PROPERTIES OF INFINITE TOEPLITZ MATRIX ON THE HILBERT SPACE

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หัวข้อวิทยานิพนธ์	สมบัติของเมทริกซ์โทปลิทฟ์ซึ่งมีมิติอนันต์บนปริภูมิฮิลแบร์ต
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บทคัดย่อ

ในงานวิจัยนี้จะแสดงเงื่อนไขที่เพียงพอ (sufficient conditions) เงื่อนไขที่จำเป็น (necessary conditions) และ เงื่อนไขที่จำเป็นและเพียงพอ (necessary and sufficient conditions) ที่ทำให้เมทริกซ์การโทปลิทฟ์ซึ่งมีมิติอนันต์ เป็นตัวดำเนินการเชิงเส้นที่มีขอบเขตบนปริภูมิ ℓ^2 นอกจากนี้ยังศึกษาสมบัติเกี่ยวกับ เซลฟ์เอดจอยท์ (self-adjoint) นอร์มอล (normal) และ ยูนิทารี (unitary) ของเมทริกซ์โทปลิทฟ์ซึ่งมีมิติอนันต์ บนปริภูมิ ℓ^2

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ABSTRACT

In this research, we will show sufficient conditions, necessary conditions and necessary and sufficient conditions for infinite Toeplitz matrix to be a bounded linear operator on ℓ^2. Furthermore, we will show properties about self-adjoint, normal and unitary of infinite Toeplitz matrix on ℓ^2.

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CHAPTER 1

INTRODUCTION

1.1 Importance and Inception

Operator theory has many applications. Most applications arise as follows. We have a problem that is formulated in a certain space, in which we have few theorems to apply to our problem. One way out to use an operator to map the space, and the problem into a new space (and a new problem) in which there are more theorems to apply. Then we solve the transformed problem in the new space and transform the solutions back to get solutions from our original problem.

Here are some examples of important spaces in operator theory.

1. $\mathbb{R}^n$ is n-dimensional Euclidean space.
2. $\mathbb{C}^n$ is unitary space.
3. ℓ^p where $1 \leq p < \infty$ is the set of all complex sequences $x = \{x_i\}_{i=1}^{\infty}$, where $x \in \ell^p$ if and only if $\sum_{i=1}^{\infty} \mid x_i \mid^p < \infty$.
4. ℓ^∞ is the set of all complex sequences $x = \{x_i\}_{i=1}^{\infty}$, where $x \in \ell^\infty$ if and only if there exists M_x such that $\mid x_i \mid \leq M_x$ for all i.
5. $C[a, b]$ is the set of all real-valued functions defined and continuous on the closed interval $[a, b]$.
6. $L^p[a, b]$ is the set of all Lebesgue measurable functions f on $[a, b]$ such that $\int_a^b \mid f(x) \mid^p dx < \infty$.

We call the spaces in 3 and 4 sequence spaces because every element in the spaces is a sequence. We call the spaces in 5 and 6 function spaces because every element in the spaces is a function.

Given an operator, we will study its properties. For example, is the operator linear? What are necessary or sufficient conditions for it bounded? compact? In case of bounded operator we will consider the norm of the operator or approximations of the norm of the operator. And in the case that the operator acts on a Hilbert space, we will try to determine conditions for it to be self-adjoint, normal or unitary.

Operator theory has been an active field of research for many years. Many interesting spaces are sequence spaces and most operators for sequence spaces form infinite dimensional matrices.

Two famous matrices in this field are Toeplitz and Hankel and these matrices have special forms that are different from general matrices.
A Toeplitz matrix T has the form

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots \\ t_1 & t_0 & t_{-1} & \cdots \\ t_2 & t_1 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and a Hankel matrix H has the form

$$H = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots \\ h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In this thesis, we are interested in properties of Toeplitz matrices that are also bounded linear operators on ℓ^2 and Hilbert-adjoint operators.

1.2 Main objectives

The main objectives of the research are the following :

1. Find sufficient conditions, necessary conditions and necessary and sufficient conditions for a Toeplitz matrix to be a bounded linear operator on ℓ^2.
2. Compare the results that are obtained with the other results.
3. Study properties about Hilbert-adjoint operators of Toeplitz matrix on ℓ^2.

1.3 Steps of research

1. To study the related research.
   In this step, we study research about Toeplitz matrices (both infinite and finite dimensions). The objectives of this step are to find out what did the people do and look for topics that are interesting.
2. Create the problems.
   Find an interesting problems to work on.
3. Solve the problems.
   Try to solve the problems that we have from (2). The problems may have been previously solved but we will look at it from a different point of view. The problems may need different assumptions to make it possible to solve.

4. Compare results.

   After the problems have been solved, we compare our results with the previous ones.

5. Conclusions.

   In this step, we conclude our results, obstructions and suggestions for some one who wants to improve our results or interested in this field.

1.4 Convention

In this thesis, we almost exclusively consider operator on ℓ^2, so the symbol $\|\cdot\|$ is norm in ℓ^2 ($\|\cdot\|_2$). Norms of other spaces we will mention and give the notation, for example norm of ℓ^p is denoted by $\|\cdot\|_p$.

1.5 Benefits

1. Obtain simple conditions for checking any Toeplitz matrix is a bounded linear operator on ℓ^2, is not it?
2. Obtain properties about Hilbert-adjoint operators of Toeplitz matrix on ℓ^2.

CHAPTER 2

DEFINITIONS,THEOREMS AND LITERATURE REVIEWS

2.1 Basic Definitions

In this section we present some important definitions that will be used throughout the thesis. These definitions refer to [1-5].

Definition 2.1.1 ***(Toeplitz matrix).***
Let $\{t_n\}_{n=-\infty}^{\infty}$ *be any sequence of complex numbers. The Toeplitz matrix* T *induced by this sequence is*

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots \\ t_1 & t_0 & t_{-1} & \cdots \\ t_2 & t_1 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Definition 2.1.2 *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers.* $\sum_{n=1}^{\infty} a_n$ *is convergent provide that there exists* α *such that* $\sum_{n=1}^{k} a_n \to \alpha$ *as* $k \to \infty$.

Definition 2.1.3 *A series is said to be divergent when it does not converge. This means that there are two ways in which a given series* $\sum a_n$ *might diverge:*

1. $\sum_{n=1}^{k} a_n$*may have no limit(finite or infinite) as* $k \to \infty$.
2. $\sum_{n=1}^{k} a_n \to \pm\infty$ *as* $k \to \infty$.

Definition 2.1.4 *A series* $\sum a_n$ *is said to converge absolutely if the series* $\sum \mid a_n \mid$ *converges.*

Definition 2.1.5 *A convergent series that is not absolutely convergent is said to be conditionally convergent.*

Definition 2.1.6 *A series* $\sum b_n$ *is said to be a rearrangement of a series* $\sum a_n$ *if there is a one to one function* p *from* $\mathbb{N}$ *onto* $\mathbb{N}$ *such that for every natural number* n *we have*

$$b_n = a_{p(n)}.$$

Axiom 2.1.7 ***(Axiom of completeness).***
Every nonempty set of real numbers which is nonempty and bounded above has a least upper bound.

Definition 2.1.8 ***(Vector space).***
A vector space *(or linear space) over a field* $\mathbb{K}$ *is a nonempty set* X *of elements* $x,y,\ldots$ *(called vectors) together with two algebraic operations. Theses operations are called vector addition (denoted by* $+$ *) and multiplication of vectors by scalars (denote by* $\cdot$*) such that* $+ : X \times X \longrightarrow X$ *and* $\cdot : \mathbb{K}\times X \longrightarrow X$. *For all scalars* α, $\beta \in \mathbb{K}$*and for all vectors* x, y, $z \in X$ *we have*

1. $x+y = y+x$,
2. $(x+y)+z=x+(y+z)$,
3. *there exists* $\theta \in X$ *such that* $x+\theta = x$,
4. *there exists* $-x \in X$ *such that* $x+(-x) = \theta$,
5. $1\cdot x = x$,
6. $\alpha\cdot(x+y) = \alpha\cdot x+\alpha\cdot y$,
7. $(\alpha+\beta)\cdot x = \alpha\cdot x+\beta\cdot x$,
8. $\alpha\cdot(\beta\cdot x) = (\alpha\beta)\cdot x$.

Definition 2.1.9 ***(Metric space).***
A metric space is a pair (X,d), *where* X *is a nonempty set and* d *is a metric on* X, *that is, a function defined on* $X \times X$ *such that for all* x, y, $z \in X$ *we have*

1. $d(x,y) \geq 0$,
2. $d(x,y) = 0$ *if and only if* $x = y$,
3. $d(x,y) = d(y,x)$,
4. $d(x,y) \leq d(x,z) + d(z,y)$.

Definition 2.1.10 ***(Convergent sequence).***
A sequence $\{x_n\}_{n=1}^{\infty}$ *in a metric space* X *is said to be* convergent *if there exists* x *in* X *such that* $\lim_{n\to\infty} d(x,x_n) = 0$, *and we say that* $\{x_n\}_{n=1}^{\infty}$ *converges to* x.

Definition 2.1.11 ***(Cauchy sequence).***
A sequence $\{x_n\}_{n=1}^{\infty}$ *in a metric space* X *is said to be* Cauchy sequence *if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in \mathbb{N}$ *such that*

$$d(x_m,x_n) < \epsilon \quad \textit{for all } m,n \geq N_\epsilon.$$

Definition 2.1.12 ***(Complete space).***
A metric space X *is said to be complete if every Cauchy sequence in* X *converges in* X.

Definition 2.1.13 ***(Normed space).***
A normed space X is a vector space with norm defined on it . A norm is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and for all x, $y \in X$, $\alpha \in \mathbb{K}$ has the following properties;

1. $\|x\| \geq 0$,
2. $\|x\| = 0$ *if and only if* $x = \theta$,
3. $\|\alpha x\| = | \alpha | \, \|x\|$,
4. $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2.1.14 ***(ℓ^p).***
ℓ^p is a sequence space of complex numbers where $x = \{x_i\}_{i=1}^{\infty} \in \ell^p$, if and only if $\sum_{i=1}^{\infty} | x_i |^p < \infty$, where $1 \leq p < \infty$.

