MATRIX TRANSFORMATIONS OF CESARO VECTOR-VALUED SEQUENCE SPACE INTO MADDOX SEQUENCE SPACE

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หัวข้อวิทยานิพนธ์	การแปลงเมทริกซ์จากปริภูมิลำดับค่าเวกเตอร์เซซาโรไปยังปริภูมิลำดับแมดดอกซ์
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บทคัดย่อ

ในงานวิจัยนี้เป็นการศึกษาถึงโครงสร้างของปริภูมิลำดับค่าเวกเตอร์เซซาโร(Cesaro Vector-Valued Sequence Space) $Ces(X,p)$ และการให้ลักษณะเฉพาะของเมทริกซ์อนันต์ที่ส่งจากปริภูมิลำดับค่าเวกเตอร์ $Ces(X,p)$ ไปยังปริภูมิลำดับแมดดอกซ์ (Maddox Sequence Space) และโดยการประยุกต์ผลลัพธ์ที่ได้ จะทำให้ได้เงื่อนไขที่จำเป็นและเพียงพอ สำหรับเมทริกซ์อนันต์ที่ส่งจาก $Ces(X,p)$ ไปยัง $\ell_\infty(q), M_\infty(q), E_r(q)$ และ $F_r(q)$ โดยที่ $p=(p_k)$ และ $q=(q_k)$ เป็นลำดับของจำนวนจริงบวกที่มีขอบเขต

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ABSTRACT

In this thesis, we study the structure of Cesaro vector-valued sequence space and characterize an infinite matrix mapping Cesaro vector-valued sequence space into Maddox sequence space. By applying these results, we can gave necessary and sufficient conditions for infinite matrices mapping from Cesaro vector-valued sequence space $Ces(X,p)$ into $\ell_\infty(q)$, $M_\infty(q)$, $E_r(q)$ and $F_r(q)$ when $p=(p_k)$ and $q=(q_k)$ are bounded sequence of positive real numbers.

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CHAPTER 1

INTRODUCTION

1.1 Importance and Inception

Sequence spaces have been investigated many fields. The examples of sequence space are geometry property, summability and matrix transformations. Matrix transformations have been studied an infinite matrix mapping between two exact sequence spaces. In the past, most sequence spaces were scalar-valued. Afterwards, matrix transformations concerned with vector-valued sequence spaces. A well- known scalar-valued sequence space is Maddox sequence space. Maddox sequence space is defined by :

$$c_0(p) = \{x = (x_k) : \lim_{k\to\infty} |x_k|^{p_k} = 0\},$$

$$c(p) = \{x = (x_k) : \lim_{k\to\infty} |x_k - a|^{p_k} = 0 \text{ for some a}\},$$

$$\ell_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

$$\ell(p) = \{x = (x_k) : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\}$$

where $p = (p_k)$ is a bounded sequence of positive real numbers.

A sequence space is a linear subspace of $W = C^N$ of all sequences. A matrix space is a linear subspace of the space $W(N^2) = C^{N^2}$ of all (complex) matrices. For a matrix $A = (a_{nk})$ and a sequence $x = (x_k)$, we put $Ax = (\sum_{k=1}^{\infty} a_{nk}x_k)_{n=1}^{\infty}$ if the series $\sum_{k=1}^{\infty} a_{nk}x_k$ converges for all $n \in N$. If E and F are sequence spaces, we say that A *maps* E into F if for each $x \in E$, Ax exists and $Ax \in F$. The metric space (E, F) is defined as

$$(E, F) = \{A \in W(N^2) : A \text{ maps } E \text{ into } F\}.$$

Let $(X, \|.\|)$ be a Banach space with a scalar field K, the space of all sequences in X is denote by $W(X)$ and let $\Phi(X)$ denote the space of all finite sequences in X. When $X = \mathbb{R}$ or $\mathcal{C}$, the corresponding spaces are written as W and Φ. Let N be the set of all natural numbers, we write $x = (x_k)$ with $x_k \in X$ for all $k \in N$. A sequence space in X is linear subspace of $W(X)$. Let $p = (p_k)$ be a bounded sequence of positive real numbers, the X-valued sequence space $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $Ces(X,p)$, $M_0(X,p)$, $M_\infty(X,p)$, $\underline{\ell_\infty}(X,p)$, $E_r(X,p)$ and $F_r(X,p)$ are defined by:

$$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\},$$

$$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\},$$

$$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\},$$

$$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\},$$

$$Ces(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} (\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k} < \infty\},$$

$$M_0(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} n^{\frac{-1}{p_k}} \|x_k\| < \infty \text{ for some } n \in N\},$$

$$M_\infty(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} n^{\frac{1}{p_k}} \|x_k\| < \infty \text{ for some } n \in N\},$$

$$\underline{\ell_\infty}(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|\delta_k x_k\| = 0 \quad \text{for each } (\delta_k) \in c_0\},$$

$$E_r(X,p) = \{x = (x_k) : \sup_k k^{-r} \|x_k\|^{p_k} < \infty\} and$$

$$F_r(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} k^r \|x_k\|^{p_k} < \infty\}.$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), \ell_\infty(p), Ces(p), M_0(p), M_\infty(p), \underline{\ell_\infty}(p), E_r(p)$ and $F_r(p)$ respectively.

1.2 Initial Idea

In 1993, F.M. Khan and M.A.Khan[5] gave characterization of infinite matrices in the classes Cesaro sequence space ($Ces(p,s)$) into the space of convergent series (cs) and the space of bounded series (bs). $Ces(p,s)$ defined by:

$$Ces(p,s) = \{x = (x_k) : \sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\}$$

where Σ denotes as a sum over the ranges $2^r \leq k < 2^{r+1}$.

This paper, Cesaro sequence space $Ces(p,s)$ is an scalar-valued and that brought *β-dual* to proved conditions of matrix transformations. However, their paper[5] have not presented the proof of *β-dual*.

1.3 Involved Research

Grosse and Erdmann, [2-3] investigated and gave characterization of infinite matrices to transform between Maddox sequence space and Maddox sequence space. Wu and Liu [16] gave the matrix transformations from X-valued sequence spaces $c_0(X,p), \ell_\infty(X,p)$ and $\ell(X,p)$ into scalar-valued sequence spaces $c_0(q)$ and $\ell_\infty(q)$. S.Suantai [12-14] gave characterization of infinite matrices mapping Nakano vector-valued sequence $\ell(X,p)$ into any

BK-space, ℓ_∞ and $\ell_\infty(q)$. In [1] C. Sudsukh characterized an infinite matrix to tranform Maddox vector-valued sequence space into Nakano sequence space and Nakano vector-valued sequence space into Maddox sequence space. In [8] S. Kongnual characterized an infinite matrices mapping bounded variation vector-valued sequence space into Maddox sequence space.

1.4 Process of the study

1.4.1 Study research about matrix transformations.

1.4.2 Consider structure of Cesaro vector-valued sequence space [$Ces(X,p)$] and prove some properties of $Ces(X,p)$.

1.4.3 Prove the condition of β-*dual* of $Ces(X,p)$, using direct proof and proposition in S. Suantai[15].

1.4.4 Use the condition of β-*dual* and Lemma 2.2.2.9 to prove conditions of matrix transformations of $Ces(X,p)$ into Maddox sequence space.

1.4.5 Apply the results of 1.4.4 to prove conditions of matrix transformations of $Ces(X,p)$ into $\underline{\ell_\infty}(q)$, $M_\infty(q)$, $E_r(q)$ and $F_r(q)$.

1.5 Main Objective

The aim of this thesis is to study matrix transformations of Cesaro vector-valued sequence space $Ces(X,p)$ into Maddox sequence space. The main purposes of this thesis are

(1) to study about structure of Cesaro vector-valued sequence and space

(2) to characterize infinite matrices such elements of the matrix are bounded linear functional on the vector space which mapping vector-valued sequence space of Cesaro into scalar-valued sequence space of Maddox, $M_\infty(q)$, $\underline{\ell_\infty}(q)$, $E_r(q)$ and $F_r(q)$.

CHAPTER 2

DEFINITIONS AND LITERATURE REVIEWS

2.1 General Concepts and Definitions

In this part, we give some general concepts which are important to discuss and prove conditions.

2.1.1 Metric space, Fre'chet space and Norm space

Definition 2.1.1.1. (Vector space(linear space))
A vector space over a field K is a nonempty set X with an operator $+$ on $X \times X$ into X and an operator $\bullet$ on $X \times X$ into X such that for all scalars α, β and elements(vectors) $x, y, z \in X$ we have

(1) x+y=y+x,
(2) (x+y)+z=x+(y+z),
(3) there exists $\theta \in X$ such that $x + \theta = x$,
(4) there exists -x$\in X$ such that $x + (-x) = \theta$,
(5) $1{\cdot}x = x$,
(6) $\alpha(x + y) = \alpha x + \alpha y$,
(7) $(\alpha + \beta)x = \alpha x + \beta x$,
(8) $\alpha(\beta x) = (\alpha\beta)x$.

Definition 2.1.1.2. (Metric space(semimetric))
A metric(semimetric) space is a pair (X, d), where X is a set and d is a metric(semimetric) on X, that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

Metric
(M1) d is real-valued, finite and nonnegative.
(M2) d(x,y)=0 if and only if x=y
(M3) d(x,y)=d(y,x)
(M4) d(x,y)$\leq d(x, z) + d(z, y)$
Semimetric
(M1) d is real-valued, finite and nonnegative.
(M2) d(x,x)=0
(M3) d(x,y)=d(y,x)
(M4) d(x,y)$\leq d(x, z) + d(z, y)$

Definition 2.1.1.3. (Normed space)
A normed space X is a vector space with a norm defined on it. A norm is a real-valued function on X whose at $x \in X$ is defined by $\|x\|$ and has the properties

(N1) $\|x\| \geq 0$
(N2) $\|x\| = 0 \Leftrightarrow x = 0$
(N3) $\|\alpha x\| = \alpha\|x\|$
(N4) $\|x + y\| \leq \|x\| + \|y\|$.

A norm on X defines a metric d on X which is given by $d(x, y) = \|x - y\|$ for all $x, y \in X$, and is called *the metric induce by the norm.* The norm space X is denoted by $(X, \|.\|)$.

Definition 2.1.1.4. (Paranorm space, Seminorm space)
A paranorm (X, g) is a linear space together with a paranorm g on it. A paranorm $g : X \to R$, satisfies

(1) $g(\theta) = 0$,
(2) g(x)=g(-x),
(3) g(x+y)$\leq g(x) + g(y)$ and
(4) $\lambda \to \lambda_0, x \to x_0$ imply $\lambda x \to x_0\lambda_0$.

A seminorm p on a linear space X, is a function $p : X \to R$ such that

(1) $p(\lambda x) = |\lambda| p(x)$, (2) $p(x + y) \leq p(x) + p(y)$

By (1) and (2) we also have $p(\theta) = 0$ and $p(x) \geq 0$. Paranorm becomes seminorm.

Definition 2.1.1.5. (Translation invariance)
A metric d is induced by a norm on a norm space X satisfies

(a) d(x+a,y+a)=d(x,y) ,(b) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.

Definition 2.1.1.6. (Convergent sequence)
A sequence (x_n) in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \in X$ such that $\lim_{n\to\infty} d(x, y) = 0$, we say that (x_n) convergent to x, imply $x_n \to x$. If (x_n) is not convergent, it is said to be divergent.

Definition 2.1.1.7. (Cauchy sequence)
A sequence (x_n) in a metric space is said to be Cauchy if for every $\varepsilon > 0$ there is an $N_\varepsilon \in N$ such that

$$d(x_m, x_n) < \varepsilon \qquad \text{for all} \quad m, n \geq N_\epsilon.$$

Definition 2.1.1.8. (complete)
A metric space X is said to be complete if every Cauchy sequence in X converges.

Definition 2.1.1.9. (Fre'chet space)
A vector space X is said to be a linear metric space if X is a metric space such that both of mapping $(x, y) \to x + y$ and $(\alpha, x) \to \alpha x$ are continuous. A linear metric space is called a *Fre'chet space* (*F-space*) if the metric d on X is complete and translation invariance.

