TRACE INEQUALITY OF HERMITIAN MATRIX

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หัวข้อวิทยานิพนธ์	Trace Inequality of Hermitian Matrix
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บทคัดย่อ

งานวิจัยฉบับนี้ศึกษาถึงหัวข้อเรื่อง Trace Inequality of Hermitian Matrix ที่เกี่ยวข้องกับการคูณ และ การยกกำลังของเมทริกซ์ที่เป็นบวกจำกัด(positive definite matrices) เมทริกซ์เฮอร์มิเชี่ยน(Hermitian matrices) และ เมทริกซ์เสมือนเฮอร์มิเชี่ยน(skew Hermitian matrices) โดยเน้นการพิสูจน์อสมการใหม่ๆ รวมถึงอสมการที่เกี่ยวข้องกับ ผลบวกเฉียง(trace) และ ตัวกำหนด(determinant) ของเมทริกซ์ที่เป็นบวกแน่นอน

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ABSTRACT

In this research, Inequalities have been presented are involving the power and the product of positive definite matrices, Hermitian matrices and skew Hermitian matrices. In particular inequalities involving the trace and the determinant of certain positive definite matrices.

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Monsicha Tipawanna

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Notation

R The real numbers.

C The complex numbers.

F A field (usually R or C).

M_n n by n complex matrices $M_{n,n}(C)$.

I Identity matrix in $M_n(F)$.

$\bar{A}$ Matrix of complex conjugates of entries of $A \in M_{m,n}(C)$.

A^T Transpose of $A \in M_{m,n}(F)$.

A^* Hermitian adjoint of $A \in M_{m,n}(C)$, $(\bar{A})^T$.

A^{-1} Inverse of a nonsingular $A \in M_n(F)$.

$A^{\frac{1}{2}}$ Unique positive (semi)definite square root of a positive definite $A \in M_n(F)$.

det A Diterminant of $A \in M_n(F)$.

λ Eigenvalue of $A \in M_n(F)$.

$\|.\|_2$ l_2 (Frobenius) matrix norm on M_n.

tr A trace of $A \in M_n(F)$.

Chapter 1

Introduction

1.1 Motivation and Inception

Recently the trace inequality of two powered Hermitian matrices was given in [2] as follows:

$$\operatorname{tr}(AB)^{2^k} \leq \operatorname{tr} A^{2^k} B^{2^k}, \quad A, B \text{ are Hermitian matrices.} \tag{1.1}$$

Furthermore, the following two results were proved in [16] when n(≥1), A, B are positive semidefinite, then

$$0 \leq \operatorname{tr}(AB)^{2n} \leq (\operatorname{tr} A)^2 (\operatorname{tr} A^2)^{n-1} (\operatorname{tr} B^2)^n \tag{1.2}$$

and

$$0 \leq \operatorname{tr}(AB)^{2n+1} \leq (\operatorname{tr} A)(\operatorname{tr} B)(\operatorname{tr} A^2)^n (\operatorname{tr} B^2)^n. \tag{1.3}$$

Another two results appeared in [19, theorem 1] and [4, theorem 1]. When A, B are positive definite, the following inequalities hold:

$$\operatorname{tr}(AB)^n \leq (\operatorname{tr} A^{2n})^{\frac{1}{2}} (\operatorname{tr} B^{2n})^{\frac{1}{2}}, \quad n \in N \tag{1.4}$$

and

$$\operatorname{tr}(AB)^n \leq (\operatorname{tr} AB)^n, \quad n \in N. \tag{1.5}$$

The above results (1.1), (1.2), (1.3), (1.4) and (1.5) are related to the work of Bellman. In 1980, Bellman [12] proved:

$$\operatorname{tr}(AB)^2 \leq \operatorname{tr} A^2 B^2 \tag{1.6}$$

and proposed the conjecture whether

$$\mathrm{tr}(AB)^n \leq \mathrm{tr}(A^nB^n), \qquad n \in N. \qquad (1.7)$$

holds when A, B are positive semidefinite.

Since then, many authors have proved that the conjecture (1.6) is correct. In [9], it was pointed out that the inequality (1.6) was also proposed by Lieb and Thiring in 1976, and a similar inequality was proposed also in [1]. R.A. Brualdi [10] commented further work of the inequality (1.6) that was constructed by Lieb and Thiring in [9] and [1].

Whether or not the inequality (1.7) as a conjecture at that time, the condition in [2] was different from that in [1, 4, 9, 16, 12, 19] which dropped the demand of "semi-definite property" for matrices in [2], and examined the trace inequality on the general Hermitian matrix powers. Of course, it increases inevitably the discussed difficulty. In 1992 Zhengming Jiao [15] generalised inequality (1.6) for Hermitian matrix A , skew Hermitian matrix B and A,B are skew Hermitian matrix, and also presented to questions as follows:

$$\mathrm{tr}(AB)^n \geq \mathrm{tr}(A^nB^n), \quad A \text{ is Hermitian, } B \text{ is skew Hermitian, } n \in N \qquad (1.8)?$$

and
$$\mathrm{tr}(AB)^n \leq \mathrm{tr}(A^nB^n), \quad A, B \text{ are skew Hermitian, } n \in N. \qquad (1.9)?$$

Zhong Peng Yang and Xiao Xia Feng [21] gave the resolution for above questions hold, as follows: if A is Hermitian matrix and B is skew Hermitian, then

$$\mathrm{tr}(AB)^n \leq \mathrm{tr}(A^2B^2)^{\frac{n}{2}} \leq \mathrm{tr}(A^nB^n), \quad n=4t, t\in N. \qquad (1.10)$$

and
$$\mathrm{tr}(AB)^n \geq \mathrm{tr}(A^2B^2)^{\frac{n}{2}} \geq \mathrm{tr}(A^nB^n), \quad n=4t+2, t\in N. \qquad (1.11)$$

if A and B are skew Hermitian , then

$$\mathrm{tr}(AB)^n \leq |\mathrm{tr}(AB)^n| \leq \mathrm{tr}(A^2B^2)^{\frac{n}{2}} \leq \mathrm{tr}(A^nB^n), \quad n=2t, t\in N. \qquad (1.12)$$

We will improve and further inequality (1.7) , (1.8) and (1.9) concerning the power index is a nonnegative number for obtain upper bound and answer completely mentioned in [15]. Thereby the results in [21] can be obtain and improve. Moreover a simpler proof for the inequality (1.7) which condition A and B are Hermitian matrices and the new trace inequalities may be presented.

1.2 Initial Idea

In 1980, R. Bellman [12] gave the trace inequality as follow ;

$$tr(AB)^2 \leq trA^2B^2$$

, when A and B are Hermitian matrices and m is nonnegative integer.

Many authors[2, 12, 21] have presented Bellman's Inequality still holds with higher order ,when order index is even number. However above authors [2, 12,21] have not present the proof of order index is odd number.

1.3 Objectives

This thesis we interested to study the matrix trace inequality of certain positve definite matrices, Hermitian matrices and skew Hermitian matrices. The main purpose of this thesis is

(1.) To show that Bellman's Inequality ;

$$\text{tr}(AB)^n \leq \text{tr}(A^nB^n), \qquad n \in N.$$

holds for any nonnegative integer m.

(2.) Further upper bound(or lower bound) of Bellman's Inequality.

(3.) Give new Inequality involving the trace and the determinant of the product and the power of Hermitian matrices.

1.4 Scope of study

In this thesis ,new inequalities were presented and prove are involving the trace and the determinant of the product and the power for certain positive definite matrices, Hermitian matrices and skew Hermitian matrices. Through out this thesis I worked with square matrices on a finite dimentional Hilbert space and work with order index is nonnegative integer.

1.5 Process of study

1.5.1 Find and study researches and document that concern about trace inequality of Hermitian matrices.

1.5.2 Consider character and property of Hermitian matrices, skew–Hermitain matrices and Hermitian matrices with condition positive definite.

1.5.3 Show the Bellman's inequality of Hermitian matrices with higher order does not holds by counterexample.

1.5.4 Prove the Bellman's inequality of Hermitian matrices with higher order holds when add some conditions, direct proof and used the Arithmetic and Geometric Mean Inequality.

1.5.5 Used the Arithmetic and Geometric Mean Inequality , Theorem 2.1.2.2 ,Theorem 2.2.2.3 , property of determinant to prove Theorem 4.1.4 -4.1.7.

1.5.6 Further upper bound of Bellman's inequality concern about product of Hermitian matrices and skew-Hermitian matrices. Direct proof and used character of those matrices.

Chapter 2

Basic Knowledge and Literature Reviews

2.1 Definition

In this part, we gave some definitions and some concepts which important to proof any inequalities

2.1.1 Matrices

Definition 2.1.1.1 (Matrix operations) Matrix addition is defined entry-wise for arrays of the same dimensions and is denoted by + ("$A+B$"). It corresponds to addition linear transformations (relative to the same basis), and it inherits commutativity and associativity from the scalar field. The zero matrix (all entries zero) is the identity under addition, and $M_{m,n}(F)$ is itself a vector space over F. Matrix multiplication is defined in the usual way, is denoted by juxtaposition, AB, and corresponds to the composition of linear transformations. As such, it is defined only when $A \in M_{m,n}(F)$, $B \in M_{p,q}(F)$, and $p = n$; it is associative. It is not, in general, commutative. There is an identity under matrix multiplication, the matrix

$I \in M_n(F)$ of the form

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & & \cdot \\ & & & & 1 \end{bmatrix}$$

This matrix and all scalar multiples of it (called scalar matrices) commute with all other matrices in $M_n(F)$ and are the only matrices with do so. Matrix multiplication is distributive over matrix addition.We note here that the symbol 0 is used throughout to denote each of the following: the zero scalar, the zero vector (all components equal to the zero scalar), and the zero matrix(all entries equal to the zero scalar). We also used the symbol I to denote the identity matrix of any size.

Definition 2.1.1.2 (The transpose and the Hermitian adjoint) If $A = \lfloor a_{ij} \rfloor \in M_{m,n}(F)$, the *transpose* of A, denoted A^T, is that matrix in $M_{m,n}(F)$ whose entries are a_{ji}; that is , rows are exch anged for columns are vice versa. Of course $(A^T)^T = A$. The *Hermitian adjiont* A^* (or A^H) of $A \in M_{m,n}(C)$ is denote by $A^* = (\overline{A})^T$, where $\overline{A}$ the component-wise conjugate.

2.1.2 Determinants

Definition 2.2.2.1 Let A be a square matrix of order n. The determinant of A, denoted by det A, is denoted by

$$\det A = \sum_j (-1)^{t(j)} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \tag{2.1}$$

where $t(j)$ is the number of inversions in the permutation $j = (j_1, j_2, \ldots, j_n)$ and j varies over $n!$ permutations of 1, 2 , ..., n.

Definition 2.1.2.2 (The trace) Let A be a square matrix of order n.The *trace* of A, denoted by *trace* A (or tr A), is denote by

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} \tag{2.2}$$

2.1.3 Nonsingularity

Definition 2.1.3.1 A linear transformation or matrix is said to be *nonsingular* if it produce the output 0 only for input 0. Otherwise, it is singular . If $A \in M_{m,n}(F)$ and $m < n$, then A is necessarily singular. If $A \in M_n(F)$,A is called *invertible* if there is a matrix $A^{-1} \in M_n(F)$ called the *inverse* of A such that $A A^{-1} = I$. If $A \in M_n(F)$ and $A A^{-1} = I$, then $A^{-1}A = I$; A^{-1} is unique whenever it exists.

2.1.4 The usual inner product

Definition 2.1.4.1 The scalar $y^* x$ is an *inner product (scalar product)* of x *and* $y \in C^n$ and is often denoted $\langle x, y \rangle \equiv y^* x$.

Definition 2.1.4.2 (Orthogonality) Two vectors x , $y \in C^n$ are called *orthogonal* if $\langle x, y \rangle = 0$.

2.1.5 The special type of matrix.

Definition 2.1.5.1 (Diagonal matrices) The matrix $D = (d_{ij}) \in M_n$ is called *diagonal* if $d_{ij} = 0$ whenever $j \neq i$. Conventionally, we denote such a matrix as $D = diag(d_{11}, d_{22}, \ldots, d_{nn})$.

2.1.6 Eigenvalued, Eigenvactors and Similarity

Definition 2.1.6.1 (The eigenvalue-eigenvector equaltion) If $A \in M_n$ and $x \in C^n$,we consider the equation

$$Ax = \lambda x , \qquad x \neq 0 \tag{2.3}$$

Where λ is a scalar. If a scalar λ and a nonzero vector x happen to satisfy this equation, then λ is called an *eigenvalue* of A and x is called *eigenvector* of A associated with λ .

Definition 2.1.6.2 The set of all $\lambda \in C$ that are eigenvalues of $A \in M_n$ is called the *spectrum* of A and is denoted by $\sigma(A)$.

2.1.7 Similarity

Definition 2.1.7.1 A matrix $B \in M_n$ is said to be *similar* to a matrix $A \in M_n$ if there exists a nonsingular matrix $S \in M_n$ such that

$$B = S^{-1}AS$$

The transformation $A \rightarrow S^{-1}AS$ is called a *similarity transformation* by the *transformation* by the *similarity* matrix S. The relation "B is similar to A" is sometimes abbreviated $B \sim A$.

Definition 2.1.7.2 If the matrix $A \in M_n$ is similar to a diagonal matrix, then A is said to be diagonalizable. Sometimes the term *diagonable* is used.

2.1.8 Hermitian matrices

Definition 2.1.8.1 A matrix $A = [a_{ij}] \in M_n$ is said to be Hermitian if $A = A^*$, where $A^* = (\overline{A})^T = [\overline{a_{ij}}]$. It is skew - Hermitian if $A = -A^*$.

Remark 2.1.8.2 It is know that $A = [a_{ij}]$ is Hermitian matrix if the matrix A have the following property

(1) A is a square matrix.

(2) Elements above the main diagonal of A are real.

(3) Two elements are positioned symmetrically with respect to the main diagonal of A, they are mutually complex conjugate.

See example:

$$A = \begin{bmatrix} 3 & i & 2-i \\ -i & 4 & 2 \\ 2+i & 2 & 0 \end{bmatrix}$$

Remark 2.1.8.3 It is know that $A = [a_{ij}]$ is skew-Hermitian matrix if the matrix A have the following properties

(1) A is a square matrix.

(2) Elements above the main diagonal of A are pure imaginary or zero.

