SOLVING SYSTEMS OF NONLINEAR EQUATIONS

SOMNUK SRISAWAT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENT FOR THE DEGREE OF
MASTER OF SCIENCE IN APPLIED MATHEMATICS
SCHOOL OF GRADUATE STUDIES
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG
2003

ISBN
974 - 324 - 321 - 6

หัวข้อวิทยานิพนธ์	การหาผลเฉลยของระบบสมการไม่เชิงเส้น
นักศึกษา	นายสมนึก ศรีสวัสดิ์
รหัสประจำตัว	39065364
ปริญญา	วิทยาศาสตรมหาบัณฑิต
สาขาวิชา	คณิตศาสตร์ประยุกต์
พ.ศ.	2546
อาจารย์ผู้ควบคุมวิทยานิพนธ์	รศ.ผ่องพรรณ รัตนธนาวันต์
อาจารย์ผู้ควบคุมวิทยานิพนธ์ร่วม	รศ.ดร.ไมตรี โพธิ์สุข

บทคัดย่อ

เนื้อหาของวิทยานิพนธ์ฉบับนี้ เป็นการเสนอวิธีการทำซ้ำเพื่อหาผลเฉลยเชิงตัวเลขของระบบสมการไม่เชิงเส้นแบบใหม่ 8 รูปแบบ โดยใช้การประมาณค่าอนุพันธ์ในจาโคเบียนเมตริกซ์ซึ่งเป็นเมตริกซ์ของระบบสมการไม่เชิงเส้นนั้นๆ ด้วยวิธีผลต่างจำกัด(Finite Difference) ซึ่งวิธีการทำซ้ำแบบใหม่นี้จะเป็นอีกทางเลือกหนึ่งในการหาผลเฉลยเชิงตัวเลขของระบบสมการไม่เชิงเส้นโดยเฉพาะกับระบบสมการไม่เชิงเส้นที่มีฟังก์ชัน $f(x)$ ยากต่อการหาอนุพันธ์ ระบบสมการไม่เชิงเส้นอยู่ในรูป

$$f(x) = 0$$

เมื่อ $x \in R^n$, f เป็นฟังก์ชันจากเซตย่อยของ R^n ไปยังเซตย่อยของ R^n

หรือ

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

Thesis Title	Solving Systems of Nonlinear Equations
Student	Mr. Somnuk Srisawat
Student ID.	39065364
Degree	Master of Science
Programme	Applied Mathematics
Year	2003
Thesis Advisor	Assoc.Prof. Pongpan Rattanathanawan
Thesis Co-Advisor	Assoc.Prof.Dr. Maitree Podisuk

ABSTRACT

In this thesis we will introduce the new formulas of iterative methods for solving systems of nonlinear equations which use to approximate the derivative in the Jacobian matrix of systems of nonlinear equations from finite difference method. The new iterative methods should use to solve the system of nonlinear equations which is difficult to find derivative in the Jacobian matrix. A system of nonlinear equations is of the form

$$f(x) = 0$$

where $x \in R^n$, f is a function from a subset of R^n to a subset of R^n.

or

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my major advisor, Assoc.Prof. Pongpan Rattanathanawan and co-advisor, Assoc.Prof.Dr. Maitree Podisuk, for their encouragement, continual suggestions and constructive criticisms to me throughout this research, which enable me to complete this thesis.

Also, I would like to thank my friends who helped and gave me suggestions about this thesis.

Somnuk Srisawat

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

A system of nonlinear equations is a n - equation and n - unknowns, which one or more of the equations are nonlinear and two or more independent variables. System of nonlinear equations can be put in the general from

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

or

$$f(x) = 0 \tag{1.1}$$

where $f(x) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

A vector $[x_1, x_2, \ldots, x_n]^T$ that satisfies $f(x) = 0$ will be called a root or solution of the system $f(x) = 0$. We need to know how to find approximate value of x, when f is real function. The method for solving systems of nonlinear equations is very important in the study of applied mathematics.

There are two groups of Numerical method for finding the root or solution of the system (1.1). First, Nonmemory method, we have iterative methods for solving the system (1.1) base on Newton method which uses only one initial vector i.e. Newton method, Newton-Like method, Modified Newton method. Another is Memory method, we can find new data by using more than one initial vector i.e Multivariate secant method.

They are complicated for some systems of nonlinear equations which derivative are not available or are difficult to calculate for finding Jacobian matrix. The purpose of this thesis is to find the new iterative method which can solve the systems of nonlinear equations which derivative are not available or are difficult to calculate. It is not popular, but it is a good and appropriate method to solve such systems.

In this thesis, we will find formulas of the same type as the Modified Newton method for solving systems of nonlinear equations. In chapter 2, we will discuss the important theorems concern with the Modified Newton method. We will introduce the new iterative methods that are developed from the Modified Newton method for solving systems of nonlinear equations in chapter 3. In chapter 4, we will find the approximating solution of examples including application problem. Finally, we will make a conclusion and suggestions in chapter 5.

Many problems require the method for solving the systems of nonlinear equations. The following examples.

The plane $x+y+z-12=0$ intersects the paraboloid $z=x^2+y^2$ in an ellipse (Figure 1.1). Find the highest and lowest point on this ellipse by Lagrange Multipliers with two constraints.

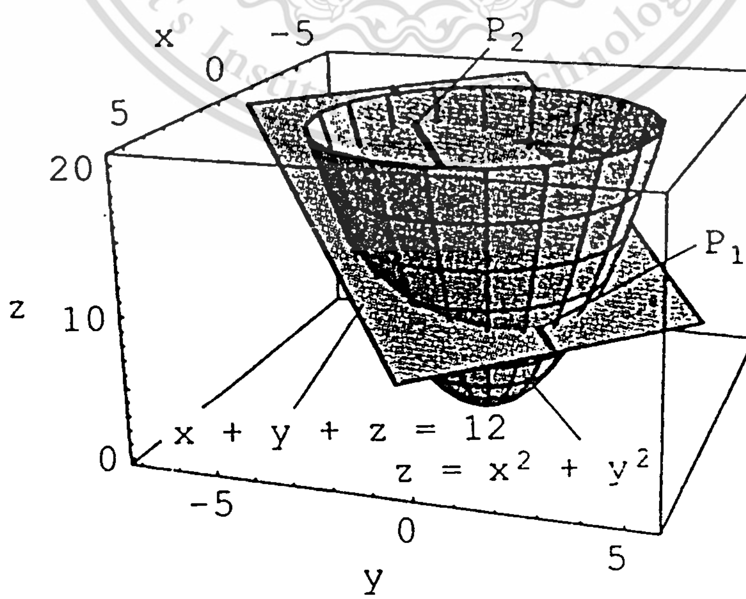

Figure 1.1 The plane and paraboloid intersecting in the ellipse

The height of the point (x,y,z) is z, so we want to find the maximum and minimum values

$$f(x,y,z) = z$$

subject to the two conditions

$$g(x,y,z) = x + y + z - 12 = 0$$

and

$$h(x,y,z) = x^2 + y^2 - z = 0.$$

By Lagrange Multipliers with two constraints, we have

$$L(x,y,z,\lambda,\mu) = z + \lambda(x + y + z - 12) + \mu(x^2 + y^2 - z)$$

and the system of nonlinear equations

$$L_x(x,y,z,\lambda,\mu) = \lambda + 2\mu x = 0$$
$$L_y(x,y,z,\lambda,\mu) = \lambda + 2\mu y = 0$$
$$L_z(x,y,z,\lambda,\mu) = 1 + \lambda - \mu = 0$$
$$L_\lambda(x,y,z,\lambda,\mu) = x + y + z - 12 = 0$$
$$L_\mu(x,y,z,\lambda,\mu) = x^2 + y^2 - z = 0$$

The next application is taken from the Bohr theory of the hydorgen spectrum in atomic physics. A certain constant known as the Rydberg constant, usually denoted by R_∞, plays an important part in that theory, but it cannot be measured directly The value must be inferred from measurements of other constants. Approximate values R_H and R_{He} can be determined from measurements of the spectrum of hydrogen and helium, respectively. If we knew the exact mass M_H of the nucleus of a hydrogen atom and the exact mass m of an electron, then theoretically. We could determine R_∞ from the equation

$$R_\infty = \left(1 + \frac{m}{M_H}\right) R_H$$

Suppose, for the moment, that m and M_H are unknown (because even our best values are inferred from other measured variables). The atomic weight A_H of hydrogen can be measured. Also, $A_H = M_H + m$. Similar equations hold for the helium atom. So we have a system of four equations:

$$R_\infty = \left(1 + \frac{m}{M_H}\right) R_H$$

$$R_\infty = \left(1 + \frac{m}{M_{He}}\right) R_{He}$$

$$A_H = M_H + m$$

$$A_{He} = M_{He} + m$$

In this system, the numbers A_H, A_{He}, R_H and R_{He} can be taken as known because their values can be determined from laboratory measurements. The number R_∞ and masses m, M_H and M_{He} will be regarded as unknown. Their values can be determined by solving the system of four equations.