Definition 2.1.1.10. (K-space)
If the X-valued sequence space E is called a *K-space* if for each $n \in N$ then n^{th} coordinate mapping p_n from E into X, defined by $p_n(x) = x_n$ is continuous on E.

Definition 2.1.1.11. (FK-space)
If the X-valued sequence space E is an *Fre'chet* and *K-space* then E is called *FK-space*.

Definition 2.1.1.12. (Banach space)
A Banach space X is a complete normed linear space. Completeness means that if $\|x_m - x_n\| \to 0$ $(m, n \to \infty)$, where $x_n \in X$, then there exists $x \in X$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$.

Definition 2.1.1.13. (Equivalent norms)
A norm $\|.\|$ on a vector space X is said to be equivalent to a norm $\|.\|_0$ on X if there are positive numbers a and b such that for all $x \in X$ we have

$$a\|x\| \leq \|x\| \leq b\|x\|_0$$

.

2.1.2 Sequence spaces

Definition 2.1.2.1. A sequence space is a vector space whose elements are sequences.

Definition 2.1.2.2. Scalar sequence space of Maddox are following:

$$c_0(p) = \{x = (x_k) : \lim_{k\to\infty} |x_k|^{p_k} = 0\},$$

$$c(p) = \{x = (x_k) : \lim_{k\to\infty} |x_k - a|^{p_k} = 0 \text{ for some a}\},$$

$$\ell_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

$$\ell(p) = \{x = (x_k) : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\}.$$

When $p = (p_k)$ is a bounded sequence of positive real numbers.

Definition 2.1.2.3. Let $(X, \|.\|)$ be a Banach space with a scalar field K and $p = (p_k)$ be a bounded sequence of positive real numbers. The Cesaro vector-valued sequence space is

$$Ces(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{k}\|x_n\|)^{p_k} < \infty\}.$$

Definition 2.1.2.4. Let E be an X-valued sequence space. For $x \in E$ we write $x = (x_k), k \in N$. For $k \in N$ denote e_k by the sequence $(0,0,0,...,0,1,0,...)$ with 1 in the k^{th} position and e by the sequence $(1,1,1,...)$. For $z \in X$ and $k \in N$, let $e^{(k)}(z)$ be the sequence $(0,0,0,...,0,z,0,...)$ with z in the k^{th} position and let $e(z)$ be the sequence $(z,z,z,...)$.

Definition 2.1.2.5. Let $(X, \|.\|)$ be a Banach space with scalar field K. E be any X-valued sequence space. For a fixed scalar sequence $u = (u_k)$ the sequence space E_u is defined by

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

Definition 2.1.2.6. (AK-property)
Suppose that E contains $\Phi(X)$. Then E is said to have **property AB** if the set $\{\Sigma_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It said to have **property AK** if $\Sigma_{k=1}^{n} e^k(x_k) \to x \in E$ as $n \to \infty$ for every $x = (x_k) \in E$.

Definition 2.1.2.7. Let $X \subset \omega$, then

$$X^\beta = \{a = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ converges for all } x \in X\}$$

is called *β-dual* or generated *Köthe-Toeplitz dual* of X.

2.1.3 Matrix and Transformations

Definition 2.1.3.1. Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to **map** E **into** F, written $A : E \to F$ if for each $x = (x_k) \in E, A_n(x) = \Sigma_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and if the sequence $Ax = (A_n(x)) \in F$.

Definition 2.1.3.2. Let (E, F) denote for the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\frac{1}{u_k})$.

Definition 2.1.3.3. (weak**convergence*)
Let(f_n) be a sequence of bounded linear functional on a norm space X. Then *weak*convergence* of (f_n) means that there is an $f \in X'$ such that $f_n(x) \to f(x)$, $\forall x \in X$. This written $f_n \longrightarrow^{w^*} f$.

2.2 Literature Reviews

2.2.1 Transformations between Scalar-Valued Sequence Spaces

The followings are some literature reviews of matrices transformations which are concerned in this thesis. In 1993, F.M.Khan and M.A.Khan[5] gave characterization of the matrices in the classes $(Ces(p,s), cs)$ and $(Ces(p,s), bs)$. The results are as follows:

Theorem 2.2.1.1. *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $A \in (Ces(p,s), cs)$ *if and only if*

(1) *there exists an integer* $E > 1$ *such that* $T = \sup_n (U_n) < \infty$
where $U_n = \sum_{r=0}^{\infty} (\max_r |b_{nk}|.2^r)^{q_r} (2^r)^{s(q_r - 1)} E^{-q_r}$ *and* $\frac{1}{p_r} + \frac{1}{q_r} = 1$,
r=0,1,2,...

(2) $\lim_{n\to\infty} b_{nk} = u_k$ *for all* k.

Theorem 2.2.1.2. *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $A \in (Ces(p,s), c_0 s)$ *if and only if*

(1) *there exists an integer* $E > 1$ *such that* $T = \sup_n (U_n) < \infty$
where $U_n = \sum_{r=0}^{\infty} (\max_r |b_{nk}|.2^r)^{q_r} (2^r)^{s(q_r - 1)} E^{-q_r}$ *and* $\frac{1}{p_r} + \frac{1}{q_r} = 1$,
r=0,1,2,...

(2) $\lim_{n\to\infty} b_{nk} = 0$.

Theorem 2.2.1.3. *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $A \in (Ces(p,s), bs)$ *if and only if there exists an integer* $E > 1$ *such that* $T = \sup_n(U_n) < \infty$ *where* $U_n = \sum_{r=0}^{\infty}(\max_r |b_{nk}|.2^r)^{q_r}(2^r)^{s(q_r-1)}E^{-q_r}$ *and* $\frac{1}{p_r} + \frac{1}{q_r} = 1$, $r = 0, 1, 2, ...$

2.2.2 Transformations between Vector-Valued Sequence Space

In [16], Wu and Liu gave the matrix characterizations of X-valued sequence spaces $c_0(X,p), \ell_\infty(X,p)$ and $\ell(X,p)$ into scalar valued sequence spaces $c_0(q)$ and $\ell_\infty(q)$. The results are as follows:

Theorem 2.2.2.1. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers with* $p_k \leq 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : \ell(X,p) \to \ell_\infty(q)$ *if and only if there exists* $M \in N$ *such that* $\|f_k^n\| \leq M^{\frac{1}{p_k}+\frac{1}{q_k}}$ *for all* $n, k \in N$.

Theorem 2.2.2.2. *Let* $p = (p_k)$ *and* $q = (q_k)$ *be bounded sequence of positive real numbers with* $p_k > 1$ *and* $\frac{1}{p_k} + \frac{1}{t_k} = 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : \ell(X,p) \to \ell_\infty(q)$ *if and only if there exists* $M \in N$ *such that* $\sup_n(\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M^{\frac{-t_k}{q_n}})^{q_n} < \infty$.

Theorem 2.2.2.3. *Let* $p = (p_k)$ *be bounded sequence of positive real numbers with* $p_k \leq 1$ *for all* $k \in N$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A : \ell(X,p) \to c_0$ *if and only if*

(1) *for each* $k \in N$, $f_k^n \longrightarrow^{w^*} 0$ *as* $n \to \infty$ *and*
(2) *there exists* $M \in N$ *such that* $\|f_k^n\|^{p_k} \leq M$ *for all* $n, k \in N$.

S.Suantai and C.Sudsukh [13] gave necessary and sufficient condition of infinite matrices which are mapping Nakano vector-valued $\ell(X,p)$ and $M_0(X,p)$ into sequence space $E_r (r \geq 0)$.

Theorem 2.2.2.4. *Let* $r \geq 0$ *and let* $p = (p_k)$ *be bounded sequence of positive real numbers with* $p_k \leq 1$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A \in (\ell(X,p), E_r)$ *if and only if there is* $m_0 \in N$ *such that*

$$\sup_{n,k} m_0^{\frac{-1}{p_k}} n^{-r} \|f_k^n\| < \infty.$$

Theorem 2.2.2.5. *Let* $r \geq 0$ *and let* $p = (p_k)$ *be bounded sequence of positive real numbers with* $p_k \leq 1$ *and* $A = (f_k^n)$ *an infinite matrix. Then* $A \in (M_0(X,p), E_r)$ *if and only if for each* $s \in N$ *such that*

$$\sup_{n,k} s^{\frac{1}{p_k}} n^{-r} \|f_k^n\| < \infty.$$

C.Sudsukh [1] characterized an infinite matrix transformations between Maddox vector-valued sequence space and Nakano sequence spaces. These are useful results

Theorem 2.2.2.6. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ and $q_k > 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to \ell(q)$ if the following two conditions hold;*

(1) *for each $n \in N$ there exists $M_n \in N$ such that $\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M_n^{-(t_k-1)} < \infty$*
(2) *there exists $M_n \in N$ such that*

$$\sup_K \sum_{n=1}^{\infty} (\sum_{k \in K} \|f_k^n\| M_0^{\frac{-1}{p_k}})^{q_n} < \infty$$

where supremum is taken over all finite subsets K of N.

Theorem 2.2.2.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to c_0(q)$ if and only if*

(1) *for all $m,k \in N$, $m^{\frac{1}{q_n}} f_k^n \longrightarrow^{w^*} 0$ as $n \to \infty$ and*
(2) *for each $m \in N$, $(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} \|f_k^n\|^{t_k} r^{-(t_k-1)}) \to 0$ as $r \to \infty$ uniformly for $n \geq 0$.*

Theorem 2.2.2.8. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : \ell(X,p) \to c_(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) *there exists $M \in N$ such that $\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} M^{-(t_k-1)} < \infty$,*
(2) *for all $m,k \in N$, $m^{\frac{1}{q_n}} (f_k^n - f_k) \longrightarrow^{w^*} 0$ as $n \to \infty$ and*
(3) *for each $m \in N$, $(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} \|f_k^n - f_k\|^{t_k} r^{-(t_k-1)}) \to 0$ as $r \to \infty$ uniformly for $n \geq 1$.*

Lemma 2.2.2.9. *Let $E \subseteq W(X)$ be an FK-space with AK property and F an FK-space of scalar sequences. Then, for an infinite matrix $A = (f_k^n), A : E \to F$ if and only if*

(1) *for each $n \in N, \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in E$,*
(2) *for each $k \in N, (f_k^n(z))_{n=1}^{\infty} \in F$ for all $z \in X$, and*
(3) *$A : \Phi(X) \to F$ is continuous when $\Phi(X)$ is considered as a subspace of E.*

S.Suantai [11] gave the matrix characterizations of Nakano vector-valued sequence space $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces E_r, ℓ_∞, $\underline{\ell_\infty}(q), bs$ and cs.

Theorem 2.2.2.10. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{q_k} = 1$ for all $k \in N$ and let $r \geq 0$. For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in N$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < \infty.$$

Theorem 2.2.2.11. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{q_k} = 1$ for all $k \in N$, $r \geq 0$ and $s \geq 0$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p), E_s)$ if and only if there is $m_0 \in N$ such that*

$$\sup_n \sum_{k=1}^{\infty} (k^{\frac{-rq_k}{p_k}} \|f_k^n\|^{q_k} n^{-sq_k} m_0^{-q_k}) < \infty.$$

Theorem 2.2.2.12. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and let $\frac{1}{p_k} + \frac{1}{q_k} = 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \ell_\infty)$ if and only if there is $m_0 \in N$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} m_0^{-q_k} < \infty.$$

Theorem 2.2.2.13. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and let $\frac{1}{p_k} + \frac{1}{q_k} = 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p), \ell_\infty)$ if and only if there is $m_0 \in N$ such that*

$$\sup_n \sum_{k=1}^{\infty} (k^{\frac{-rq_k}{p_k}} \|f_k^n\|^{q_k} m_0^{-q_k}) < \infty.$$

Theorem 2.2.2.14. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and let $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \underline{\ell_\infty}(q))$ if and only if for each $r \in N$, there is $m_r \in N$ such that*

$$\sup_{n,k} \sum_{k=1}^{\infty} r^{\frac{t_k}{q_n}} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty.$$

Theorem 2.2.2.15. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and let $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p), \underline{\ell_\infty}(q))$ if and only if for each $i \in N$, there is $m_i \in N$ such that*

$$\sup_n \sum_{k=1}^{\infty} i^{\frac{t_k}{q_n}} k^{\frac{-rt_k}{p_k}} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty.$$

Theorem 2.2.2.16. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and let $\frac{1}{p_k} + \frac{1}{q_k} = 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), bs)$ if and only if there is $m_0 \in N$ such that*

$$\sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i \|^{q_k} m_0^{-q_k} < \infty.$$

Theorem 2.2.2.17. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ and let $\frac{1}{p_k} + \frac{1}{q_k} = 1$ for all $k \in N$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), cs)$ if and only if*

(1) *there is $m_0 \in N$ such that $\sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i \|^{q_k} m_0^{-q_k} < \infty$ and*
(2) *for each $k \in N$ and $x \in X, \sum_{n=1}^{\infty} f_k^n(x)$ converges.*

CHAPTER 3

STRUCTURE AND PROPERTY

In this chapter we give general form of matrix transformations and explain about the relationship of this thesis. Then we consider structure and some properties of $Ces(X,p)$ which are useful for the later chapter.