(3) Two elements are positioned symmetrically with respect to the main diagonal of A, they are mutually inverse addition of its complex conjugate.

See example:

$$A = \begin{bmatrix} 3i & i & 2+i \\ i & 4i & 2 \\ -2+i & -2 & 0 \end{bmatrix}$$

2.1.9 Positive definite matrices

Definition 2.1.9.1 An *n-by-n* Hermitian matrix A is said to be *positive definite* if

$$x^* Ax > 0 \quad \text{for all nonzero } x \in C^n, \qquad (2.4)$$

If the strict inequality required in (1.5.9.1) is weakened to $x^* Ax \geq 0$, then A is said to be *positive semidefinite*. Implicit in these defining inequality is the observation that if A is Hermitian, the left- hand side of (1.5.9.1) is always a real number. Of course, if A is positive definite , then it is also positive semidefinite.

2.1.10 Nonnegative matrices

Definition 2.1.10.1 (Inequalities and generalities) Let $B = [b_{ij}] \in M_n$ and $A = [a_{ij}] \in M_n$. We write

$$B \geq 0 \quad \text{if all} \quad b_{ij} \geq 0$$
$$B > 0 \quad \text{if all} \quad b_{ij} > 0$$
$$A \geq B \quad \text{if all} \quad A - B \geq 0$$
$$A > B \quad \text{if all} \quad A - B > 0$$

The reverse relation $\leq$ and $<$ are defined similarly. We define $|A| \equiv [|a_{ij}|]$. If $A \geq 0$, we say A is nonnegative matrix, and if $A > 0$, we say that A is a positive definite matrix.

2.2 General Properties and Theorems

In this part, we give some general concepts which important to discuss and proof any inequalities.

2.2.1 Properties

Property 2.2.1.1 (The Trace) For $A, B \in M_n(F)$ and α , $\beta \in F$, then

(1.) $$\operatorname{tr}(\alpha A + \beta B) = \alpha \operatorname{tr} A + \beta \operatorname{tr} B \tag{2.5}$$

(2.) $$\operatorname{tr}(AB) = \operatorname{tr}(BA) \tag{2.6}$$

(3.) $$\operatorname{tr}(A^T) = \operatorname{tr}(A) \tag{2.7}$$

(4.) $$\operatorname{tr} A^* = \overline{(trA)} \tag{2.8}$$

(5.) $\operatorname{tr} A = \operatorname{tr}(C^{-1}AC)$, if C is invertible squre matrix dimension as A (2.9)

(6.) $$\operatorname{tr} A^*A = \operatorname{tr} AA^* = \sum_{i=1}^{n} a_{i,i} \tag{2.10}$$

Property 2.2.1.2 (Determinant) The most crucial and important property of the determinant function is that it is multiplicative: For $A, B \in M_n(F)$

$$\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA) \tag{2.11}$$

Property 2.2.1.3 (Positive definite matrices) If $A = [a_{ij}] \in M_n$ is positive definite, then so are $\overline{A}$, A^T , A^* , A^{-1}.

2.2.2 Theorems

Theorem 2.'2.2.1 (The arithmetic – Geometric Mean Inequality (or (A-G) inequality)).
If $x_1, x_2, \ldots, x_n$ are positive real numbers and if $\delta_1, \delta_2, \ldots, \delta_n$ are positive numbers whose sum is one, then

$$\prod_{i=1}^{n}(x_i)^{\delta_i} \leq \sum_{i=1}^{n}\delta_i x_i \tag{2.12}$$

with equality in (A-G) if and only if $x_1 = x_2 = \ldots = x_n$.

Theorem 2.1.2.2 If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A \in M_n(F)$, then

$$\operatorname{tr} A = \sum_{i=1}^{n}\lambda_i \tag{2.13}$$

and

$$\det A = \prod_{i=1}^{n}\lambda_i \tag{2.14}$$

Theorem 2.2.2.3 Let $A, B \in M_n(F)$. if B is similar to A, then the characteristic polynomial of B is the same as that of A.

Corollary 2.2.2.4 If $A, B \in M_n(F)$ and if A and B are similar, then they have the same eigenvalues , counting multiplicity.

Theorem 2.2.2.5 If the matrix $A \in M_n(F)$. Then A is diagonalizable if and only if there is a set of n linearly independent vectors, each of which is an eigenvector of A .

Theorem 2.2.2.6 If the matrix $A \in M_n(F)$ has n distinct eigenvalues, then A is diagonalizable.

Theorem 2.2.2.7 Let $A \in M_n$ be Hermitian. Then

(a) $x^* Ax$ is real for all $x \in C^n$;

(b) All the eigenvalues of A are real ; and

(c) $S^* AS$ is Hermitian for all $S \in M_n$.

Theorem 2.2.2.8 If A is symmetric matrix (or Hemitian) , then:

(a) the matrix A is *positive definite(ressp. Negative definite)* if and only if all the eigenvalues of A are *positive real number(resp. negative);*

(b) the matrix A is *positive semidefinite(ressp. Negative semidefinite)* if and only if all the eigenvalues of A are *nonnegative real number(resp. nonpositive);*

(c) the matrix A is *in definite* if and only if A has at least one positive eigenvalue and at least one negative eigenvalue.

Theorem 2.2.2.9 If $A \in M_n$ be Hermitian. Then A is positive definite if and only if $\det A_i > 0$ for $i=1, 2, \ldots ,n$. More generally, the positive of any nested sequence of n principal minors of A(not just the leading principal minors) is necessary and sufficient for A to be positive definite.

Theorem 2.2.2.10 (Hadamard's inequality) If $A = [a_{ij}] \in M_n$ is positive semidefinite, then

$$\det A \leq \prod_{i=1}^{n} a_{ii} \tag{2.15}$$

Furthermore, when A is positive definite, then equality holds if and only if A is diagonal.

2.3 Literature Reviews

2.3.1 The trace inequalities for Hermitian matrices.

These are some literature reviews of trace inequalities for Hermitian matrices which concern this thesis. In 1980, Bellman [11] had proved the trace inequality for products of Hermitian matrix, results that:

Theorem 2.3.1.1 For Hermitian matrices A and B of the same order, then

$$\operatorname{tr}(AB)^2 \leq \operatorname{tr}A^2B^2 \tag{2.16}$$

Further he asked: "Does the above inequality hold for higher order?

In 1999, Da-Wei Chang [2] had proved the inequality also partly answers a conjecture in Bellman[12], the results that :

Theorem 2.3.1.2 For Hermitian matrices A and B of the same order, then

$$\operatorname{tr}(AB)^{2^k} \leq \operatorname{tr} A^{2^k} B^{2^k} \tag{2.17}$$

where k is an integer.

In 2002, Zhong Peng Yang and Xiao Xia Peng [21] had proved inequality for the products and the powers on Hermitian matrices and skew Hermitian matrices ,results of them are:

Theorem 2.3.1.3 For Hermitian matrices A and B of the same order, then

$$\operatorname{tr}(AB)^{2k} \leq \operatorname{tr} A^{2k} B^{2k} \tag{2.18}$$

where k is an integer

Theorem 2.3.1.4

For Hermitian matrix A and skew Hermitian matrix B ,then when $m = 4t$ or $m=4t+2$, $t \in N$, $\operatorname{tr}(AB)^m$ and $\operatorname{tr}A^m B^m$ are all real numbers, and

$$\text{if } m = 4t,\ t \in N; \qquad \operatorname{tr}(AB)^m \leq \operatorname{tr}(A^2 B^2)^{\frac{m}{2}} \leq \operatorname{tr}A^m B^m \tag{2.19}$$

$$\text{if } m = 4t+2,\ t \in N; \qquad \operatorname{tr}(AB)^m \geq \operatorname{tr}(A^2 B^2)^{\frac{m}{2}} \geq \operatorname{tr}A^m B^m \tag{2.20}$$

Theorem 2.3.1.4

For skew Hermitian matrices A and B ,then when $m = 2t$, $t \in N$, $\operatorname{tr}(AB)^m$ and $\operatorname{tr}A^m B^m$ are all real numbers, and

$$\operatorname{tr}(AB)^{m} \leq \operatorname{tr}(A^{2}B^{2})^{\frac{m}{2}} \leq \operatorname{tr}A^{m}B^{m} \qquad (2.21)$$

2.3.2 The trace inequalities for Positive (semi)definite matrices.

These are some literature reviews of trace inequalities for (semi)definite matrices which concern this thesis. In 1980, Bellman [12] had already proved(very neatly) the stronger results that:

Theorem 2.3.2.1 For positive semidefinite matrices A , B of the same order ,then

$$2\operatorname{tr}(AB) \leq \operatorname{tr}(A^{2} + B^{2}) \qquad (2.22)$$

$$\operatorname{tr}(AB) \leq (\operatorname{tr} A^{2})^{\frac{1}{2}} (\operatorname{tr} B^{2})^{\frac{1}{2}} \qquad (2.23)$$

$$\operatorname{tr}(AB)^{2} \leq \operatorname{tr}(A^{2}B^{2}) \qquad (2.24)$$

In 1992, Heinz Neudecker [6] gave inequality for the product of positive (semi)definite matrices which different method of Yang's[20] . Result of them are:

Theorem 2.3.2.2 If A and B are two positive semidefinite matrices of the same order ,then

$$\operatorname{tr}(AB) \geq 0 \qquad (2.25)$$

and

$$(\operatorname{tr} AB)^{\frac{1}{2}} \leq \frac{1}{2}(\operatorname{tr} A + \operatorname{tr} B). \qquad (2.26)$$

In 1994, I.D. Coope[7] gave some trace inequalities, about alternative proofs of some simple of Bellman[12], Neudecker[6] and Yang[20] are considered and further properties of products of Hermitian and positive (semi)definite matrices are investigated. Result of him is:

Theorem 2.3.2.3 For positive semidefinite matrices A and B of the same order, then

$$0 \leq \mathrm{tr}(AB) \leq \mathrm{tr}(A)\mathrm{tr}(B) \tag{2.27}$$

if A and B are both positive definite then inequalities (2.17) be strengthened to

$$0 < \mathrm{tr}(AB) \leq \mathrm{tr}(A)\mathrm{tr}(B). \tag{2.28}$$

Theorem 2.3.2.4 If A, B are positive definite then AB has positive eigenvalues.

Theorem 2.3.2.5 The trace of a product of two Hermitian matrices of the same order is real.

In 1995, Xin Min Yang [17] gave a new proof of Y.Yang's result and generalize it to a generalized positive definite matrix. Result of him is:

Lemma 2.3.2.6 If A is a positive semidefinite matrix, then

$$\mathrm{tr}\ (A^2) \leq (\mathrm{tr}\ A)^2 \tag{2.29}$$

Lemma 2.3.2.7 If A and B are two positive definite matrices of the same order, then

$$\mathrm{tr}\ (AB) \geq 0 \tag{2.30}$$

and

$$(\mathrm{tr}\ AB)^{\frac{1}{2}} \leq \frac{1}{2}(\mathrm{tr}\ A + \mathrm{tr}\ B) \tag{2.31}$$

Lemma 2.3.2.8 Let $A \in R^{n \times n}$, $S(A) := \frac{1}{2}(A + A^T)$, and $T(A) := \frac{1}{2}(A - A^T)$, then

$$\mathrm{tr}\ S(A) = \mathrm{tr}\ A \tag{2.32}$$

andtr

$$T(A) = 0 \tag{2.33}$$

Lemma 2.3.2.9 Let $A \in R^{n\times n}$ be a symmetric matrix , $B \in R^{n\times n}$, then

$$\text{tr}\,(AB) \quad = \quad \text{tr}\,A[\,S\,(B)] \tag{2.34}$$

and

$$\text{tr}\,A[\,T\,(B)] \quad = \quad 0 \tag{2.35}$$

Theorem 2.3.2.10 Let $A \in R^{n\times n}$ is a positive semidefinite (symmetric) matrix and $B \in R^{n\times n}$ is a generalized positive semidefinite matrix, then

$$\text{tr}\,(AB) \quad \geq \quad 0 \tag{2.36}$$

and

$$(\text{tr}\,AB)^{\frac{1}{2}} \quad \leq \quad \frac{1}{2}(\text{tr}\;A \;+\; \text{tr}\;B\;) \tag{2.37}$$

Remark If A and B are two generalized positive semidefinite matrices of the same order, Theorem 2.2.2.10 is not true.

In 2000, Xiaojing Yang[16] had proved inequalities, which is a continuation of the work of Bellman[12] , Coope[7], Neudecker[6], Yang[20] and Cheng[2]. Results of him is:

Lemma 2.3.2.11 If A and B are positive semidefinite matrices of the same order , then for $n = 1, 2, \ldots$, $(AB)^n B$ and $(BA)^n B$ are positive semidefinite matrices.

Theorem 2.3.2.12 If A and B are *positive semidefinite matrices* of the same order , then for n = 1, 2, ...

$$0 \quad \leq \quad \text{tr}(AB)^{2n} \quad \leq \quad (\text{tr}\,A)^2(\text{tr}\;A^2)^{n-1}\;(\text{tr}\,B^2)^n \tag{2.38}$$

and

$$0 \quad \leq \quad \text{tr}(AB)^{2n+1} \quad \leq \quad (\text{tr}\,A)(\text{tr}\;B)(\text{tr}\,A^2)^n(\text{tr}\,B^2)^n\,. \tag{2.39}$$

Corollary 2.3.2.13 If A and B are *positive semidefinite matrices* of the same order, then for n = 1, 2, ...

$$0 \leq \operatorname{tr}(AB)^n \leq (\operatorname{tr} A)^n (\operatorname{tr} B)^n \qquad (2.40)$$

In 2001, Xin Min Yang[17] had proved the trace inequality which improves the results given by Yang, results of him is:

Theorem 2.3.2.14 Let $A \in C^{n\times n}$ and $B \in C^{n\times n}$ be positive semidefinite matrices, then

$$\operatorname{tr}(AB)^m \leq \{\operatorname{tr}(A)^{2m} \operatorname{tr}(B)^{2m}\}^{\frac{1}{2}} \qquad (2.41)$$

where m is an integer.

In 2001, Fozi M. Dannan[4] gave certain inequalities involving matrices on the Hilbert space. In particular inequalities involving the trace and the determinant of the product of positive definite matrix. Results of him is:

Theorem 2.3.2.15 If A and B are positive definite matrices , then

$$0 < \operatorname{tr}(AB)^m < [\operatorname{tr}(AB)]^m \qquad (2.42)$$

for any integer $m > 0$.