The work is simplified if we first make some algebraic changes in the system. Let $x_1 = R_\infty, x_2 = m, x_3 = M_H$ and $x_4 = M_{He}$. Then our system becomes

$$x_1 = \left(1 + \frac{x_2}{x_3}\right) R_H$$

$$x_1 = \left(1 + \frac{x_2}{x_4}\right) R_{He}$$

$$x_2 + x_3 = A_H$$

$$2x_2 + x_4 = A_{He}$$

Since $x_1 = R_\infty$ and R_∞ is approximated by the measured number R_H, we have an initial estimate of x_1. Similarly, since x_2 is the mass of the electron and we know that the value is somewhere around $\frac{1}{2000}$ of a unit of atomic mass, an initial estimate of 0.0005 for x_2 seems reasonable. We need values of A_H, A_{He}, R_H and R_{He} to perform the calculations. Suppose laboratory measurements produced the following values: A_H = 1.00812, A_{He} = 4.00388, R_H = 109677.68, and R_{He} = 109722.34. What are the corresponding values for Rydberg constant R_∞ and masses m, M_H and M_{He}?. A machine implementation of the iterations suggested here produced the following results.

In order to find the solutions of this problem, we need to solve the systems of nonlinear equations and their numerical solution that will be stated in chapter 4.

CHAPTER 2

LITERATURE REVIEWS

In this chapter, we will consider multivariable problems by discussing the systems of nonlinear equations in real variable. It is

$$\text{Given } f : X \to X$$
$$\text{find } x \in X \text{ for which } f(x) = 0 \qquad (2.1)$$

where $X \equiv R^n$ is n-dimensional real vector space, and f is assumed to be continuously differentiable.

Of course, (2.1) is just the standard way of denoting a system of n-nonlinear equations in n-unknowns, with the convention that right-hand side of each equation is zero, and throughout this study we let

$D \subset X$ denotes the domain of $f(x)$,

D_0 denotes closed convex subset of D,

$N(x,t)$ denotes the open neighborhood of radius t around x, i.e.

$$N(x,t) = \{y \in X : \|x - y\| < t\}$$

$L(X,Y)$ denotes a real Banach space,

$f \in Lip_k(D)$ if $\|f(x) - f(y)\| \leq K\|x - y\|$ for some constant K,

$\{x_k\}$ denotes sequence of vectors in R^n,

$B(x) = f'(x) - A(x)$,

and $M(x) = f'(x) - H(x)$.

In the analysis of Newton's method, it will be necessary to assume that the Jacobian matrix,

$$J(x)=\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \tag{2.2}$$

where f_{ij} denotes the partial derivative of $f_i(x)$ with respect to the j^{th} variable and evaluated at x, is at least continuous at the solution x^* ; that is

$$\|f'(x^*+h)-f'(x^*)\| \to 0 \text{ as } h \to 0 .$$

On occasion, it will also be useful to assume, instead of continuity of f' that the derivative satisfies the following property.

Definition 2.1 The mapping $f : R^n \to R^n$ is (Totally or Frechet) differentiable at x if the Jacobian matrix (2.2) exists at x and

$$\lim_{h\to 0}\frac{\|f(x+h)-f(x)-f'(x)h\|}{\|h\|}=0 \tag{2.3}$$

Note that if $n=1$, Definition 2.1 reduces to the usual definition of differentiability. Note also that if f is differentiable at x, then f is continuous at x; this follows from the inequality

$$\|f(x+h)-f(x)\| \le \|f(x+h)-f(x)-f'(x)h\| + \|f'(x)h\| . \tag{2.4}$$

One of the basic tools of nonlinear analysis is the mean value theorem. If f is a differentiable function from R^1 to R^1, this state that

$$f(x)-f(y)=f'(z)(x-y) \tag{2.5}$$

for some point z between x and y. Unfortunately, this result does not extend verbatim to mapping from R^n to R^n. However, we are able to prove some results which are often just as useful. First, we define the integral $\int_a^b g_i(t)dt$, $i = 1,2,\ldots,n$ where $g_1, g_2, \ldots, g_n$ are the components of G. Thus, for example, if $f : R^n \to R^n$ the relation

$$f(y) - f(x) = \int_0^1 f'(x + t(y - x))(y - x)\,dt \qquad (2.6)$$

is equivalent to

$$f_i(y) - f_i(x) = \int_0^1 \sum_{j=1}^{n} f_{ij}(x + t(y - x))(y_j - x_j)\,dt \qquad (2.7)$$

for $i = 0,1,2,\ldots,n$

For $n = 1$, (2.6) is simply the fundamental theorem of the integral calculus. Hence the next result extension of the theorem to n - dimensions.

Lemma 2.1 Assume that $f : R^n \to R^n$ is continuously differentiable on a convex set $D \subset R^n$. Then for any $x, y \in D$, (2.6) holds.

Proof: For fixed $x, y \in D$ define the functions $g_i : [0,1] \subset R^1 \to R^1$ by

$$g_i(t) = f_i(x + t(y - x)), \quad t \in [0,1],\ i = 1,2,\ldots,n. \qquad (2.8)$$

By the convexity of D, it follows that g_i is continuously differentiable on [0,1] and thus fundamental theorem of the integral calculus implies that

$$g_i(1) - g_i(0) = \int_0^1 g_i'(t)\,dt \qquad (2.9)$$

But a simple calculation shown that

$$g_i'(t) = \sum_{j=1}^{n} f_{ij}(x + t(y - x))(y_j - x_j) \qquad (2.10)$$

so that (2.9) is equivalent to (2.7).

For the next result, we first need a lemma on integration.

Lemma 2.2 Assume that $G : [a,b] \subset R^1 \to R^n$ is continuous. Then

$$\left\| \int_a^b G(t)\,dt \right\| \le \int_a^b \|G(t)\|\,dt\,. \qquad (2.11)$$

proof : Since any norm is a continuous function on R^n, both integrals of (2.11) exist and therefore any $\varepsilon > 0$ there is a partition $a < t_0 < \ldots < t_p < b$ of $[a,b]$ such that

$$\left\| \int_a^b G(t)\,dt - \sum_{i=1}^{p} G(t_i)(t_i - t_{i-1}) \right\| < \varepsilon \qquad (2.12)$$

and

$$\left| \int_a^b \|G(t)\| dt - \sum_{i=1}^{p} \|G(t_i)\| (t_i - t_{i-1}) \right| < \varepsilon$$

Hence

$$\left\| \int_a^b G(t)dt \right\| \le \left\| \sum_{i=1}^{p} G(t_i)(t_i - t_{i-1}) \right\| + \varepsilon \le \sum_{i=1}^{p} \|G(t_i)\|(t_i - t_{i-1}) + \varepsilon \le \int_a^b \|G(t)\| dt + 2\varepsilon$$

and, since ε was arbitrary, (2.11) must be valid.

By means of Lemma 2.1 and 2.2 we can prove the following useful alternative of the mean value theorem.