3.1 General Form of Matrix Transformation

By definition 2.1.3.1 is the general matrix transformations. We first give a basic examples of matrix transformations.

Example 3.1.1. *Let infinite matrix* $A = (f_k^n) = \frac{n}{k^2}$ *and* $x_k = e = (1,1,...)$. *Then* $A : E \to F$, *if for each* $x = (e) \in E$, E *and* F *are any sequence space*

$$Ax = \sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{k=1}^{\infty} \frac{n}{k^2}.(e) = n\sum_{k=1}^{\infty} \frac{1}{k^2} \in F.$$

Previous example if we specified sequence space, that $E = Ces(X,p)$ and $F = Maddox\ (\ell(q), \ell_\infty(q), c_0(q), c(q))$. we can consider the similar of general example but we must be aware of the condition of each sequence space.

Example 3.1.2. *Let infinite matrix* $A = (f_k^n) = \frac{1}{2^n}$ *and* $x_k = e_k = (0,0,1,0,...)$ *with 1 in the* k^{th} *position. Then* $A : Ces(X,p) \to \ell(q)$, *if for each* $x_k = e_k \in Ces(X,p)$,

$$Ax = \sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{k=1}^{\infty} \frac{1}{2^n}.(e_k) = \frac{1}{2^n} \in \ell(q) = \sum_{n=1}^{\infty} |\frac{1}{2^n}|^{q_n} < \infty.$$

Example 3.1.3. *Let infinite matrix* $A = (f_k^n) = \frac{1}{2^n}$ *and* $x_k = e_k = (0,0,1,0,...)$ *with 1 in the* k^{th} *position. Then* $A : Ces(X,p) \to \ell_\infty(q)$, *for each* $x_k = e_k \in Ces(X,p)$,

$$Ax = \sum_{k=1}^{\infty} \frac{1}{2^n}.(e_k) = \frac{1}{2^n} \in \ell_\infty(q) = \sup_n |\frac{1}{2^n}|^{q_n} < \infty.$$

By example, we define an infinite matrix with scalar members and consider simple elements of sequence space. In complicate case, we study any element of infinite matrix and any sequence that belong to sequence space. In definition 2.1.2.7, β-*dual* is important for matrix transformation between sequence space and sequence space. We must prove the condition of β-*dual* of $Ces(X,p)$ for finding condition of matrix transformation later.

3.2 FK and AK properties

In this section we give and prove consequence properties will be mentioned and related in the next chapter.

Proposition 3.2.1. *$Ces(X,p)$ contains $\Phi(X)$.*

Proof. Let $x = (x_k) \in \Phi(X)$, this means there exists $N_0 \in N$ such that $x_k = 0, \forall k \geq N_0$. Then $\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|x_j\|)^{p_k} = \sum_{k=1}^{n}(\frac{1}{n}\sum_{j=1}^{n}\|x_j\|)^{p_k} < \infty$. Therefore $Ces(X,p)$ contains $\Phi(X)$. □

Proposition 3.2.2. *$(\sum_{j=1}^{\infty}\|\zeta_j+\eta_j\|^{p_j})^{\frac{1}{M}} \leq (\sum_{j=1}^{\infty}\|\zeta_j\|^{p_j})^{\frac{1}{M}}+(\sum_{j=1}^{\infty}\|\eta_j\|^{p_j})^{\frac{1}{M}}$, when $M = \max(1, \sup_j p_j)$.*

Proof. For this proof, we can see in [8, Proposition 3.1.2]. □

Proposition 3.2.3. *$Ces(X,p)$ is an Fre'chet space when $p = (p_k)$ be bounded sequence of positive real number and $p_k > 1$ for all $k \in N$.*

Proof. Let $(x_m)_{m=1}^{\infty}$ be Cauchy sequence in $Ces(X,p)$ such that $x_m = (r_1^{(m)}, r_2^{(m)}, ...)$. Since $Ces(X,p)$ have paranormed, that is defined by

$$g(x) = (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j\|)^{p_k})^{\frac{1}{M}}$$

when $M = \sup_k p_k$, $p_k > 1$ then the semimetric, which is induced by norm $g(x)$ is

$$d(x,y) = g(x-y) = (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j - u_j\|)^{p_k})^{\frac{1}{M}}$$

when $x = (r_j)$ and $y = (u_j)$. Since (x_m) is Cauchy sequence in $Ces(X,p)$. Let $\varepsilon_1 > 0$, $\exists N(\varepsilon_1) \ni n, m > N(\varepsilon_1)$

$$d(x_m, x_n) = (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)} - r_j^{(n)}\|)^{p_k})^{\frac{1}{M}} < \varepsilon_1 \qquad (3.1)$$

$$\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)} - r_j^{(n)}\|)^{p_k}) < \varepsilon_1{}^M$$

fixed for each k $\quad (\frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)} - r_j^{(n)}\|)^{p_k} < \varepsilon_1{}^M$

$$\frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)} - r_j^{(n)}\| < \varepsilon_1{}^{\frac{M}{p_k}}$$

since $1 < \frac{M}{p_k} < M$ then ${\varepsilon_1}^{\max\frac{M}{p_k}} < {\varepsilon_1}^{\frac{M}{p_k}} < {\varepsilon_1}^{\min\frac{M}{p_k}}$ when $\varepsilon_1 < 1$. Let $l = \min\frac{M}{p_k}$, $L = \max\frac{M}{p_k}$ then ${\varepsilon_1}^L < {\varepsilon_1}^{\frac{M}{p_k}} < {\varepsilon_1}^l$ hence

$$\frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)} - r_j^{(n)}\| < {\varepsilon_1}^{\frac{M}{p_k}} < {\varepsilon_1}^l$$

$$\sum_{j=1}^{k}\|r_j^{(m)} - r_j^{(n)}\| < k{\varepsilon_1}^l$$

fixed for each j, $\quad \|r_j^{(m)} - r_j^{(n)}\| < k{\varepsilon_1}^l.$

We want to show that $(r_j^{(m)})$ is Cauchy sequence.

$\forall\varepsilon > 0$, $\exists N(\varepsilon) \ni n, m > N(\varepsilon)$ such that $\|r_j^{(m)} - r_j^{(n)}\| < \varepsilon$. Let $\varepsilon_1 < (\frac{\varepsilon}{k})^{\frac{1}{l}}$ for fixed k and since

$$\|r_j^{(m)} - r_j^{(n)}\| < k{\varepsilon_1}^l \ \ \forall n, m > N(\varepsilon)$$

$$\|r_j^{(m)} - r_j^{(n)}\| < k{\varepsilon_1}^l < k\{(\frac{\varepsilon}{k})^{\frac{1}{l}}\}^l = \varepsilon$$

so $\quad \|r_j^{(m)} - r_j^{(n)}\| < \varepsilon.$

Hence $(r_j^{(m)})$ is Cauchy sequence in Banach space X for each $j = 1, 2, \ldots$. Since X is any Banach space thus it converges, say $(r_j) \ni (r_j^{(m)}) \to (r_j)$, $m \to \infty$. Using these infinite limits $r_1, r_2, r_3, \ldots$, we define $x = (r_1, r_2, \ldots) = (r_j)$ and we want to show that $x \in Ces(X, p)$ and $x_m \to x$. From (3.1) with $n \to \infty$ we have

$$d(x_m, x) = (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)} - r_j\|)^{p_k})^{\frac{1}{M}} \le \varepsilon_1 \tag{3.2}$$

$$\begin{aligned}
(\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j\|)^{p_k})^{\frac{1}{M}} &= (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j - r_j^{(m)} + r_j^{(m)}\|)^{p_k})^{\frac{1}{M}} \\
&\le (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}(\|r_j - r_j^{(m)}\| + \|r_j^{(m)}\|)^{p_k})^{\frac{1}{M}} \\
&= (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j - r_j^{(m)}\| + \frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)}\|)^{p_k})^{\frac{1}{M}} \\
&\le (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j - r_j^{(m)}\|)^{p_k})^{\frac{1}{M}} + (\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j^{(m)}\|)^{p_k})^{\frac{1}{M}} \\
&\le \varepsilon_1 + K_0 \qquad ; by\ (3.2) \text{ and } K_0 < \infty \\
&< \infty
\end{aligned}$$

$$(\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|r_j\|)^{p_k})^{\frac{1}{M}} < \infty.$$

This show that $x = (r_j) \in Ces(X,p)$ and from (3.2) represents $d(x_m, x)$. Since $d(x_m, x) \to 0,\ m \to \infty,\ (r_j^{(m)} \to r_j,\ \forall j \in N)$ hence $x_m \to x$ and (x_m) was any Cauchy sequence in $Ces(X,p)$, this prove completeness of $Ces(X,p)$. Since the semimetric d is induced by the paranorm $g(x)$ so that semimetric d is translation invariance thus we have $Ces(X,p)$, when $p_k > 1$ is a *Fre'chet space.* □

Proposition 3.2.4. *$Ces(X,p)$ is an K-space when $p = (p_k)$ be bounded sequence of positive real number and $p_k > 1$ for all $k \in N$.*

Proof. For each $n \in N$, n^{th} coordinate mapping $p_n : Ces(X,p) \to X$, define by $p_n(x) = x_n$, is continuous on $Ces(X,p)$.