Theorem 2.3.2.16 If A and B are positive definite matrices , then

$$0 < \operatorname{tr}(AB)^m < [\operatorname{tr}(AB)^s]^{\frac{m}{s}} \qquad (2.43)$$

provided that m and s are positive integer and $m > s$.

Theorem 2.3.2.17 If A_i and B_i are positive definite for $i = 1, 2, \ldots, k$,then

$$(\operatorname{tr} \sum_{i=1}^{n} A_i B_i)^2 \leq (\operatorname{tr} \sum_{i=1}^{n} A_i^2)(\operatorname{tr} \sum_{i=1}^{n} B_i^2) \qquad (2.44)$$

if $A_i\, B_i$ are positive definite for $i = 1, 2, \ldots, k$,then

$$(\text{tr} \sum_{i=1}^{n} A_i B_i)^2 < (\text{tr} \sum_{i=1}^{n} A_i^2)(\text{tr} \sum_{i=1}^{n} B_i^2) \tag{2.45}$$

Theorem 2.3.2.18 If A_i and B_i are positive definite and $0 < A_i \le B_i$ for $i = 1, 2$,then

$$0 < A_1 B_1 \le A_2 B_2 . \tag{2.46}$$

Theorem 2.3.2.19 If A and B are positive definite , then

$$n(\det A \det B)^{\frac{m}{n}} \le \text{tr}\,(A^m B^m) \tag{2.47}$$

Corollary 2.3.2.20 If A and X are positive definite $n \times n$ matrices such that $\det X = 1$, then

$$n(\det A)^{\frac{1}{n}} \le \text{tr}\,(AX) \tag{2.48}$$

Theorem 2.3.2.21 If A and B are positive semidefinite and $AB = BA$, then

$$2^{(m-1)n} \det(A^m + B^m) \ge [\det(A + B)]^m \tag{2.49}$$

and

$$2^{(m-1)}\, tr\,(A^m + B^m) \ge tr\,(A + B)^m \tag{2.50}$$

Chapter 3

Research Methodology

3.1 Research Methodology

This thesis is an Exploratory Research. I have prove new trace inequality concerning about product and power of Hermitian matrices and skew-Hermitian matrices with order index is nonnegative integer and consider about Bellman's inequality of Hermitian matrices with order index is nonnegative integer as well as further the upper bound of that inequality.

3.2 Steps of Study

The steps of study and made this thesis as follows;

3.2.1 Find and Study researches and document that concern about trace inequality of Hermitian matrices.

I have found researches involving a matrix trace inequality base on Bellman's inequality ;

$$tr(AB)^m \leq tr A^m B^m \tag{3.1}$$

from internet, and I have found relate document from the library

3.2.2 Consider structure and property of Hermitian matrices, skew –Hermitain matrices and Hermitian matrices with condition positive definite.

I study definition , properties and characterize of Hermitian matrices , skew-Hermitian matrices and positive definite as well as study definition , properties and characterize of eigenvalues and eigenvector of positive definite matrices. Then I consider about character of power and product of those matrices with order index is nonnegative integer and comparison with order index is odd number and even number.

3.2.3 Show the Bellman's inequality of Hermitian matrices with higher order does not holds by counterexample.

In this thesis I used Math Lab version 6.1 to calculate solutions of counterexample(see example 4.1.2) to show that Bellman's inequality can not generalize to general Hermitian matrices for higher order .

3.2.4 Prove the Bellman's inequality of Hermitian matrices with higher order holds when add some conditions, direct proof and used the Arithmetic and Geometric Mean Inequality.

After I can find the counterexample to show that Bellman's inequality does not holds for higher order .But when I add some conditions that can make inequality (3.1) holds ,step of prove as follows;

(a) Find and prove Lemma 4.1.1 to prove Theorem 4.1.2 .Because it is know that the character of product and powers of Hermitian matrices is hard to verify but diagonal is to easy . I prove Lemma 4.1.1 by used definition 1.6.7.1(similarity) , characterize of eigenvalues and eigenvectors , Diagonalization of a matrix, and property of trace(2.2).

(b) Prove Theorem 4.1.2 , direct prove by used Lemma 4.1.1 and used Theorem 2.1.2.1(Arithmetic and Geometric Mean Inequality).In this step I add conditions positive definite and $x_{p(i)} \geq 0$ (see Theorem 4.1.2) for consistent conditions to used Theorem 2.1.2.1.

(c) Give Corollary 4.1.3 for special case of Theorem 4.1.2; Corollary 4.13 still have the condition positive definite but replace condition $x_{p(i)} \geq 0$ by A and B is nonnegative matrices because that also give $x_{p(i)} \geq 0$.Then direct prove Corollary 4.1.3 by the same way Theorem 4.1.2.

(d) Give some example(see example4.1.1) to support Theorem 4.1.1-4.1.2 and Corollary 4.1.3. Used Math Lab version 6.0 to calculate solutions of that example.

3.2.5 Used the Arithmetic and Geometric Mean Inequality , Theorem 2.1.2.2 ,Theorem 2.2.2.3 , property of determinant to prove Theorem 4.1.4 -4.1.7.

In this step I have found and prove new inequality involving trace and determinant . Consider about product and power of positive definite step of prove those theorem as follows;

(a) Direct prove Theorem 4.1.4. First of all I used definition 1.6.7.1(similarity) , characterize of eigenvalues and eigenvectors and Diagonalization of a matrix, property of trace(2.2) to show $\det AB$ is equal to the product of eigenvalues of A and B . Then consider about $(\det AB)^{\frac{1}{n}}$ and used Theorem 2.1.2.1(Arithmetic and Geometric Mean Inequality) , Theorem 2.2.2.3 and Theorem 2.1.2.2 to compare with $\frac{1}{n}(trA)(trB)$ .so obtain the result of Theorem 4.1.4.

(b) Prove Theorem 4.1.5 by the same way in step 3.2.5(a) by replace A and B by A^m and B^m respectively because A and B are positive definite matrices such that A^m and B^m are also positive definite matrices , then I obtain the result of Theorem 4.1.4 with higher order and further upper bound by used Theorem 2.2.2.3 ,then I obtain the result of Theorem 4.1.5.

(c) Direct prove Theorem 4.1.6. First of all I used definition 1.6.7.1(similarity) , characterize of eigenvalues and eigenvectors and Diagonalization of a matrix, property of trace(2.2) to show $\det AB$ is equal to the product of eigenvalues of A and B . Then consider about $(\det AB)^{\frac{1}{2n}}$ and used Theorem 2.1.2.1(Arithmetic and Geometric Mean Inequality) , Theorem 2.2.2.3 and Theorem 2.1.2.2 to compare with $\frac{1}{2n}(trA+trB)$ .so obtain the result of Theorem 4.1.6.

(d) Prove Theorem 4.1.7 by the same way in step 3.2.5(a) by replace A and B by A^m and B^m respectively because A and B are positive definite matrices such that A^m and B^m are also positive definite matrices , then I obtain the result of Theorem

4.1.6 with higher order and further upper bound by used Theorem 2.2.2.3 ,then I obtain the result of Theorem 4.1.7.

(e) Give some example(see example4.1.2) to support Theorem 4.1.4 -4.1.7. Used Math Lab version 6.0 to calculate solutions of that example.

3.2.6 Further upper bound of Bellman's inequality concern about product of Hermitian matices and skew-Hermitian matrices. Direct proof and used structure of those matrices.

After I obtain Theorem 4.1.2 , then I further the upper bound of that result , consider in three case ;first case consider about product of skew-Hermitian matrices , second case consider about product of Hermitian and final case consider about product of Hermitian matrices with skew-Hermitain matrices . Step of proof as follows;

First case, product of skew-Hermitian matrices A and B;

(a) Consider the character of (A - B) is skew – Hermitian , such that used that character obtain that $tr(A-B)^2 \leq 0$ .Then by direct prove I obtain Theorem 4.2.3.

(b) Direct prove Theorem 4.2.7 by used character of (A - B) is skew – Hermitian , such that used that character obtain that $tr(A-B)^2 \leq 0$ and used property 2.1.1(1-2) as well as theorem 4.2.3 ,I obtain theorem 4.2.7.

(c) Prove that Theorem 4.2.9 holds , direct prove by used character of $(A^m - B^m)$. For t $\in N$. , consider m=2t+1 , character of $(A^m - B^m)$ is skew-Hermitian matrices, obtain that $tr(A^m - B^m)^2 \leq 0$.. Used that character and prove by the same way theorem 4.2.3 ,then I obtain Theorem 4.2.9(4.15). Consider m= 2t , character of $(A^m - B^m)$ is Hermitian matrices, obtain that $tr(A^m - B^m)^2 \geq 0$ Used that character and prove by the same way theorem 4.2.4 ,then I obtain Theorem 4.2.9(4.16).

(d) Prove that Theorem 4.2.12 holds , direct prove by used character of $(A^m - B^m)$. For t $\in N$. , consider m=2t+1 , character of $(A^m - B^m)$ is skew-Hermitian matrices ,obtain that $tr(A^m - B^m)^2 \leq 0$. Used that character and prove by the same way theorem 4.2.7 ,then I obtain Theorem 4.2.12(4.19). Consider m= 2t , character of $(A^m - B^m)$ is Hermitian matrices, obtain that $tr(A^m - B^m)^2 \geq 0$.. Used that character and prove by the same way theorem 4.2.8 ,then I obtain Theorem 4.2.12(4.20).

Second case, product of Hermitian matrices *A* and *B*;

(e) Consider the character of (*A* - *B*) is Hermitian , such that used that character obtain that $tr(A-B)^2 \leq 0$ .Then by direct prove I obtain Theorem 4.2.4.

(f) Direct prove Theorem 4.2.8 by used character of (*A* - *B*) is Hermitian , such that used that character obtain that $tr(A-B)^2 \geq 0$ and used property 2.1.1(1-2) as well as theorem 4.2.4 ,I obtain theorem 4.2.8.

(g) Prove that Theorem 4.2.10 holds , direct prove by used character of $(A^m - B^m)$, when m $\in N$.., character of $(A^m - B^m)$ is Hermitian matrices, obtain that $tr(A^m - B^m)^2 \geq 0$... Used that character and prove by the same way theorem 4.2.4 ,then I obtain Theorem 4.2.10 .

(h) Prove that Theorem 4.2.13 holds , direct prove by used character of $(A^m - B^m)$. For m $\in N$. , character of $(A^m - B^m)$ is Hermitian matrices , obtain that $tr(A^m - B^m)^2 \geq 0$.. Used that character and prove by the same way theorem 4.2.8 ,then I obtain Theorem 4.2.13(4.21).

Third Case, product of Hermitian matrices *A* and skew-Hermitian matrices *B*;

(i) Used character of tr *AB* (see [4, lemma 2.5]) to show that tr *AB* can not compare with each other .Such that I obtain Theorem 4.2.5.

(j) Prove that Theorem 4.2.11 holds for , direct prove by used character of $(A^m - B^m)$. For t $\in N$. , consider m=2t+1 , Used Lemma 2.5 (see [4]) to show that $tr\, A^m B^m$ can not compare with each other . Consider m= 2t , character of

$(A^m - B^m)$ is Hermitian matrices, obtain that $tr(A^m - B^m)^2 \geq 0$ ... Used that character and prove by the same way theorem 4.2.4 ,then I obtain Theorem 4.2.11(4.18).

(k) Prove that Theorem 4.2.14 holds , direct prove by used character of $(A^m - B^m)$. For t $\in N$. , consider m=2t+1 , used Lemma 2.5 (see [4]) to show that $tr\, A^m B^m$ can not compare with each other. Consider m= 2t , character of $(A^m - B^m)$ is Hermitian matrices, obtain that $tr(A^m - B^m)^2 \geq 0$. Used that character and prove by the same way theorem 4.2.8 ,then I obtain Theorem 4.2.14(4.22).

(l) Give some example(see example4.2.1-4.2.6) to support Theorem 4.2.3-4.2.14 . Used Math Lab version 6.0 to calculate solutions of that example.

3.2.7 Summarize main results

I shall summarized what I obtain from our study trace inequality of .the product of Hermitian matrices and skew-Hermitian matrices .And give our results reach the objective of this thesis.

3.2.8 Write thesis

I used Microsoft Word to wrote this thesis. This thesis contains of 5 chapters. The first chapter is introduction. The importance and inception , objectives , scope of study , and process of study are contain in this chapter. Chapter 2 contents about some relate basic definitions involving with the study and proof of this research, properties , relate theorem and literature reviews. In this chapter contains some relate properties , relate theorem involve the prove of main result of this Thesis as well as contains some researches involve with trace inequality of the product and the powers Hermitian matrices . Chapter 3 contains research methodology and steps of study , that is about how to make this thesis .Chapter 4 is the main results , in this chapter contains some Theorem involving Trace inequality of product and power of Hermitain matrices and skew-Hermitian matrices with a finite dimensional Hilbert space and with order index is nonnegative integer, and contain the prove of above Theorems and also contain some examples to support that Theorems. Chapter 5, contains conclusion , suggestions and application , In this chapter I concluded results in chapter 4 , application and suggestions about trace inequality of Hermitian matrices.

Chapter 4

Main results

4.1 Trace inequality of power on Hermitian matrices

In this part, inequalities are presented involving the trace of Hermitian matrices. In particular inequalities involving the trace and the determinant of certain positive definite matrices.

Lemma 4.1.1 For positive definite matrices A and B of the same order $n \times n$, then

(a) $$trAB = trDX \quad (4.1)$$

(b) $$tr(AB)^m = tr(DX)^m \quad (4.2)$$

and(c) $$tr\left(A^m B^m\right) = tr\left(D^m X^m\right) \quad (4.3)$$

proof Let $A = PDP^T$, $X = P^T BP$,

$D = dig(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n)$ and $P = [u_1, u_2, u_3, \cdots, u_n]$

when λ_i are eigenvalue of A and u_i are eigenvactor of A.

consider
$$\begin{aligned} DX &= DP^T BP \\ &= IDP^T BP \\ &= P^T PDP^T BP \\ &= P^T ABP \end{aligned}$$

such that
$$\begin{aligned} trDX &= trP^T ABP \\ &= trAB \end{aligned}$$
giving (4.1)

Consider
$$\begin{aligned} tr(DX)^m &= tr\left(P^T ABP\right)^m \\ &= tr\left(P^T (AB)^m P\right) \\ &= tr(AB)^m \end{aligned}$$
giving (4.2)

Consider
$$tr\left(A^m B^m\right) = tr\left((PDP^T)^m B^m\right)$$

$$= tr\left(PD^m P^T B^m\right)$$
$$= tr\left(D^m P^T B^m P\right)$$
$$= tr\left(D^m (P^T BP)^m\right)$$
$$= tr\left(D^m X^m\right) \qquad \text{giving (4.3)} \qquad \square$$

Theorem 4.1.2 For positive definite matrices A and B of the same order $n \times n$,

If $$x_{p(i)} \geq 0 \quad ,$$

Then $$tr\,(AB)^m \leq trA^m B^m \qquad (4.4)$$

holds for m = 1, 2, 3, ...