Theorem 2.3 Assume that $f : R^n \to R^n$ is continuously differentiable on the convex set $D \subset R^n$. Then for any $x, y \in D$,

$$\|f(x) - f(y)\| \le \sup_{0 \le t \le 1} \|f'(x + t(y - x))\| \|y - x\|. \tag{2.13}$$

Proof: By Lemma 2.1 and Lemma 2.2, we have

$$\|f(x) - f(y)\| = \left\|\int_0^1 f'(x + t(y - x))(y - x)dt\right\|$$

$$\le \int_0^1 \|f'(x + t(y - x))\| \|y - x\| dt$$

$$\le \sup_{0 \le t \le 1} \|f'(x + t(y - x))\| \int_0^1 \|y - x\| dt$$

which is (2.13).

Lemma 2.4 Let $f' \in Lip_k(D_0)$ and let $x, y \in D_0$, then

$$\|f(y) - f(x) - f'(x)(y - x)\| \le \frac{1}{2} K \|x - y\|^2 \tag{2.14}$$

Proof: From $f : R^n \to R^n$ is differentiable and satisfied $f' \in Lip_k(D_0)$, that means

$\|f'(x) - f'(y)\| \le K\|x - y\|$ for all x, y in closed convex subset D_0 of D.

Since $f'(x)(y - x)$ is constant with respect to the integration, we have

$$f(y) - f(x) - f'(x)(y - x) = \int_0^1 [(f'(x + t(y - x))(y - x)) - f'(x)(y - x)]dt$$

$$= \int_0^1 [f'(x + t(y - x) - f'(x)](y - x)dt$$

Hence, the result follows by taking norms of both sides,

$$\begin{aligned}\|f(y)-f(x)-f'(x)(y-x)\| &= \left\|\int_0^1 [f'(x+t(y-x)-f'(x)](y-x)dt\right\| \\ &\le \int_0^1 \|f'(x+t(y-x)-f'(x)\|\|y-x\|dt \\ &\le \int_0^1 K\|x+t(y-x)-x\|\|y-x\|dt \\ &= \int_0^1 K\|t(y-x)\|\|y-x\|dt \\ &= \int_0^1 K\|\,t\,\|\|y-x\|^2\,dt \\ &= K\|y-x\|^2\int_0^1 t\,dt \\ &= \frac{1}{2}K\|y-x\|^2 .\end{aligned}$$

Definition 2.2 Let t_0 and t' be non-negative real numbers, g be a continuously differentiable real function on $[t_0, t_0+t']$ and G a continuously differentiable operator on $\overline{N}(x_0, t) \subset X$ into X. Then the equation $t = g(t)$ will be said to majorize the equation $x = G(x)$, or g majorize G, on $N(x_0, t')$ if

$$\|G(x_0)-x_0\| \le g(t_0)-t_0 \tag{2.15}$$

and

$$\|G'(x)\| \le g'(t) \text{ where } \|x-x_0\| \le t-t_0 < t' \tag{2.16}$$

Definition 2.3 Let t_0 and t' be non-negative real numbers, g be a real function on $[t_0, t_0+t']$ and G an operator sending $N(x_0, t')$ into X. Then the equation $t = g(t)$ will be said to weakly majorize the equation $x = G(x)$, or g weakly majorize G, if (2.15) holds and in addition

$$\|G(G(x))-G(x)\| \le g(g(t))-g(t) \tag{2.17}$$

when $\|x-x_0\| \le t-t_0 < t'$

and $$\|G(x)-x\| \le g(t)-t\,.$$

We remark that Lemma 2.4 and Lemma 2.5 are known results in mathematical analysis and Lemma 2.6 is given by Ortega [4].

Lemma 2.5 (The Banach Lemma) Let $M \in L(X,X)$ and $\|I-J\| \le \delta < 1$, then J^{-1} exists in $L(X,X)$ and $J^{-1} \le (1-\delta)^{-1}$.

Lemma 2.6 Let $\{x_k\}$ be a sequence in X and $\{t_k\}$ a sequence of non-negative real numbers such that

$$\|x_{k+1}-x_k\| \le t_{k+1}-t_k,\quad k=0,1,2,\ldots \tag{2.18}$$

and $t_k \to t^* < \infty$. By these conditions, there exists a point $s \in X$ such that $x_k \to s$ and

$$\|s-x_k\| \le t^*-t_k,\quad k=0,1,2,\ldots \tag{2.19}$$

Proof: The proof is immediate from

$$\begin{aligned}\|x_{k+p}-x_k\| &\le \sum_{i=1}^{p}\|x_{k+i}-x_{k+i-1}\| \\ &\le t_{k+p}-t_k \\ &\le t^*-t_k,\end{aligned}$$

which show that $\{x_k\}$ is a Cauchy Sequence.

We shall say that $\{t_k\}$ majorizes $\{x_k\}$ if $\|x_{k+1}-x_k\| \le t_{k+1}-t_k$, $k=0,1,2,\ldots$ holds.

The following theorem is Kantorovich Theorem. It is one of the fundamental theorems in numerical mathematics. The idea for proof this theorem in [5].

Theorem 2.7 (The Kantorovich Theorem) Let x_0 be in D_0 and let $\Gamma_0 = [f'(x_0)]^{-1}$ exist with $\Gamma_0 \in Lip_k(D_0)$,

$$\left\|\Gamma_0 f(x_0)\right\| \le \eta \text{ and } h = K\eta \le \frac{1}{2}. \tag{2.20}$$

Define
$$r_0(h) = \frac{1}{h}(1-\sqrt{1-2h})\eta \tag{2.21}$$

$$r_1(h) = \frac{1}{h}(1+\sqrt{1-2h})\eta. \tag{2.22}$$

Then if $N(x_0, r_0(h)) \subset D_0$, the sequence of iterates defined by Newton's method exists, remains in $N(x_0, r_0(h))$ and converges to s in $N(x_0, r_0(h))$ such that $f(s) = 0$. If $h < \frac{1}{2}$, s is the only root in $N(x_0, r_1(h)) \cap D_0$, and if $h = \frac{1}{2}$, s is unique in $N(x_0, r_1(h)) \cap D_0$. Furthermore, the sequence of the iterates satisfy the error bounds

$$\left\|s - x_m\right\| \le \frac{1}{h}\frac{1}{2^m}(1-\sqrt{1-2h})^{2m}\,\eta. \tag{2.23}$$

The following theorem is given by Kantorovich [5]. This theorem together with Lemma 2.6 and Definition 2.2 give the convergence of the sequence $\{x_k\}$ in X when the sequence of $\{t_k\}$ converges.

Theorem 2.8 If g majorizes G on $\overline{N}(x_0,t')$ and g has a fixed point in $[t_0,t_0+t']$, then G has a fixed point s in $\overline{N}(x_0,t')$. Furthermore, $x_{k+1}=G(x_k)$ and $t_{k+1}=g(t_k)$, $k=0,1,2,\ldots$ converges to s and t^* respectively with the real sequence majorizing the vector sequence.

Next lemma uses the weakly majorizing property and it is given by Dennis [6].

Lemma 2.9 If $g(t)\in(t,t_0+t')$ when $t\in(t_0,t_0+t')$ and g weakly majorizing G on $N(x_0,t')$, then there are elements $t^*\in[t_0,t_0+t'],s\in\overline{N}(x_0,t')$ such that

$$x_{k+1}=G(x_k) \tag{2.24}$$

and

$$t_{k+1}=g(t_k),\quad k=0,1,2,\ldots \tag{2.25}$$

converge to s and t^* respectively with the t sequence majorizing in x sequence.

Theorems 2.10-2.12 in this section are given by Dennis [6]. These theorems give the convergence of Newton-like method.

Theorem 2.10 Let A be a function on $N(x_0,r)$ such that $A(x)\in L(X,Y)$ for each x and $A(x)$ is invertible for each x in $N(x_0,r)$ and that there is a real, nonvanishing, nonincreasing function $a(t)$ on $[0,r)$ such that

$$\|A^{-1}(x)\|\le a(\|x-x_0\|)^{-1} \tag{2.26}$$

If $f'\in Lip_k(\overline{N}(x_0,r))$ then if $\sigma\ge 1$ and $\delta>0$ are real numbers such that

$$a(t)+\sigma kt \tag{2.27}$$

is isotonic on $(0,r)$, and

$$\|B(x)\| \le a(\|x-x_0\|) + \sigma k\|x-x_0\| - \delta \tag{2.28}$$

for every $x \in N(x_0,r)$

then $$g(t) = t + (a(t))^{-1}(0.5\sigma kt^2 - \delta t - a(0)\|(A(x_0))^{-1} - f(x_0)\|) \tag{2.29}$$

weakly majorizes $G(x) = x - (A(x))^{-1} f(x)$ on $N(x_0,r)$.