Since we have $\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n} < \infty$ then $\lim_{n\to\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n} = 0$ this implies $\lim_{n\to\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|) = 0$. Then there exists $k_0 \in N$ such that $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|) < 1$, $\forall n > k_0$. We have $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} < (\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}$, $p_n > 1$ for $\forall n > k_0$. For each $1 < n < k_0$, if $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|) < 1$ implies

$$(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} < (\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n},\ p_n > 1. \tag{3.3}$$

If $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|) > 1$ we have

$$(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} = (\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2}\frac{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}}{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}}$$

$$= L_n(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n} \quad , L_n = \frac{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2}}{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}}$$

because $\max_n L_n < \infty$, $\max_n L_n > 1$ and $L_n > 1$ thus

$$(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} \le (\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}$$

when $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|) > 1$

$$(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} \le \max_n L_n(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}. \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} \le L(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}\ ,\ L = \max_n(1, \max_n L_n)$$

$$\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} \leq L\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}. \tag{3.5}$$

Since $\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n} < \infty,\ p_n > 1$

fixed for each $n = 1, 2, ...$ $(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n} < \infty$

$$\frac{1}{n}\sum_{k=1}^{n}\|x_k\| < \infty$$

let $\frac{1}{n}\sum_{k=1}^{n}\|x_k\| = L_0$

$$\sum_{k=1}^{n}\|x_k\| = n.L_0 < \infty \text{ , for n fixed}$$

so $$\sum_{k=1}^{n}\|x_k\| = L_1 \text{ , } L_1 = n.L_0 \text{ and n fixed} \tag{3.6}$$

$$\|p_n(x)\| = \|x_n\| \leq \sum_{k=1}^{n}\|x_k\| = \frac{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)}{\frac{1}{n}}$$

$$= \frac{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)}{\frac{1}{n}} \cdot \frac{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2}}{(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2}}$$

$$= (\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} . \frac{1}{(\frac{1}{n})^{p_n^2}(\sum_{k=1}^{n}\|x_k\|)^{p_n^2-1}}$$

$$= (\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} . \frac{1}{((\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}}\sum_{k=1}^{n}\|x_k\|)^{p_n^2-1}}$$

$$= (\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} . ((\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}}\sum_{k=1}^{n}\|x_k\|)^{1-p_n^2}$$

$$\leq \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} . ((\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}}\sum_{k=1}^{n}\|x_k\|)^{1-p_n^2}$$

$$\leq \sup_n((\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}}\sum_{k=1}^{n}\|x_k\|)^{1-p_n^2} . \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2} \text{ ; } \Sigma|ab| \leq \sup_n b\Sigma|a| \tag{3.7}$$

Because $\sup_n((\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}}\sum_{k=1}^{n}\|x_k\|)^{1-p_n^2}$ can find supremum. This to show, from $(\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}} \leq 1$ when $p_n > 1$ and by (3.6) we have
$\sum_{k=1}^{n}\|x_k\| = L_1$, $L_1 = n.L_0$ for n fixed hence

$$((\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}}\sum_{k=1}^{n}\|x_k\|)^{1-p_n^2} \leq (1.L_1)^{1-p_n^2} \text{ , by } 1-p_n^2 < 1$$

and so

$$\sup_n((\frac{1}{n})^{\frac{p_n^2}{p_n^2-1}}\sum_{k=1}^{n}\|x_k\|)^{1-p_n^2} \leq \sup_n(L_1)^{1-p_n^2}.$$

Since (3.7) we have

$$\leq \sup_n(L_1)^{1-p_n^2}\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n^2}$$

$$\leq \sup_n(L_1)^{1-p_n^2}.L\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n} \text{ ; from (3.5)}$$

$$= V\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n} \quad ; V = L\sup_n(L_1)^{1-p_n^2}$$

$$= V\|x\| \quad ; \|x\| = \sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}\|x_k\|)^{p_n}$$

$$\|p_n(x)\| = \|x_n\| \leq V\|x\| \text{ , } V = L\sup_n(L_1)^{1-p_n^2} \text{ , } L = \max_n(1, \max_n L_n)$$

Hence p_n is bounded linear operator and continuous linear operator and we have $p_n(x) = x_n$ is continuous coordinate. Therefore, $Ces(X,p)$ is K-*space.* □

Corollary 3.2.5. *$Ces(X,p)$ is an FK-space when $p = (p_k)$ be bounded sequence of positive real number and $p_k > 1$ for all $k \in N$.*

Proposition 3.2.6. *$Ces(X,p)$ has property AK.*

Proof. By definition, $Ces(X,p)$ has property AK if $\sum_{k=1}^{n}e^k(x_k) \to x$, $\forall x \in Ces(X,p)$ as $n \to \infty$. Let $x = (x_k)$ be any sequence in $Ces(X,p)$. Since $x = \sum_{k=1}^{\infty}e^k(x_k)$, so $\lim_{n\to\infty}\|x - \sum_{k=1}^{n}e^k(x_k)\| = \|x - \sum_{k=1}^{\infty}e^k(x_k)\| = 0$, Hence we have that $\sum_{k=1}^{n}e^k(x_k) \to x$, $\forall x \in Ces(X,p)$ as $n \to \infty$. □

3.3 Köthe-Toeplitz dual of $Ces(X,p)$

This section we first give useful results that concern with *β-dual* of $Ces(X,p)$. We present two ways to prove *β-dual*, direct proof and used the proposition in S.Suantai [15]. Following definition 2.1.2.7 (*β-dual*) we obtained direct proof.

Proposition 3.3.1. *Let (f_k) be a sequence of continuous linear functional on X and $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in Ces(X,p)$ if and only if $\sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} < \infty$ for some $B \in N$.*

Proof. Suppose that $\sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} < \infty$ for some $B \in N$ then for each $x = (x_k) \in Ces(X,p)$

$$\begin{aligned}
\sum_{k=1}^{\infty} |f_k(x_k)| &\le \sum_{k=1}^{\infty} \|f_k\|.k.B^{-1}B.\frac{1}{k}\|x_k\| \\
&\le \sum_{k=1}^{\infty}[(\|f_k\|.k)^{t_k} B^{-t_k} + B^{p_k}(\frac{1}{k}\|x_k\|)^{p_k}] \quad ; ab \le \frac{a^{t_k}}{t_k} + \frac{b^{p_k}}{p_k} \le a^{t_k} + b^{p_k} \\
&\le \sum_{k=1}^{\infty}[(\sup_k \|f_k\|.k)^{t_k} B^{-t_k} + B^{p_k}(\frac{1}{k}\|x_k\|)^{p_k}] \quad ; \|f_k\| \le \sup_k \|f_k\| \\
&= \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} + \sum_{k=1}^{\infty} B^{p_k}(\frac{1}{k}\|x_k\|)^{p_k} \\
&\le \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} + B^{\sup_k p_k}\sum_{k=1}^{\infty}(\frac{1}{k}\|x_k\|)^{p_k} \quad ; B^{p_k} \le B^{\sup_k p_k} \\
&\le \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} + B^{G}\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{k=1}^{k}\|x_k\|)^{p_k} \quad ; \|x_k\| \le \sum_{k=1}^{k}\|x_k\| \\
&< \infty
\end{aligned}$$

Thus $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

Assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges $\forall x = (x_k) \in Ces(X,p)$ for each $x = (x_k) \in Ces(X,p)$. Choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$, $\forall k \in N$. Since $(t_k x_k) \in Ces(X,p)$ by assumption we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges then

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all} \quad x = (x_k) \in Ces(X,p) \tag{3.8}$$

We want to show that $\exists B \in N$ such that $\sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} < \infty$. On contrary, suppose that

$$\sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} b^{-t_k} = \infty, \quad \forall b \in N \tag{3.9}$$

By (3.9) implies that for each $k_0 \in N$

$$\sum_{k>k_0}(\sup_n \|f_n\|.k)^{t_k} b_1{}^{-t_k} = \infty, \quad \forall b_1 \in N \tag{3.10}$$

From (3.10) we can choose $b_2 > b_1$ and $b_2 > 2^2$ and $k_2 > k_1$ such that

$$\sum_{k_1<k\leq k_2}(\sup_n \|f_n\|.k)^{t_k} b_2{}^{-t_k} > k^2. \tag{3.11}$$

Doing in this way go on, we have sequence $1 = k_0 < k_1 < k_2 < \ldots$ and $b_1 < b_2 <, \ldots, b_i > 2^i$ such that

$$\sum_{k_{i-1}<k\leq k_i}(\sup_n \|f_n\|.k)^{t_k} b_i{}^{-t_k} > k^2.$$

Choose x_k in X with $\|x_k\| = 1$ such that $\sum_{k_{i-1}<k\leq k_i}(\sup_n |f_n(x_n)|.k)^{t_k} b_i{}^{-t_k} > k^2, \forall i \in N$. Let $a_i = \sum_{k_{i-1}<k\leq k_i}(\sup_n |f_n(x_n)|.k)^{t_k} b_i{}^{-t_k}$ and $y = (y_k)$, $y_k = a_i^{-1}(\sup_n |f_n(x_n)|.k)^{t_k}|f_k(x_k)|^{-1}x_k$. Then $y \in Ces(X,p)$. Let $\alpha = (\sup_n |f_n(x_n)|)$ and $G = \sup_k p_k$, we can separate two cases.
Case $\alpha < 1$;

$$[\sum_{k_{i-1}<k\leq k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k}]^{\frac{1}{G}} = [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}\|a_i^{-1}\alpha^{t_j} j^{t_j}|f_j(x_j)|^{-1}.x_j\|)^{p_k}]^{\frac{1}{G}}$$

$$= [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}\alpha^{t_j} j^{t_j}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \qquad ; \|x_j\| = 1$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}1.j^{t_j}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \qquad ; \alpha^{t_j} \leq \alpha < 1,\ t_j > 1$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}1.k^{\sup_k t_k}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \quad ; j^{t_j} \leq k^{t_j} \leq k^{\sup_k t_k}, \forall k \in N$$

Let $K_{1,k}$ and $K_{2,k}$ are partitions of $\{1, 2, \ldots, k\}$, if $j \in K_{1,k}$ then $|f_j(x_j)| < 1$ and if $j \in K_{2,k}$ then $|f_j(x_j)| \geq 1$ for all $j = 1, 2, \ldots, k$. So we have

$$= [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}} a_i^{-1}k^L|f_j(x_j)|^{-1} + \sum_{j\in K_{2,k}} a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}; L = \sup_k t_k$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}} a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$+ [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}} a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$\leq [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}} a_i^{-1}k^L C_j)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}} a_i^{-1}k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$; |f_j(x_j)|^{-1} \le \max_j |f_j(x_j)|^{-1} = C_j$$

$$\le [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}} a_i^{-1}k^L C_j)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}} a_i^{-1}k^L.1)^{p_k}]^{\frac{1}{G}}$$

$$; |f_j(x_j)|^{-1} \le 1$$

$$\le [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}k^L C_j)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}k^L.1)^{p_k}]^{\frac{1}{G}}$$

$$; \sum_{j\in K_{1,k}} + \sum_{j\in K_{2,k}} \le \sum_{j=1}^{k} + \sum_{j=1}^{k}$$

$$= [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}.k^{p_k}(a_i^{-1}k^L C_j)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}.k^{p_k}(a_i^{-1}k^L)^{p_k}]^{\frac{1}{G}}$$

$$; (\sum_{j=1}^{k} \square)^{p_k} = (k.\square)^{p_k}$$

$$= [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}k^{L(p_k)}C_j^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}k^{L(p_k)}]^{\frac{1}{G}}$$

$$\le [\sum_{k_{i-1}<k\le k_i} a_i^{-1}k^{Lp_k}C_j^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}k^{Lp_k}]^{\frac{1}{G}} \qquad ; a_i^{-p_k} \le a_i^{-1}$$

$$\le [\sum_{k_{i-1}<k\le k_i} a_i^{-1}k^{L\sup_k p_k}C_j^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}k^{Lp_k}]^{\frac{1}{G}} \qquad ; k^{Lp_k} \le k^{L\sup_k p_k}$$

$$\le [C_j^G \sum_{k_{i-1}<k\le k_i} a_i^{-1}k^{L.G}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}k^{Lp_k}]^{\frac{1}{G}} \; ; C_j^{p_k} \le C_j^{\sup_k p_k},\ G = \sup_k p_k$$

$$\le [C_j^G k_i^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}k^{Lp_k}]^{\frac{1}{G}} \qquad ; k^{L.G} \le k_i^{L.G}$$

$$\le [C_j^G k_i^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-1}k^{Lp_k}]^{\frac{1}{G}} \qquad ; a_i^{-p_k} \le a_i^{-1}$$

$$\le [C_j^G k_i^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-1}k^{L.G}]^{\frac{1}{G}} \; ; k^{Lp_k} \le k^{L\sup_k p_k}, G = \sup_k p_k$$

$$\le [C_j^G k_i^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} + [k_i^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} \qquad ; k^{L.G} \le k_i^{L.G}$$

$$< [C_j^G k_i^{L.G} \sum_{k_{i-1}<k\le k_i} \frac{1}{k^2}]^{\frac{1}{G}} + [k_i^{L.G} \sum_{k_{i-1}<k\le k_i} \frac{1}{k^2}]^{\frac{1}{G}} \qquad ; a_i > k^2 \Rightarrow a_i^{-1} < k^{-2}$$

$$= (C_j^G k_i^{L.G})^{\frac{1}{G}} (\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}} + (k_i^{L.G})^{\frac{1}{G}} (\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}}$$

$$= [(C_j k_i^L) + k_i^L](\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}}$$

$$= [k_i^L(C_j+1)](\sum_{k_{i-1}<k\le k_i} \frac{1}{k^2})^{\frac{1}{G}}$$

Hence, $[\sum_{k_{i-1}<k\le k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k}]^{\frac{1}{G}} \le T_{i,1}(\sum_{k_{i-1}<k\le k_i}\frac{1}{k^2})^{\frac{1}{G}}$ then $\sum_{k_{i-1}<k\le k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k} \le T_{i,1}^G\sum_{k_{i-1}<k\le k_i}\frac{1}{k^2}$. Therefore, $\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k} \le T_{i,1}^G\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$, where $T_{i,1} = [k_i^L(C_j+1)]$.
Case $\alpha \ge 1$;

$$[\sum_{k_{i-1}<k\le k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k}]^{\frac{1}{G}} = [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}\|a_i^{-1}\alpha^{t_j}j^{t_j}|f_j(x_j)|^{-1}.x_j\|)^{p_k}]^{\frac{1}{G}}$$

$$= [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}\alpha^{t_j}j^{t_j}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \qquad ; \|x_j\| = 1$$

$$\le [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}\alpha^{\sup_k t_k}.j^{t_j}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \qquad ; \alpha^{t_j} \le \alpha^{\sup_k t_k}, \alpha \ge 1$$

$$\le [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k}a_i^{-1}\alpha^{\sup_k t_k}.k^{\sup_k t_k}|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \qquad ; j^{t_j} \le k^{t_j} \le k^{\sup_k t_k}$$