Definition 2.1.15 ***(ℓ^∞).***
ℓ^∞ is a sequence space of complex numbers where $x = \{x_i\}_{i=1}^{\infty} \in \ell^\infty$, if for all $i = 1, 2, \ldots$ we have $|x_i| \leq c_x$ where c_x is a real number which may depend on x, but does not depend on i.

A mapping (or function) from a vector space to vector space is called an operator.

Definition 2.1.16 ***(Linear operator).***
A linear operator T is an operator such that

1. *the domain $D(T)$ of T is a vetor space and range $R(T)$ of T lies in a vector space over the same field,*
2. *for all $x, y \in D(T)$ and for all scalar α*

$$T(x + y) = T(x) + T(y),$$

$$T(\alpha x) = \alpha T(x).$$

Definition 2.1.17 ***(Bounded linear operator).***
Let X and Y be normed spaces and $T : D(T) \longrightarrow Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$

$$\|Tx\| \leq c\|x\|,$$

and we define

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

Definition 2.1.18 ***(Linear functional).***
A linear functional f is a linear operator with domain in a vector space X and range in the scalar field $\mathbb{K}$ of X; thus,

$$f : D(f) \longrightarrow \mathbb{K}.$$

Definition 2.1.19 ***(Bounded linear functional).***
A bounded linear functional f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $D(f)$ lies. Thus there exists a real number c such that for all $x \in D(f)$,

$$\mid f(x) \mid \leq c\|x\|$$

and we define

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{\mid f(x) \mid}{\|x\|}.$$

Definition 2.1.20 ***(Algebraic dual space).***
The set of all linear functionals defined on a vector space X can itself be made into a vector space. This space is denoted by X^* and is called the algebraic dual space of X.

Definition 2.1.21 ***(Dual space).***
Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with norm defined by

$$\|f\| = sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|},$$

which is called dual space of X, and is denoted by X'.

Definition 2.1.22 ***(Inner product space).***
An inner product space is a vector space X with an inner product defined on it. An inner product on X is mapping of $X \times X$ into the scalar field $\mathbb{K}$ of X such that for all x,$y \in X$ and $\alpha \in \mathbb{K}$ the inner product of x and y, denoted by $\langle x, y \rangle$, has the properties

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
4. $\langle x, x \rangle \geq 0$,
5. $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

An inner product on X *defines a norm on* X *given by* $\|x\| = \sqrt{\langle x, x\rangle}$. *Hence inner product spaces are normed spaces.*

Definition 2.1.23 ***(Hilbert space).***
A Hilbert space X *is a complete inner product space with metric*

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}.$$

Definition 2.1.24 ***(Hilbert-adjoint operator).***
Let $T : H_1 \longrightarrow H_2$ *be a bounded liner operator, where* H_1 *and* H_2 *are Hilbert spaces.Then the Hilbert-adjoint operator* T^* *of* T *is the operator*

$$T^* : H_2 \longrightarrow H_1,$$

such that for all $x \in H_1$ *and* $y \in H_2$

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

Definition 2.1.25 ***(Self-adjoint, unitary and normal operators).***
A bounded linear operator $T : H \longrightarrow H$ *on Hilbert space* H *is said to be*

self-adjoint or Hermitian if $T^* = T$,

unitary if T *is bijective and* $T^* = T^{-1}$,

normal if $TT^* = T^*T$.

2.2 Basic Theorems

In this section we present important basic theorems that will be used in this research.

Theorem 2.2.1 *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are sequences,* c *is a real number and both series* $\sum a_n$ *and* $\sum b_n$ *converge. Then*

1. *The series* $\sum(a_n + b_n)$ *also converges, and we have*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

2. *The series* $\sum ca_n$ *converges, and we have*

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

Proof. See [1].

Theorem 2.2.2 *Suppose that* $\sum a_n$ *converges absolutely, and that* b_n *is a rearrangement of* $\sum a_n$. *Then* $\sum b_n$ *also converges absolutely and has the same sum.*

Proof. See [1].

Theorem 2.2.3 *If* $\sum a_n$ *is a convergent, but not an absolutely convergent series, then there are rearrangements that diverge.*

Proof. See [7].

Theorem 2.2.4 *Every absolutely convergent series is convergent. Furthermore, if* $\sum a_n$ *is absolutely convergent, then*

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} | a_n | .$$

Proof. See [1].

Theorem 2.2.5 *If* $a_{ij} \geq 0$ *for all* $i, j = 1, 2, \ldots$ *then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof. See [2].

Theorem 2.2.6 ***(Auxiliary inequality).*** *If α, β are any positive numbers then*

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \textit{where} \quad p > 1 \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. See [3].

Theorem 2.2.7 ***(Hölder's inequality).***
For all sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}$. We have,

$$\sum_{i=1}^{\infty} | x_i y_i | \leq \Big(\sum_{i=1}^{\infty} | x_i |^p \Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{\infty} | y_i |^q \Big)^{\frac{1}{q}}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. See [3].

Theorem 2.2.8 ***(Minkowski's inequality).***
For all sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}$ we have,

$$\left(\sum_{i=1}^{\infty} | x_i + y_i |^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} | x_i |^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} | y_i |^p\right)^{\frac{1}{p}}$$

where

$$\sum_{i=1}^{\infty} | x_i |^p, \sum_{i=1}^{\infty} | y_i |^p < \infty.$$

Proof. See [3].

Theorem 2.2.9 *For $q > 1$, $z_k \geq 0$, and $k \leq N$,*

$$\left(\sum_{n=k}^{N} z_n\right)^q \leq q \sum_{n=k}^{N} z_n \left(\sum_{j=n}^{N} z_j\right)^{q-1}$$

(Here, N may be ∞)

Proof. See [8].

Theorem 2.2.10 *ℓ^2 is a normed space with norm given by*

$$\|x\| = (\sum_{i=1}^{\infty} | x |^2)^{\frac{1}{2}}$$

for all $x \in \ell^2$.

Proof. See [3].

Theorem 2.2.11 *ℓ^2 is a Hilbert space, with inner product given by*

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

for all $x, y \in \ell^2$.

Proof. See [3] .

Theorem 2.2.12 *The dual space of ℓ^p is isometric to ℓ^q where $1 \geq p$ and $\frac{1}{p}+\frac{1}{q} = 1$, with the convention $\frac{1}{\infty} = 0$.*

Proof. See [3].

Theorem 2.2.13 *Let A be a bounded linear operator on a Hilbert space H and let A^* be the Hilbert-adjoint of A.*

1. *A is normal if and only if $\langle A(x), A(y)\rangle = \langle A^*(x), A^*(y)\rangle$ for all x,y in H.*
2. *A is unitary if and only if $\langle A(x), A(y)\rangle = \langle x, y\rangle = \langle A^*(x), A^*(y)\rangle$ for all x, y in H.*

Proof. See [4].

Theorem 2.2.14 *Let A be a bounded linear operator on a Hilbert space H and let A^* be the Hilbert-adjoint of A.*

1. *Let A be a self-adjoint. If $\langle A(x), x\rangle = 0$ for all $x \in H$ then A=0.*
2. *If $\mathbb{K} = \mathbb{C}$, then A is self-adjoint if and only if $\langle A(x), x\rangle \in \mathbb{R}$ for all $x \in H$. If $\mathbb{K} = \mathbb{R}$, then A is self-adjoint if and only if $\langle A(x), y\rangle = \langle A(y), x\rangle$ for all $x, y \in H$.*
3. *A is unitary if and only if $\|A(x)\| = \|x\|$ for all $x \in H$ and A is onto.*
4. *A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$.*

Proof. See [4].

Theorem 2.2.15 *The adjoint of matrix operator is its conjugate transpose.*

Proof. See [3].

2.3 Literature Reviews

Many references about Toeplitz matrices have been included in [5]. Here is list of theorems that will be used in this thesis.

Theorem 2.3.1 *Let*

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\{a_n\}_{n=-\infty}^{\infty}$ *is a sequence of complex numbers. The matrix A defines a bounded operator on* ℓ^2 *if and only if the numbers* $\{a_n\}_{n=-\infty}^{\infty}$ *are the Fourier coefficients of some function* $a \in L^\infty(\mathbb{T})$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-in\theta}d\theta, \qquad n \in \mathbb{N}.$$

A is denoted by T(a). In this case the norm of A equals $\|a\|_\infty$.

Theorem 2.3.2 *The only compact Toeplitz operator is the zero operator.*

Theorem 2.3.3 *The Toeplitz operator* $T(a)$ *is self-adjoint if and only if a is a real-valued function.*

Maddox [6] proved the following theorems :

Theorem 2.3.4 *Let A be an infinite matrix. A is bounded linear operator on* ℓ^∞ *if and only if* $\sup_n \sum_k |a_{nk}| < \infty$. *And* $\|A\|_\infty = \sup_n \sum_k |a_{nk}| < \infty$

Theorem 2.3.5 *Let A be an infinite matrix. A is bounded linear operator on* ℓ^1 *if and only if* $\sup_k \sum_n |a_{nk}| < \infty$. *And* $\|A\|_1 = \sup_k \sum_n |a_{nk}| < \infty$

Theorem 2.3.6 *Let A be an infinite matrix and let* $1 < p < \infty$. *If A is a bounded linear operator on* ℓ^∞ *and* ℓ^1, *then A is a bounded linear operator on* ℓ^p.

Proof. Consider,

$$\left|\sum_k a_{nk}x_k\right| \leq \sum_k |a_{nk}|^{\frac{1}{p}} |a_{nk}|^{\frac{1}{q}} |x_k|,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality,

$$\sum_k |a_{nk}|^{\frac{1}{p}} |a_{nk}|^{\frac{1}{q}} |x_k| \leq \left(\sum_k |a_{nk}| |x_k|^p\right)^{\frac{1}{p}} \left(\sum_k |a_{nk}|\right)^{\frac{1}{q}}$$

and so

$$\left|\sum_k a_{nk}x_k\right|^p \leq \left(\sum_k |a_{nk}|\,|x_k|^p\right)\left(\sum_k |a_{nk}|\right)^{\frac{p}{q}}.$$

Hence

$$\begin{aligned}
\|Ax\|^p &= \sum_n \left|\sum_k a_{nk}x_k\right|^p \\
&\leq \sum_n \sum_k |a_{nk}|\,|x_k|^p \,\|A\|_\infty^{\frac{p}{q}} \\
&\leq \|A\|_\infty^{\frac{p}{q}} \sum_k |x_k|^p \sum_n |a_{nk}| \\
&\leq \|A\|_\infty^{\frac{p}{q}} \|x\|^p \|A\|_1.
\end{aligned}$$

Therefore $\|Ax\| \leq \|A\|_\infty^{\frac{1}{q}} \|A\|_1^{\frac{1}{p}} \|x\|$, whenever $x \in \ell^p$.

