do

$$y0_i = c$$

$$f\_prime_i = f(x_i, y0_i, 0, \ldots, 0)$$

endfor

Repeat

$error = 0$

for $i = 0$ to $n$ - 1 do

$$k1 = f(x_i, y0_i, y_i(y0_i), \ldots, y_i^{m-1}(y0_i))$$

$$k2 = f(x_i + \frac{h}{2}, y0_i + \frac{h * k1}{2}, y_i(y0_i + \frac{h * k1}{2}), \ldots, y_i^{m-1}(y0_i + \frac{h * k1}{2}))$$

$$k3 = f(x_i + \frac{h}{2}, y0_i + \frac{h * k2}{2}, y_i(y0_i + \frac{h * k2}{2}), \ldots, y_i^{m-1}(y0_i + \frac{h * k2}{2}))$$

$$k4 = f(x_i + h, y0_i + h * k3, y_i(y0_i + h * k3), \ldots, y_i^{m-1}(y0_i + h * k3))$$

$$y_{i+1} = y_i + \frac{h}{6}(k1 + 2 * k2 + 2 * k3 + k4)$$

$$error = error + |y0_{i+1} - y_{i+1}|$$

$$f0_i = k1$$

endfor

for $i = 0$ to $n$ do

$$y0_i = y_i$$

$$f\_prime = f0_i$$

endfor

until $error < 0.000005$.

**Notice** We find $y_i^k(value)$ in $k1, k2, k3, k4$ from Interpolations and use $f\_prime, f0_i$ for A2.2 Cubic Hermite Interpolation.

**A1.2 Simpson's Rule**

The algorithm is

for $i = 0$ to $n$ do

$$y0_i = c$$

$$f\_prime_i = f(x_i, y0_i, 0, \dots, 0)$$

endfor

for $i = 1$ to $n$ do

$$x_{i/2} = x_i - \frac{h}{2}$$

endfor

Repeat

*error* = 0

for $i = 0$ to $n$-1 do

$$f1 = f(x_i, y0_i, y_i(y0_i), \dots, y_i^{m-1}(y0_i))$$

$$y\_half = y_i(x_{i/2})$$

$$f2 = f(x_{i/2}, y\_half, y_i(y\_half), \dots, y_i^{m-1}(y\_half))$$

$$f3 = f(x_{i+1}, y0_{i+1}, y_i(y0_{i+1}), \dots, y_i^{m-1}(y0_{i+1}))$$

$$y_{i+1} = y_i + \frac{h}{6}(f1 + 4 * f2 + f3)$$

$$error = error + |y0_{i+1} - y_{i+1}|$$

$$f0_i = f1$$

endfor

for $i = 0$ to $n$ do

$$y0_i = y_i$$

$$f\_prime = f0_i$$

endfor

until *error* < 0.000005.

**Notice** We find $y_i^k(value)$ in $y\_half, f1, f2, f3$ from Interpolations and use $f\_prime, f0_i$ for A2.2 Cubic Hermite Interpolation.

## A2 Interpolations

We shall use these algorithm to find $y_i^k(value)$ from A1.

### A2.1 Four Points Newton's Divided Difference Method

This algorithm uses the four points of $x$ to find $y_i^k(value)$ that when $x_{l-1} < value < x_l$, we uses the points $x_{l-2}, x_{l-1}, x_l, x_{l+1}$ for $x_1 < value < x_{n-1}$. If $value < x_1$ then we uses the points $x_0, x_1, x_2, x_3$ and if $value > x_{n-1}$ then we uses the points $x_{n-3}, x_{n-2}, x_{n-1}, x_n$.

The algorithm is

for $j = 1$ to $k$ do

$l = 2$

while $((value > x_{l+1})$ and $(l < n-2))$ do

$l = l + 1$

endwhile

$p1 = value - x_{l-2}$

$p2 = value - x_{l-1}$

$p3 = value - x_l$

$$f\_1 = \frac{y0_{l-1} - y0_{l-2}}{h}$$

$$f\_2 = \frac{y0_l - y0_{l-1}}{h}$$

$$f\_12 = \frac{f\_2 - f\_1}{2 * h}$$

$$f\_3 = \frac{y0_{l+1} - y0_l}{h}$$

$$f\_23 = \frac{f\_3 - f\_2}{2 * h}$$

$$f\_123 = \frac{f\_23 - f\_12}{3 * h}$$

$$value = y0_{l-2} + f\_1 * p1 + f\_12 * p2 + f\_123 * p3$$

endfor

$y_i^k(value) = value$.

### A2.2 Cubic Hermite Interpolation

The algorithm is

for $j$ =1 to $k$ do

$l = 0$

while $((value > x_{l+1})$ and $(l < n - 2))$ do

$l = l + 1$

endwhile

$c1 = y0_l$

$c2 = f\_prime_l$

$$c4 = \frac{f\_prime_{l+1}}{h^2} + \frac{c2}{h^2} - \frac{2 * (y0_{l+1} - y0_l)}{h^3}$$

$$c3 = \frac{y0_{l+1} - y0_l}{h^2} - \frac{c2}{h} - c4 * h$$

$$value = c1 + c2 * (value - x_l) + c3 * (value - x_l)^2 + c4 * (value - x_l)^3$$

endfor

$y_i^k(value) = value$ .

# AUTHOR BIOGRAPHY

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