Now doing same as case $\alpha < 1$, which separate by partition. We have

$$= [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}} J + \sum_{j\in K_{2,k}} J)^{p_k}]^{\frac{1}{G}} \qquad ; J = a_i^{-1}\alpha^{\sup_k t_k}k^{\sup_k t_k}|f_j(x_j)|^{-1}$$

$$\le [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}} a_i^{-1}\alpha^L k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}} a_i^{-1}\alpha^L k^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}}$$

$$\le [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{1,k}} a_i^{-1}\alpha^L k^L C_j)^{p_k}]^{\frac{1}{G}}$$

$$+ [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j\in K_{2,k}} a_i^{-1}(\alpha.k)^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \; ; |f_j(x_j)|^{-1} \le \max_j |f_j(x_j)|^{-1} = C_j$$

$$\le [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}(\alpha.k)^L C_j)^{p_k}]^{\frac{1}{G}}$$

$$+ [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}(\alpha.k)^L|f_j(x_j)|^{-1})^{p_k}]^{\frac{1}{G}} \; ; \sum_{j\in K_{1,k}} + \sum_{j\in K_{2,k}} \le \sum_{j=1}^{k} + \sum_{j=1}^{k}$$

$$\le [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}(\alpha.k)^L C_j)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i}\frac{1}{k^{p_k}}(\sum_{j=1}^{k} a_i^{-1}(\alpha.k)^L.1)^{p_k}]^{\frac{1}{G}}$$

$$;|f_j(x_j)|^{-1} \le 1, |f_j(x_j)| \ge 1$$

$$= [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}.k^{p_k}(a_i^{-1}(\alpha.k)^L C_j)^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} \frac{1}{k^{p_k}}.k^{p_k}(a_i^{-1}(\alpha.k)^L)^{p_k}]^{\frac{1}{G}}$$

$$;(\sum_{j=1}^{k} \Box)^{p_k} = (k.\Box)^{p_k}$$

$$= [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}(\alpha.k)^{L.p_k} C_j^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}(\alpha.k)^{L.p_k}]^{\frac{1}{G}}$$

$$\le [\sum_{k_{i-1}<k\le k_i} a_i^{-1}(\alpha.k)^{L.p_k} C_j^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}(\alpha.k)^{L.p_k}]^{\frac{1}{G}} \qquad ;a_i^{-p_k} \le a_i^{-1}$$

$$\le [\sum_{k_{i-1}<k\le k_i} a_i^{-1}(\alpha.k)^{L.G} C_j^{p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}(\alpha.k)^{L.p_k}]^{\frac{1}{G}} \; ;(\alpha.k)^{L.p_k} \le (\alpha.k)^{L.G}$$

$$\le [C_j^G \sum_{k_{i-1}<k\le k_i} a_i^{-1}(\alpha.k)^{L.\sup_k p_k}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}(\alpha.k)^{L.p_k}]^{\frac{1}{G}} \; ;C_j^{p_k} \le C_j^G$$

$$\le [C_j^G \sum_{k_{i-1}<k\le k_i} a_i^{-1}(\alpha.k_i)^{L.G}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-p_k}(\alpha.k)^{L.p_k}]^{\frac{1}{G}} \qquad ;k \le k_i$$

$$\le [C_j^G(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-1}(\alpha.k)^{L.p_k}]^{\frac{1}{G}} \qquad ;a_i^{-p_k} \le a_i^{-1}$$

$$\le [C_j^G(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} + [\sum_{k_{i-1}<k\le k_i} a_i^{-1}(\alpha.k)^{L.G}]^{\frac{1}{G}}$$

$$\le [C_j^G(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}} + [(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} a_i^{-1}]^{\frac{1}{G}}$$

$$< [C_j^G(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} k^{-2}]^{\frac{1}{G}} + [(\alpha.k_i)^{L.G} \sum_{k_{i-1}<k\le k_i} k^{-2}]^{\frac{1}{G}}$$

$$= [(\alpha.k_i)^L(C_j+1)](\sum_{k_{i-1}<k\le k_i} k^{-2})^{\frac{1}{G}}$$

Thus, $[\sum_{k_{i-1}<k\le k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k}]^{\frac{1}{G}} \le T_{i,2}(\sum_{k_{i-1}<k\le k_i}\frac{1}{k^2})^{\frac{1}{G}}$ then $\sum_{k_{i-1}<k\le k_i}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k} \le T_{i,2}^G\sum_{k_{i-1}<k\le k_i}\frac{1}{k^2}$. Therefore, $\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{j=1}^{k}\|y_j\|)^{p_k} \le T_{i,2}^G\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$, where $T_{i,2} = [(\alpha.k_i)^L(C_j+1)]$.
We obtained $y \in Ces(X,p)$.
For each $i \in N$, we have

$$\sum_{k_{i-1}<k\le k_i} |f_k(y_k)| = \sum_{k_{i-1}<k\le k_i} |f_k(a_i^{-1}(\sup_n |f_n(x_n).k|)^{t_k}.|f_k(x_k)|^{-1}x_k)|$$

$$= a_i^{-1} \sum_{k_{i-1}<k\le k_i} (\sup_n |f_n(x_n).k|)^{t_k}.b_i^{-t_k}.b_i^{t_k} \quad ;b_i^{-t_k}.b_i^{t_k} = 1$$

$$\ge a_i^{-1} \sum_{k_{i-1}<k\le k_i} (\sup_n |f_n(x_n).k|)^{t_k}.b_i^{-t_k} \quad ;b_i^{t_k} \ge 1, b_i > 2^i$$

$$= 1 .$$

Then , $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$ which is a contradiction. The proof is complete. □

Remark 3.3.2. For each $T_{i,1}$ and $T_{i,2}$ in Proposition 3.3.1 that are bounded.

$$T_{i,1} = [k_i^L(C_j + 1)] = [k_i^{\sup_k t_k}(\max_j |f_j(x_j)|^{-1} + 1)]$$

$$T_{i,2} = [(\alpha.k_i)^L(C_j + 1)] = [(\sup_n |f_n(x_n)|)^{\sup_k t_k} k_i^{\sup_k t_k}(\max_j |f_j(x_j)|^{-1} + 1)].$$

Proof. Since (f_k) be a sequence of continuous linear functional and in the proof we choose sequence x_k in X with $\|x_k\| = 1$, so $|f_k(x_k)| \leq \|f_k\|.\|x_k\| = \|f_k\|.1 < \infty$. We have $|f_k(x_k)|$ bounded, thus $\max_j |f_j(x_j)|^{-1}$ and $(\sup_n |f_n(x_n)|)$ are bounded for all $j = 1, 2, ...k$. Therefore, $T_{i,1}$ and $T_{i,2}$ are bounded for all $i \in N$. □

Another way, we use proposition in [15] that give a relationship between vector-valued and scalar-valued sequence spaces. If we know *β-dual* of $Ces(p)$ then we obtain $\beta - dual$ of $Ces(X,p)$.

Proposition 3.3.3. *Let (a_k) be a sequence in the space of all sequence ω and $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$. Then $\sum_{k=1}^{\infty} a_k x_k$ converges for all $x = (x_k) \in Ces(p)$ if and only if $\sum_{k=1}^{\infty}(\sup_n |a_n|.k)^{t_k}.B^{-t_k} < \infty$ for some $B \in N$.*

Proof. Suppose that $\sum_{k=1}^{\infty}(\sup_n |a_n|.k)^{t_k} B^{-t_k} < \infty$ for some $B \in N$ then for each $x = (x_k) \in Ces(p)$.

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} |a_k.k.B^{-1}\frac{1}{k}.B.x_k|$$

$$\leq \sum_{k=1}^{\infty} |\frac{(a_k.k.B^{-1})^{t_k}}{t_k} + \frac{(\frac{1}{k}.B.x_k)^{p_k}}{p_k}|$$

$$\leq \sum_{k=1}^{\infty} |(a_k.k.B^{-1})^{t_k} + (\frac{1}{k}.B.x_k)^{p_k}|$$

$$\leq \sum_{k=1}^{\infty} |(a_k.k.B^{-1})^{t_k}| + |(\frac{1}{k}.B.x_k)^{p_k}|$$

$$= \sum_{k=1}^{\infty} |(a_k.k.B^{-1})^{t_k}| + \sum_{k=1}^{\infty} |(\frac{1}{k}.B.x_k)^{p_k}|$$

$$= \sum_{k=1}^{\infty} (|a_k|.k)^{t_k} B^{-t_k} + \sum_{k=1}^{\infty} B^{p_k}(\frac{1}{k}.|x_k|)^{p_k}$$

$$\leq \sum_{k=1}^{\infty}(\sup_k |a_k|.k)^{t_k} B^{-t_k} + \sum_{k=1}^{\infty} B^{p_k}(\frac{1}{k}.|x_k|)^{p_k}$$

$$\leq \sum_{k=1}^{\infty}(\sup_n |a_n|.k)^{t_k} B^{-t_k} + \sum_{k=1}^{\infty} B^{\sup_k p_k}(\frac{1}{k}.|x_k|)^{p_k}$$

$$\leq \sum_{k=1}^{\infty}(\sup_n |a_n|.k)^{t_k} B^{-t_k} + B^G \sum_{k=1}^{\infty}(\frac{1}{k}.\sum_{n=1}^{k}|x_n|)^{p_k}$$

$$< \infty$$

Thus $\sum_{k=1}^{\infty} a_k x_k$ converges.