□

Next we will present literature about finite dimensional normal Toeplitz matrices. Let T be the $(N+1)\times(N+1)$ Toeplitz matrix where $N \in \mathbb{N}$, as follows.

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-N} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-N+1} \\ t_2 & t_1 & t_0 & \cdots & t_{-N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N-1} & t_{N-2} & \cdots & t_0 \end{pmatrix}.$$

T is called normal if $TT^* - T^*T = 0$ where T^* is transposed conjugate of T. Farenick and Lee [9], Ito [10], Farenick,Krupnik,Krupnik and Lee [11] and Arimoto [12] classified T as follows :

1. T is called **type I**, if $[t_{-1}, t_{-2}, \ldots, t_{-N}] = \alpha_0[\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_N]$ for some α_0 such that $|\alpha_0| = 1$. That is

$$T = \begin{pmatrix} t_0 & \alpha_0\bar{t}_1 & \alpha_0\bar{t}_2 & \cdots & \alpha_0\bar{t}_N \\ t_1 & t_0 & \alpha_0\bar{t}_1 & \cdots & \alpha_0\bar{t}_{N-1} \\ t_2 & t_1 & t_0 & \cdots & \alpha_0\bar{t}_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N-1} & t_{N-2} & \cdots & t_0 \end{pmatrix}.$$

2. T is called **type II**, if $[t_{-1}, t_{-2}, \ldots, t_{-N}] = \beta_0[t_N, t_{N-1}, \ldots, t_1]$ for some β_0 such that $|\beta_0| = 1$. That is

$$T = \begin{pmatrix} t_0 & \beta_0 t_N & \beta_0 t_{N-1} & \cdots & \beta_0 t_1 \\ t_1 & t_0 & \beta_0 t_N & \cdots & \beta_0 t_2 \\ t_2 & t_1 & t_0 & \cdots & \beta_0 t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N-1} & t_{N-2} & \cdots & t_0 \end{pmatrix}.$$

3. T is called **symmetric**, if $[t_{-1}, t_{-2}, \dots, t_{-N}] = [t_1, t_2, \dots, t_N]$. That is

$$T = \begin{pmatrix} t_0 & t_1 & t_2 & \dots & t_N \\ t_1 & t_0 & t_1 & \dots & t_{N-1} \\ t_2 & t_1 & t_0 & \dots & t_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N-1} & t_{N-2} & \dots & t_0 \end{pmatrix}.$$

4. T is called **skew-symmetric**, if $[t_{-1}, t_{-2}, \dots, t_{-N}] = -[t_1, t_2, \dots, t_N]$. That is

$$T = \begin{pmatrix} t_0 & -t_1 & -t_2 & \dots & -t_N \\ t_1 & t_0 & -t_1 & \dots & -t_{N-1} \\ t_2 & t_1 & t_0 & \dots & -t_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N-1} & t_{N-2} & \dots & t_0 \end{pmatrix}.$$

5. T is called **circulant**, if $[t_{-1}, t_{-2}, \dots, t_{-N}] = [t_N, t_{N-1}, \dots, t_1]$. That is

$$T = \begin{pmatrix} t_0 & t_N & t_{N-1} & \dots & t_1 \\ t_1 & t_0 & t_N & \dots & t_2 \\ t_2 & t_1 & t_0 & \dots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N-1} & t_{N-2} & \dots & t_0 \end{pmatrix}.$$

6. T is called **skew-circulant**, if $[t_{-1}, t_{-2}, \dots, t_{-N}] = -[t_N, t_{N-1}, \dots, t_1]$. That is

$$T = \begin{pmatrix} t_0 & -t_N & -t_{N-1} & \dots & -t_1 \\ t_1 & t_0 & -t_N & \dots & -t_2 \\ t_2 & t_1 & t_0 & \dots & -t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_{N-1} & t_{N-2} & \dots & t_0 \end{pmatrix}.$$

Farenick and Lee [9] proved the following theorems :

Theorem 2.3.7 *For* $1 \leq N \leq 4$ *every normal Toeplitz matrix* T_N *is either of type I or of type II.*

T. Ito [10] extended this result to arbitrary order, that is

Theorem 2.3.8 *Every normal Toeplitz matrix* T_N *with* $N \geq 5$ *is either of type I or of type II .*

The following are results of Farenick, Krupnik, Krupnik and Lee [11].

Theorem 2.3.9 *Every finite complex normal Toeplitz matrix is a generalized circulant or rotation and translation of a Hermitian Toeplitz matrix. In particular, every finite real normal Toeplitz matrix is symmetric, skew-symmetric, circulant or skew-circulant.*

Theorem 2.3.10 *A Toeplitz matrix T of the form*

$$T = \begin{pmatrix} 0 & b_1 & b_2 & \dots & b_N \\ a_1 & 0 & b_1 & \dots & b_{N-1} \\ a_2 & a_1 & 0 & \dots & b_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_N & a_{N-1} & a_{N-2} & \dots & 0 \end{pmatrix}$$

is normal if and only if for each p and q the following equalities hold :

$$\bar{a}_q a_p + a_{N-q+1}\bar{a}_{N-p+1} = b_q\bar{b}_p + \bar{b}_{N-q+1}b_{N-p+1}.$$

Arimoto [12] presented a new and much simpler proof of the result of Ito and the previous theorem (in the real number case).

CHAPTER 3

PROCESS OF RESEARCH

In this research we conclude the following 7 steps.

1. We use theorem 2.3.4, 2.3.5 and 2.3.6 to find sufficient conditions for Toeplitz matrix to be a bounded linear operator on ℓ^2. Results of this step are theorem 4.1 and corollary 4.2.

2. We find a necessary condition for a bounded linear Toeplitz matrix on ℓ^2. We define a functional by multiplication between each row of matrix and x in ℓ^2, then using theorem 2.2.11, we have the first property of theorem 4.5. Next we study property of the norm of Te_n where e_n is the sequence with 1 in the n^{th} position and 0 everywhere else. We have the second property of theorem 4.5.

3. In this step, we find necessary and sufficient conditions for a bounded linear Toeplitz matrix on ℓ^2. Since the conditions in step one and two are not the same, we will add the following assumptions.
   - we add some assumptions for theorem 2.2.9, we have lemma 4.7,
   - we use lemma 4.7 and results in step 1 and 2, we have theorem 4.8 which are necessary and sufficient condition.

4. We find examples to guarantee that our conditions in theorem 4.8 are make sense, they are theorem 4.9, proposition 4.10, corollary 4.11 and corollary 4.12.

5. We study properties of Hilbert adjoint operator of Toeplitz matrix on ℓ^2.
   - **Unitary.** We have necessary and sufficient conditions, in theorem 4.16. Furthermore, we give examples that one that is consistent the theorem and one that is not consistent.
   - **Normal.** We study in general case and the real case. In the general case, we have sufficient conditions (theorem 4.19) and necessary conditions (theorem 4.20). For the proof of real case, we use result in general case, we have necessary and sufficient conditions(theorem 4.22).

6. In this step, we compare our results and related results (in literature reviews).

    - Compare sufficient conditions (theorem 4.1 and corollary 4.2) with theorem 2.3.1.
      The aim of this step is to decide what conditions are weaker. We found that conditions of theorem 4.1 are weaker than those of theorem 2.3.1, for a proof, we give a function that satisfies the conditions of theorem 2.3.1, but its fourier coefficients are not in ℓ^1. This proof appears in section 5.2.1.
    - Compare our conditions in the normal case (theorem 4.19 and 4.22)with theorem 2.3.7-2.3.9. The aim of this step is to decide what types of Toeplitz matrices are normal in both in finite and infinite dimension, you can see the details in section 5.2.3.

7. Finally, we conclude our results and give some suggestions to improve our results.

CHAPTER 4

MAIN RESULTS

In this chapter we present results of this research and we give some illustrations for the results.

Theorem 4.1 *Let* $\{t_k\}_{k=-\infty}^{\infty}$ *be any sequence of complex numbers and* T *be a Toeplitz matrix induced by this sequence.*
If $\sum_{k=-\infty}^{-1} \mid t_k \mid < \infty$ *and* $\sum_{k=0}^{\infty} \mid t_k \mid < \infty$*, then* T *is a bounded linear operator on* ℓ^2*.*

Proof.

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots \\ t_1 & t_0 & t_{-1} & \cdots \\ t_2 & t_1 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Consider each row of T we have

row 1:

$$\mid t_0 \mid + \mid t_{-1} \mid + \mid t_{-2} \mid + \ldots \leq \sum_{k=-\infty}^{-1} \mid t_k \mid + \sum_{k=0}^{\infty} \mid t_k \mid < \infty$$

row 2:

$$\mid t_1 \mid + \mid t_0 \mid + \mid t_{-1} \mid + \ldots \leq \sum_{k=-\infty}^{-1} \mid t_k \mid + \sum_{k=0}^{\infty} \mid t_k \mid < \infty$$

row 3:

$$\mid t_2 \mid + \mid t_1 \mid + \mid t_0 \mid + \ldots \leq \sum_{k=-\infty}^{-1} \mid t_k \mid + \sum_{k=0}^{\infty} \mid t_k \mid < \infty$$

$$\vdots$$

That is $\sum_{i=1}^{\infty} \mid t_{n-i} \mid \leq \sum_{k=-\infty}^{-1} \mid t_k \mid + \sum_{k=0}^{\infty} \mid t_k \mid < \infty$ for all $n = 1, 2, \ldots$ By the axiom of completeness we have $\sup_n \sum_{i=1}^{\infty} \mid t_{n-i} \mid < \infty$. Similary we will have $\sup_i \sum_{n=1}^{\infty} \mid t_{n-i} \mid < \infty$, so by theorem 2.3.6 and lemma 4.1, T is a bounded linear operator on ℓ^2.