Theorem 2.11 Let $f' \in Lip_k(\overline{N}(x_0,r))$ and $[A(x_0)]^{-1}$ exist and be bounded in the norm by $[a(0)]^{-1}$, if $\|B(x_0)\| < a(0)$ and

$$h' = \frac{K}{a(0)}\|[A(x_0)]^{-1} f(x_0)\|a(0) - (\|B(x_0)\|)^2 \le \frac{1}{2} \tag{2.30}$$

and $$r_0' = \frac{1}{K}\left(1-\sqrt{1-2h'}\right)\left(a(0) - \|B(x_0)\|\right) \le r \tag{2.31}$$

Then if f has a unique zero $s \in \overline{N}(x_0,r_0')$, and

$$x'_{m+1} = x'_m - [A(x_0)]^{-1} f(x'_m), \qquad m = 0,1,2,\dots$$

converges to s from any $x_0' \in \overline{N}(x_0,r_0')$ such that

$$\|x_0' - x_0\| < r_1' = \frac{1}{K}\left(1-\sqrt{1-2h'}\right)\left(a(0) - \|B(x_0)\|\right).$$

If, in addition, σ, δ and a satisfy the conditions of theorem 2.10 and

$$h = \frac{1}{\delta^2}\sigma K \left\| [A(x_0)]^{-1} f(x_0) \right\| a(0) \le \frac{1}{2} \tag{2.32}$$

and

$$r_0 = \frac{1}{\sigma K}\left(1 - \sqrt{1-2h}\right)\delta < r, \tag{2.33}$$

then

$$x_{m+1} = x'_m - [A(x_m)]^{-1} f(x'_m), \qquad m = 0,1,2,\ldots \tag{2.34}$$

converges to s.

In the following theorem, we impose one more condition on $A(x)$ and one condition on $B(x)$ instead of $B(x_0)$.

Theorem 2.12 Let $f' \in Lip_k(\overline{D})$ where $x_0 \in D$ and D is an open convex subset of X. Assume that

$$\left\| [A(x_0)]^{-1} f(x_0) \right\| \le \alpha \tag{2.35}$$

$$\left\| [A(x_0)]^{-1} \right\| \le \beta \tag{2.36}$$

$$\left\| A(x) - A(x_0) \right\| \le \eta_0 + \eta_1 \left\| x - x_0 \right\|, \ \forall x \in D \tag{2.37}$$

$$\left\| B(x) \right\| \le \delta_0 + \delta_1 \left\| x - x_0 \right\|, \ \forall x \in D. \tag{2.38}$$

Then

$$\beta\delta_0 < 1, \ h' = \frac{\beta K \alpha}{(1-\beta\delta_0)^2} \le \frac{1}{2}$$

and $N(x_0, r'_0) \subset D$ where

$$r_0' = \frac{1-\sqrt{1-2h'}}{\beta K}(1-\beta\delta_0)$$

implied that f has a solution $r \in \overline{N}(x_0, r_0')$ which is unique in $D \cap N(x_0, r_0')$ where

$$r_1' = \frac{1-\sqrt{1+2h'}}{\beta K}(1-\beta\delta_0)$$

Furthermore $$x_{m+1}' = x_m' - [A(x_0)]^{-1} f(x_m')$$

converges to s from any $x_0' \in \overline{D} \cap N(x_0, r_1')$.

If, in addition, $\beta(\delta_0 + \eta_0) < 1$ and $h = \frac{\sigma\beta K\alpha}{(1-\beta\eta_0-\beta\delta_0)^2} \le \frac{1}{2}$ where $\sigma = \max\left(1, \frac{\delta_1+\eta_1}{K}\right)$, and $N(x_0, r_0) \subset D$,

$$r_0' = \frac{1-\sqrt{1-2h}}{\sigma\beta K}(1-\beta\eta_0-\beta\delta_0)$$

then

$$x_{m+1} = x_m - [A(x_m)]^{-1} f(x_m)$$

converges to s.

Main Results

The following theorem will ensure that the convergence of the Modified Newton method for solving systems of nonlinear equations which is a special kind of the Newton-like method. The proof of this theorem in [7].

Theorem 2.13 Let D be an open convex subset of the space X and $f' \in Lip_k(\overline{D})$. Assuming that $f(x)$ and $H(x)$ satisfy all the conditions of the previous theorems, then there exists a unique zero s in D so that for any point x_0 in D the sequence $\{x_m\}$ where

$$x_{m+1} = x_m - [H(x_m)]^{-1} f(x_m)$$

converges to s.

The last theorem of this section, Theorem 2.14, shows that the order of the convergence of the Modified Newton method for solving a system of nonlinear equations which is of second order. The proof of this theorem in [7].

Theorem 2.14 Let the conditions of Theorem 2.12 be satisfied and $\delta_0 = 0$, that is

$$\|M(x)\| \le \delta_1 \|x - x_0\|, \forall x \in D \tag{2.39}$$

Then the order of the convergence of the method is equal to 2.

CHAPTER 3

SOLVING SYSTEMS OF NONLINEAR EQUATIONS

In this chapter, we will introduce iterative method for solving systems of nonlinear equations.

The systems of nonlinear equations is

$$f(x) = 0 \tag{3.1}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $f(x) = \begin{bmatrix} f_1(x_1, x_n, \ldots, x_n) \\ f_2(x_1, x_n, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_n, \ldots, x_n) \end{bmatrix}$

First, we can find the solution of systems of nonlinear equations (3.1) by the Newton method which iterative equation is

$$x^{k+1} = x^k - [J(x^k)]^{-1} f(x^k), \qquad k = 0,1,2,\ldots \tag{3.2}$$

where $J(x)$ is Jacobian matrix of $f(x)$ defined by

$$J(x) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

where $f_{ij}(x)$ are partial derivative of $f_i(x)$ with respect to the j^{th} variable and evaluated at x.

Second, we can find the solution of systems of nonlinear equations (3.1) by the Modified Newton method which iterative equation is

$$x^{k+1} = x^k - [H(x^k)]^{-1} f(x^k), \qquad k = 0,1,2,\ldots \qquad (3.3)$$

where $H(x)$ is the diagonal matrix defined by

$$H(x) = \begin{bmatrix} f_{11} & 0 & \cdots & 0 \\ 0 & f_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & f_{nn} \end{bmatrix}$$

where $f_{ii}(x)$ are partial derivative of $f_i(x)$ with respect to the i^{th} variable and evaluated at x.

Next, we will introduce new iterative method for solving systems of nonlinear equations which do not use any derivative and developed from the Modified Newton method which iterative equation is

$$x^{k+1} = x^k - [H(x^k)]^{-1} f(x^k), \qquad k = 0,1,2,\ldots \qquad (3.4)$$

where $H(x)$ is the diagonal matrix with value (from upper left to lower right) of $f_{ii}(x)$, $i = 1,2,\ldots,n$ which approximate from The Numerical Analysis by next theorem.

3.1 Definition and theorem

Theorem 3.1 Let $(x_0, y_0), (x_1, y_1)$ and $x_0 = x$, $x_1 = x + h$. The first derivative at $x_0 = x$ is

$$f'(x) \cong \frac{f(x+h) - f(x)}{h} \tag{3.5}$$

and

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^k)}{x_i^{k+1} - x_i^k} \tag{3.6}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Proof : From Lagrange Interpolating polynomial, we can find polynomial degree n such that approximation $y = f(x)$.