When
$$x_{p(i)} = x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} \cdots x_{i_{m-1} i_m} x_{i_m i_1}$$
$$d_{p(i)} = d_{i_1} d_{i_2} d_{i_3} \cdots d_{i_m}$$
$$d_{p(t_k)} = d_k^m$$

$$p(i) = \left\{ i = \left(i_1, i_2, i_3, \cdots, i_m\right) \mid \forall i_j \in \left(1, 2, 3, \ldots, n\right), j = 1, 2, \cdots, m \right\}$$

and $$p(t_k) = \left\{ t = \left(t_1 = k, t_2, t_3, \cdots, t_m\right) \mid k, \forall t_j \in \left(1, 2, 3, \ldots, n\right), j = 2, \cdots, m \right\}$$

proof Let $A = PDP^T$, $X = P^T BP = \left(x_{ij}\right)_{n \times n}$ and $Y = DX$

Let

$$X = \begin{array}{c} \\ X_{1.} \\ X_{2.} \\ \vdots \\ X_{n.} \end{array} \begin{array}{c} \begin{matrix} X_{.1} & X_{.2} & \cdots & X_{.n} \end{matrix} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \end{array}$$

and

$$
Y = DX = \begin{array}{c} \\ Y_{1.} \\ Y_{2.} \\ \vdots \\ Y_{n.} \end{array}
\begin{array}{c} \begin{array}{cccc} Y_{.1} & Y_{.2} & \cdots & Y_{.n} \end{array} \\
\begin{bmatrix} d_1x_{11} & d_1x_{12} & \cdots & d_1x_{1n} \\ d_2x_{21} & d_2x_{22} & \cdots & d_2x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_nx_{n1} & d_nx_{n2} & \cdots & d_nx_{nn} \end{bmatrix} \end{array}
$$

From lemma 4.1.1(b) we have

$$
tr(AB)^m = tr(DX)^m
$$

Then ,consider

$$
Y^2 = \begin{array}{c} \\ Y_{1.}^{(2)} \\ Y_{2.}^{(2)} \\ \vdots \\ Y_{n.}^{(2)} \end{array}
\begin{array}{c} \begin{array}{cccc} Y_{.1}^{(2)} & Y_{.2}^{(2)} & \cdots & Y_{.n}^{(2)} \end{array} \\
\begin{bmatrix} \langle Y_{1.}^T, Y_{.1} \rangle & \langle Y_{1.}^T, Y_{.2} \rangle & \cdots & \langle Y_{1.}^T, Y_{.n} \rangle \\ \langle Y_{2.}^T, Y_{.1} \rangle & \langle Y_{2.}^T, Y_{.2} \rangle & \cdots & \langle Y_{2.}^T, Y_{.n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle Y_{n.}^T, Y_{.1} \rangle & \langle Y_{n.}^T, Y_{.2} \rangle & \cdots & \langle Y_{n.}^T, Y_{.n} \rangle \end{bmatrix} \end{array}
$$

and

$$
Y^3 = \begin{array}{c} \\ Y_{1.}^{(3)} \\ Y_{2.}^{(3)} \\ \vdots \\ Y_{n.}^{(3)} \end{array}
\begin{array}{c} \begin{array}{cccc} Y_{.1}^{(3)} & Y_{.2}^{(3)} & \cdots & Y_{.n}^{(3)} \end{array} \\
\begin{bmatrix} \langle Y_{1.}^T, Y_{.1}^{(2)} \rangle & \langle Y_{1.}^T, Y_{.2}^{(2)} \rangle & \cdots & \langle Y_{1.}^T, Y_{.n}^{(2)} \rangle \\ \langle Y_{2.}^T, Y_{.1}^{(2)} \rangle & \langle Y_{2.}^T, Y_{.2}^{(2)} \rangle & \cdots & \langle Y_{2.}^T, Y_{.n}^{(2)} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle Y_{n.}^T, Y_{.1}^{(2)} \rangle & \langle Y_{n.}^T, Y_{.2}^{(2)} \rangle & \cdots & \langle Y_{n.}^T, Y_{.n}^{(2)} \rangle \end{bmatrix} \end{array}
$$

By the same fashion , we have

$$Y^m = \begin{array}{c} \\ Y_{1.}^{(m)} \\ Y_{2.}^{(m)} \\ \vdots \\ Y_{n.}^{(m)} \end{array} \begin{array}{c} \begin{array}{cccc} Y_{.1}^{(m)} & Y_{.2}^{(m)} & \cdots & Y_{.n}^{(m)} \end{array} \\ \begin{bmatrix} \left\langle Y_{1.}^T, Y_{.1}^{(m-1)} \right\rangle & \left\langle Y_{1.}^T, Y_{.2}^{(m-1)} \right\rangle & \cdots & \left\langle Y_{1.}^T, Y_{.n}^{(m-1)} \right\rangle \\ \left\langle Y_{2.}^T, Y_{.1}^{(m-1)} \right\rangle & \left\langle Y_{2.}^T, Y_{.2}^{(m-1)} \right\rangle & \cdots & \left\langle Y_{2.}^T, Y_{.n}^{(m-1)} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle Y_{n.}^T, Y_{.1}^{(m-1)} \right\rangle & \left\langle Y_{n.}^T, Y_{.2}^{(m-1)} \right\rangle & \cdots & \left\langle Y_{n.}^T, Y_{.n}^{(m-1)} \right\rangle \end{bmatrix} \end{array}$$

Then

$$\begin{aligned} tr(AB)^m &= tr(DX)^m \\ &= \sum_{i=1}^{n} \left\langle Y_{i.}^T, Y_{.i}^{(m-1)} \right\rangle \\ &= \sum_{p(i)} d_{p(i)} x_{p(i)} \\ &= \sum_{p(i)} (d_{i_1} d_{i_2} d_{i_3} \cdots d_{i_m}) x_{p(i)} \end{aligned} \qquad (*)$$

And from lemma 4.1.1(c) we also have

$$tr\left(A^m B^m\right) = tr\left(D^m X^m\right)$$

Then, consider

$$X^2 = \begin{array}{c} \\ X_{1.}^{(2)} \\ X_{2.}^{(2)} \\ \vdots \\ X_{n.}^{(2)} \end{array} \begin{array}{c} \begin{array}{cccc} X_{.1}^{(2)} & X_{.2}^{(2)} & \cdots & X_{.n}^{(2)} \end{array} \\ \begin{bmatrix} \left\langle X_{1.}^T, X_{.1} \right\rangle & \left\langle X_{1.}^T, X_{.2} \right\rangle & \cdots & \left\langle X_{1.}^T, X_{.n} \right\rangle \\ \left\langle X_{2.}^T, X_{.1} \right\rangle & \left\langle X_{2.}^T, X_{.2} \right\rangle & \cdots & \left\langle X_{2.}^T, X_{.n} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle X_{n.}^T, X_{.1} \right\rangle & \left\langle X_{n.}^T, X_{.2} \right\rangle & \cdots & \left\langle X_{n.}^T, X_{.n} \right\rangle \end{bmatrix} \end{array}$$

then

$$X^3 = \begin{array}{c} \\ X_{1.}^{(3)} \\ X_{2.}^{(3)} \\ \vdots \\ X_{n.}^{(3)} \end{array} \begin{array}{c} \begin{array}{cccc} X_{.1}^{(3)} & X_{.2}^{(3)} & \cdots & X_{.n}^{(3)} \end{array} \\ \begin{bmatrix} \left\langle X_{1.}^T, X_{.1}^{(2)} \right\rangle & \left\langle X_{1.}^T, X_{.2}^{(2)} \right\rangle & \cdots & \left\langle X_{1.}^T, X_{.n}^{(2)} \right\rangle \\ \left\langle X_{2.}^T, X_{.1}^{(2)} \right\rangle & \left\langle X_{2.}^T, X_{.2}^{(2)} \right\rangle & \cdots & \left\langle X_{2.}^T, X_{.n}^{(2)} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle X_{n.}^T, X_{.1}^{(2)} \right\rangle & \left\langle X_{n.}^T, X_{.2}^{(2)} \right\rangle & \cdots & \left\langle X_{n.}^T, X_{.n}^{(2)} \right\rangle \end{bmatrix} \end{array}$$

By the same fashion , we have

$$X^m = \begin{array}{c} \\ X_{1.}^{(m)} \\ X_{2.}^{(m)} \\ \vdots \\ X_{n.}^{(m)} \end{array} \begin{array}{c} \begin{array}{cccc} X_{.1}^{(m)} & X_{.2}^{(m)} & \cdots & X_{.n}^{(m)} \end{array} \\ \begin{bmatrix} \left\langle X_{1.}^T, X_{.1}^{(m-1)} \right\rangle & \left\langle X_{1.}^T, X_{.2}^{(m-1)} \right\rangle & \cdots & \left\langle X_{1.}^T, X_{.n}^{(m-1)} \right\rangle \\ \left\langle X_{2.}^T, X_{.1}^{(m-1)} \right\rangle & \left\langle X_{2.}^T, X_{.2}^{(m-1)} \right\rangle & \cdots & \left\langle X_{2.}^T, X_{.n}^{(m-1)} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle X_{n.}^T, X_{.1}^{(m-1)} \right\rangle & \left\langle X_{n.}^T, X_{.2}^{(m-1)} \right\rangle & \cdots & \left\langle X_{n.}^T, X_{.n}^{(m-1)} \right\rangle \end{bmatrix} \end{array}$$

Let $$D = dig(d_1, d_2, d_3, \cdots, d_n)$$

Then
$$D^m X^m = \begin{bmatrix} d_1^m & 0 & 0 & \cdots & 0 \\ 0 & d_2^m & 0 & \cdots & \vdots \\ 0 & 0 & d_3^m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & d_n^m \end{bmatrix} X^m$$

$$= \begin{bmatrix} d_1^m \left\langle X_{1.}^T, X_{.1}^{(m-1)} \right\rangle & d_1^m \left\langle X_{1.}^T, X_{.2}^{(m-1)} \right\rangle & \cdots & d_1^m \left\langle X_{1.}^T, X_{.n}^{(m-1)} \right\rangle \\ d_2^m \left\langle X_{2.}^T, X_{.1}^{(m-1)} \right\rangle & d_2^m \left\langle X_{2.}^T, X_{.2}^{(m-1)} \right\rangle & \cdots & d_2^m \left\langle X_{2.}^T, X_{.n}^{(m-1)} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ d_n^m \left\langle X_{n.}^T, X_{.1}^{(m-1)} \right\rangle & d_n^m \left\langle X_{n.}^T, X_{.2}^{(m-1)} \right\rangle & \cdots & d_n^m \left\langle X_{n.}^T, X_{.n}^{(m-1)} \right\rangle \end{bmatrix}$$

Such that $$tr\left(A^m B^m\right) = tr\left(D^m X^m\right)$$

$$trD^m X^m = \sum_{i=1}^{n} d_i^m \left\langle X_{i.}^T, X_{.i}^{(m-1)} \right\rangle$$

$$= \sum_{k=1}^{n} \sum_{p(t_k)} d_{t_k} \, x_{p(t_k)}$$

$$= \sum_{k=1}^{n} \sum_{p(t_k)} \frac{1}{m}(d_{k(1)}^m + d_{k(2)}^m + \cdots + d_{k(m)}^m) \, x_{p(t_k)}$$

$$= \sum_{p(i)} \frac{1}{m}(d_{i_1}^m + d_{i_2}^m + d_{i_3}^m + \cdots + d_{i_m}^m) x_{p(i)} \qquad (**)$$

From (∗) , (∗∗) and Theorem 2.1.2.1 [(A-G) inequality]

Such that
$$tr(AB)^m = \sum_{p(i)} (d_{i_1} d_{i_2} d_{i_3} \cdots d_{i_m}) x_{p(i)}$$
$$\le \sum_{p(i)} \frac{1}{m}(d_{i_1}^m + d_{i_2}^m + d_{i_3}^m + \cdots + d_{i_m}^m) x_{p(i)}$$
$$= tr\left(A^m B^m\right)$$

giving (4.4) □

Corollary 4.1.3 For positive definite matrices A and B of the same order $n \times n$, If A and B are Nonnegative matrices (4.4) holds for any m = 1, 2, 3, ...

Proof Let $A = PDP^T$ and $X = P^T BP = \left(x_{ij}\right)_{n\times n}$

Since A and B are Nonnegative matrices. So we have know that $x_{p(i)} \ge 0$.

By Theorem 4.1.2 we get the result. □

Example 4.1.1

Let
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

It is know that
$$AB = \begin{bmatrix} 4 & 3 \\ 7 & 9 \end{bmatrix} ; (AB)^2 = \begin{bmatrix} 37 & 39 \\ 91 & 102 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 5 \\ 5 & 17 \end{bmatrix} \text{ and } B^2 = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix} ; (A^2B^2) = \begin{bmatrix} 45 & 35 \\ 135 & 110 \end{bmatrix}$$

Such that
$$tr(AB)^2 = 139 < 155 = tr(A^2B^2)$$

Remark 4.1 Inequality $tr(AB)^m \le trA^m B^m$ can not generalize to generalized Hermitian matrix for any integer m .The following counterexample show.

Example 4.1.2

Let $A = \begin{bmatrix} -2 & -i \\ i & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix}$

Then $tr(AB)^3 = -132 > -458 = trA^3B^3$ □

Theorem 4.1.4 For positive definite matrices A and B of the same order $n\times n$, then

$$n\left(\det AB\right)^{\frac{1}{n}} \leq (trA)(trB) \tag{4.5}$$

proof Let $A = PDP^T$, $B=SCS^T$

$D=dig(\lambda_1, \lambda_2, \lambda_3,\cdots.\lambda_n)$, $C=dig(\beta_1, \beta_2, \beta_3,\cdots.\beta_n)$

$P=[u_1, u_2, u_3,\cdots,u_n]$ and $C=[v_1, v_2, v_3,\cdots,v_n]$

when λ_i and β_i are eigenvalue of A and B respectively

P and S are eigenvactor of A and B respectively.