□

Corollary 4.2 *Let $\{t_k\}_{k=-\infty}^{\infty}$ be any sequence of complex numbers and T is a Toeplitz matrix induced by this sequence. If $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$, then T is a bounded linear operator on ℓ^2.*

Proof.
Since $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$, $\sum_{k=-\infty}^{\infty} \mid t_k \mid < \infty$.
Therefore $\sum_{k=-\infty}^{-1} \mid t_k \mid < \infty$ and $\sum_{k=0}^{\infty} \mid t_k \mid < \infty$.
By Theorem 4.1, we have T is a bounded linear operator on ℓ^2.

□

The next examples are illustative of the previous results.

Example 4.3

Let

$$t_n = \begin{cases} -\frac{1}{2^{n+1}} & , n = 0, 1, 2, \ldots \\ \frac{1}{2^{-n}} & , n = -1, -2, \ldots \end{cases}$$

So the Toeplitz matrix T induced by this sequence is

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \cdots \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \mid t_n \mid &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1 \\ \sum_{n=-\infty}^{-1} \mid t_n \mid &= \mid -\frac{1}{2} \mid + \mid -\frac{1}{4} \mid + \mid -\frac{1}{8} \mid + \ldots = 1 \end{aligned}$$

So T is a bounded linear operator. Suppose $x = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \ldots\right\}$. It easy to see that $x \in \ell^2$.

$$
\begin{aligned}
Tx &= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \cdots \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ -\frac{1}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \\ \vdots \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{3}{4} \\ 0 \\ -\frac{3}{4} \\ -\frac{3}{8} \\ \vdots \end{pmatrix}.
\end{aligned}
$$

Consider

$$
\begin{aligned}
\left| \ \frac{3}{4} \ \right|^2 + |\,0\,|^2 + \left| -\frac{3}{4} \ \right|^2 + \left| -\frac{3}{8} \ \right|^2 + \ldots &= \frac{9}{16} + 0 + \frac{9}{16} + \frac{9}{64} + \ldots \\
&= \frac{9}{16} + \left(\frac{\frac{9}{16}}{1 - \frac{1}{4}} \right) \\
&= \frac{9}{16} + \left(\frac{9}{16} \times \frac{4}{3} \right) \\
&= \frac{21}{16} < \infty.
\end{aligned}
$$

That is $Tx \in \ell^2$. Suppose $y = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \ldots \right\}$.

$$
\begin{aligned}
\left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{1}{\sqrt{4}} \right|^2 + \left| \frac{1}{\sqrt{8}} \right|^2 + \ldots &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \\
&= 1.
\end{aligned}
$$

So $\ y \in \ell^2$.

$$
\begin{aligned}
Ty &= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \cdots \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{8}} \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=1}^{\infty} \frac{1}{2^n \left(\sqrt{2}\right)^n} \\ -\frac{1}{2\sqrt{2}} + \sum_{n=2}^{\infty} \frac{1}{2^{(n-1)} \left(\sqrt{2}\right)^n} \\ -\frac{1}{4\sqrt{2}} - \frac{1}{2\sqrt{4}} + \sum_{n=3}^{\infty} \frac{1}{2^{(n-2)} \left(\sqrt{2}\right)^n} \\ \vdots \end{pmatrix} \\
&< \begin{pmatrix} \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^n} \\ \sum_{n=2}^{\infty} \frac{1}{(\sqrt{2})^n} \\ \sum_{n=3}^{\infty} \frac{1}{(\sqrt{2})^n} \\ \vdots \end{pmatrix},
\end{aligned}
$$

consider

$$\left|\sum_{n=1}^{\infty}\frac{1}{(\sqrt{2})^n}\right|^2+\left|\sum_{n=2}^{\infty}\frac{1}{(\sqrt{2})^n}\right|^2+\left|\sum_{n=3}^{\infty}\frac{1}{(\sqrt{2})^n}\right|^2+\ldots=\frac{4}{\left(-2+\sqrt{2}\right)^2}.$$

That is, $Ty\in\ell^2$.

□

Example 4.4

Let

$$t_n=\begin{cases}\left(\frac{1+i}{3}\right)^n & ,n=0,1,2,\ldots\\ \left(\frac{-n}{-2n+1}\right)^{-n} & ,n=-1,-2,\ldots\end{cases}$$

For $n=0,1,2,\ldots$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{1+i}{3}\right)^n\right|} &= \lim_{n\to\infty}\sqrt[n]{\left|\frac{1+i}{3}\right|^n}\\ &= \lim_{n\to\infty}\left|\frac{1+i}{3}\right|\\ &= \frac{\sqrt{2}}{3}<1.\end{aligned}$$

By the root test we have,

$$\sum_{n=0}^{\infty}\left|\left(\frac{1+i}{3}\right)^n\right|<\infty.$$

For $n=-1,-2,\ldots$

$$\begin{aligned}\sum_{n=-\infty}^{-1}\left|\left(\frac{-n}{-2n+1}\right)^{-n}i^{-n}\right| &= \left|\frac{1}{3}i\right|+\left|\frac{2}{5}i^2\right|+\ldots\\ &= \sum_{k=1}^{\infty}\left|\left(\frac{k}{2k+1}\right)^k i^k\right|.\end{aligned}$$

Consider

$$\begin{aligned}\lim_{k\to\infty}\sqrt[k]{\left|\left(\frac{k}{2k+1}\right)^k i^k\right|} &= \lim_{n\to\infty}\left|\frac{k}{2k+1}\right||i|\\ &= \lim_{n\to\infty}\left|\frac{k}{2k+1}\right|\\ &= \frac{1}{2}<1.\end{aligned}$$

Therefore

$$\sum_{n=-\infty}^{-1}\left|\left(\frac{-n}{-2n+1}\right)^{-n}i^{-n}\right|<\infty.$$

Hence the Toeplitz matrix induced by this sequence is a bounded linear operator on ℓ^2.

□

Theorem 4.5 *Let* $\{t_k\}_{k=-\infty}^{\infty}$ *be any sequence of complex number and* T *be the Toeplitz matrix induced by* $\{t_k\}_{k=-\infty}^{\infty}$ *. If* T *is a bounded linear operator on* ℓ^2*, then*

1. $\sup_n(\sum_{i=1}^{\infty} \mid t_{n-i} \mid^2)^{\frac{1}{2}} < \infty,$
2. $\sup_i(\sum_{n=1}^{\infty} \mid t_{n-i} \mid^2)^{\frac{1}{2}} < \infty.$

Proof.
Let $x = \{x_i\}_{i=1}^{\infty} \in \ell^2$.

$$\begin{aligned} Tx &= \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \dots \\ t_1 & t_0 & t_{-1} & \dots \\ t_2 & t_1 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{\infty} t_{1-i}x_i \\ \sum_{i=1}^{\infty} t_{2-i}x_i \\ \sum_{i=1}^{\infty} t_{3-i}x_i \\ \vdots \end{pmatrix}. \end{aligned}$$

Define $T_n(x) = \sum_{i=1}^{\infty} t_{n-i}x_i$. Hence $T_n : \ell^2 \longrightarrow \mathbb{C}$.
Let $\{y_i\}_{i=1}^{\infty} \in \ell^2$ and $\alpha \in \mathbb{C}$,

$$\begin{aligned} T_n(x+y) &= \sum_{i=1}^{\infty} t_{n-i}(x_i + y_i) \\ &= \sum_{i=1}^{\infty}(t_{n-i}x_i + t_{n-i}y_i) \\ &= \sum_{i=1}^{\infty}(t_{n-i}x_i) + \sum_{i=1}^{\infty}(t_{n-i}y_i) \\ &= T_n(x) + T_n(y), \end{aligned}$$

and

$$\begin{aligned} T_n(\alpha x) &= \sum_{i=1}^{\infty} t_{n-i}(\alpha x) \\ &= \alpha \sum_{i=1}^{\infty} t_{n-i}(x_i) \\ &= \alpha T_n(x). \end{aligned}$$

So T_n is a linear functional for all $n = 1, 2, \dots$.

$$\begin{aligned} | T_n(x) |^2 &= | \sum_{i=1}^{\infty} t_{n-i}x_i |^2 \\ &\leq \sum_{n=1}^{\infty} | \sum_{i=1}^{\infty} t_{n-i}x_i |^2 \\ &= \|Tx\|^2 \\ &\leq (\|T\| \, \|x\|)^2. \end{aligned}$$

Hence $| T_n(x) | \leq \|T\| \, \|x\|$. That is, T_n is a bounded linear functional and $\|T_n\| \leq \|T\|$ for all n, $T_n \in (\ell^2)'$, *where* $(\ell^2)'$ is the set of bounded linear functionals on ℓ^2. By proof of the theorem 2.2.11 we will have $A : \ell^{2'} \longrightarrow \ell^2$, defined by

$$A(f) = \{f_i\}_{i=1}^{\infty}$$

where

$$\begin{aligned} f_1 &= f(e_1), & e_1 &= \{1,0,0,\ldots\} \\ f_2 &= f(e_2), & e_2 &= \{0,1,0,\ldots\} \\ & & &\vdots \end{aligned}$$

and we have

$$\|f\| = \|A(f)\|.$$

Therefore

$$\|T_n\| = \|A(T_n)\|. \tag{4.1}$$

Suppose $A(T_n) = \{z_i\}_{i=1}^{\infty}$, from above we have

$$z_1 = T_n(e_1) = (t_{n-1} \times 1) + (t_{n-2} \times 0) + (t_{n-3} \times 0) + \cdots = t_{n-1}$$

$$z_2 = T_n(e_2) = (t_{n-1} \times 0) + (t_{n-2} \times 1) + (t_{n-3} \times 0) + \cdots = t_{n-2}$$

$$\vdots$$

That is $A(T_n) = \{t_{n-i}\}_{i=1}^{\infty}$, from (4.1) we have

$$\|T_n\| = (\sum_{i}^{\infty} | t_{n-i} |^2)^{\frac{1}{2}} \leq \|T\| \qquad \text{for all } n.$$

Therefore

$$\sup_n \left(\sum_{i=1}^{\infty} | t_{n-i} |^2 \right)^{\frac{1}{2}} \leq \|T\| \leq \infty. \tag{4.2}$$

Consider

$$T(e_1) = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots \\ t_1 & t_0 & t_{-1} & \cdots \\ t_2 & t_1 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \end{pmatrix}.$$

Since $T : \ell^2 \longrightarrow \ell^2$, $T(e_1) = \{t_0, t_1, t_2, \ldots\} \in \ell^2$, $T(e_2) = \{t_{-1}, t_0, t_1, \ldots\} \in \ell^2$ and so on. Since T is a bounded linear operator, $\|Tx\| \leq \|T\|$ for all $x \in \ell^2$ and $\|x\| = 1$.
Hence

$$\|T(e_i)\| = \left(\sum_{n=1}^{\infty} \mid t_{n-i} \mid^2\right)^{\frac{1}{2}} \leq \|T\| \qquad \text{for all } i.$$

Therefore

$$\sup_i \left(\sum_{n=1}^{\infty} \mid t_{n-i} \mid^2\right)^{\frac{1}{2}} \leq \infty. \tag{4.3}$$

This proof is complete.