Conversely, assume that $\sum_{k=1}^{\infty} a_k x_k$ converges $\forall x \in Ces(p)$. We want to show that $\exists B \in N$ such that $\sum_{k=1}^{\infty}(\sup_n |a_n|.k)^{t_k} B^{-t_k} < \infty$. On contrary, suppose that

$$\sum_{k=1}^{\infty}(\sup_n |a_n|.k)^{t_k} b^{-t_k} = \infty, \quad \forall b \in N \tag{3.12}$$

For each $k_0 \in N$

$$\sum_{k>k_0}(\sup_n |a_n|.k)^{t_k} b^{-t_k} = \infty \tag{3.13}$$

by (3.13), let $b_1 = 1$ we have $k_1 \in N$ such that $\sum_{k\leq k_1}(\sup_n |a_n|.k)^{t_k} {b_1}^{-t_k} > 1$. From (3.13), we can choose $b_2 > b_1$ and $k_2 > k_1$ such that

$$\sum_{k_1<k\leq k_2}(\sup_n |a_n|.k)^{t_k} {b_2}^{-t_k} > 1. \tag{3.14}$$

Doing this way go on, we have sequence $1 = k_0 < k_1 < k_2 < \ldots$ and $b_1 < b_2 < \ldots, b_i > 2^i$ such that

$$\sum_{k_{i-1}<k\leq k_i}(\sup_n |a_n|.k)^{t_k} {b_i}^{-t_k} > 1. \tag{3.15}$$

Let $c_i = \sum_{k_{i-1}<k\leq k_i}(\sup_n |a_n|.k)^{t_k} {b_i}^{-t_k}$ and $x = (x_k)$, $x_k = (sgn\ a_k)|a_k|^{-1}{c_i}^{-1}$. We want to show that $x \in Ces(p)$.

$$\sum_{k_{i-1}<k\leq k_i}(\frac{1}{k}\sum_{n=1}^{k}|x_n|)^{p_k} = \sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}(\sum_{n=1}^{k}|a_n|^{-1}.{c_i}^{-1})^{p_k}.$$

Let $K_{n,1}$ and $K_{n,2}$ are partition of $1, 2, \ldots, k$, if $n \in K_{n,1}$ then $|a_n| < 1$ and if $n \in K_{n,2}$ then $|a_n| \geq 1$ for all $n = 1, 2, \ldots, k$. So we have

$$= \sum_{k_{i-1}<k\leq k_i}\frac{1}{k^{p_k}}.\frac{1}{{c_i}^{p_k}}(\sum_{n\in K_{n,1}}|a_n|^{-1} + \sum_{n\in K_{n,2}}|a_n|^{-1})^{p_k}$$

$$\leq \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{p_k}}.\frac{1}{c_i^{p_k}}(\sum_{n\in K_{n,1}} (\max_n |a_n|^{-1}) + \sum_{n\in K_{n,2}} |a_n|^{-1})^{p_k}$$

$$\leq \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{p_k}}.\frac{1}{c_i^{p_k}}(\sum_{n\in K_{n,1}} (\max_n |a_n|^{-1}) + \sum_{n\in K_{n,2}} 1)^{p_k}$$

$$\leq \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{p_k}}.\frac{1}{c_i^{p_k}}(\sum_{n=1}^{k}(\max_n |a_n|^{-1}) + \sum_{n=1}^{k} 1)^{p_k}$$

$$= \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{p_k}}.\frac{1}{c_i^{p_k}}(k(\max_n |a_n|^{-1}) + k)^{p_k}$$

$$\leq \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{p_k}}.\frac{1}{c_i^{p_k}}(k_i\{(\max_n |a_n|^{-1}) + 1\})^{p_k}$$

$$\leq (k_i\{(\max_n |a_n|^{-1}) + 1\})^{\sup_k p_k} \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{p_k}}.\frac{1}{c_i^{p_k}}$$

$$\leq (k_i\{(\max_n |a_n|^{-1}) + 1\})^{G} \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{p_k}}.1$$

$$\leq (k_i\{(\max_n |a_n|^{-1}) + 1\})^{G} \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^{\inf_k p_k}} \quad ; k^{\inf_k p_k} < k^{p_k} \Rightarrow \frac{1}{k^{\inf_k p_k}} \geq \frac{1}{k^{p_k}}$$

$$= M_i \sum_{k_{i-1}<k\leq k_i} \frac{1}{k^L} \quad ; L = \inf_k p_k,\ L > 1,\ M_i = (k_i\{(\max_n |a_n|^{-1}) + 1\})^{G}.$$

Hence $\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{k}|x_n|)^{p_k} \leq M_i \sum_{k=1}^{\infty} \frac{1}{k^L} < \infty$. Then $x \in Ces(p)$. Since $x = (x_k)$, $x_k = (sgn\, a_k)|a_n|^{-1}c_i^{-1}$ for each $i \in N$ we have

$$\sum_{k_{i-1}<k\leq k_i} a_k x_k = \sum_{k_{i-1}<k\leq k_i} |a_k|.|a_k|^{-1}c_i^{-1}$$

$$\geq \sum_{k_{i-1}<k\leq k_i} \frac{1}{c_i^{p_k}}$$

$$\geq \sum_{k_{i-1}<k\leq k_i} \frac{1}{c_i^{\sup_k p_k}}$$

$$= \frac{1}{c_i^G} \sum_{k_{i-1}<k\leq k_i} 1 \quad ; G = \sup_k p_k$$

Hence, $\sum_{k=1}^{\infty} a_k x_k \geq \frac{1}{c_i^G}\sum_{k=1}^{\infty} 1 = \infty$ which is a contradiction. The proof is complete. □

The next proposition give a relationship between the β-dual of vector-valued and scalar-valued sequence spaces which obtained from S.Suantai [15].

Proposition 3.3.4. *Let X be Banach space and F a normal scalar-valued sequence space and define $F(X) = \{(x_k) \in W(X) : (\|x_k\|) \in F\}$. Then for $(f_k) \subset X'$, the topological dual of X, $(f_k) \in F(X)^\beta$ if and only if $(\|f_k\|) \in F^\beta$.*

We used previous proposition to prove β-dual of $Ces(X,p)$ as follows.

Proposition 3.3.5. *Let X be Banach space and $Ces(p)$ a normal scalar-valued sequence space and define $Ces(X,p) = \{(x_k) \in W(X) : (\|x_k\|) \in Ces(p)\}$. Then for $(f_k) \subset X'$, the topological dual of X, $(f_k) \in Ces(X,p)^\beta$ if and only if $(\|f_k\|) \in Ces(p)^\beta$.*

Proof. If $(\|f_k\|) \in Ces(p)^\beta$, then $\sum_{k=1}^{\infty}(\sup_k \|f_k\|.k)^{q_k} B^{-q_k} < \infty$. For $x = (x_k) \in Ces(X,p)$, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| \le \sum_{k=1}^{\infty} \|f_k\|.\|x_k\|$$

$$= \sum_{k=1}^{\infty} \|f_k\|.k.B^{-1}.B.\frac{1}{k}\|x_k\|$$

$$\le \sum_{k=1}^{\infty}((\|f_k\|.k)^{q_k}.B^{-q_k} + B^{p_k}.(\frac{1}{k}\|x_k\|)^{p_k}) \qquad ;ab \le \frac{a^{q_k}}{q_k} + \frac{b^{p_k}}{p_k}$$

$$= \sum_{k=1}^{\infty}(\|f_k\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{p_k}.(\frac{1}{k}\|x_k\|)^{p_k}$$

$$\le \sum_{k=1}^{\infty}(\sup_k \|f_k\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{p_k}.(\frac{1}{k}\|x_k\|)^{p_k} \qquad ;\|f_k\| \le \sup_k \|f_k\|$$

$$\le \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{p_k}.(\frac{1}{k}\sum_{k=1}^{k}\|x_j\|)^{p_k} \qquad ;\|x_k\| \le \sum_{k=1}^{k}\|x_j\|$$

$$\le \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{\sup_k p_k}.(\frac{1}{k}\sum_{k=1}^{k}\|x_j\|)^{p_k} \quad ;B^{p_k} \le B^{\sup_k p_k}$$

$$= \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{q_k}.B^{-q_k} + B^{G}\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{k=1}^{k}\|x_j\|)^{p_k} \qquad ;G = \sup_k p_k$$

$$< \infty$$

,so that $(f_k) \in Ces(X,p)^\beta$.

On the other hand, suppose that $(f_k) \in Ces(X,p)^\beta$ and $a = (a_k) \in Ces(p)$. Since $Ces(p)$ is normal, $(|a_k|) \in Ces(p)$. We assume that $f_k \ne 0$ for all $k \in N$. Choose a sequence $(x_k) \subset X$ such that $\|x_k\| = 1$ and $f_k(x_k) \ne 0$ for all $k \in N$. Let $y = (a_k x_k)$, then $y \in Ces(X,p)$. Choose a sequence (t_k)

of scalars such that $|t_k| \leq 1$ and $f_k(t_k y_k) = \|f_k\||a_k|$ for all $k \in N$. Since $Ces(p)$ is normal, $(t_k y_k) \in Ces(X,p)$, so we have that

$$\sum_{k=1}^{\infty} |f_k(t_k y_k)| \leq \sum_{k=1}^{\infty} \|f_k\|.\|t_k y_k\|$$

$$= \sum_{k=1}^{\infty} \|f_k\|.k.B^{-1}.B.\frac{1}{k}\|t_k y_k\|$$

$$\leq \sum_{k=1}^{\infty}((\|f_k\|.k)^{q_k}.B^{-q_k} + B^{p_k}.(\frac{1}{k}\|t_k y_k\|)^{p_k}) \qquad ; ab \leq \frac{a^{q_k}}{q_k} + \frac{b^{p_k}}{p_k}$$

$$= \sum_{k=1}^{\infty}(\|f_k\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{p_k}.(\frac{1}{k}\|t_k y_k\|)^{p_k}$$

$$\leq \sum_{k=1}^{\infty}(\sup_k \|f_k\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{p_k}.(\frac{1}{k}\|t_k y_k\|)^{p_k} \qquad ; \|f_k\| \leq \sup_k \|f_k\|$$

$$\leq \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{p_k}.(\frac{1}{k}\sum_{k=1}^{k} \|t_j y_j\|)^{p_k} \; ; \|t_k y_k\| \leq \sum_{k=1}^{k} \|t_j y_j\|$$

$$\leq \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{q_k}.B^{-q_k} + \sum_{k=1}^{\infty} B^{\sup_k p_k}.(\frac{1}{k}\sum_{k=1}^{k} \|t_j y_j\|)^{p_k} \; ; B^{p_k} \leq B^{\sup_k p_k}$$

$$= \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{q_k}.B^{-q_k} + B^{G}\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{k=1}^{k} \|t_j y_j\|)^{p_k} \qquad ; G = \sup_k p_k$$

$$< \infty$$

Thus, $\sum_{k=1}^{\infty} |f_k(t_k y_k)|$ converges. This implies that $\sum_{k=1}^{\infty} \|f_k\||a_k| < \infty$, therefore $(\|f_k\|) \in Ces(p)^{\beta}$. □

Proposition 3.3.6. *If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then*

$$Ces(X,p)^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k} B^{-t_k} < \infty \text{ for some } B \in N\}$$

where $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$.

Proposition 3.3.7. *A norm $\|x\|_{Ces(X,p)}$ on a Banach space X is said to be equivaent to a norm $\|x\|_{Ces(p)}$ on X if there are positive numbers a and b such that for all $x \in X$ we have*

$$\|x\|_{Ces(p)} \leq \|x\|_{Ces(X,p)} \leq b\|x\|_{Ces(p)}$$

where $b = (\max_n |t_n|)^G$, $G = \sup_k p_k$.

Proof. For each $x \in X$ we have

$$\begin{aligned}\|x\|_{Ces(p)} &= \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{\infty}|x_n|)^{p_k} \quad , p_k > 1\\ &\leq \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{\infty}\|x_n\|)^{p_k} \quad ; |x_n| \leq \|x_n\|\\ &= \|x\|_{Ces(p)}.\end{aligned}$$

Thus,

$$\|x\|_{Ces(p)} \leq \|x\|_{Ces(X,p)}. \tag{3.16}$$

Since $\|x\|_{Ces(X,p)} = \sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{k}\|x_n\|)^{p_k}$. Choose scalar sequence (t_k) in Banach space X with $\|t_x\| = 1$ such that $\|x_n\| = |x_n t_n|$.

$$\begin{aligned}\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{k}\|x_n\|)^{p_k} &= \sum_{k=1}^{\infty}\frac{1}{k^{p_k}}(\sum_{n=1}^{k}|x_n t_n|)^{p_k}\\ &\leq \sum_{k=1}^{\infty}\frac{1}{k^{p_k}}(\max_n |t_n|.\sum_{n=1}^{k}|x_n|)^{p_k}\\ &\leq (\max_n |t_n|)^{\sup_k p_k}\sum_{k=1}^{\infty}(\frac{1}{k}\sum_{n=1}^{k}|x_n|)^{p_k}\\ &= b\|x\|_{Ces(p)} \quad ; b = (\max_n |t_n|)^G,\ G = \sup_k p_k.\end{aligned}$$

Hence,

$$\|x\|_{Ces(X,p)} \leq b\|x\|_{Ces(p)}. \tag{3.17}$$

By (3.16) and (3.17) we obtained $\|x\|_{Ces(p)} \leq \|x\|_{Ces(X,p)} \leq b\|x\|_{Ces(p)}$. □

CHAPTER 4

MAIN RESULTS

4.1 Mapping into Maddox Sequence Spaces

In this chapter, we use Lemma 2.2.2.9 and Proposition 3.3.1 to characterize matrix transformations of $Ces(X,p)$ into Maddox sequence spaces. We obtained these theorems.