Consider

$$\begin{aligned}
\det AB &= (\det A)(\det B) \\
&= \det(PDP^T)\det(SCS^T) \\
&= \det(DP^TP)\det(CS^TS) \\
&= \det(DI)\det(CI) \\
&= (\prod_{i=1}^{n}\lambda_i)(\prod_{i=1}^{n}\beta_i) \\
&= (\prod_{i=1}^{n}\lambda_i\beta_i)
\end{aligned}$$

such that

$$\begin{aligned}
(\det AB)^{\frac{1}{n}} &= (\prod_{i=1}^{n}\lambda_i\beta_i)^{\frac{1}{n}} \\
&\leq \frac{1}{n}(\sum_{i=1}^{n}\lambda_i\beta_i) && \textit{(by Theorem 2.1.2.1)} \\
&= \frac{1}{n}(\mathrm{tr}\ DC) \\
&\leq \frac{1}{n}(\mathrm{tr}\ D)(\mathrm{tr}\ C) && \textit{(by Theorem 2.2.2.3)} \\
&= \frac{1}{n}(\mathrm{tr}\ A)(\mathrm{tr}\ B) && \textit{(by Theorem 2.1.2.2)}
\end{aligned}$$

giving (4.1). □

Theorem 4.1.5 For positive definite matrices A and B of the same order $n\times n$, then

$$n\left(\det A^m B^m\right)^{\frac{1}{n}} \le \left(trA^m\right)\left(trB^m\right) \le (trA)^m(trB)^m \qquad (4.6)$$

holds for any m = 1, 2, 3, . . .

proof Since A and B are positive definite matrices. Such that A^m and B^m are also positive definite matrices. By Theorem 4.1.4 giving

$$n\left(\det A^m B^m\right)^{\frac{1}{n}} \le \left(trA^m\right)\left(trB^m\right)$$

$$\le (trA)^m(trB)^m \quad \text{(By Theorem 2.2.2.3)}$$

giving (4.6) □

Theorem 4.1.6 For positive definite matrices A and B of the same order $n\times n$, then

$$2n\left(\det AB\right)^{\frac{1}{2n}} \le trA + trB \qquad (4.7)$$

proof Let $A = PDP^T$, $B=SCS^T$

$D=dig(\lambda_1, \lambda_2, \lambda_3,\cdots,\lambda_n)$, $C=dig(\beta_1,\beta_2,\beta_3,\cdots,\beta_n)$

$P=[u_1, u_2, u_3,\cdots,u_n]$ and $C=[v_1, v_2, v_3,\cdots,v_n]$

when λ_i and β_i are eigenvalue of A and B respectively

P and S are eigenvactor of A and B respectively.

Consider

$$\begin{aligned} \det AB &= (\det A)(\det B) \\ &= \det(PDP^T)\det(SCS^T) \\ &= \det(DP^TP)\det(CS^TS) \\ &= \det(DI)\det(CI) \\ &= \left(\prod_{i=1}^n \lambda_i\right)\left(\prod_{i=1}^n \beta i_i\right) \end{aligned}$$

such that

$$\begin{aligned} (\det AB)^{\frac{1}{2n}} &= \left[\left(\prod_{i=1}^n \lambda_i\right)\left(\prod_{i=1}^n \beta i_i\right)\right]^{\frac{1}{2n}} \\ &\le \frac{1}{2n}\left(\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \beta_i\right) \qquad \textit{(by Theorem 2.1.2.1)} \end{aligned}$$

$$= \frac{1}{2n}(trD + trC)$$

$$= \frac{1}{2n}(trA + trB) \qquad \textit{(by Theorem 2.1.2.2)}$$

giving (4.7). □

Theorem 4.1.7 For positive definite matrices A and B of the same order $n \times n$, then

$$2n\left(\det A^m B^m\right)^{\frac{1}{2n}} \leq trA^m + trB^m \leq (trA)^m + (trB)^m \tag{4.8}$$

holds for any m = 1, 2, 3, . . .

proof Since A and B are positive definite matrices. Such that A^m and B^m are also positive definite matrices. By Theorem 4.1.6 giving

$$2n\left(\det A^m B^m\right)^{\frac{1}{2n}} \leq trA^m + trB^m$$

$$\leq (trA)^m + (trB)^m \qquad \text{(By Theorem 2.2.2.3)}$$

giving (4.6) □

Example 4.1.2

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

It is know that $AB = \begin{bmatrix} 4 & 3 \\ 7 & 9 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & 5 \\ 5 & 17 \end{bmatrix} \quad \textit{and} \quad B^2 = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$$

and n=2

so that $2(\det AB)^{\frac{1}{2}} = 7.746 < 25 = (trA)(trB)$

$$2(\det A^2B^2)^{\frac{1}{2}} = 30 < 285 = (trA^2)(trB^2) < 625 = (trA)^2(trB)^2$$

and

$$4(\det AB)^{\frac{1}{4}} = 7.872 < 10 = (trA + trB)$$

$$4(\det A^2B^2)^{\frac{1}{4}} = 15.4919 < 34 = (trA^2 + trB^2) < 50 = (trA)^2 + (trB)^2$$

4.2 Trace of the power on the Hermitian matrix and skew Hermitian Matrix

In this part inequalities are presented involving the trace of Hermitian matrices and skew Hermitian matrices.

Lemma 4.2.1 For Hermitian matrix $A = [a_{ij}]_{n\times n}$, then

$$trA^2 \geq 0 \qquad (4.9)$$

Proof Consider Its power ;

Let

$$A = \begin{matrix} & A_{.1} & A_{.2} & \cdots & A_{.n} \\ A_{1.} & a_{11} & a_{12} & \cdots & a_{1n} \\ A_{2.} & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n.} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$$

Since A is Hermitian matrix , we have know that

$$A_{1.} = A^*_{.1}$$

$$A_{2.} = A^*_{.2}$$

$$A_{3.} = A^*_{.3}$$

$$\vdots$$

$$A_{n.} = A^*_{.n}$$

such that $A^T_{i.} = \overline{A_{.1}}$ for i = 1, 2, ..., n (*)

Consider
$$A^2 = \begin{bmatrix} \left\langle A_{1.}^T, A_{.1} \right\rangle & \left\langle A_{1.}^T, A_{.2} \right\rangle & \cdots & \left\langle A_{1.}^T, A_{.n} \right\rangle \\ \left\langle A_{2.}^T, A_{.1} \right\rangle & \left\langle A_{2.}^T, A_{.2} \right\rangle & \cdots & \left\langle A_{n.}^T, A_{.n} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle A_{n.}^T, A_{.1} \right\rangle & \left\langle A_{n.}^T, A_{.2} \right\rangle & \cdots & \left\langle A_{n.}^T, A_{.n} \right\rangle \end{bmatrix}$$

Then

$$\begin{aligned} trA^2 &= \sum_{i=1}^{n} \left\langle A_{i.}^T, A_{.i} \right\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} a_{ji} \qquad \text{from}(*) \\ &\geq 0 \end{aligned}$$

giving (4.9) □

Lemma 4.2.2 For skew-Hermitian matrix $A = [a_{ij}]_{n \times n}$, then

$$trA^2 \leq 0 \tag{4.10}$$

Proof Consider Its power ;

Let
$$A = \begin{matrix} & A_{.1} & A_{.2} & \cdots & A_{.n} \\ A_{1.} & a_{11} & a_{12} & \cdots & a_{1n} \\ A_{2.} & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n.} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$$

Since A is skew-Hermitian matrix , we have know that

$$A_{1.} = -A_{.1}^{*}$$

$$A_{2.} = -A_{.2}^{*}$$

$$A_{3.} = -A_{.3}^{*}$$

$$\vdots$$

$$A_{n.} = -A_{.n}^{*}$$

such that $A_{i.}^{T} = -(\overline{A_{.1}})$ for i = 1, 2, …, n (∗)

Consider
$$A^2 = \begin{bmatrix} \langle A_{1.}^{T}, A_{.1}\rangle & \langle A_{1.}^{T}, A_{.2}\rangle & \cdots & \langle A_{1.}^{T}, A_{.n}\rangle \\ \langle A_{2.}^{T}, A_{.1}\rangle & \langle A_{2.}^{T}, A_{.2}\rangle & \cdots & \langle A_{n.}^{T}, A_{.n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle A_{n.}^{T}, A_{.1}\rangle & \langle A_{n.}^{T}, A_{.2}\rangle & \cdots & \langle A_{n.}^{T}, A_{.n}\rangle \end{bmatrix}$$

Then

$$trA^2 = \sum_{i=1}^{n}\langle A_{i.}^{T}, A_{.i}\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \qquad \text{from}(*)$$

$$\leq 0$$

giving (4.10) □

Theorem 4.2.3 For skew Hermitian matrices A and B of the same order , then

$$\text{tr}(AB) \geq \frac{1}{2}\text{tr}(A^2 + B^2) \tag{4.11}$$

Proof Since A and B are skew Hermitian matrix.

Such that $(A - B)$ is skew Hermitian matrix .From lemma 4.2.2, It is known that

$$\text{tr}(A - B)^2 \leq 0 \qquad (*)$$

Consider

$$\text{tr}(A - B)^2 = \text{tr}(A^2 + B^2 - AB - BA)$$

$$= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) - \operatorname{tr}(AB) - \operatorname{tr}(BA)$$

$$= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) - \operatorname{tr}(AB) - \operatorname{tr}(AB)$$

$$= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) - 2\operatorname{tr}(AB)$$

From ($*$),such that

$$2\operatorname{tr}(AB) \geq \operatorname{tr}(A^2 + B^2)$$

$$\operatorname{tr}(AB) \geq \frac{1}{2}\operatorname{tr}(A^2 + B^2)$$

giving (4.11) □

Theorem 4.2.4 For Hermitian matrices A and B of the same order , then

$$\operatorname{tr}(AB) \leq \frac{1}{2}\operatorname{tr}(A^2 + B^2) \qquad (4.12)$$

Proof Since A and B are Hermitian matrix.

Such that $(A - B)$ is Hermitian matrix .From lemma 4.2.1,it is known that

$$\operatorname{tr}(A - B)^2 \geq 0 \qquad (*)$$

Consider

$$\operatorname{tr}(A - B)^2 = \operatorname{tr}(A^2 + B^2 - AB - BA)$$

$$= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) - \operatorname{tr}(AB) - \operatorname{tr}(BA)$$

$$= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) - \operatorname{tr}(AB) - \operatorname{tr}(AB)$$

$$= \operatorname{tr}(A^2) + \operatorname{tr}(B^2) - 2\operatorname{tr}(AB)$$

From ($*$),such that

$$2\operatorname{tr}(AB) \leq \operatorname{tr}(A^2 + B^2)$$

$$\operatorname{tr}(AB) \leq \frac{1}{2}\operatorname{tr}(A^2 + B^2)$$

giving (4.12) □

Theorem 4.2.5 A and B are Hermitian matrix and skew Hermitian matrix respectively tr AB are zero or pure imaginary numbers can't be compared with each other.

Proof Since A and B are Hermitian matrix and skew Hermitian matrix respectively, tr AB are zero or pure imaginary numbers (see [4, Lemma 2.5]) . □

Remark 4.2.6 From Example 4.2.1 we have $tr\,AB = -3 < -2 = trA \cdot trB$,then Example 4.2.1 indicates that $0 \leq trAB \leq trAtrB$ for positive definite matrices (see[6]) cannot be generalized to general skew –Hermitian matrices.

Theorem 4.2.7 For skew-Hermitian matrices A and B of the same order, then

$$trAB \geq \frac{1}{4}\,tr(A+B)^2 \geq \frac{1}{2}tr(A^2+B^2) \tag{4.13}$$

Proof Since $(A-B)$ is skew-Hermitian matrix .From lemma 4.2.2, it is know that

$$tr(A-B)^2 \leq 0 \qquad (*)$$

Consider

$$\begin{aligned} tr(A-B)^2 &= tr(A^2+B^2-AB-BA) \\ &= tr(A^2+B^2)-trAB-trBA && \text{(by (2.1))} \\ &= tr(A^2+B^2)-2trAB && \text{(by (2.2))} \\ &= tr(A^2+B^2)-2trAB+(2trBA-2trAB) \\ &= tr(A^2+B^2)+2trAB-4trAB \\ &= tr(A+B)^2 - 4trAB \end{aligned}$$

from $(*)$ we have

$$trAB \geq \frac{1}{4}tr(A+B)^2 \qquad (**)$$

Consider $(A+B)^2 = (A^2+B^2+AB+BA)$

Then

$$\begin{aligned} \frac{1}{4}tr(A+B)^2 &= \frac{1}{4}(trA^2+trB^2+2trAB) \\ &\geq \frac{1}{4}(trA^2+trB^2+trA^2+trB^2) && \text{(by Theorem4.2.3)} \\ &= \frac{1}{2}(trA^2+trB^2) && (***) \end{aligned}$$

On combining $(**)$ with $(***)$ giving (4.13) □

Theorem 4.2.8 For Hermitian matrices A and B of the same order, then

$$trAB \leq \frac{1}{4}\,tr(A+B)^2 \leq \frac{1}{2}tr(A^2+B^2) \tag{4.14}$$

Proof Since $(A-B)$ is Hermitian matrix and from lemma 4.2.1, it is know that

$$tr(A-B)^2 \geq 0 \qquad (*)$$

Consider
$$\begin{aligned} tr(A-B)^2 &= tr(A^2+B^2-AB-BA) \\ &= tr(A^2+B^2)-trAB-trBA && \text{(by (2.1))} \\ &= tr(A^2+B^2)-2trAB && \text{(by (2.2))} \\ &= tr(A^2+B^2)-2trAB+(2trBA-2trAB) \\ &= tr(A^2+B^2)+2trAB-4trAB \\ &= tr(A+B)^2 - 4trAB \end{aligned}$$

from $(*)$ we have

$$trAB \leq \frac{1}{4}tr(A+B)^2 \qquad (**)$$

Consider
$$(A+B)^2 = (A^2+B^2+AB+BA)$$

Then
$$\begin{aligned} \frac{1}{4}tr(A+B)^2 &= \frac{1}{4}(trA^2+trB^2+2trAB) \\ &\leq \frac{1}{4}(trA^2+trB^2+trA^2+trB^2) && \text{(by Theorem4.2.4)} \\ &= \frac{1}{2}(trA^2+trB^2) && (***) \end{aligned}$$

On combining $(**)$ with $(***)$ giving (4.14) □

Example 4.2.1

Let
$$A = \begin{bmatrix} i & -1+i \\ 1+i & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$$

and so from

$$AB = \begin{bmatrix} -2+i & -1-2i \\ -1+i & -1-i \end{bmatrix}, \quad (A+B)^2 = \begin{bmatrix} -9 & -3-6i \\ -3+6i & -6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -3 & -1-i \\ -1+i & -2 \end{bmatrix} \quad \text{and} \quad B^2 = \begin{bmatrix} -2 & -2i \\ 2i & -2 \end{bmatrix}$$

It follows that

$$\text{tr } AB = -3 > -3.75 = \frac{1}{4}tr(A+B)^2 > -4.5 = \frac{1}{2}\text{tr}(A^2+B^2)$$

Example 4.2.2

Let $A = \begin{bmatrix} -1 & -1-i \\ -1+i & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix}$

and so from

$$AB = \begin{bmatrix} 2-i & 1+2i \\ 1-i & 1+i \end{bmatrix}, \ (A+B)^2 = \begin{bmatrix} 9 & 3+6i \\ 3-6i & 6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 1+i \\ 1-i & 2 \end{bmatrix} \quad \text{and} \quad B^2 = \begin{bmatrix} 2 & 2i \\ -2i & 2 \end{bmatrix}$$

It follows that

$$\text{tr}\, AB = 3 < 3.75 = \frac{1}{4} tr(A+B)^2 < 4.5 = \frac{1}{2}\text{tr}(A^2 + B^2)$$

Example 4.2.3

Let $A = \begin{bmatrix} i & -1+i \\ 1+i & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

and so from

$$AB = \begin{bmatrix} 1+2i & -3+2i \\ 1+i & -1+i \end{bmatrix}$$

It follows that $\text{tr}\, AB = 3i$.