□

Corollary 4.6 *By the hypothesis of theorem 4.5, we will have* $\{t_n\}_{n=-\infty}^{\infty} \in \ell^2$.

Proof.
By theorem4.5, we have rows and columns of T are in ℓ^2. So

$$\left(\sum_{j=0}^{\infty} \mid t_j \mid^2\right)^{\frac{1}{2}} < \infty,$$

and

$$\left(\sum_{j=-\infty}^{-1} \mid t_j \mid^2\right)^{\frac{1}{2}} < \infty.$$

And we have

$$\left(\sum_{n=-\infty}^{\infty} \mid t_n \mid^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=-\infty}^{-1} \mid t_j \mid^2\right)^{\frac{1}{2}} + \left(\sum_{j=0}^{\infty} \mid t_j \mid^2\right)^{\frac{1}{2}} < \infty.$$

Therefore $\{t_n\}_{n=-\infty}^{\infty} \in \ell^2$.

□

Lemma 4.7 *Let* $\{a_k\}_{k=1}^{\infty}$ *be in* ℓ^2.
$\forall k$, *if* $| a_k | \geq \sum_{n=k+1}^{\infty} | a_n |$ *then* $(\sum_{n=1}^{\infty} | a_n |) \leq 2\,(\sum_{n=1}^{\infty} | a_n |^2)^{\frac{1}{2}}$.

Proof.
By theorem 2.2.9, we will have,

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} | a_n |\right)^2 &\leq 2\sum_{n=1}^{\infty}\left[| a_n | \left(\sum_{j=n}^{\infty} | a_j |\right)\right] \\
&= 2\,[| a_1 | \,(| a_1 | + | a_2 | + \ldots) + | a_2 | \,(| a_2 | + | a_3 | + \ldots) + \ldots] \\
&= 2\left[(| a_1 |^2 + | a_1 || a_2 | + \ldots) + (| a_2 |^2 + | a_2 || a_3 | + \ldots) + \ldots\right] \\
&= 2\left[(| a_1 |^2 + | a_2 |^2 + \ldots) + (| a_1 || a_2 | + | a_1 || a_3 | + \ldots) + \ldots\right] \\
&= 2\left[(| a_1 |^2 + | a_2 |^2 + \ldots) + | a_1 | \,(| a_2 | + | a_3 | + \ldots) + \ldots\right]
\end{aligned}$$

Since $| a_k | \geq \sum_{n=k+1}^{\infty} | a_n |$. Therefore we have $| a_1 | \geq | a_2 | + | a_3 | + \ldots,$ $| a_2 | \geq | a_3 | + | a_4 | + \ldots,$. So So

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} | a_n |\right)^2 &\leq 2\left[(| a_1 |^2 + | a_2 |^2 + \ldots) + | a_1 || a_1 | + | a_2 || a_2 | + \ldots\right] \\
&= 2[2\,| a_1 |^2 + 2\,| a_2 |^2 + \ldots] \\
&= 4[| a_1 |^2 + | a_2 |^2 + \ldots].
\end{aligned}$$

Therefore

$$\left(\sum_{n=1}^{\infty} | a_n |\right) \leq 2\left(\sum_{n=1}^{\infty} | a_n |^2\right)^{\frac{1}{2}}.$$

□

Theorem 4.8 *Let* $\{t_k\}_{k=-\infty}^{\infty}$ *be any sequence of complex numbers,* T *a Toeplitz matrix induced by* $\{t_n\}_{n=-\infty}^{\infty}$ *and* $\sum_{n=k+1}^{\infty} | t_n | \leq | t_k |$ *for all* $n = 1, 2, \ldots$ *and* $\sum_{n=-\infty}^{n-1} | t_n | \leq | t_k |$ *for all* $n = -1, -2, \ldots$. *Then the following statements are equivalent.*

1. *T is a bounded linear operator on ℓ^2.*

2. $\{t_k\}_{k=-\infty}^{\infty} \in \ell^2$.

3. $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$.

Proof.
(1 ⇒ 2) By corollary 4.6.
(2 ⇒ 3) Since $\{t_k\}_{k=-\infty}^{\infty} \in \ell^2, \sum_{k=-\infty}^{\infty} | t_k |^2 < \infty$.
So $\sum_{k=-\infty}^{-1} | t_k |^2 < \infty$ and $\sum_{k=0}^{\infty} | t_k |^2 < \infty$.

By lemma 4.7 we have

$$\sum_{k=-\infty}^{-1} | t_k | \leq 2 \left(\sum_{k=-\infty}^{-1} | t_k |^2 \right)^{\frac{1}{2}} < \infty$$

$$\sum_{k=0}^{\infty} | t_k | \leq 2 \left(\sum_{k=0}^{\infty} | t_k |^2 \right)^{\frac{1}{2}} < \infty$$

Hence $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$.
$(3 \Rightarrow 1)$ By corollary 4.2 we have T is a bounded linear operator on ℓ^2.

□

Next we will study the norm of special Toeplitz matrices, upper and lower triangular matrix, denoted by

$$T^- = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \dots \\ 0 & t_0 & t_{-1} & \dots \\ 0 & 0 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ and}$$

$$T_- = \begin{pmatrix} t_0 & 0 & 0 & \dots \\ t_1 & t_0 & 0 & \dots \\ t_2 & t_1 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 4.9 *Let $\{t_k\}_{k=0}^{\infty}$ be a sequence that satisfies the hypothesis of lemma 4.7. If Toeplitz matrix T_- is induced by the sequence, then T_- is bounded linear operator on ℓ^2 and*

$$\frac{1}{2}\Big(\sum_{k=0}^{\infty} | t_k | \Big) \leq \|T_-\| \leq \Big(\sum_{k=0}^{\infty} | t_k | \Big).$$

Proof. From the proof of theorem 2.3.6 we have

$$\|T_-\| \leq \Big(\sup_i \sum_{n=0}^{\infty} | t_{n-i} | \Big)^{\frac{1}{2}} \Big(\sup_n \sum_{i=0}^{\infty} | t_{n-i} | \Big)^{\frac{1}{2}}.$$

We know

$$\Big(\sup_i \sum_{n=0}^{\infty} | t_{n-i} | \Big) = \Big(\sup_n \sum_{i=0}^{\infty} | t_{n-i} | \Big) = \sum_{k=0}^{\infty} | t_k | .$$

Therefore

$$\|T_-\| \leq \Big(\sum_{k=0}^{\infty} | t_k | \Big). \tag{4.4}$$

Since T_- is bounded linear operator on ℓ^2, by (4.2) $\sup_n \big(\sum_{i=1}^{\infty} | t_{n-i} |^2\big)^{\frac{1}{2}} \leq \|T_-\| \leq \infty$.

And we know

$$\sup_n \left(\sum_{i=1}^{\infty} | t_{n-i} |^2 \right)^{\frac{1}{2}} = \Big(\sum_{k=0}^{\infty} | t_k |^2 \Big)^{\frac{1}{2}}.$$

Therefore

$$\Big(\sum_{k=0}^{\infty} | t_k |^2 \Big)^{\frac{1}{2}} \leq \|T_-\|. \tag{4.5}$$

By lemma 4.7 we have

$$\sum_{k=0}^{\infty} | t_k | \ \leq \ 2\Big(\sum_{k=0}^{\infty} | t_k |^2 \Big)^{\frac{1}{2}}.$$

So

$$\frac{1}{2}\sum_{k=0}^{\infty} | t_k | \ \leq \ \Big(\sum_{k=0}^{\infty} | t_k |^2 \Big)^{\frac{1}{2}}. \tag{4.6}$$

From (4.4), (4.5), (4.6), we have

$$\frac{1}{2}\sum_{k=0}^{\infty} | t_k | \leq \|T_-\| \leq \Big(\sum_{k=0}^{\infty} | t_k | \Big).$$

□

This theorem also holds for T^-.

Proposition 4.10 *For all geometric sequences* $\{ar^k\}_{k=0}^{\infty}$ *such that* $| r | \leq \frac{1}{2}$,

$$\sum_{k=n+1}^{\infty} | ar^k | \ \leq \ | ar^n |, \quad \forall\, n = 0, 1, 2, \ldots .$$

Proof.
We will prove this by induction .
For $n = 0$:

$$\begin{aligned} \sum_{k=1}^{\infty} | ar^k | &= | a | \sum_{k=1}^{\infty} | r^k | \\ &= | a | \sum_{k=1}^{\infty} | r |^k \\ &= | a | \Big(\frac{| r |}{1- | r |} \Big). \end{aligned} \tag{4.7}$$

From $| r | \leq \frac{1}{2}$, we have

$$\begin{aligned} | r | &\leq 1- | r | \\ \frac{| r |}{1- | r |} &\leq 1. \end{aligned}$$

Hence, $\sum_{k=1}^{\infty} \mid ar^k \mid \;\leq\; \mid a \mid \;=\; \mid ar^0 \mid$.
For $n \;=\; m$, if

$$\sum_{k=m+1}^{\infty} \mid ar^k \mid \;\leq\; \mid ar^m \mid . \tag{4.8}$$

Then for $n \;=\; m+1$, we have from (4.8),

$$\begin{aligned} \mid ar^{m+1} \mid + \mid ar^{m+2} \mid + \ldots &\leq \mid ar^m \mid \\ \mid r \mid (\mid ar^{m+1} \mid + \mid ar^{m+2} \mid + \ldots) &\leq \mid r \mid\mid ar^m \mid \\ \mid ar^{m+2} \mid + \mid ar^{m+3} \mid + \ldots &\leq \mid ar^{m+1} \mid \\ \sum_{k=m+2}^{\infty} \mid ar^k \mid &\leq \mid ar^{m+1} \mid . \end{aligned}$$

Therefore, $\sum_{k=n+1}^{\infty} \mid ar^k \mid \leq \mid ar^n \mid$ for all n.