Theorem 4.1.1. *Let $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers such that $p_k>1$ and $\frac{1}{p_k}+\frac{1}{t_k}=1$ for all $k\in N$, and $A=(f_k^n)$ an infinite matrix. Then $A:Ces(X,p)\to \ell(q)$ if and only if*

(1) *for each $n\in N$ there exists $B_n\in N$ such that*

$$\sum_{k=1}^{\infty}(\sup_j \|f_j^n\|.k)^{t_k}B_n^{-t_k}<\infty,$$

(2) *for each $k\in N$, $\sum_{n=1}^{\infty}|f_k^n(x)|^{q_n}<\infty$ for every $x\in X$ and*
(3) *for each $r\in N$ there exists $B_r\in N$ such that*

$$\sum_{k\in K}(\frac{1}{k}\sum_{j=1}^{k}\|x_j\|)^{p_k}<\frac{1}{B_r}\ \Rightarrow\ \sum_{n=1}^{\infty}|\sum_{k\in K}f_k^n(x_k)|^{q_n}<\frac{1}{r}$$

for all $x=(x_k)\in\Phi(X)$ and all finite subsets K of N.

Theorem 4.1.2. *Let $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers with $p_k>1$ and $\frac{1}{p_k}+\frac{1}{t_k}=1$ for all $k\in N$ and $A=(f_k^n)$ an infinite matrix. Then $A:Ces(X,p)\to\ell_\infty(q)$ if and only if there exists $B\in N$ such that*

$$\sup_n(\sum_{k=1}^{\infty}(\sup_j\|f_j^n\|.k)^{t_k}B^{-t_k})^{q_n}<\infty.$$

Proof. By Proposition 3.3.1, we have $\sum_{k=1}^{\infty}(\sup_j\|f_j\|.k)^{t_k}B^{-t_k}<\infty$ for all $x=(x_k)\in Ces(X,p)$. Since $A=(f_k^n):Ces(X,p)\to\ell_\infty(q)$ and definition of $\ell_\infty(q)$, we have that $\sup_n(\sum_{k=1}^{\infty}(\sup_j\|f_j^n\|.k)^{t_k}B^{-t_k})^{q_n}<\infty$. □

Theorem 4.1.3. *Let $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers with $p_k>1$ and $\frac{1}{p_k}+\frac{1}{t_k}=1$ and $q_k>1$ for all $k\in N$ and $A=(f_k^n)$ an infinite matrix. Then $A:Ces(X,p)\to\ell_\infty(q)$ if the following two conditions hold;*

(1) *for each* $n \in N$ *there exists* $B_n \in N$ *such that*

$$\sum_{k=1}^{\infty} (\sup_j \|f_j^n\|.k)^{t_k}.B_n^{-t_k} < \infty, \quad \textit{and}$$

(2) *for each* $n \in N$,

$$\sup_K \sum_{n=1}^{\infty} (\sum_{k \in K} \|f_k^n\|.k)^{q_n} < \infty,$$

where supremum is taken over all finite subsets K *of* N.

Proof. Suppose that two conditions hold. Then by Proposition 3.3.1, condition (1) implies condition (1) of Lemma 2.2.2.9. Since condition (2), we have that there exists $L \in N$ such that

$$\sum_{n=1}^{\infty} (\sum_{k \in K} \|f_k^n\|.k)^{q_n} < L \tag{4.1}$$

for all the finite subsets K of N. Then for each $x \in X - \{0\}$ and (1) we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} |f_k^n(x)|^{q_n} &\leq \sum_{n=1}^{\infty} (\|f_k^n\| \cdot \|x\|)^{q_n} \\
&\leq \sum_{n=1}^{\infty} (\|f_k^n\|.k)^{q_n} \|x\|^{q_n} \quad ; \|f_k^n\| \cdot \|x\| \leq k.\|f_k^n\| \cdot \|x\|, \forall k \in N \\
&\leq \|x\|^{\sup_n q_n} \sum_{n=1}^{\infty} (\|f_k^n\|.k)^{q_n} \\
&\leq \|x\|^{\beta}.L \qquad \beta = \sup_n q_n \\
&< \infty.
\end{aligned}$$

Thus, we have $(f_k^n(x))_{n=1}^{\infty} \in \ell(q)$ for all $x \in X$ and $k \in N$. Hence condition (2) of Lemma 2.2.2.9 holds. We shall now show that condition (3) of Lemma 2.2.2.9 is satisfied. Let $\varepsilon > 0$ and $x = (x_k) \in \Phi(X)$. Recall that $\|x\| = (\sum_{k=1}^{\infty} (\frac{1}{k} \sum_{n=1}^{k} \|x_n\|)^{p_k})^{\frac{1}{M}}$,where $M = \sup_k p_k$. If $\|x\| \leq 1$, then for all $k \in N$ we have

$$\frac{1}{k}\|x_k\| \leq \|x\|^{\frac{M}{p_k}} \leq \|x\| \;\Rightarrow\; \|x_k\| \leq k.\|x\| \tag{4.2}$$

Since $x = (x_k) \in \Phi(X)$, there is a finite subset K_0 of N such that

$$\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{k \in K_0} f_k^n(x_k) \quad \text{for all } n \in N. \tag{4.3}$$

Therefore, we have (1),(2) and (3) that

$$\begin{aligned}
\|Ax\| &= (\sum_{n=1}^{\infty} |\sum_{k=1}^{\infty} f_k^n(x_k)|^{q_n})^{\frac{1}{G}} \qquad ; G = \max_n\{1, \sup_n q_n\} = \sup_n q_n \\
&= (\sum_{n=1}^{\infty} |\sum_{k\in K_0} f_k^n(x_k)|^{q_n})^{\frac{1}{G}} \\
&\leq (\sum_{n=1}^{\infty} (\sum_{k\in K_0} \|f_k^n\| \|x_k\|)^{q_n})^{\frac{1}{G}} \\
&\leq (\sum_{n=1}^{\infty} (\sum_{k\in K_0} \|f_k^n\|.k\|x\|)^{q_n})^{\frac{1}{G}} \\
&\leq (\|x\| \sum_{n=1}^{\infty} (\sum_{k\in K_0} \|f_k^n\|.k)^{q_n})^{\frac{1}{G}} \quad ; \|x\|^{q_n} \leq \|x\| \\
&\leq (\|x\|.L)^{\frac{1}{G}}. \qquad (4.4)
\end{aligned}$$

By (4.4) that implies $A : \Phi(X) \to \ell(q)$. Choose $\delta = \min\{1, \frac{\varepsilon^G}{L}\}$ and follows by (4.4) that

$$\|x\| < \delta \Rightarrow \|Ax\| < \varepsilon.$$

Hence, we obtain $A : \Phi(X) \to \ell(q)$ is continuous. Thus, by Lemma 2.2.2.9 we have that $A : Ces(X,p) \to \ell(q)$. □

Theorem 4.1.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to c_0(q)$ if and only if*

(1) *for all m,k ∈ N, $m^{\frac{1}{q_n}} f_k^n \to^{w^*} 0$ as $n \to \infty$ and*
(2) *for each m∈ N,*

$$(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} (\sup_j \|f_j^n\|)^{t_k} k^{t_k} r^{-t_k}) \to 0 \text{ as } r \to \infty \text{ uniformly for } n \geq 1.$$

Proof. By [1, Proposition 2.3(i)], we have $c_0(q) = \bigcap_{m=1}^{\infty} c_{0(m^{\frac{1}{q_n}})}$. By [1, Proposition 2.2(ii),(iv) and Theorem 1.6], we have

$$\begin{aligned}
A : Ces(X,p) \to c_0(q) &\Leftrightarrow A : Ces(X,p) \to \bigcap_{m=1}^{\infty} c_{0(m^{\frac{1}{q_n}})} \\
&\Leftrightarrow A : Ces(X,p) \to c_{0(m^{\frac{1}{q_n}})} \quad \text{, for all } m \in N \\
&\Leftrightarrow (m^{\frac{1}{q_n}} f_k^n) : Ces(X,p) \to c_0 \quad \text{, for all } m \in N \\
&\Leftrightarrow \text{the conditions (1) and (2) hold.}
\end{aligned}$$

□

Theorem 4.1.5. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) *there exists $B \in N$ such that $\sum_{k=1}^{\infty} (\sup_n \|f_n\|.k)^{t_k}.B^{-t_k} < \infty$*

(2) *for all $m,k \in N$, $m^{\frac{1}{q_n}}(f_k^n - f_k) \to^{w^*} 0$ as $n \to \infty$ and*

(3) *for each $m \in N$,*

$$(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}}(\sup_j \|f_j^n - f_j\|)^{t_k}.k^{t_k}.r^{-t_k}) \to 0 \text{ as } r \to \infty \text{ uniformly for } n \geq 1.$$

Proof. If $A : Ces(X,p) \to c(q)$, we have $A : Ces(X,p) \to (c_0(q) \oplus < e >)$, since $c(q) = (c_0(q) \oplus < e >)$. It follows from [1, Theorem 3.1.2] that $A = B + C$, where $B : Ces(X,p) \to c_0(q)$ and $C : Ces(X,p) \to < e >$. Let $C = (h_k^n)$. Since $\Phi(X) \subseteq Ces(X,p)$, we have

$$(h_k^n(x))_{n=1}^{\infty} \in < e > \text{ for all } x \in X \text{ and } k \in N,$$

which implies that $h_k^n = h_k^{n+1}$ for all $n, k \in N$. For each $k \in N$, let $f_k = h_k^1$. Then we have $B = (f_k^n - f_k)_{n,k} : Ces(X,p) \to c_0(q)$. Thus the conditions (2) and (3) are hold by Theorem 4.1.4 and the condition (1) holds by Proposition 3.3.1.

Conversely, assume that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that the conditions (1),(2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. The condition (1) implies by Proposition 3.3.1 that $C : Ces(X,p) \to < e >$. The conditions (2) and (3), by Theorem 4.1.4, imply that $B : Ces(X,p) \to c_0(q)$. By [1, Theorem 3.1.2], we obtain that $A : Ces(X,p) \to c(q)$. This completes the proof. □

4.2 Mapping into $M_\infty(q), \underline{\ell_\infty}(q), E_r(q)$ and $F_r(q)$

We give characterization of the infinite matrix transformations of $Ces(X,p)$ *into $M_\infty(q)$, $\underline{\ell_\infty}(q)$, $E_r(q)$ and $F_r(q)$.*

Theorem 4.2.1. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to M_\infty(q)$ if and only if*

(1) *for each $m,n \in N$ there exists $B \in N$ such that*

$$\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}}(\sup_j \|f_j^n\|)^{t_k}.k^{t_k}B^{-t_k} < \infty$$

(2) *for all $m,k \in N$, $\sum_{n=1}^{\infty} m^{\frac{1}{q_n}}|f_k^n(x)| < \infty$ for every $x \in X$ and*

(3) *for each $m,r \in N$ there exists $S \in N$ such that*

$$\sum_{k \in K}(\frac{1}{k}\sum_{j=1}^{k}\|x_j\|)^{p_k} < \frac{1}{S} \quad \Rightarrow \quad \sum_{n=1}^{\infty} m^{\frac{1}{q_n}}|\sum_{k \in K} f_k^n(x_k)| < \frac{1}{r},$$

for all $x = (x_k) \in \Phi(X)$ and all finite subsets K of N.