Theorem 4.2.9 For skew-Hermitian matrices A and B of the same order,

If $m = 2t$,

then $$tr\, A^m B^m \le \frac{1}{2} tr(A^{2m} + B^{2m}) \tag{4.15}$$

and If $m = 2t+1$,

then $$tr\, A^m B^m \ge \frac{1}{2} tr(A^{2m} + B^{2m}) \tag{4.16}$$

When t is nonnegative integer.

Proof

Case m=2t+1 ; it is know that A^m and B^m also skew-Hermitian matrices.

Then by Theorem 4.2.3 giving (4.15).

Case m=2t ; it is know that A^m and B^m are Hermitian matrices.

Then by Theorem 4.2.4 giving (4.16). □

Theorem 4.2.10 For Hermitian matrices A and B of the same order, then

$$tr\,A^m B^m \le \frac{1}{2} tr(A^{2m} + B^{2m}) \tag{4.17}$$

for any integer m = 1, 2, 3, ...

Proof Since A and B are Hermitian matrices, it is know that A^m and B^m are Hermitian matrices.Then by Theorem 4.2.4 giving (4.17). □

Theorem 4.2.11 For Hermitian matrix A and skew-Hermitian matrix B of the same order, If m = 2t ,then

$$tr\,A^m B^m \le \frac{1}{2} tr(A^{2m} + B^{2m}) \tag{4.18}$$

When t is nonnegative integer. Further, when m= 2t+1 , $tr\,A^m B^m$ are all zeros or pure imaginary numbers cannot be compared with each other.

Proof

Case m=2t+1,

It is know that tr $A^m B^m$ are zero or pure imaginary numbers (see [4,Lemma 2.5]) .

Case m=2t,

Since A and B are Hermitian matrix and skew-Hermitian matrix respectively. Then A^m and B^m are Hermitian matrices, from lemma 4.2.1.

Such that $$tr(A^m - B^m)^2 \ge 0$$

Hence by Theorem 4.2.4 giving (4.18)

Theorem 4.2.12 For skew-Hermitian matrices A and B of the same order,

If m = 2t ,

then $$tr\,A^m B^m \le \frac{1}{4} tr(A^m + B^m)^2 \le \frac{1}{2} tr(A^{2m} + B^{2m}) \qquad (4.19)$$

and If m = 2t+1 ,

then $$tr\,A^m B^m \ge \frac{1}{4} tr(A^m + B^m)^2 \ge \frac{1}{2} tr(A^{2m} + B^{2m}) \qquad (4.20)$$

When t is nonnegative integer.

Proof

Case m = 2t+1 ,

it is know that A^m and B^m is skew-Hermitian matrices and from lemma 4.2.2.

So that $$tr(A^m - B^m)^2 \le 0$$

Hence by Theorem 4.2.7 and Theorem 4.2.9(4.16) giving (4.20).

Case m=2t ,

it is know that A^m and B^m is Hermitian matrices and from lemma 4.2.1 .

So that $$tr(A^m - B^m)^2 \ge 0$$

Hence by Theorem 4.2.8 and Theorem 4.2.9(4.15) giving (4.19) □

Theorem 4.2.13 For Hermitian matrices A and B of the same order, then

$$tr\,A^m B^m \le \frac{1}{4} tr(A^m + B^m)^2 \le \frac{1}{2} tr(A^{2m} + B^{2m}) \qquad (4.21)$$

For any nonnegative integer t = 1, 2, 3, . . .

Proof Since A and B are Hermitian matrices, it is know that A^m and B^m are also Hermitian matrices and from lemma 4.2.1.

So that $$tr(A^m - B^m)^2 \ge 0$$

Hence by Theorem 4.2.8 and Theorem 4.2.10 giving (4.21) □

Theorem 4.2.14 For Hermitian matrix A and skew-Hermitian matrix B of the same order,

If m = 2t ,

then $$tr\,A^m B^m \le \frac{1}{4} tr(A^m + B^m)^2 \le \frac{1}{2} tr(A^{2m} + B^{2m}) \qquad (4.22)$$

when t is nonnegative integer. Further, when m= 2t+1 , $tr\,A^m B^m$ are all zeros or pure imaginary numbers cannot be compared with each other.

Proof

Case m=2t+1,

It is know that tr $A^m B^m$ are zero or pure imaginary numbers (see [4,Lemma 2.5]) .

Case m=2t,

it is know that A^m and B^m is Hermitian matrices and from lemma 4.2.1.

So that $$tr(A^m - B^m)^2 \ge 0$$

Hence by Theorem 4.2.8 and Theorem 4.2.11 giving (4.22) □

Example 2.2.4(Product of skew-Hermitian matrices)

Let $$A = \begin{bmatrix} i & 2 \\ -2 & 2i \end{bmatrix}, \quad B = \begin{bmatrix} i & 3+i \\ -3+i & i \end{bmatrix}$$

And so from

$$(AB)^3 = \begin{bmatrix} -1179+414i & -207+1035i \\ -414-1656i & -1386-414i \end{bmatrix},$$

$$(A^3B^3) = \begin{bmatrix} -1385+286i & -221+1345i \\ -364-1774i & -1726-286i \end{bmatrix}$$

$$(A^3 + B^3)^2 = \begin{bmatrix} -6194 & -1391+6527i \\ -1391-6527i & -7371 \end{bmatrix}$$

$$(A^6+B^6)=\begin{bmatrix} -3424 & -860+3408i \\ -806-3408i & -3919 \end{bmatrix}$$

$$(AB)^4=\begin{bmatrix} 16119-5670i & 2835+14175i \\ 5670+22680i & 18954-5670i \end{bmatrix},$$

$$(A^4B^4)=\begin{bmatrix} 20117-3432i & 2684-20610i \\ 4400+25758i & 26396+3432i \end{bmatrix}$$

$$(A^4+B^4)^2=\begin{bmatrix} 95320 & 21250-101430i \\ 21250+101430i & 114160 \end{bmatrix}$$

$$(A^8+B^8)=\begin{bmatrix} 55086 & 14168-55062i \\ 14168+55062i & 61365 \end{bmatrix}$$

It follows that

$$\begin{aligned} tr(AB)^3 = -2565 > \ & tr(A^3B^3) = -3111 \\ > \ & \frac{1}{4}tr(A^3+B^3)^2 = -3391.3 \\ > \ & \frac{1}{2}tr(A^6+B^6) = -3671.5 \end{aligned}$$

and

$$\begin{aligned} tr(AB)^4 = 35073 < \ & tr(A^4B^4) = 46513 \\ < \ & \frac{1}{4}tr(A^4+B^4)^2 = 52370 \\ < \ & \frac{1}{2}tr(A^8+B^8) = 58225 \end{aligned}$$

Example 2.2.5(Product of Hermitian matrices)

Let $A=\begin{bmatrix} -1 & 2i \\ -2i & -2 \end{bmatrix}$, $B=\begin{bmatrix} -1 & -1+3i \\ -1-3i & -1 \end{bmatrix}$

It follows that

$$\begin{aligned} tr(AB)^4 = 35073 < \ & tr(A^4B^4) = 46513 \\ < \ & \frac{1}{4}tr(A^4+B^4)^2 = 52370 \\ < \ & \frac{1}{2}tr(A^8+B^8) \ = 58225 \end{aligned}$$

Example 2.2.6(Product of Hermitian matrices and skew-Hermitian matrices)

Let $A=\begin{bmatrix} -1 & 2i \\ -2i & -2 \end{bmatrix}$, $B=\begin{bmatrix} i & 3+i \\ -3+i & i \end{bmatrix}$

It follows that

$$\begin{aligned} tr(AB)^4 = 35073 < \ & tr(A^4B^4) = 46513 \\ < \ & \frac{1}{4}tr(A^4+B^4)^2 = 52370 \\ < \ & \frac{1}{2}tr(A^8+B^8) \ = 58225 \end{aligned}$$

and

$$tr(AB)^3 = 2565i \quad and \quad tr(A^3B^3) = 3111i$$

can not compare with each other.

Chapter 5

Conclusion and Suggestion

In this part I concluded the resultes in Chater 4 , application and suggestion about trace inequality of Hermitian matrix

5.1 Conclusion

In this thesis I let A and B are square matrices on a finite dimentional Hilbert space, m ≥ 1 and t ≥ 1. We have the main resultes;

(5.1.1) For positive definite matrices A and B of the same order $n \times n$,

If
$$x_{p(i)} \geq 0 ,$$

Then
$$tr(AB)^m \leq trA^m B^m$$

holds for m = 1, 2, 3, ...

When
$$x_{p(i)} = x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} \cdots x_{i_{m-1} i_m} x_{i_m i_1}$$
$$d_{p(i)} = d_{i_1} d_{i_2} d_{i_3} \cdots d_{i_m}$$
$$d_{p(t_k)} = d_k^m$$

$$p(i) = \left\{ i = \left(i_1, i_2, i_3, \cdots, i_m \right) | \forall i_j \in \left(1, 2, 3, ..., n \right) , j = 1, 2, \cdots, m \right\}$$

and
$$p(t_k) = \left\{ t = \left(t_1 = k, t_2, t_3, \cdots, t_m \right) | k, \forall t_j \in \left(1, 2, 3, ..., n \right), j = 2, \cdots, m \right\}$$

(5.1.2) For positive definite matrices A and B of the same order $n \times n$,If A and B are Nonnegative matrices ,then

$$tr(AB)^m \leq trA^m B^m$$

holds for any m = 1, 2, 3, ...

(5.1.3) For positive definite matrices A and B of the same order $n\times n$, then

$$n\left(\det A^m B^m\right)^{\frac{1}{n}} \leq \left(trA^m\right)\left(trB^m\right) \leq (trA)^m (trB)^m$$

holds for any m = 1, 2, 3, . . .

(5.1.4) For positive definite matrices A and B of the same order $n \times n$, then

$$2n\left(\det A^m B^m\right)^{\frac{1}{2n}} \leq trA^m + trB^m \leq (trA)^m + (trB)^m$$

holds for any m = 1, 2, 3, . . .

(5.1.5) For skew-Hermitian matrices A and B of the same order,

If m = 2t ,then

(1) $$tr\,A^m B^m \leq \frac{1}{4}tr(A^m + B^m)^2 \leq \frac{1}{2}tr(A^{2m} + B^{2m})$$

and If m = 2t+1 ,then

(2) $$tr\,A^m B^m \geq \frac{1}{4}tr(A^m + B^m)^2 \geq \frac{1}{2}tr(A^{2m} + B^{2m})$$

(5.1.6) For Hermitian matrices A and B of the same order, then

$$tr\,A^m B^m \leq \frac{1}{4}tr(A^m + B^m)^2 \leq \frac{1}{2}tr(A^{2m} + B^{2m})$$

For any nonnegative integer t = 1, 2, 3, . . .

(5.1.7) For Hermitian matrix A and skew-Hermitian matrix B of the same order,

If m = 2t ,

then $$tr\,A^mB^m \le \frac{1}{4}tr(A^m + B^m)^2 \le \frac{1}{2}tr(A^{2m} + B^{2m}) \qquad (4.20)$$

when t is nonnegative integer. Further, when m= 2t+1 , $tr\,A^mB^m$ are all zeros or pure imaginary numbers cannot be compared with each other.

Main results and the comparison with another author have been present in table 5.1-5.3

5.2 Application

Most application of Trace inequality Hermitian matrices be finite dimensional, which to be powerful tool in mathematics and to applied in many fields. For example, in modeling error analysis for filtering and estimation problems, in adaptive stochastic control and for investigation of quantum mechanical Hamiltonians [3, 8, 13, 14]

Hopefully, application of trace inequalities of Hermitian matrices have prove in this thesis will arise in the future. I will expect that have the educators interested to study and apply knowledge in this thesis.

5.3 Problem

Bellman's Inequality with order index is nonnegative odd number is so hard for verify and give certain character of that inequality . Because we can not explain certain character of $(AB)^m$ *and* A^mB^m when m is odd number, so that we can not used the basic property , definition and characterize to matrices to proof that inequality.

In this thesis author choose to used property, definition and characterize of eigenvalues and eigenvector to give and prove character of Bellman's Inequality, But that condition can explain for some case of Hermitian matrices.

5.4 Suggestion

The research about a matrix trace inequality of the product and power of Hermitian matrices has been widely studied. Furthermore, mathematicians are study , generalize Bellman's inequality to general Hermitians matrices and define new inequality with for better upper bound (or lower bound) and check equality .