Let $n+1$ order pairs, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ we have

$$f(x) \cong \sum_{k=0}^{n} y_k \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)} \tag{3.7}$$

or
$$f(x) \cong y_0 \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$
$$+ y_1 \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}$$
$$+\ldots$$
$$+ y_k \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)}$$
$$+\ldots$$

$$+ y_n \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}. \tag{3.8}$$

In this case, let $(x_0, y_0), (x_1, y_1)$ we have

$$f(x) \cong y_0 \frac{(x-x_1)}{(x_0-x_1)} + y_1 \frac{(x-x_0)}{(x_1-x_0)}$$
$$= f(x_0)\frac{(x-x_1)}{(x_0-x_1)} + f(x_1)\frac{(x-x_0)}{(x_1-x_0)}$$

and

$$f'(x) \cong \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0}$$
$$= -\frac{f(x_0)}{x_1-x_0} + \frac{f(x_1)}{x_1-x_0}$$
$$= \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

at x_0 we have $$f'(x_0) \cong \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

assume that $x_0 = x, x_1 = x+h$

$$f'(x) \cong \frac{f(x+h)-f(x)}{h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1})-f_i(x^k)}{x_i^{k+1}-x_i^k}.$$

Definition 3.1 The method is called 2-point forward method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^k)}{x_i^{k+1} - x_i^k}, \qquad i = 1,2,3,\ldots,n \text{ and } k = 1,2,3,\ldots$$

Theorem 3.2 Let $(x_0, y_0), (x_1, y_1)$ and $x_0 = x - h,\ x_1 = x$. The first derivative at $x_1 = x$ is

$$f'(x) \cong \frac{f(x) - f(x-h)}{h} \tag{3.9}$$

and

$$f_{ii}(x^k) \approx \frac{f_i(x^k) - f_i(x^{k-1})}{x_i^k - x_i^{k-1}} \tag{3.10}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Proof: Since, the proof of theorem 3.1 we have

$$f'(x) \cong \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

at x_1 we have
$$f'(x_1) \cong \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

assume that $x_0 = x - h,\ x_1 = x$

$$f'(x) \cong \frac{f(x) - f(x-h)}{h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k) \approx \frac{f_i(x^k) - f_i(x^{k-1})}{x_i^k - x_i^{k-1}}.$$

Definition 3.2 The method is called 2-point backward method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{f_i(x^k) - f_i(x^{k-1})}{x_i^k - x_i^{k-1}}, \qquad i = 1,2,3,\ldots,n \text{ and } k = 1,2,3,\ldots$$

Theorem 3.3 Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and $x_0 = x - h,\ x_1 = x,\ x_2 = x + h$. The first derivative at $x_1 = x$ is

$$f'(x) \cong \frac{f(x+h) - f(x-h)}{2h} \tag{3.11}$$

and

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_i^{k+1} - x_i^{k-1}} \tag{3.12}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Proof: Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and equation (3.8) we have

$$\begin{aligned} f(x) &\cong y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ &\quad + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \end{aligned}$$

and

$$f'(x) \cong f(x_0)\frac{[(x-x_1)+(x-x_2)]}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{[(x-x_0)+(x-x_2)]}{(x_1-x_0)(x_1-x_2)}$$
$$+ f(x_2)\frac{[(x-x_0)+(x-x_1)]}{(x_2-x_0)(x_2-x_1)}$$

at x_1 we have

$$f'(x_1) \cong f(x_0)\frac{x_1-x_2}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{2x_1-x_0-x_2}{(x_1-x_0)(x_1-x_2)}$$
$$+ f(x_2)\frac{x_1-x_0}{(x_2-x_0)(x_2-x_1)}$$

assume that $x_0 = x-h$, $x_1 = x$, $x_2 = x+h$

$$f'(x) \cong \frac{f(x+h)-f(x-h)}{2h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1})-f_i(x^{k-1})}{x_i^{k+1}-x_i^{k-1}}.$$

Definition 3.3 The method is called 3-point central method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1})-f_i(x^{k-1})}{x_i^{k+1}-x_i^{k-1}}, \qquad i = 1,2,3,\ldots,n \text{ and } k = 1,2,3,\ldots$$

Theorem 3.4 Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and $x_0 = x$, $x_1 = x + h$, $x_2 = x + 2h$. The first derivative at $x_0 = x$ is

$$f'(x) \cong \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} \tag{3.13}$$

and

$$f_{ii}(x^k) \approx \frac{-3f_i(x^k) + 4f_i(x^{k+1}) - f_i(x^{k+2})}{x_i^{k+2} - x_i^k} \tag{3.14}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Proof: Since, the proof of theorem 3.3 we have

$$f'(x) \cong f(x_0)\frac{[(x-x_1)+(x-x_2)]}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{[(x-x_0)+(x-x_2)]}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{[(x-x_0)+(x-x_1)]}{(x_2-x_0)(x_2-x_1)}$$

at x_0 we have

$$f'(x_0) \cong f(x_0)\frac{2x_0 - x_1 - x_2}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{x_0 - x_2}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{x_0 - x_1}{(x_2-x_0)(x_2-x_1)}$$

assume that $x_0 = x$, $x_1 = x + h$, $x_2 = x + 2h$

$$f'(x) \cong \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k) \approx \frac{-3f_i(x^k)+4f_i(x^{k+1})-f_i(x^{k+2})}{x_i^{k+2}-x_i^k}.$$

Definition 3.4 The method is called 3-point forward method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{-3f_i(x^k)+4f_i(x^{k+1})-f_i(x^{k+2})}{x_i^{k+2}-x_i^k}, \qquad i=1,2,3,\ldots,n \text{ and } k=1,2,3,\ldots$$

Theorem 3.5 Let $(x_0,y_0),(x_1,y_1),(x_2,y_2)$ and $x_0=x-2h,\ x_1=x-h,\ x_2=x$. The first derivative at $x_2=x$ is

$$f'(x) \cong \frac{f(x-2h)+4f(x-h)+3f(x)}{2h} \tag{3.15}$$

and

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2})-4f_i(x^{k-1})+3f_i(x^k)}{x_i^k-x_i^{k-2}} \tag{3.16}$$

when $i=1,2,3,\ldots,n$ and $k=1,2,3,\ldots$

Proof: Since, the proof of theorem 3.3 we have

$$f'(x) \cong f(x_0)\frac{[(x-x_1)+(x-x_2)]}{(x_0-x_1)(x_0-x_2)}+f(x_1)\frac{[(x-x_0)+(x-x_2)]}{(x_1-x_0)(x_1-x_2)}$$
$$+f(x_2)\frac{[(x-x_0)+(x-x_1)]}{(x_2-x_0)(x_2-x_1)}$$

at x_2 we have

$$f'(x_0) \cong f(x_0)\frac{x_2 - x_1}{(x_0 - x_1)(x_0 - x_2)} + f(x_1)\frac{x_2 - x_0}{(x_1 - x_0)(x_1 - x_2)}$$
$$+ f(x_2)\frac{2x_2 - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}$$

assume that $x_0 = x - 2h$, $x_1 = x - h$, $x_2 = x$

$$f'(x) \cong \frac{f(x-2h) + 4f(x-h) + 3f(x)}{2h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2}) - 4f_i(x^{k-1}) + 3f_i(x^k)}{x_i^k - x_i^{k-2}}.$$

Definition 3.5 The method is called 3-point backward method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2}) - 4f_i(x^{k-1}) + 3f_i(x^k)}{x_i^k - x_i^{k-2}}, \qquad i = 1,2,3,\ldots,n \text{ and } k = 1,2,3,\ldots$$