Proof. By [1, Proposition 2.3(vii)],we have $M_\infty(q) = \bigcap_{m=1}^{\infty} \ell_{(m^{\frac{1}{q_n}})}$. By [1, Proposition 2.2(ii) and (iv)] and Theorem 4.1.1, we have

$$A : Ces(X,p) \to M_\infty(q) \Leftrightarrow A : Ces(X,p) \to \bigcap_{m=1}^{\infty} \ell_{(m^{\frac{1}{q_n}})}$$

$$\Leftrightarrow A : Ces(X,p) \to \ell_{(m^{\frac{1}{q_n}})}, \text{ for all } m \in N$$

$$\Leftrightarrow (m^{\frac{1}{q_n}} f_k^n)_{n,k} : Ces(X,p) \to \ell, \text{ for all } m \in N$$

$$\Leftrightarrow \text{the conditions (1),(2) and (3) hold.}$$

□

Theorem 4.2.2. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to \underline{\ell_\infty}(q)$ if and only if for each $m \in N$ there exists $B_m \in N$ such that*

$$\sup_n \Big(\sum_{k=1}^{\infty} s^{\frac{t_k}{q_n}} (\sup_j \|f_j^n\|)^{t_k} . k^{t_k} B_m^{-t_k}\Big) < \infty.$$

Proof. By [1, Proposition 2.3(vi)], we have $\underline{\ell_\infty}(q) = \bigcap_{s=1}^{\infty} \ell_{\infty_{(s^{\frac{1}{q_n}})}}$ and [1, Proposition 2.2(ii) and (iv)], we have

$$A : Ces(X,p) \to \underline{\ell_\infty}(q) \Leftrightarrow A : Ces(X,p) \to \bigcap_{s=1}^{\infty} \ell_{\infty_{(s^{\frac{1}{q_n}})}}$$

$$\Leftrightarrow A : Ces(X,p) \to \ell_{\infty_{(s^{\frac{1}{q_n}})}}, \quad \text{for all } s \in N$$

$$\Leftrightarrow (s^{\frac{1}{q_n}} f_k^n)_{n,k} : Ces(X,p) \to \ell_\infty, \quad \text{for all } s \in N$$

$$\Leftrightarrow \text{the condition holds.}$$

□

Theorem 4.2.3. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to E_r(q)$ if and only if there exists $B \in N$ such that*

$$\sup_n \Big(\sum_{k=1}^{\infty} n^{\frac{-rt_k}{q_n}} (\sup_j \|f_j^n\| . k)^{t_k} . B^{-t_k}\Big)^{q_n} < \infty$$

Proof. By [1, Proposition 2.3(viii)], we have $E_r(q) = \ell_\infty(q)_{(n^{\frac{-r}{q_n}})}$. By [1, Proposition 2.2(iv)], we have

$$A : Ces(X,p) \to E_r(q) \Leftrightarrow A : Ces(X,p) \to \ell_\infty(q)_{(n^{\frac{-r}{q_n}})}$$

$$\Leftrightarrow (n^{\frac{-r}{q_n}} f_k^n)_{n,k} : Ces(X,p) \to \ell_\infty(q)$$

$\Leftrightarrow$ the condition holds.

□

Theorem 4.2.4. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers such that $p_k > 1$ and $\frac{1}{p_k} + \frac{1}{t_k} = 1$ for all $k \in N$, and $A = (f_k^n)$ an infinite matrix. Then $A : Ces(X,p) \to F_s(q)$ if and only if*

(1) *for each $n \in N$ there exists $B_n \in N$ such that*

$$\sum_{k=1}^{\infty} n^{\frac{st_k}{q_n}} (\sup_j \|f_j^n\|.k)^{t_k}.B^{-t_k})^{q_n} < \infty,$$

(2) *for each $k \in N$, $\sum_{n=1}^{\infty} n^s |f_k^n(x)|^{q_n} < \infty$ for every $x \in X$ and*
(3) *for each $r \in N$, there exists $M_r \in N$ such that*

$$\sum_{k\in K} (\frac{1}{k} \sum_{j=1}^{k} \|x_j\|)^{p_k} < \frac{1}{M_r} \Rightarrow \sum_{n=1}^{\infty} n^s |\sum_{k\in K} f_k^n(x_k)|^{q_n} < \frac{1}{r},$$

for all $x = (x_k) \in \Phi(X)$ and all finite subsets K of N.

Proof. By [1,Proposition 2.3(ix)], we have $F_s(q) = \ell(q)_{(n^{\frac{s}{q_n}})}$. By [1, Proposition 2.2(iv)], we have

$$A : Ces(X,p) \to F_s(q) \Leftrightarrow A : Ces(X,p) \to \ell(q)_{(n^{\frac{s}{q_n}})}$$

$$\Leftrightarrow (n^{\frac{s}{q_n}} f_k^n)_{n,k} : Ces(X,p) \to \ell(q)$$

and we have the Theorem 4.1.1, $A : Ces(X,p) \to \ell(q)$. Therefore the conditions (1)-(3) hold. □

CHAPTER 5

CONCLUSION AND SUGGESTIONS

In this chapter we conclude the results from chapter 4, then an applications and suggestions about matrix transformations.

5.1 Conclusion

In this thesis, we let $p = (p_k)$, $p_k > 1$ and $q = (q_k)$ be bounded sequences of positive real numbers. $A = (f_k^n)$ be an infinite matrix of continuous functional and $j, r, s \geq 0$. The main result can be concluded as follows:

(5.1.1) $A : Ces(X,p) \to \ell(q)$ *if and only if*

(1) *for each* $n \in N$ *there exists* $B_n \in N$ *such that*

$$\sum_{k=1}^{\infty}(\sup_j \|f_j^n\|.k)^{t_k} B_n^{-t_k} < \infty,$$

(2) *for each* $k \in N$, $\sum_{n=1}^{\infty} |f_k^n(x)|^{q_n} < \infty$ *for every* $x \in X$ *and*
(3) *for each* $r \in N$ *there exists* $B_r \in N$ *such that*

$$\sum_{k\in K}(\frac{1}{k}\sum_{j=1}^{k}\|x_j\|)^{p_k} < \frac{1}{B_r} \quad \Rightarrow \quad \sum_{n=1}^{\infty}|\sum_{k\in K} f_k^n(x_k)|^{q_n} < \frac{1}{r}$$

for all $x = (x_k) \in \Phi(X)$ *and all finite subsets* K *of* N.

(5.1.2) $A : Ces(X,p) \to \ell_\infty(q)$ *if and only if there exists* $B \in N$ *such that*

$$\sup_n(\sum_{k=1}^{\infty}(\sup_j \|f_j^n\|.k)^{t_k} B^{-t_k})^{q_n} < \infty.$$

(5.1.3) $A : Ces(X,p) \to \ell_\infty(q)$ *if and only if the following two conditions hold*;

(1) *for each* $n \in N$ *there exists* $B_n \in N$ *such that*

$$\sum_{k=1}^{\infty}(\sup_j \|f_j^n\|.k)^{t_k}.B_n^{-t_k} < \infty, \quad \text{and}$$

(2) *for each* $n \in N$, $\sup_K \sum_{n=1}^{\infty}(\sum_{k\in K} \|f_k^n\|.k)^{q_n} < \infty$,
where supremum is taken over all finite subsets K *of* N.

(5.1.4) $A : Ces(X,p) \to c_0(q)$ *if and only if*

(1) *for all* $m,k \in N$, $m^{\frac{1}{q_n}} f_k^n \to^{w^*} 0$ as $n \to \infty$ *and*
(2) *for each* $m \in N$,

$$(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} (\sup_j \|f_j^n\|)^{t_k} k^{t_k} r^{-t_k}) \to 0 \text{ as } r \to \infty \text{ uniformly for } n \geq 1.$$

(5.1.5) $A : Ces(X,p) \to c(q)$ *if and only if there is a sequence* (f_k) *with* $f_k \in X'$ *for all* $k \in N$ *such that*

(1) *there exists* $B \in N$ *such that* $\sum_{k=1}^{\infty}(\sup_n \|f_n\|.k)^{t_k}.B^{-t_k} < \infty$
(2) *for all* $m,k \in N$, $m^{\frac{1}{q_n}}(f_k^n - f_k) \to^{w^*} 0$ *as* $n \to \infty$ *and*
(3) *for each* $m \in N$,

$$(\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} (\sup_j \|f_j^n - f_j\|)^{t_k}.k^{t_k}.r^{-t_k}) \to 0 \text{ as } r \to \infty \text{ uniformly for } n \geq 1.$$

(5.1.6) $A : Ces(X,p) \to M_\infty(q)$ *if and only if*

(1) *for each* $m,n \in N$ *there exists* $B \in N$ *such that*

$$\sum_{k=1}^{\infty} m^{\frac{t_k}{q_n}} (\sup_j \|f_j^n\|)^{t_k}.k^{t_k} B^{-t_k} < \infty$$

(2) *for all* $m,k \in N$, $\sum_{n=1}^{\infty} m^{\frac{1}{q_n}} |f_k^n(x)| < \infty$ *for every* $x \in X$ *and*
(3) *for each* $m,r \in N$ *there exists* $S \in N$ *such that*

$$\sum_{k\in K}(\frac{1}{k}\sum_{j=1}^{k} \|x_j\|)^{p_k} < \frac{1}{S} \quad \Rightarrow \quad \sum_{n=1}^{\infty} m^{\frac{1}{q_n}} |\sum_{k\in K} f_k^n(x_k)| < \frac{1}{r},$$

for all $x = (x_k) \in \Phi(X)$ *and all finite subsets* K *of* N.

(5.1.7) $A : Ces(X,p) \to \ell_\infty(q)$ *if and only if for each* $m \in N$ *there exists* $B_m \in N$ *such that*

$$\sup_n\Big(\sum_{k=1}^{\infty} s^{\frac{t_k}{q_n}} (\sup_j \|f_j^n\|)^{t_k}.k^{t_k} B_m^{-t_k}\Big) < \infty.$$

(5.1.8) $A : Ces(X,p) \to E_r(q)$ *if and only if there exists* $B \in N$ *such that*

$$\sup_n\Big(\sum_{k=1}^{\infty} n^{\frac{-rt_k}{q_n}} (\sup_j \|f_j^n\|.k)^{t_k}.B^{-t_k})^{q_n} < \infty$$

(5.1.9) $A : Ces(X,p) \to F_s(q)$ *if and only if*

(1) *for each* $n \in N$ *there exists* $B_n \in N$ *such that*

$$\sum_{k=1}^{\infty} n^{\frac{st_k}{q_n}} (\sup_j \|f_j^n\|.k)^{t_k}.B^{-t_k})^{q_n} < \infty,$$

(2) *for each* $k \in N$, $\sum_{n=1}^{\infty} n^s |f_k^n(x)|^{q_n} < \infty$ *for every* $x \in X$ *and*
(3) *for each* $r \in N$, *there exists* $M_r \in N$ *such that*

$$\sum_{k\in K}\Big(\frac{1}{k}\sum_{j=1}^{k}\|x_j\|\Big)^{p_k} < \frac{1}{M_r} \Rightarrow \sum_{n=1}^{\infty} n^s \Big|\sum_{k\in K} f_k^n(x_k)\Big|^{q_n} < \frac{1}{r},$$

for all $x = (x_k) \in \Phi(X)$ *and all finite subsets* K *of* N.

5.2 Applications of Infinite Matrix Transformations

Most applications of matrix transformations, the elements of matrix are finite matrix. There are applied in many fields. For example, the matrix transform chip (MTC) [9], wavelet matrix transform approach for the solution of electromagnetic integral equations [4] and matrix methods for switched capacitor filter design [10]. Another part are sequence spaces. The example of sequence spaces are theory on sequence spaces and shift register generators [6]and applications of sequence space and SRG theories to distributed sample scrambling [7]. In this thesis, we consider elements of matrix are an infinite matrix and sequence space in Banach space.

Hopefully, the applications of infinite matrix transformations between sequence space will arise in the future. We expect that many educators will be interested in applying this knowledge.

5.3 Suggestions

Research about matrix transformations between sequence spaces are study and define new sequence space. For example, concern with difference sequence space. When a new sequence space will arise that can study some properties, β-dual and matrix transformations. And we can extend the knowledge of any scalar-valued sequence space to vector-valued sequence space. Finally, there are many open problems about matrix transformations for further study.

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