Table 5.1: Bellman's Inequality with higher order.

AUTHER	Inequality		
	Product of Hermitian matrices	Product of skew-Hermitian matrices	Product of Hermitian matrices and skew-Hermitian matrices
Bellman(1980)	$tr(AB)^2 \leq trA^2B^2$		
Da-wei Cheng(1999)	$tr(AB)^{2^m} \leq trA^{2^m}B^{2^m}$		
Zhong Peng Yang and Xiao Xia Feng(2002)	$tr(AB)^{2m} \leq trA^{2m}B^{2m}$	$tr(AB)^m \leq trA^mB^m$,m=2t ,$t \in N$	(1) $tr(AB)^m \leq trA^mB^m$,m=2t ,$t \in N$ (2) $tr(AB)^m \geq trA^mB^m$,m=2t+1 ,$t \in N$
Monsicha(2004)	$tr(AB)^m \leq trA^mB^m$ when add conditions 1. A and B are positive definite 2. $x_{p(i)} \geq 0$		

Table 5.2: Upper bound(or lower bound) of $tr(A^m B^m)$.

AUTHER	INEQUALITY		
	Product of Hermitian matrices	Product of skew-Hermitian matrices	Product of Hermitian matrices and skew-Hermitian matrices
1.Bellman(1980) 2 Yang(1988) 3. Neudecker (1992)	$0 \le tr(AB)$ $(trAB)^{\frac{1}{2}} \le \frac{1}{2}tr(A+B)$ $(trAB) \le \frac{1}{2}tr(A^2+B^2)$ with condition: A and B are positive definite.		
Monsicha(2004)	$tr A^m B^m \le \frac{1}{4}tr(A^m+B^m)^2 \le \frac{1}{2}tr(A^{2m}+B^{2m})$	$tr A^m B^m \le \frac{1}{4}tr(A^m+B^m)^2 \le \frac{1}{2}tr(A^{2m}+B^{2m})$ when m = 2t $tr A^m B^m \ge \frac{1}{4}tr(A^m+B^m)^2 \ge \frac{1}{2}tr(A^{2m}+B^{2m})$ when m=2t+1	$tr A^m B^m \le \frac{1}{4}tr(A^m+B^m)^2 \le \frac{1}{2}tr(A^{2m}+B^{2m})$ when m=2t

Table 5.3: New inequality involving trace and determinant

AUTHOR	INEQUALITY
	Product of positive definite matrices
Fozi M. Dannan(2001)	$n(\det AB)^{\frac{m}{n}} \leq (trA)^m (trB)^m$
	$n(\det A)^{\frac{1}{n}} \leq trAX$, when $\det X = 1$
Monsicha(2004)	$n\left(\det A^m B^m\right)^{\frac{1}{n}} \leq \left(trA^m\right)\left(trB^m\right) \leq (trA)^m (trB)^m$
	$2n\left(\det A^m B^m\right)^{\frac{1}{2n}} \leq trA^m + trB^m \leq (trA)^m + (trB)^m$

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Appendix A

An Application to Data Fitting.

I.D.Coope and P.F. Renaud have presented an application of a matrix trace inequality as follow;

Suppose that data is available as a set of m points in R^n represented by the columns of the n x m matrix A and it is required to find the best k-dimensional linear manifold $L_k \in R^n$ approximating the set of points in the sense that the sum of squares of the distances of each data point from its orthogonal projection onto the linear manifold is minimized. A general point in L_k can be expressed in parametric form as

$$x(t)=z+Z_k t,\ t \in R^n \tag{1}$$

where z is a fixed point in L_k and the columns of the $n \times k$ matrix Z_k can be taken to be orthonormal . The problem is now to identify a suitable z and Z_k . Now the orthogonal projection of a point $a \in R^n$ onto L_k can be written as

$$proj(a,L_k)=z+Z_k Z_k^T(a-z),$$

and hence the Euclidean distance from a to L_k is

$$dist(a,L_k)=\left\|a-proj(a,L_k)\right\|_2=\left\|(I-Z_k Z_k^T)(a-z)\right\|_2.$$

Therefore, the total least square data-fitting problem is reduced to finding a suitable z and corresponding Z_k to minimize the sum-of –squares function

$$SS=\sum_{j=1}^{m}\left\|(I-Z_k Z_k^T)(a_j-z)\right\|_2^2,$$

where a_j is the jth data point(jth column of A) . A necessary condition for SS to be minimized with respect to z is

$$0=\sum_{j=1}^{m}(I-Z_kZ_k^T)(a_j-z)=(I-Z_kZ_k^T)\sum_{j=1}^{m}(a_j-z).$$

Therefore, $\sum_{j=1}^{m}(a_j-z)$ lies in the null space of $(I-Z_kZ_k^T)$ or equivalently the column space of Z_k . The parametric representation (1) shows that there is no loss of generality in letting $\sum_{j=1}^{m}(a_j-z)=0$ or

$$z=\frac{1}{m}\sum_{j=1}^{m}a_j. \qquad (2)$$

Thus, a suitable z has been determined and it should be noted that the value (2) solves the zero-dimensional case corresponding to k=0. It remains to find Z_k when k>0, which is the problem

$$\min\sum_{j=1}^{m}\left\|(I-Z_kZ_k^T)(a_j-z)\right\|_2^2, \qquad (3)$$

subject to the constraint that the columns of Z_k are orthonormal and z satisfies equation (2) . Using the properties of orthogonal projections and definition of vector 2-norm, (3) can be rewritten

$$\min\sum_{j=1}^{m}(a_j-z)^T(I-Z_kZ_k^T)(a_j-z), \qquad (4)$$

Ignoring the terms in (4) independent of Z_k then reduces the problem to

$$\min \sum_{j=1}^{m} -(a_j - z)^T Z_k Z_k^T (a_j - z),$$

or equivalently

$$\max\ tr \sum_{j=1}^{m} (a_j - z)^T Z_k Z_k^T (a_j - z). \qquad (5)$$

The introduction of the trace operator in (5) is allowed because the argument to the trace function is a matrix with only one element . The commutative property of the trace then shows that problem (5) is equivalent to

$$\max tr \sum_{j=1}^{m} Z_k^T (a_j - z)(a_j - z)^T Z_k \equiv \max tr Z_k^T \hat{A}\hat{A}^T Z_k \qquad (6)$$

where $\hat{A}$ is the matrix

$$\hat{A} = [a_1 - z, a_2 - z, \ldots, a_m - z]$$

Again, the commutative property of the trace then shows that problem(6) is equivalent to

$$\max\ tr Z_k^T \hat{A}\hat{A}^T Z_k \equiv \max tr \hat{A}\hat{A}^T Z_k Z_k^T \qquad (7)$$

Since $\hat{A} \in R^{nxm}$ and $Z_k \in R^n$,then $\hat{A}\hat{A}^T$ *and* $Z_k Z_k^T$ are Hermitian matrices. Such that

$$tr \hat{A}\hat{A}^T Z_k Z_k^T \le \frac{1}{4} tr(\hat{A}\hat{A}^T + Z_k Z_k^T)^2 \le \frac{1}{2} tr[(\hat{A}\hat{A}^T)^2 + (Z_k Z_k^{T\,2})]$$

It remains to find Z_k when k>0, which is the problem

$$\max tr\hat{A}\hat{A}^T Z_k Z_k^T$$

or

$$\max \frac{1}{4} tr(\hat{A}\hat{A}^T + Z_k Z_k^T)^2$$

$$\max \frac{1}{4} tr[(\hat{A}\hat{A}^T)^2 + (Z_k Z_k^T)^2]$$

s.t.

1. The columns of Z_k are orthonormal

2. $z = \frac{1}{m}\sum_{j=1}^{m} a_j$.

Because the problem is required to find the best k-dimensional linear manifold $L_k \in R^n$ (from Z_k)

Appendix B

Direct proof of Theorem 4.1.2 with for some case.

In this part we give the direct proof of Theorem 4.1.2 with case Hermitian matrices A and B of the order 2×2, and power index is 2.

Example Direct proof for some case of Theorem 4.1.2

Let
$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A=PDP^{T}, \quad D=diag[d_1, d_2]$$

$$X=P^{T}BP=\left[x_{ij}\right]_{2\times 2} \quad ,and \quad Y=DX$$

Consider

$$Y=DX=\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}=\begin{bmatrix} d_1x_{11} & d_1x_{12} \\ d_2x_{21} & d_2x_{22} \end{bmatrix}$$

then
$$Y^2=\begin{bmatrix} (d_1d_1)x_{11}^2+d_1d_2(x_{12}x_{21}) & d_1d_1(x_{11}x_{12})+d_1d_2(x_{12}x_{22}) \\ d_2d_1(x_{21}x_{11})+d_2d_2(x_{22}x_{21}) & d_2d_1(x_{21}x_{12})+(d_2d_2)x_{22}^2 \end{bmatrix}$$

Hence
$$tr(DX)^2=(d_1d_1)x_{11}^2+(d_1d_2)x_{12}x_{21}+(d_2d_1)x_{21}x_{12}+(d_2d_2)x_{22}^2 \qquad (*)$$

And consider

$$X^2=\begin{bmatrix} x_{11}^2+x_{12}x_{21} & x_{11}x_{12}+x_{12}x_{22} \\ x_{21}x_{11}+x_{22}x_{21} & x_{21}x_{12}+x_{22}^2 \end{bmatrix} \text{ and } D^2=\begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix}$$

such that

$$D^2X^2 = \begin{bmatrix} d_1^2(x_{11}^2 + x_{12}x_{21}) & d_1^2(x_{11}x_{12} + x_{12}x_{22}) \\ d_2^2(x_{21}x_{11} + x_{22}x_{21}) & d_2^2(x_{21}x_{12} + x_{22}^2) \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{d_1^2 + d_1^2}{2})(x_{11}^2 + x_{12}x_{21}) & d_1^2(x_{11}x_{12} + x_{12}x_{22}) \\ d_2^2(x_{21}x_{11} + x_{22}x_{21}) & (\frac{d_2^2 + d_2^2}{2})(x_{21}x_{12} + x_{22}^2) \end{bmatrix}$$

Hence

$$trD^2X^2 = (\frac{d_1^2 + d_1^2}{2})(x_{11}^2 + x_{12}x_{21}) + (\frac{d_2^2 + d_2^2}{2})(x_{21}x_{12} + x_{22}^2)$$

$$= (\frac{d_1^2 + d_1^2}{2})x_{11}^2 + (\frac{d_1^2 + d_1^2}{2})x_{12}x_{21} + (\frac{d_2^2 + d_2^2}{2})x_{21}x_{12} + (\frac{d_2^2 + d_2^2}{2})x_{22}^2$$

$$= (\frac{d_1^2 + d_1^2}{2})x_{11}^2 + (\frac{d_1^2}{2})x_{12}x_{21} + (\frac{d_1^2}{2})x_{12}x_{21} + (\frac{d_2^2}{2})x_{21}x_{12} + (\frac{d_2^2}{2})x_{21}x_{12} + (\frac{d_2^2 + d_2^2}{2})x_{22}^2$$

$$= (\frac{d_1^2 + d_1^2}{2})x_{11}^2 + (\frac{d_1^2}{2})x_{12}x_{21} + (\frac{d_1^2}{2})x_{21}x_{12} + (\frac{d_2^2}{2})x_{12}x_{21} + (\frac{d_2^2}{2})x_{21}x_{12} + (\frac{d_2^2 + d_2^2}{2})x_{22}^2$$

$$= (\frac{d_1^2 + d_1^2}{2})x_{11}^2 + (\frac{d_1^2}{2})x_{12}x_{21} + (\frac{d_2^2}{2})x_{12}x_{21} + (\frac{d_1^2}{2})x_{21}x_{12} + (\frac{d_2^2}{2})x_{21}x_{12} + (\frac{d_2^2 + d_2^2}{2})x_2^2$$

$$= (\frac{d_1^2 + d_1^2}{2})x_{11}^2 + (\frac{d_1^2 + d_2^2}{2})x_{12}x_{21} + (\frac{d_2^2 + d_1^2}{2})x_{21}x_{12} + (\frac{d_2^2 + d_2^2}{2})x_{22}^2$$

Therefore

$$trD^2X^2 = (\frac{d_1^2 + d_1^2}{2})x_{11}^2 + (\frac{d_1^2 + d_2^2}{2})x_{12}x_{21} + (\frac{d_2^2 + d_1^2}{2})x_{21}x_{12} + (\frac{d_2^2 + d_2^2}{2})x_{22}^2 \qquad (**)$$

By Arithmetic and Geometric Mean Inequality , we have

$$d_1d_1 \le (\frac{d_1^2 + d_1^2}{2})$$

$$d_1d_2 \le (\frac{d_1^2 + d_2^2}{2})$$

$$d_2d_1 \le (\frac{d_2^2 + d_1^2}{2})$$

$$d_2d_2 \le (\frac{d_2^2 + d_2^2}{2})$$

and from ($*$) and ($**$).Such that

$$tr(AB)^2 \le trA^2B^2$$

If $x_{11}^2, \ x_{12}x_{21}, \ x_{21}x_{12}, \ x_{22}^2 \ge 0$

Appendix C

Example of the main results.

In this part we give some example to support the main results , as follow;

Let A and B are square matrices on a finite dimentional Hilbert space, m ≥ 1 and t ≥ 1. We have the main resultes;

(1) Example of inequality (Theorem 4.1.2) ;

If $$x_{p(i)} \geq 0 ,$$

Then $$tr(AB)^m \leq trA^m B^m$$

When A and B are positive definite matrices of the same order $n \times n$, and nonnegative integer m.