Theorem 3.6 Let $(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ and $x_0 = x - 2h$, $x_1 = x - h$, $x_2 = x$, $x_3 = x + h$, $x_4 = x + 2h$. The first derivative at $x_2 = x$ is

$$f'(x) \cong \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} \qquad (3.17)$$

and

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2}) - 8f_i(x^{k-1}) + 8f_i(x^{k+1}) - f_i(x^{k+2})}{12(x_i^{k+2} - x_i^{k+1})} \qquad (3.18)$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Proof: Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ and equation (3.8) we have

$$
\begin{aligned}
f(x) \cong\ & y_0 \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
& + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
& + y_2 \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
& + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\
=\ & f(x_0) \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
& + f(x_1) \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
& + f(x_2) \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
& + f(x_3) \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}
\end{aligned}
$$

and

$$
\begin{aligned}
f'(x) \cong\ & \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}[(x-x_1)(x-x_2)(x-x_3) \\
& + (x-x_1)(x-x_2)(x-x_4) + (x-x_1)(x-x_3)(x-x_4) + (x-x_2)(x-x_3)(x-x_4)] \\
& + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}[(x-x_0)(x-x_2)(x-x_3) \\
& + (x-x_0)(x-x_2)(x-x_4) + (x-x_0)(x-x_3)(x-x_4) + (x-x_2)(x-x_3)(x-x_4)] \\
& + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}[(x-x_0)(x-x_1)(x-x_3) \\
& + (x-x_0)(x-x_1)(x-x_4) + (x-x_0)(x-x_3)(x-x_4) + (x-x_1)(x-x_3)(x-x_4)] \\
& + \frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}[(x-x_0)(x-x_1)(x-x_2) \\
& + (x-x_0)(x-x_1)(x-x_4) + (x-x_0)(x-x_2)(x-x_4) + (x-x_1)(x-x_2)(x-x_4)]
\end{aligned}
$$

$$+\frac{f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}[(x-x_0)(x-x_1)(x-x_2)$$
$$+(x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)]$$

at x_2 we have

$$\begin{aligned}
f'(x_2) &\cong \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}(x_2-x_1)(x_2-x_3)(x_2-x_4) \\
&+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}(x_2-x_0)(x_2-x_3)(x_2-x_4) \\
&+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}[(x_2-x_0)(x_2-x_1)(x_2-x_3) \\
&+(x_2-x_0)(x_2-x_1)(x_2-x_4)+(x_2-x_0)(x_2-x_3)(x_2-x_4) \\
&+(x_2-x_1)(x_2-x_3)(x_2-x_4)] \\
&+\frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}(x_2-x_0)(x_2-x_1)(x_2-x_4) \\
&+\frac{f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}(x_2-x_0)(x_2-x_1)(x_2-x_3)
\end{aligned}$$

assume that $x_0 = x-2h$, $x_1 = x-h$, $x_2 = x$, $x_3 = x+h$, $x_4 = x+2h$

$$f'(x) \cong \frac{f(x-2h)-8f(x-h)+8f(x+h)-f(x+2h)}{12h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2})-8f_i(x^{k-1})+8f_i(x^{k+1})-f_i(x^{k+2})}{12(x_i^{k+2}-x_i^{k+1})}.$$

Definition 3.6 The method is called 5-point central method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2}) - 8f_i(x^{k-1}) + 8f_i(x^{k+1}) - f_i(x^{k+2})}{12(x_i^{k+2} - x_i^{k+1})}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Theorem 3.7 Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ and $x_0 = x, x_1 = x + h$, $x_2 = x + 2h$, $x_3 = x + 3h$, $x_4 = x + 4h$. The first derivative at $x_0 = x$ is

$$f'(x) \cong \frac{-25f(x) + 48f(x+h) - 36f(x+2h) + 16f(x+3h) - 3f(x+4h)}{12h} \tag{3.19}$$

and

$$f_{ii}(x^k) \approx \frac{-25f_i(x^k) + 48f_i(x^{k+1}) - 36f_i(x^{k+2}) + 16f_i(x^{k+3}) - 3f_i(x^{k+4})}{12(x_i^{k+1} - x_i^k)} \tag{3.20}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Proof: Since, the proof of theorem 3.6 we have

$$\begin{aligned} f'(x) \cong & \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}[(x - x_1)(x - x_2)(x - x_3) \\ & + (x - x_1)(x - x_2)(x - x_4) + (x - x_1)(x - x_3)(x - x_4) + (x - x_2)(x - x_3)(x - x_4)] \\ & + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}[(x - x_0)(x - x_2)(x - x_3) \\ & + (x - x_0)(x - x_2)(x - x_4) + (x - x_0)(x - x_3)(x - x_4) + (x - x_2)(x - x_3)(x - x_4)] \end{aligned}$$

$$+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}[(x-x_0)(x-x_1)(x-x_3)$$
$$+(x-x_0)(x-x_1)(x-x_4)+(x-x_0)(x-x_3)(x-x_4)+(x-x_1)(x-x_3)(x-x_4)]$$
$$+\frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}[(x-x_0)(x-x_1)(x-x_2)$$
$$+(x-x_0)(x-x_1)(x-x_4)+(x-x_0)(x-x_2)(x-x_4)+(x-x_1)(x-x_2)(x-x_4)]$$
$$+\frac{f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}[(x-x_0)(x-x_1)(x-x_2)$$
$$+(x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)]$$

at x_0 we have

$$f'(x_0)\cong\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}[(x_0-x_1)(x_0-x_2)(x_0-x_3)$$
$$+(x_0-x_1)(x_0-x_2)(x_0-x_4)+(x_0-x_1)(x_0-x_3)(x_0-x_4)$$
$$+(x_0-x_2)(x_0-x_3)(x_0-x_4)]$$
$$+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}(x_0-x_2)(x_0-x_3)(x_0-x_4)$$
$$+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}(x_0-x_1)(x_0-x_3)(x_0-x_4)$$
$$+\frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}(x_0-x_1)(x_0-x_2)(x_0-x_4)$$
$$+\frac{f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}(x_0-x_1)(x_0-x_2)(x_0-x_3)$$

assume that $x_0=x$, $x_1=x+h$, $x_2=x+2h$, $x_3=x+3h$, $x_4=x+4h$

$$f'(x)\cong\frac{-25f(x)+48f(x+h)-36f(x+2h)+16f(x+3h)-3f(x+4h)}{12h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k)\approx\frac{-25f_i(x^k)+48f_i(x^{k+1})-36f_i(x^{k+2})+16f_i(x^{k+3})-3f_i(x^{k+4})}{12(x_i^{k+1}-x_i^k)}.$$

Definition 3.7 The method is called 5-point forward method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{-25f_i(x^k) + 48f_i(x^{k+1}) - 36f_i(x^{k+2}) + 16f_i(x^{k+3}) - 3f_i(x^{k+4})}{12(x_i^{k+1} - x_i^k)}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Theorem 3.8 Let $(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ and $x_0 = x-4h$, $x_1 = x-3h$, $x_2 = x-2h$, $x_3 = x-h$, $x_4 = x$. The first derivative at $x_0 = x$ is

$$f'(x) \cong \frac{3f(x-4h) - 16f(x-3h) + 36f(x-2h) - 48f(x-h) + 25f(x)}{12h} \tag{3.21}$$

and

$$f_{ii}(x^k) \approx \frac{3f_i(x^{k-4}) - 16f_i(x^{k-3}) + 36f_i(x^{k-2}) - 48f_i(x^{k-1}) + 25f_i(x^k)}{12(x_i^k - x_i^{k-1})} \tag{3.22}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Proof: Since, the proof of theorem 3.6 we have

$$\begin{aligned}
f'(x) \cong{}& \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}[(x-x_1)(x-x_2)(x-x_3) \\
&+ (x-x_1)(x-x_2)(x-x_4) + (x-x_1)(x-x_3)(x-x_4) + (x-x_2)(x-x_3)(x-x_4)] \\
&+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}[(x-x_0)(x-x_2)(x-x_3) \\
&+ (x-x_0)(x-x_2)(x-x_4) + (x-x_0)(x-x_3)(x-x_4) + (x-x_2)(x-x_3)(x-x_4)]
\end{aligned}$$

$$+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}[(x-x_0)(x-x_1)(x-x_3)$$
$$+(x-x_0)(x-x_1)(x-x_4)+(x-x_0)(x-x_3)(x-x_4)+(x-x_1)(x-x_3)(x-x_4)]$$
$$+\frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}[(x-x_0)(x-x_1)(x-x_2)$$
$$+(x-x_0)(x-x_1)(x-x_4)+(x-x_0)(x-x_2)(x-x_4)+(x-x_1)(x-x_2)(x-x_4)]$$
$$+\frac{f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}[(x-x_0)(x-x_1)(x-x_2)$$
$$+(x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)]$$

at x_4 we have

$$f'(x_4)\cong\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}(x_4-x_1)(x_4-x_2)(x_4-x_3)$$
$$+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}(x_4-x_0)(x_4-x_2)(x_4-x_3)$$
$$+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}(x_4-x_0)(x_4-x_1)(x_4-x_3)$$
$$+\frac{f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}(x_4-x_0)(x_4-x_1)(x_4-x_2)$$
$$+\frac{f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}[(x_4-x_0)(x_4-x_1)(x_4-x_2)$$
$$+(x_4-x_0)(x_4-x_1)(x_4-x_3)+(x_4-x_0)(x_4-x_2)(x_4-x_3)$$
$$+(x_4-x_1)(x_4-x_2)(x_4-x_3)]$$

assume that $x_0=x-4h$, $x_1=x-3h$, $x_2=x-2h$, $x_3=x-h$, $x_4=x$

$$f'(x)\cong\frac{3f(x-4h)-16f(x-3h)+36f(x-2h)-48f(x-h)+25f(x)}{12h}.$$

Therefore, we can approximate $f_{ii}(x)$ from

$$f_{ii}(x^k)\approx\frac{3f_i(x^{k-4})-16f_i(x^{k-3})+36f_i(x^{k-2})-48f_i(x^{k-1})+25f_i(x^k)}{12(x_i^k-x_i^{k-1})}.$$