□

Corollary 4.11 *If T is a lower(or upper) triangle Toeplitz matrix induced by $\{t_k\}_{k=0}^{\infty}$ and T satisfies the assumption of proposition 4.12, then T is bounded linear operator on ℓ^2.*

Every Toeplitz matrix T can be written in form,

$$T = T^{-} + T_{-} - T_d,$$

where

$$T_d = \begin{pmatrix} t_0 & 0 & 0 & \ldots \\ 0 & t_0 & 0 & \ldots \\ 0 & 0 & t_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 4.12 *If a Toeplitz matrix T is written in term of T^{-} and T_{-} as above and T^{-} and T_{-} satisfy the hypothesis of theorem 4.10, then T is bounded linear operator on ℓ^2.*

Proof. Let $x \in \ell^2$

$$\begin{aligned} \|Tx\| &= \|(T^{-} + T_{-} - T_d)x\| \\ &= \|T^{-}x + T_{-}x - T_d x\| \\ &\leq \|T^{-}x\| + \|T_{-}x\| + \|T_d x\| \\ &\leq (\|T^{-}\| + \|T_{-}\| + |t_0|)\|x\|. \end{aligned}$$

□

Example 4.13

$$T^{-} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7\times 3} & \frac{1}{7\times 3^2} & \cdots \\ 0 & \frac{1}{7} & \frac{1}{7\times 3} & \cdots \\ 0 & 0 & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is, the Toeplitz matrix is induced from the sequence $\left\{\frac{1}{7\times 3^k}\right\}_{k=0}^{\infty}$, and we have $|r| = \frac{1}{3} < \frac{1}{2}$, so T^{-} is bounded linear operator.

□

Example 4.14

$$T_{-} = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \cdots \\ \frac{1}{8\times 5} & \frac{1}{8} & 0 & \cdots \\ \frac{1}{8\times 5^2} & \frac{1}{8\times 5} & \frac{1}{8} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is, the Toeplitz matrix is induced from the sequence $\left\{\frac{1}{8\times 5^k}\right\}_{k=0}^{\infty}$, and we have $|r| = \frac{1}{5} < \frac{1}{2}$, so T_{-} is bounded linear operator.

□

Example 4.15

$$T = \begin{pmatrix} i & \frac{i}{2+2i} & \frac{i}{(2+2i)^2} & \cdots \\ \frac{i}{2+3i} & i & \frac{i}{2+2i} & \cdots \\ \frac{i}{(2+3i)^2} & \frac{i}{2+3i} & i & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Both $\left|\frac{1}{2+2i}\right|$ and $\left|\frac{1}{2+3i}\right|$ are less than $\frac{1}{2}$, so by corollary 4.13, we have this Toeplitz matrix is bounded linear operator on ℓ^2.

Theorem 4.16 *Let T, a Toeplitz matrix, be a bounded linear operator on ℓ^2. T is unitary if and only if T is of the form,*

$$T = \begin{pmatrix} t_0 & 0 & 0 & \dots \\ 0 & t_0 & 0 & \dots \\ 0 & 0 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and $|t_0| = 1$.

Proof. We recall that T is unitary if $TT^* = T^*T = I$ where I is identity and T^* is conjugate transpose of T.
Suppose T is unitary.

$$\begin{aligned}
(TT^*)e_1 &= T(T^*e_1) \qquad (4.9)\\
&= \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \dots \\ t_1 & t_0 & t_{-1} & \dots \\ t_2 & t_1 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \left(\begin{pmatrix} \bar{t}_0 & \bar{t}_1 & \bar{t}_2 & \dots \\ \bar{t}_{-1} & \bar{t}_0 & \bar{t}_1 & \dots \\ \bar{t}_{-2} & \bar{t}_{-2} & \bar{t}_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \right)\\
&= \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \dots \\ t_1 & t_0 & t_{-1} & \dots \\ t_2 & t_1 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \bar{t}_0 \\ \bar{t}_{-1} \\ \bar{t}_{-2} \\ \vdots \end{pmatrix}\\
&= \begin{pmatrix} t_0\bar{t}_0 + t_{-1}\bar{t}_{-1} + t_{-2}\bar{t}_{-2} + \dots \\ t_1\bar{t}_0 + t_0\bar{t}_{-1} + t_{-1}\bar{t}_{-2} + \dots \\ t_2\bar{t}_0 + t_1\bar{t}_{-1} + t_0\bar{t}_{-2} + \dots \\ \vdots \end{pmatrix}\\
&= \begin{pmatrix} |t_0|^2 + |t_{-1}|^2 + |t_{-2}|^2 + \dots \\ t_1\bar{t}_0 + t_0\bar{t}_{-1} + t_{-1}\bar{t}_{-2} + \dots \\ t_2\bar{t}_0 + t_1\bar{t}_{-1} + t_0\bar{t}_{-2} + \dots \\ \vdots \end{pmatrix}. \qquad (4.10)
\end{aligned}$$

In the same way,

$$(T^*T)e_1 = \begin{pmatrix} |t_0|^2 + |t_1|^2 + |t_2|^2 + \dots \\ \bar{t}_1 t_0 + \bar{t}_0 t_{-1} + \bar{t}_{-1} t_{-2} + \dots \\ \bar{t}_2 t_0 + \bar{t}_1 t_{-1} + \bar{t}_0 t_{-2} + \dots \\ \vdots \end{pmatrix}. \qquad (4.11)$$

Since T is unitary, (4.10) = (4.11) = e_1. That is

$$|t_0|^2 + |t_{-1}|^2 + |t_{-2}|^2 + \dots = |t_0|^2 + |t_1|^2 + |t_2|^2 + \dots = 1. \qquad (4.12)$$

We replace e_1 by e_2, so we will obtain

$$|t_1|^2 + |t_0|^2 + |t_{-1}|^2 + \dots = |t_{-1}|^2 + |t_0|^2 + |t_1|^2 + \dots = 1. \qquad (4.13)$$

By this fashion, we have

$$\left.\begin{array}{rcrcl} |t_0|^2+|t_{-1}|^2+|t_{-2}|^2+\dots & = & |t_0|^2+|t_1|^2+|t_2|^2+\dots & = & 1 \\ |t_1|^2+|t_0|^2+|t_{-1}|^2+\dots & = & |t_{-1}|^2+|t_0|^2+|t_1|^2+\dots & = & 1 \\ |t_2|^2+|t_1|^2+|t_0|^2+\dots & = & |t_{-2}|^2+|t_{-1}|^2+|t_0|^2+\dots & = & 1 \\ & \vdots & & & \end{array}\right\}. \qquad (4.14)$$

We can conclude that $|t_0| = 1$ and $t_k = 0, k \in \mathbb{I} - \{0\}$.
Conversely, let $x \in \ell^2$.

$$\begin{aligned} (TT^*)x &= T(T^*x) \\ &= \begin{pmatrix} t_0 & 0 & 0 & \dots \\ 0 & t_0 & 0 & \dots \\ 0 & 0 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \left(\begin{pmatrix} \bar{t}_0 & 0 & 0 & \dots \\ 0 & \bar{t}_0 & 0 & \dots \\ 0 & 0 & \bar{t}_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \right) \\ &= \begin{pmatrix} t_0 & 0 & 0 & \dots \\ 0 & t_0 & 0 & \dots \\ 0 & 0 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \bar{t}_0 x_1 \\ \bar{t}_0 x_2 \\ \bar{t}_0 x_3 \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} t_0\bar{t}_0 x_1 \\ t_0\bar{t}_0 x_2 \\ t_0\bar{t}_0 x_3 \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} |t_0|^2 x_1 \\ |t_0|^2 x_2 \\ |t_0|^2 x_3 \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}. \end{aligned}$$

And similarly, we will have $(T^*T)x = x$. That is $TT^* = T^*T = I$.

□

Example 4.17 *Let*

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} & 0 & \cdots \\ 0 & 0 & \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

So

$$T^* = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} & 0 & \cdots \\ 0 & 0 & \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

And it is easy to see that

$$T^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} & 0 & \cdots \\ 0 & 0 & \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is $T^* = T^{-1}$, T is unitary. □

Now we will show example in which T is not unitary.

Example 4.18 *Let T, a bounded linear operator on ℓ^2, be in form*

$$T = \begin{pmatrix} t_0 & 0 & 0 & \cdots \\ rt_0 & t_0 & 0 & \cdots \\ r^2t_0 & rt_0 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$t_0 \neq 0$

Define

$$A = \begin{pmatrix} \frac{1}{t_0} & 0 & 0 & \cdots \\ \frac{-r}{t_0} & \frac{1}{t_0} & 0 & \cdots \\ 0 & \frac{-r}{t_0} & \frac{1}{t_0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We will show that A is the inverse of T.

Let $x \in \ell^2$.

$$\begin{aligned} T(Ax) &= \begin{pmatrix} t_0 & 0 & 0 & \cdots \\ rt_0 & t_0 & 0 & \cdots \\ r^2t_0 & rt_0 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \left(\begin{pmatrix} \frac{1}{t_0} & 0 & 0 & \cdots \\ \frac{-r}{t_0} & \frac{1}{t_0} & 0 & \cdots \\ 0 & \frac{-r}{t_0} & \frac{1}{t_0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \right) \\ &= \begin{pmatrix} t_0 & 0 & 0 & \cdots \\ rt_0 & t_0 & 0 & \cdots \\ r^2t_0 & rt_0 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \frac{x_1}{t_0} \\ \frac{-rx_1}{t_0} + \frac{x_2}{t_0} \\ \frac{-rx_2}{t_0} + \frac{x_3}{t_0} \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}. \end{aligned}$$

Consider $\{a_n\}$, where

$$a_1 = t_0\frac{x_1}{t_0} = x_1$$

$$a_n = r^{n-1}t_0\frac{x_1}{t_0}+r^{n-2}t_0(\frac{-rx_1}{t_0}+\frac{x_2}{t_0})+r^{n-3}t_0(\frac{-rx_2}{t_0}+\frac{x_3}{t_0})+\ldots+t_0(\frac{-rx_{n-1}}{t_0}+\frac{x_n}{t_0}) = x_n.$$

That is $T(Ax) = x$, and similarly we can show that $A(Tx) = x$. Therefore $A = T^{-1}$.
But

$$T^* = \begin{pmatrix} \bar{t_0} & \overline{rt_0} & \overline{r^2t_0} & \ldots \\ 0 & \bar{t_0} & \overline{rt_0} & \ldots \\ 0 & 0 & \bar{t_0} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence $T^* \neq T^{-1}$, so T is not unitary.