<u>Example 1.1</u> Let

$$A=\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

Then $$tr(AB)^2 = 144 < 204 = trA^2B^2$$

<u>Example 1.2</u> Let

$$A=\begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 7 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

Then $$tr(AB)^3 = 2.1311\times 10^5 < 2.1495\times 10^5 = trA^3B^3$$

$$tr(AB)^6 = 4.4714\times 10^{10} < 4.5916\times 10^{10} = trA^6B^6$$

$$tr(AB)^{15} = 4.2271\times 10^{26} < 4.6027\times 10^{26} = trA^{15}B^{15}$$

Example 1.3 Let

$$A=\begin{bmatrix}20 & 1 & 2 & 3 & 1\\ 1 & 10 & 1 & 2 & 2\\ 2 & 1 & 15 & 1 & 1\\ 3 & 2 & 1 & 5 & 1\\ 1 & 2 & 1 & 1 & 2\end{bmatrix} \quad and \quad B=\begin{bmatrix}10 & 1 & 2 & 1 & 1\\ 1 & 10 & 3 & 2 & 2\\ 2 & 3 & 5 & 3 & 1\\ 1 & 2 & 3 & 4 & 2\\ 1 & 2 & 1 & 2 & 4\end{bmatrix}$$

Then

$$tr(AB)^3 = 2.3275\times 10^7 < 2.4528\times 10^7 = trA^3B^3$$

$$tr(AB)^6 = 4.5946\times 10^{14} < 6.1673\times 10^{14} = trA^6B^6$$

$$tr(AB)^{15} = 4.4417\times 10^{36} < 2.3851\times 10^{37} = trA^{15}B^{15}$$

$$tr(AB)^{20} = 7.3012\times 10^{48} < 5.9180\times 10^{49} = trA^{20}B^{20}$$

(2) Example of inequality (Theorem 4.2.12);

If m = 2t ,then

(1) $$tr\, A^mB^m \le \frac{1}{4}tr(A^m+B^m)^2 \le \frac{1}{2}tr(A^{2m}+B^{2m})$$

and If m = 2t+1 ,then

(2) $$tr\, A^mB^m \ge \frac{1}{4}tr(A^m+B^m)^2 \ge \frac{1}{2}tr(A^{2m}+B^{2m})$$

When A and B are skew-Hermitian matrices of the same order, and m is nonnegative integer.

Example 2.1 Let

$$A=\begin{bmatrix} 2i & -2+2i \\ 2+2i & 0 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}$$

Then

$$tr\,AB=-12 \;>-15=\frac{1}{4}tr(A+B)^2$$
$$>-18=\frac{1}{2}tr(A^2+B^2)$$

Example 2.2 Let

$$A=\begin{bmatrix} 3i & 1+2i & i \\ -1+2i & 2i & 2 \\ i & -2 & i \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 4i & 4 & 1+i \\ -4 & 3i & 2i \\ -1+i & 2i & 2i \end{bmatrix}$$

Then

$$tr(A^3B^3)=-5.2905\times 10^4 \;>-11.3995\times 10^4=\frac{1}{4}tr(A^3+B^3)^2$$
$$>-17.5087\times 10^4=\frac{1}{2}tr(A^6+B^6)$$
$$tr(A^{10}B^{10})=14.3203\times 10^{15} \;<6.1933\times 10^{17}=\frac{1}{4}tr(A^{10}+B^{10})^2$$
$$<12.5299\times 10^{17}=\frac{1}{2}tr(A^{20}+B^{20})$$

Example 2.3 Let

$$A=\begin{bmatrix} 5i & 4i & 3 & -2+2i & 1+i \\ 4i & 4i & 2i & i & 1 \\ -3 & 2i & 3i & -1+i & -i \\ -2+2i & i & -1+i & 2i & 2 \\ -1+i & -1 & -1 & -2 & i \end{bmatrix}$$

and

$$B=\begin{bmatrix} 10i & 5i & 2 & i & 1+i \\ 5i & 7i & 4i & 3 & 1 \\ -2 & 4i & 2i & 1+i & -i \\ i & -3 & -1+i & i & i \\ -2 & 3i & -2+i & i & i \end{bmatrix}$$

Then

$$tr(A^3B^3) = -3.6370\times10^6 \quad > -5.5460\times10^6 = \frac{1}{4}tr(A^3+B^3)^2$$
$$> -7.4551\times10^6 = \frac{1}{2}tr(A^6+B^6)$$

$$tr(A^{10}B^{10}) = 6.5621\times10^{21} \quad < 15.8182\times10^{22} = \frac{1}{4}tr(A^{10}+B^{10})^2$$
$$< 3.0981\times10^{23} = \frac{1}{2}tr(A^{20}+B^{20})$$

(3) Example of inequality(Theorem 4.2.13);

$$tr\, A^mB^m \le \frac{1}{4}tr(A^m+B^m)^2 \le \frac{1}{2}tr(A^{2m}+B^{2m})$$

When A and B are Hermitian matrices of the same order, and t is nonnegative integer.

<u>Example 3.1</u> Let

$$A=\begin{bmatrix} -2 & -2-2i \\ -2+2i & 0 \end{bmatrix} \quad and \quad B=\begin{bmatrix} -2 & -2i \\ 2i & -2 \end{bmatrix}$$

Then

$$tr\, AB = 12 \; > \; 15 = \frac{1}{4}tr(A+B)^2$$
$$> \; 18 = \frac{1}{2}tr(A^2+B^2)$$

Example 3.2 Let

$$A=\begin{bmatrix} 3 & 2-i & 1 \\ 2+i & 2 & -2i \\ 1 & 2i & 1 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 4 & -4i & 1-i \\ 4i & 3 & 2 \\ 1+i & 2 & 2 \end{bmatrix}$$

Then

$$tr(A^3B^3) = 5.2905\times10^4 \quad < 11.3995\times10^4 = \frac{1}{4}tr(A^3+B^3)^2$$

$$< 17.5087\times10^4 = \frac{1}{2}tr(A^6+B^6)$$

$$tr(A^{10}B^{10}) = 14.3203\times10^{15} \quad < 6.1933\times10^{17} = \frac{1}{4}tr(A^{10}+B^{10})^2$$

$$< 12.5299\times10^{17} = \frac{1}{2}tr(A^{20}+B^{20})$$

Example 3.3 Let

$$A=\begin{bmatrix} 5 & 4 & -3i & 2-2i & 1-i \\ 4 & 4 & 2 & 1 & i \\ 3i & 2 & 3 & 1+i & -1 \\ 2+2i & 1 & 1+i & 2 & -2i \\ 1+i & i & -1 & 2i & 1 \end{bmatrix}$$

and

$$B=\begin{bmatrix} 10 & 5 & -2i & 1 & -2i \\ 5 & 7 & 4 & -3i & 3 \\ 2i & 4 & 2 & 1-i & 1-2i \\ 1 & 3i & 1+i & 1 & 1 \\ 2i & 3 & 1+2i & 1 & 1 \end{bmatrix}$$

Then

$$tr(A^3B^3) = 3.6370\times10^6 \quad < 5.5460\times10^6 = \frac{1}{4}tr(A^3+B^3)^2$$

$$< 7.4551\times10^6 = \frac{1}{2}tr(A^6+B^6)$$

$$tr(A^{10}B^{10}) = 6.5621\times 10^{21} \quad < 15.8182\times 10^{22} = \frac{1}{4}tr(A^{10}+B^{10})^2$$

$$< 3.0981\times 10^{23} = \frac{1}{2}tr(A^{20}+B^{20})$$

(4) Example of inequality(Theorem 4.2.14) ;

If m = 2t ,

then $$tr\, A^mB^m \le \frac{1}{4}tr(A^m+B^m)^2 \le \frac{1}{2}tr(A^{2m}+B^{2m})$$

If m= 2t+1 ,

$tr\, A^mB^m$ are all zeros or pure imaginary numbers cannot be compared with each other.

when Hermitian matrix A and skew-Hermitian matrix B of the same order,

t is nonnegative integer.

Example 4.1 Let

$$A = \begin{bmatrix} 2i & -2+2i \\ 2+2i & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 2 & 2i \\ -2i & 4 \end{bmatrix}$$

Then

$$tr\, AB = 12i$$

Example 4.2 Let

$$A = \begin{bmatrix} 3 & 2-i & 1 \\ 2+i & 2 & -2i \\ 1 & 2i & 1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 4i & 4 & 1+i \\ -4 & 3i & 2i \\ -1+i & 2i & 2i \end{bmatrix}$$

Then

$$tr(A^3B^3) \doteq 5.2905i\times 10^4$$

$$tr(A^{10}B^{10}) = -14.3203\times 10^{15} < 6.1933\times 10^{17} = \frac{1}{4}tr(A^{10}+B^{10})^2$$
$$< 12.5299\times 10^{17} = \frac{1}{2}tr(A^{20}+B^{20})$$

Example 4.3 Let

$$A = \begin{bmatrix} 5 & 4 & -3i & 2-2i & 1-i \\ 4 & 4 & 2 & 1 & i \\ 3i & 2 & 3 & 1+i & -1 \\ 2+2i & 1 & 1+i & 2 & -2i \\ 1+i & i & -1 & 2i & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 10i & 5i & 2 & i & 2 \\ 5i & 7i & 4i & 3 & 3i \\ -2 & 4i & 2i & 1+i & 2+i \\ i & -3 & -1+i & i & i \\ -2 & 3i & -2+i & i & i \end{bmatrix}$$

Then $$tr(A^3B^3) = -4.2481i\times 10^6$$

$$tr(A^{10}B^{10}) = -15.1375\times 10^{21} < 14.7455\times 10^{22} = \frac{1}{4}tr(A^{10}+B^{10})^2$$
$$< 31.0040\times 10^{23} = \frac{1}{2}tr(A^{20}+B^{20})$$

(5) Example of inequality(Theorem 4.2.15) ;

$$n\left(\det A^m B^m\right)^{\frac{1}{n}} \leq \left(trA^m\right)\left(trB^m\right) \leq (trA)^m (trB)^m$$

When A and B are positive definite matrices of the same order $n\times n$, and m is nonnegative integer.

<u>Example 5.1</u> Let

$$A=\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Then

$$2(\det AB)^{\frac{1}{2}} = 15.492 < 50 = (tr\,A)(trB)$$

$$2(\det A^2B^2)^{\frac{1}{2}} = 120 < 1140 = (tr\,A^2)(tr\,B^2)$$

$$< 2500 = (tr\,A)^2(tr\,B)^2$$

<u>Example 5.2</u> Let

$$A=\begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 7 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

Then

$$3(\det A^3B^3)^{\frac{1}{3}} = 1.2180\times 10^3 < 2.7040\times 10^5 = (tr\,A^3)(tr\,B^3)$$

$$< 3.6523\times 10^6 = (tr\,A)^3(tr\,B)^3$$

$$3(\det A^6B^6)^{\frac{1}{3}} = 4.9451\times 10^5 < 4.8003\times 10^{10} = (tr\,A^6)(tr\,B^6)$$

$$< 1.3338\times 10^{13} = (tr\,A)^6(tr\,B)^6$$

Example 5.3 Let

$$A=\begin{bmatrix}20&1&2&3&1\\1&10&1&2&2\\2&1&15&1&1\\3&2&1&5&1\\1&2&1&1&2\end{bmatrix} \quad and \quad B=\begin{bmatrix}10&1&2&1&1\\1&10&3&2&2\\2&3&5&3&1\\1&2&3&4&2\\1&2&1&2&4\end{bmatrix}$$

$$5(\det A^3B^3)^{\frac{1}{5}} = 1.8173\times 10^5 < 5.8343\times 10^7 = (tr\,A^3)(tr\,B^3)$$

$$< 5.0530\times 10^9 = (tr\,A)^3(tr\,B)^3$$

$$5(\det A^6B^6)^{\frac{1}{5}} = 6.5790\times 10^9 < 1.1824\times 10^{15} = (tr\,A^6)(tr\,B^6)$$

$$< 2.5533\times 10^{19} = (tr\,A)^6(tr\,B)^6$$

$$5(\det A^{15}B^{15})^{\frac{1}{5}} = 5.4996\times 10^{27} < 3.2719\times 10^{37} = (tr\,A^{15})(tr\,B^{15})$$

$$< 3.2943\times 10^{48} = (tr\,A)^{15}(tr\,B)^{15}$$

(6) Example of inequality(Theorem 4.2.16) ;

$$2n\left(\det A^m B^m\right)^{\frac{1}{2n}} \leq trA^m + trB^m \leq (trA)^m + (trB)^m$$

When A and B are positive definite matrices of the same order $n\times n$, and m is nonnegative integer.

<u>Example 6.1</u> Let

$$A=\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Then

$$4(\det AB)^{\frac{1}{4}} = 11.13 < 15 = (tr\,A) + (trB)$$

$$4(\det A^2B^2)^{\frac{1}{4}} = 30.9838 < 91 = (tr\,A^2) + (tr\,B^2)$$

$$< 125 = (tr\,A)^2 + (tr\,B)^2$$

<u>Example 6.2</u> Let

$$A=\begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 7 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

Then

$$6(\det A^3B^3)^{\frac{1}{6}} = 1.2090\times 10^2 < 1.1410\times 10^3 = (tr\,A^3) + (tr\,B^3)$$

$$< 4.0750\times 10^3 = (tr\,A)^3 + (tr\,B)^3$$

$$6(\det A^6B^6)^{\frac{1}{6}} = 2.4360\times 10^3 < 6.0028\times 10^5 = (tr\,A^6) + (tr\,B^6)$$

$$< 9.3010\times 10^6 = (tr\,A)^6 + (tr\,B)^6$$

Example 6.3 Let

$$A=\begin{bmatrix} 20 & 1 & 2 & 3 & 1 \\ 1 & 10 & 1 & 2 & 2 \\ 2 & 1 & 15 & 1 & 1 \\ 3 & 2 & 1 & 5 & 1 \\ 1 & 2 & 1 & 1 & 2 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 10 & 1 & 2 & 1 & 1 \\ 1 & 10 & 3 & 2 & 2 \\ 2 & 3 & 5 & 3 & 1 \\ 1 & 2 & 3 & 4 & 2 \\ 1 & 2 & 1 & 2 & 4 \end{bmatrix}$$

$$10(\det A^3B^3)^{\frac{1}{10}} = 1.9046\times 10^{3} < 1.8502\times 10^{4} = (tr\,A^3)+(tr\,B^3)$$

$$< 1.7654\times 10^{5} = (tr\,A)^3+(tr\,B)^3$$

$$10(\det A^6B^6)^{\frac{1}{10}} = 3.6274\times 10^{5} < 1.2525\times 10^{8} = (tr\,A^6)+(tr\,B^6)$$

$$< 2.1062\times 10^{10} = (tr\,A)^6+(tr\,B)^6$$

$$10(\det A^{15}B^{15})^{\frac{1}{10}} = 3.3165\times 10^{14} < 1.1225\times 10^{20} = (tr\,A^{15})+(tr\,B^{15})$$

$$< 5.5020\times 10^{15} = (tr\,A)^{15}+(tr\,B)^{15}$$

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