Definition 3.8 The method is called 5-point backward method if we approximate the derivative in Jacobian matrix from

$$f_{ii}(x^k) \approx \frac{3f_i(x^{k-4}) - 16f_i(x^{k-3}) + 36f_i(x^{k-2}) - 48f_i(x^{k-1}) + 25f_i(x^k)}{12(x_i^k - x_i^{k-1})}$$

when $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots$

Thus, we have new eight iterative methods for solving systems of nonlinear equations.

CHAPTER 4

SOME EXAMPLES AND APPLICATION PROBLEMS OF SYSTEMS OF NONLINEAR EQUATIONS

In this chapter, we shall find the solutions by using new method obtained from chapter 3 for solving systems of nonlinear equations.

Example 4.1 Find the solution of system

$$f(x,y) = 0.5\cos x + 0.35\cos(x+y) - 0.2 = 0$$
$$g(x,y) = 0.5\sin x + 0.35\sin(x+y) - 0.4 = 0\,. \qquad (4.1)$$

Iterative equation is

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \end{bmatrix} - [H(x^k,y^k)]^{-1} \begin{bmatrix} f(x^k,y^k) \\ g(x^k,y^k) \end{bmatrix} k = 0,1,2,\ldots$$

and the matrix H is

$$H(x^k,y^k) = \begin{bmatrix} f_x(x^k,y^k) & 0 \\ 0 & g_y(x^k,y^k) \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k+1}, y^k) - f(x^k, y^k)}{x^{k+1} - x^k}$$

and

$$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k+1}) - g(x^k, y^k)}{y^{k+1} - y^k}$$

2-point backward method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^k, y^k) - f(x^{k-1}, y^k)}{x^k - x^{k-1}}$$

and

$$g_y(x^k, y^k) \approx \frac{g(x^k, y^k) - g(x^k, y^{k-1})}{y^k - y^{k-1}}$$

3-point central method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k+1}, y^k) - f(x^{k-1}, y^k)}{x^{k+1} - x^{k-1}}$$

and

$$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k+1}) - g(x^k, y^{k-1})}{y^{k+1} - y^{k-1}}$$

3-point forward method, we have

$$f_x(x^k, y^k) \approx \frac{-3f(x^k, y^k) + 4f(x^{k+1}, y^k) - f(x^{k+2}, y^k)}{x^{k+2} - x^k}$$

and

$$g_y(x^k, y^k) \approx \frac{-3g(x^k, y^k) + 4g(x^k, y^{k+1}) - g(x^k, y^{k+2})}{y^{k+2} - y^k}$$

3-point backward method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k-2}, y^k) - 4f(x^{k-1}, y^k) + 3f(x^k, y^k)}{x^k - x^{k-2}}$$

and
$$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k-2}) - 4g(x^k, y^{k-1}) + 3g(x^k, y^k)}{y^k - y^{k-2}}$$

5-point central method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k-2}, y^k) - 8f(x^{k-1}, y^k) + 8f(x^{k+1}, y^k) - f(x^{k+2}, y^k)}{12(x^{k+2} - x^{k+1})}$$

and
$$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k-2}) - 8g(x^k, y^{k-1}) + 8g(x^k, y^{k+1}) - g(x^k, y^{k+2})}{12(y^{k+2} - y^{k+1})}$$

5-point forward method, we have

$$f_x(x^k, y^k) \approx \frac{1}{12(x^{k+1} - x^k)}[-25f(x^k, y^k) + 48f(x^{k+1}, y^k) - 36f(x^{k+2}, y^k) + 16f(x^{k+3}, y^k) - 3f(x^{k+4}, y^k)]$$

and
$$g_y(x^k, y^k) \approx \frac{1}{12(y^{k+1} - y^k)}[-25g(x^k, y^k) + 48g(x^k, y^{k+1}) - 36g(x^k, y^{k+2}) + 16g(x^k, y^{k+3}) - 3g(x^k, y^{k+4})]$$

5-point backward method, we have

$$f_x(x^k, y^k) \approx \frac{1}{12(x^k - x^{k-1})}[3f(x^{k-4}, y^k) - 16f(x^{k-3}, y^k) + 36f(x^{k-2}, y^k) - 48f(x^{k-1}, y^k) + 25f(x^k, y^k)]$$

and $$g_{y}(x^k, y^k) \approx \frac{1}{12(y^k - y^{k-1})}[3g(x^k, y^{k-4}) - 16g(x^k, y^{k-3}) + 36g(x^k, y^{k-2}) - 48g(x^k, y^{k-1}) + 25g(x^k, y^k)]$$

Table 4.1 The result of example 4.1

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	15	x = 1.8562334 y = -2.0861669
2-point backward	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	14	x = 1.8562344 y = -2.0861669
3-point central	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	14	x = 1.8562342 y = -2.0861672
3-point forward	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	14	x = 1.8562338 y = -2.0861675
3-point backward	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	14	x = 1.8562340 y = -2.0861672
5-point central	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	14	x = 1.8562341 y = -2.0861673
5-point forward	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	15	x = 1.8562336 y = -2.0861675
5-point backward	(0.6,0.6) and $\Delta x = 0.1$, $\Delta y = 0.1$	15	x = 1.8562334 y = -2.0861674

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.1 in appendix.

Example 4.2 Find the solution of system

$$f(x,y) = e^x + xy - 1 = 0$$
$$g(x,y) = \sin(xy) + x + y - 1 = 0. \tag{4.2}$$

Iterative equation is

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \end{bmatrix} - [H(x^k, y^k)]^{-1} \begin{bmatrix} f(x^k, y^k) \\ g(x^k, y^k) \end{bmatrix} \quad k = 0,1,2,\ldots$$

and the matrix H is

$$H(x^k, y^k) = \begin{bmatrix} f_x(x^k, y^k) & 0 \\ 0 & g_y(x^k, y^k) \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k+1}, y^k) - f(x^k, y^k)}{x^{k+1} - x^k}$$

and
$$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k+1}) - g(x^k, y^k)}{y^{k+1} - y^k}$$

2-point backward method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^k, y^k) - f(x^{k-1}, y^k)}{x^k - x^{k-1}}$$

and $$g_y(x^k, y^k) \approx \frac{g(x^k, y^k) - g(x^k, y^{k-1})}{y^k - y^{k-1}}$$

3-point central method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k+1}, y^k) - f(x^{k-1}, y^k)}{x^{k+1} - x^{k-1}}$$

and $$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k+1}) - g(x^k, y^{k-1})}{y^{k+1} - y^{k-1}}$$

3-point forward method, we have

$$f_x(x^k, y^k) \approx \frac{-3f(x^k, y^k) + 4f(x^{k+1}, y^k) - f(x^{k+2}, y^k)}{x^{k+2} - x^k}$$

and $$g_y(x^k, y^k) \approx \frac{-3g(x^k, y^k) + 4g(x^k, y^{k+1}) - g(x^k, y^{k+2})}{y^{k+2} - y^k}$$