□

Theorem 4.19 *Let T be a bounded linear Toeplitz matrix on ℓ^2. T is normal, if T is Hermitian matrix.*

Proof. Since T is Hermitian,

$$T = \begin{pmatrix} t_0 & \bar{t}_1 & \bar{t}_2 & \ldots \\ t_1 & t_0 & \bar{t}_1 & \ldots \\ t_2 & t_1 & t_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad t_0 \in \mathbb{R}.$$

So $T^* = T$, that is $T^*T = TT^*$.
Therefore T is normal.

□

Theorem 4.20 *If T is normal, then $|t_k| = |t_{-k}|$ for all k.*

Proof. Since T is normal, $TT^*(e_n) = T^*T(e_n)$. From(4.14), we have

$$\left.\begin{array}{rcl} |t_0|^2+|t_{-1}|^2+|t_{-2}|^2+\ldots & = & |t_0|^2+|t_1|^2+|t_2|^2+\ldots \\ |t_1|^2+|t_0|^2+|t_{-1}|^2+\ldots & = & |t_{-1}|^2+|t_0|^2+|t_1|^2+\ldots \\ |t_2|^2+|t_1|^2+|t_0|^2+\ldots & = & |t_{-2}|^2+|t_{-1}|^2+|t_0|^2+\ldots \\ & \vdots & \end{array}\right\}. \tag{4.15}$$

If we subtract the second equation with the first equation, then we have $|t_1|^2 = |t_{-1}|^2$ that is, $|t_1| = |t_{-1}|$. Similarly, we have $|t_k|^2 = |t_{-k}|^2$ for all k, that is, $|t_k| = |t_{-k}|$ for all k.

□

Corollary 4.21 *Let T be the Toeplitz matrix T induecd by $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$. If T is normal, then $\|T\| \leq 2\left(\sum_{k=0}^{\infty} |t_k|\right)$.*

Proof.

$$\begin{aligned}
\|T\| &\leq \left(\sum_{k=-\infty}^{\infty} |t_k|\right)^{\frac{1}{2}} \left(\sum_{k=-\infty}^{\infty} |t_k|\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=-\infty}^{\infty} |t_k|\right) \\
&= (\ldots + |t_{-1}| + |t_0| + |t_1| + \ldots) \\
&\leq (\ldots + |t_{-1}| + |t_0| + |t_0| + |t_1| + \ldots) \\
&= (2\,|t_0| + 2\,|t_1| + \ldots) \\
&= 2\left(\sum_{k=0}^{\infty} |t_k|\right).
\end{aligned}$$

□

Theorem 4.22 *Let T be real Toeplitz matrix and $t_0 = 0$. T is normal if and only if T is a symmetric or skew-symmetric matrix.*

Proof. Let T be normal. By theorem 4.20 we have $|t_n| = |t_{-n}|$. Since T is a real matrix, $t_n = t_{-n}$ or $t_n = -t_{-n}$. Hence T is symmetric or skew-symmetric matrix, from the classification of Toeplitz matrix in literature reviews.

Conversely,

Case 1 : T is symmetric matrix, in this case T is Hermitian, so T is normal, by theorem 4.19.

Case 2 : T is skew-symmetric. Let $x \in \ell^2$.

$$\begin{aligned}
TT^*x &= \begin{pmatrix} 0 & -t_1 & -t_2 & \ldots \\ t_1 & 0 & -t_1 & \ldots \\ t_2 & t_1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & t_1 & t_2 & \ldots \\ -t_1 & 0 & t_1 & \ldots \\ -t_2 & -t_1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} 0 & -t_1 & -t_2 & \ldots \\ t_1 & 0 & -t_1 & \ldots \\ t_2 & t_1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 + t_1x_2 + t_2x_3 + \ldots \\ -t_1x_1 + 0 + t_1x_x + \ldots \\ -t_2x_1 + -t_1x_2 + 0 + \ldots \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} 0 & -t_1 & -t_2 & \ldots \\ t_1 & 0 & -t_1 & \ldots \\ t_2 & t_1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{\infty} t_jx_{1+j} \\ \sum_{i=1}^{2-1} -t_{2-i}x_i + \sum_{j=1}^{\infty} t_jx_{2+j} \\ \sum_{i=1}^{3-1} -t_{3-i}x_i + \sum_{j=1}^{\infty} t_jx_{3+j} \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} 0 & -t_1 & -t_2 & \ldots \\ t_1 & 0 & -t_1 & \ldots \\ t_2 & t_1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},
\end{aligned}$$

where

$$a_n = \begin{cases} \sum_{j=1}^{\infty} t_j x_{n+j} & n \ = 1 \\ \sum_{j=1}^{n-1} -t_{n-j} x_j \ + \sum_{j=1}^{\infty} t_j x_{n+j} & n \ > 1 \end{cases} .$$

$$\begin{aligned} TT^{*}x &= \begin{pmatrix} \sum_{j=1}^{\infty} -t_j a_{1+j} \\ \sum_{i=1}^{2-1} t_{2-i} a_i + \sum_{j=1}^{\infty} -t_j a_{2+j} \\ \sum_{i=1}^{3-1} t_{3-i} a_i + \sum_{j=1}^{\infty} -t_j a_{3+j} \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}, \end{aligned}$$

where

$$b_n = \begin{cases} \sum_{j=1}^{\infty} -t_j a_{n+j} & n \ = 1 \\ \sum_{j=1}^{n-1} t_{n-j} a_j \ + \sum_{j=1}^{\infty} -t_j a_{n+j} & n \ > 1 \end{cases} .$$

Consider,

$$\begin{aligned} T^{*}Tx &= \begin{pmatrix} 0 & t_1 & t_2 & \dots \\ -t_1 & 0 & t_1 & \dots \\ -t_2 & -t_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & -t_1 & -t_2 & \dots \\ t_1 & 0 & -t_1 & \dots \\ t_2 & t_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} 0 & t_1 & t_2 & \dots \\ -t_1 & 0 & t_1 & \dots \\ -t_2 & -t_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{\infty} -t_j x_{1+j} \\ \sum_{j=1}^{2-1} t_{2-j} x_j + \sum_{j=1}^{\infty} -t_j x_{2+j} \\ \sum_{j=1}^{3-1} t_{3-j} x_j + \sum_{j=1}^{\infty} -t_j x_{3+j} \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} 0 & t_1 & t_2 & \dots \\ -t_1 & 0 & t_1 & \dots \\ -t_2 & -t_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \\ \vdots \end{pmatrix} \end{aligned}$$

where

$$a'_n = \begin{cases} \sum_{j=1}^{\infty} -t_j x_{n+j} & n \ = 1 \\ \sum_{j=1}^{n-1} t_{n-j} x_j \ + \sum_{j=1}^{\infty} -t_j x_{n+j} & n \ > 1 \end{cases} .$$

Therefore

$$\begin{aligned} T^{*}Tx &= \begin{pmatrix} \sum_{j=1}^{\infty} t_j a'_{1+j} \\ \sum_{j=1}^{2-1} -t_{2-j} a'_j + \sum_{j=1}^{\infty} t_j a'_{2+j} \\ \sum_{j=1}^{3-1} -t_{3-j} a'_j + \sum_{j=1}^{\infty} t_j a'_{3+j} \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \\ \vdots \end{pmatrix}, \end{aligned}$$

where

$$b'_n = \begin{cases} \sum_{j=1}^{\infty} t_j a'_{n+j} & n = 1 \\ \sum_{j=1}^{n-1} -t_{n-j} a'_j + \sum_{j=1}^{\infty} t_j a'_{n+j} & n > 1 \end{cases} .$$

We will have $a'_n = -a_n$. Replace $a'_n = -a_n$ in b'_n,

$$\begin{aligned} b'_n &= \begin{cases} \sum_{j=1}^{\infty} t_j (-a_{n+j}) & n = 1 \\ \sum_{j=1}^{n-1} -t_{n-j} (-a_j) + \sum_{j=1}^{\infty} t_j (-a_{n+j}) & n > 1 \end{cases} \\ &= \begin{cases} \sum_{j=1}^{\infty} -t_j a_{n+j} & n = 1 \\ \sum_{j=1}^{n-1} t_{n-j} a_j + \sum_{j=1}^{\infty} -t_j a_{n+j} & n > 1 \end{cases} \\ &= b_n. \end{aligned}$$

That is $TT^* = T^*T$, thus T is normal.

$\square$

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS

In this chapter we summarize the results from chapter 4. The explanation and the future research are also presented.

5.1 Conclusions

5.1.1

Let $\{t_k\}_{k=-\infty}^{\infty}$ *be any sequence of complex numbers and* T *be a Toeplitz matrix induced by this sequence.*
If $\sum_{k=-\infty}^{-1} \mid t_k \mid < \infty$ *and* $\sum_{k=0}^{\infty} \mid t_k \mid < \infty$, *then* T *is a bounded linear operator on* ℓ^2.

5.1.2

Let $\{t_k\}_{k=-\infty}^{\infty}$ *be any sequence of complex numbers and* T *is a Toeplitz matrix induced by this sequence. If* $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$, *then* T *is a bounded linear operator on* ℓ^2.

5.1.3

Let $\{t_k\}_{k=-\infty}^{\infty}$ *be any sequence of complex number and* T *be the Toeplitz matrix induced by* $\{t_k\}_{k=-\infty}^{\infty}$. *If* T *is a bounded linear operator on* ℓ^2. *then*

1. $\sup_n(\sum_{i=1}^{\infty} \mid t_{n-i} \mid^2)^{\frac{1}{2}} < \infty.$
2. $\sup_i(\sum_{n=1}^{\infty} \mid t_{n-i} \mid^2)^{\frac{1}{2}} < \infty.$

5.1.4

By the hypothesis of 5.1.3, we will have $\{t_n\}_{n=-\infty}^{\infty} \in \ell^2$.

5.1.5

Let $\{a_k\}_{k=1}^{\infty}$ *be in* ℓ^2 .
$\forall k$ *if* $\mid a_k \mid \geq \sum_{n=k+1}^{\infty} \mid a_n \mid$ *then* $(\sum_{n=1}^{\infty} \mid a_n \mid) \leq 2\,(\sum_{n=1}^{\infty} \mid a_n \mid^2)^{\frac{1}{2}}$.

5.1.6

Let $\{t_k\}_{k=-\infty}^{\infty}$ *be any sequence of complex numbers,* T *be a Toeplitz matrix induced by* $\{t_n\}_{n=-\infty}^{\infty}$ *and* $\sum_{n=k+1}^{\infty} \mid t_n \mid \leq \mid t_k \mid$ *for all* $n = 1, 2, \ldots$ *and* $\sum_{n=-\infty}^{n-1} \mid t_n \mid \leq \mid t_k \mid$ *for all* $n = -1, -2, \ldots$. *Then the following statements are equivalent*

1. T *is a bounded linear operator on* ℓ^2,

2. $\{t_k\}_{k=-\infty}^{\infty} \in \ell^2$,

3. $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$.

5.1.7

Let $\{t_k\}_{k=0}^{\infty}$ *be a sequence that satisfies the hypothesis of lemma 4.8. If Toeplitz matrix* T_- *is induced by the sequence, then* T_- *is bounded linear operator on* ℓ^2 *and*

$$\frac{1}{2}\Big(\sum_{k=0}^{\infty} \mid t_k \mid\Big) \leq \|T_-\| \leq \Big(\sum_{k=0}^{\infty} \mid t_k \mid\Big).$$

5.1.8

For all geometric squences $\{ar^k\}_{k=0}^{\infty}$ *such that* $\mid r \mid \leq \frac{1}{2}$, *they have property*

$$\sum_{k=n+1}^{\infty} \mid ar^k \mid \quad \leq \quad \mid ar^n \mid \quad \forall\, n = 0, 1, 2, \ldots.$$

5.1.9

If T *is a lower(or upper) triangle Toeplitz matrix induced by* $\{t_k\}_{k=0}^{\infty}$ *and* T *satisfies the assumption of 5.1.8, then* T *is bounded linear operator on* ℓ^2.

5.1.10

If a Toeplitz matrix T *is written in term of* T^- *and* T_- *as above and* T^- *and* T_- *satisfy the hypothesis 5.1.8, then* T *is bounded linear operator on* ℓ^2.

5.1.11

Let T, a Toeplitz matrix, be a bounded linear operator on ℓ^2. T is unitary if and only if T is of the form,

$$T = \begin{pmatrix} t_0 & 0 & 0 & \dots \\ 0 & t_0 & 0 & \dots \\ 0 & 0 & t_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and $|t_0| = 1$.

5.1.12

Let T be a bounded linear Toeplitz matrix on ℓ^2. T is normal, if T is Hermitian matrix.

5.1.13

If T is normal, then $|t_k| = |t_{-k}|$ for all k.

5.1.14

Let T be the Toeplitz matrix T induced by $\{t_k\}_{k=-\infty}^{\infty} \in \ell^1$. If T is normal, then $\|T\| \leq 2\left(\sum_{k=0}^{\infty} |t_k|\right)$.

5.1.15

Let T be real Toepliz matrix and $t_0 = 0$. T is normal if and only if T is a symmetric or skew-symmetric matrix.

5.2 Related with previous results

5.2.1 Sufficient condition

Let $a(z) = \frac{1}{e^{\frac{\ln z}{i}}}$ we will show that $a(z) \in L^{\infty}(\mathbb{T})$ where $\mathbb{T}$ is unit circle on complex. So

$$z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} a(z) &= a(e^{i\theta}) \\ &= \frac{1}{e^{\frac{\ln e^{i\theta}}{i}}} \\ &= \frac{1}{e^{\theta}}. \end{aligned}$$

By the definition $f(z) \in L^{\infty}(\mathbb{T})$, if

$$\|f\|_{\infty} = ess \sup_{z \in \mathbb{T}} |f(z)| = \inf\{\beta : \mu(f^{-1}(\beta, \infty]) = 0\} < \infty.$$

Since $0 < \frac{1}{e^{\theta}} \leq 1, 0 \leq \theta \leq 2\pi$,

$$a(z) \leq 1,$$

that is

$$\mu(a^{-1}(1, \infty]) = 0.$$

Therefore,

$$\|a\|_{\infty} \leq 1 < \infty.$$

Hence, $a(z) \in L^{\infty}(\mathbb{T})$. By calculating compute Fourier coefficients of $a(z)$ we have for $n \in \mathbb{N}$,

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-in\theta}}{e^{\theta}}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta-\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{(-1-in)\theta}d\theta \\
&= \frac{1}{2\pi}\left[\frac{e^{(-1-in)\theta}}{-1-in}\Big|_0^{2\pi}\right] \\
&= \frac{1}{2\pi}\left[\frac{e^{(-1-in)2\pi} - e^{0}}{-1-in}\right] \\
&= \frac{1}{2\pi}\left[\frac{e^{-2\pi}e^{-i2n\pi} - 1}{-(1+in)}\right] \\
&= \frac{1}{2\pi}\left[\frac{1 - e^{-2\pi}e^{-i2n\pi}}{(1+in)}\right] \\
&= \frac{1}{2\pi}\left[\frac{1 - e^{-2\pi}(\cos(-2n\pi) + i\sin(-2n\pi))}{(1+in)}\right] \\
&= \frac{1}{2\pi}\left[\frac{1 - e^{-2\pi}}{(1+in)}\frac{1-in}{1-in}\right] \\
&= \frac{1}{2\pi}\left[\frac{(1 - e^{-2\pi}))(1-in)}{1+n^2}\right] \\
&= \frac{1}{2\pi}\left[\frac{1-in}{1+n^2} - \frac{e^{-2\pi}(1-in)}{1+n^2}\right] \\
&= \frac{1}{2\pi}\left[\left(\frac{1}{n^2+1} - \frac{in}{n^2+1}\right) - \left(\frac{1}{n^2+1} - \frac{in}{n^2+1}\right)e^{-2\pi}\right] \\
&= \frac{1}{2\pi}\left[\left(\frac{1}{n^2+1} - \frac{e^{-2\pi}}{n^2+1}\right) + i\left(\frac{-n}{n^2+1} + \frac{ne^{-2\pi}}{n^2+1}\right)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty}|a_n| &= \sum_{n=1}^{\infty}\left[\frac{\sqrt{\left(\frac{1}{n^2+1}-\frac{e^{-2\pi}}{n^2+1}\right)^2+\left(\frac{-n}{n^2+1}+\frac{ne^{-2\pi}}{n^2+1}\right)^2}}{2\pi}\right] \\
&\geq \sum_{n=1}^{\infty}\left[\frac{\sqrt{\left(\frac{-n}{n^2+1}+\frac{ne^{-2\pi}}{n^2+1}\right)^2}}{2\pi}\right] \\
&= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left|\frac{-n}{n^2+1}+\frac{ne^{-2\pi}}{n^2+1}\right| \\
&= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left|\frac{n(-1+e^{-2\pi})}{n^2+1}\right| \\
&= \frac{|-1+e^{-2\pi}|}{2\pi}\sum_{n=1}^{\infty}\left|\frac{n}{n^2+1}\right| \\
&\geq \frac{|-1+e^{-2\pi}|}{2\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+1}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{n}{n^2+1}=\infty$ and $\frac{|-1+e^{-2\pi}|}{2\pi}>0$, therefore

$$\sum_{n=1}^{\infty}|a_n| \geq \frac{|-1+e^{-2\pi}|}{2\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+1}=\infty.$$

So

$$\sum_{n=-\infty}^{\infty}|a_n|=\infty,$$

that is $\{a_n\}_{n=-\infty}^{\infty}\notin\ell^1$.
This means that our conditions are weaker than the conditions in theorem 2.3.1.

5.2.2 Conditions for Normality

In this thesis we consider several types of Toeplitz matrix in literature reviews which are known as typeI, symmetric and skew-symmetric. Others types are not to be considered because the last term of sequence.
In the general case, as in 5.1.13, our condition is special case of TypeI, in which α_0 is zero. That mean Toeplitz matrix which in form hermitian is normal not only finite but also infinite dimension. In real case, as in 5.1.16, Toeplitz matrix is normal if and only if it is symmetric or skew-symmetric. In both case the Toeplitz matrix is normal in both the finite and infinite dimensional case.

5.3 Suggestions

In this section, we give some future works to improve the results of this research.

1. You can improve necessary and sufficient condition, in theorem 4.9, by finding other conditions which imply that if x is in ℓ^2, then x is in ℓ^1.
2. You can improve theorem 4.10, by finding the best approximation for the norm of Toeplitz operator.
3. You can improve theorem 4.20, by finding other sufficient conditions for the normal operator.

Remark : Since this research concern infinite series. When rearrange them, We are sure that they converge absolutely. For example, let $T : \ell^2 \longrightarrow \ell^2$. Consider $< Tx, y >$ and $< x, T^* y >$, where

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots \\ t_1 & t_0 & t_{-1} & \cdots \\ t_2 & t_1 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$ and T^* is conjugate and transpose matrix of T.

We have

$$\begin{aligned} < Tx, y > &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_{n-i} x_i \right) \bar{y}_n \\ &= (t_0 x_1 + t_{-1} x_2 + \ldots) \bar{y}_1 + (t_1 x_1 + t_0 x_2 + \ldots) \bar{y}_2 + \ldots, \qquad (5.1) \end{aligned}$$

and

$$\begin{aligned} < x, T^* y > &= \sum_{m=1}^{\infty} x_m \overline{\left(\sum_{j=1}^{\infty} \bar{t}_{j-m} y_j \right)} \\ &= x_1 (t_0 \bar{y}_1 + t_1 \bar{y}_2 + \ldots) + x_2 (t_{-1} \bar{y}_1 + t_0 \bar{y}_2 + \ldots) + \ldots. \qquad (5.2) \end{aligned}$$

If $< Tx, y > = < x, T^* y >$, we can not conclud that the equations (5.1) and (5.2) are equal by rearrangement from (5.1) to (5.2) or (5.2) to (5.1). Because the equations may be not absolute convergence.

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