3-point backward method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k-2}, y^k) - 4f(x^{k-1}, y^k) + 3f(x^k, y^k)}{x^k - x^{k-2}}$$

and $$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k-2}) - 4g(x^k, y^{k-1}) + 3g(x^k, y^k)}{y^k - y^{k-2}}$$

5-point central method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k-2}, y^k) - 8f(x^{k-1}, y^k) + 8f(x^{k+1}, y^k) - f(x^{k+2}, y^k)}{12(x^{k+2} - x^{k+1})}$$

and $$g_y(x^k,y^k) \approx \frac{g(x^k,y^{k-2}) - 8g(x^k,y^{k-1}) + 8g(x^k,y^{k+1}) - g(x^k,y^{k+2})}{12(y^{k+2}-y^{k+1})}$$

5-point forward method, we have

$$f_x(x^k,y^k) \approx \frac{1}{12(x^{k+1}-x^k)}[-25f(x^k,y^k) + 48f(x^{k+1},y^k) - 36f(x^{k+2},y^k) + 16f(x^{k+3},y^k) - 3f(x^{k+4},y^k)]$$

and $$g_y(x^k,y^k) \approx \frac{1}{12(y^{k+1}-y^k)}[-25g(x^k,y^k) + 48g(x^k,y^{k+1}) - 36g(x^k,y^{k+2}) + 16g(x^k,y^{k+3}) - 3g(x^k,y^{k+4})]$$

5-point backward method, we have

$$f_x(x^k,y^k) \approx \frac{1}{12(x^k-x^{k-1})}[3f(x^{k-4},y^k) - 16f(x^{k-3},y^k) + 36f(x^{k-2},y^k) - 48f(x^{k-1},y^k) + 25f(x^k,y^k)]$$

and $$g_y(x^k,y^k) \approx \frac{1}{12(y^k-y^{k-1})}[3g(x^k,y^{k-4}) - 16g(x^k,y^{k-3}) + 36g(x^k,y^{k-2}) - 48g(x^k,y^{k-1}) + 25g(x^k,y^k)]$$

Table 4.2 The result of example 4.2

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	6	x = 0.0000000 y = 1.0000000
2-point backward	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	6	x = 0.0000000 y = 1.0000001

Table 4.2 (cont.)

Method	Initial vectors	Number of iterations	Solutions
3-point central	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	5	x = 0.0000000 y = 0.9999998
3-point forward	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	5	x = 0.0000000 y = 1.0000000
3-point backward	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	6	x = 0.0000000 y = 1.0000000
5-point central	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	5	x = 0.0000000 y = 1.0000000
5-point forward	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	6	x = 0.0000000 y = 1.0000000
5-point backward	(1.5,2.0) and $\Delta x = 0.5,\ \Delta y = 1.0$	6	x = 0.0000000 y = 0.9999998

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.2 in appendix.

Example 4.3 Find the solution of system

$$f(x,y)=\frac{1}{2}\sin(xy)-\frac{y}{4\pi}-\frac{1}{2}x=0$$
$$g(x,y)=\left(1-\frac{1}{4\pi}\right)\left(e^{x}-e\right)+\frac{ey}{\pi}-2ex=0. \tag{4.3}$$

Iterative equation is

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix}=\begin{bmatrix} x^{k} \\ y^{k} \end{bmatrix}-[H(x^k,y^k)]^{-1}\begin{bmatrix} f(x^k,y^k) \\ g(x^k,y^k) \end{bmatrix} \quad k=0,1,2,\ldots$$

and the matrix H is

$$H(x^k,y^k)=\begin{bmatrix} f_x(x^k,y^k) & 0 \\ 0 & g_y(x^k,y^k) \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_x(x^k,y^k)\approx\frac{f(x^{k+1},y^k)-f(x^k,y^k)}{x^{k+1}-x^k}$$

and
$$g_y(x^k,y^k)\approx\frac{g(x^k,y^{k+1})-g(x^k,y^k)}{y^{k+1}-y^k}$$

2-point backward method, we have

$$f_x(x^k,y^k)\approx\frac{f(x^k,y^k)-f(x^{k-1},y^k)}{x^k-x^{k-1}}$$

and
$$g_y(x^k, y^k) \approx \frac{g(x^k, y^k) - g(x^k, y^{k-1})}{y^k - y^{k-1}}$$

3-point central method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k+1}, y^k) - f(x^{k-1}, y^k)}{x^{k+1} - x^{k-1}}$$

and
$$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k+1}) - g(x^k, y^{k-1})}{y^{k+1} - y^{k-1}}$$

3-point forward method, we have

$$f_x(x^k, y^k) \approx \frac{-3f(x^k, y^k) + 4f(x^{k+1}, y^k) - f(x^{k+2}, y^k)}{x^{k+2} - x^k}$$

and
$$g_y(x^k, y^k) \approx \frac{-3g(x^k, y^k) + 4g(x^k, y^{k+1}) - g(x^k, y^{k+2})}{y^{k+2} - y^k}$$

3-point backward method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k-2}, y^k) - 4f(x^{k-1}, y^k) + 3f(x^k, y^k)}{x^k - x^{k-2}}$$

and
$$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k-2}) - 4g(x^k, y^{k-1}) + 3g(x^k, y^k)}{y^k - y^{k-2}}$$

5-point central method, we have

$$f_x(x^k, y^k) \approx \frac{f(x^{k-2}, y^k) - 8f(x^{k-1}, y^k) + 8f(x^{k+1}, y^k) - f(x^{k+2}, y^k)}{12(x^{k+2} - x^{k+1})}$$

and $$g_y(x^k, y^k) \approx \frac{g(x^k, y^{k-2}) - 8g(x^k, y^{k-1}) + 8g(x^k, y^{k+1}) - g(x^k, y^{k+2})}{12(y^{k+2} - y^{k+1})}$$

5-point forward method, we have

$$f_x(x^k, y^k) \approx \frac{1}{12(x^{k+1} - x^k)}[-25f(x^k, y^k) + 48f(x^{k+1}, y^k) - 36f(x^{k+2}, y^k) + 16f(x^{k+3}, y^k) - 3f(x^{k+4}, y^k)]$$

and $$g_y(x^k, y^k) \approx \frac{1}{12(y^{k+1} - y^k)}[-25g(x^k, y^k) + 48g(x^k, y^{k+1}) - 36g(x^k, y^{k+2}) + 16g(x^k, y^{k+3}) - 3g(x^k, y^{k+4})]$$

5-point backward method, we have

$$f_x(x^k, y^k) \approx \frac{1}{12(x^k - x^{k-1})}[3f(x^{k-4}, y^k) - 16f(x^{k-3}, y^k) + 36f(x^{k-2}, y^k) - 48f(x^{k-1}, y^k) + 25f(x^k, y^k)]$$

and $$g_y(x^k, y^k) \approx \frac{1}{12(y^k - y^{k-1})}[3g(x^k, y^{k-4}) - 16g(x^k, y^{k-3}) + 36g(x^k, y^{k-2}) - 48g(x^k, y^{k-1}) + 25g(x^k, y^k)]$$

Table 4.3 The result of example 4.3

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	11	x = 0.4999999 y = 3.1415927
2-point backward	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	11	x = 0.5000004 y = 3.1415927

Table 4.3 (cont.)

Method	Initial vectors	Number of iterations	Solutions
3-point central	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	11	x = 0.5000003 y = 3.1415929
3-point forward	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	10	x = 0.5000004 y = 3.1415929
3-point backward	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	11	x = 0.5000004 y = 3.1415929
5-point central	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	12	x = 0.5000001 y = 3.1415930
5-point forward	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	12	x = 0.5000001 y = 3.1415930
5-point backward	(0.5,2.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	10	x = 0.5000004 y = 3.1415928

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.4 in appendix.

Example 4.4 Find the solution of system

$$f(x,y,z) = x^3 - 10x + y - z + 3 = 0$$
$$g(x,y,z) = y^2 + 10y - 2x - 2z - 5 = 0$$
$$h(x,y,z) = x + y - 10z + 2\sin z + 5 = 0\,. \qquad (4.4)$$

Iterative equation is

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} - [H(x^k,y^k,z^k)]^{-1} \begin{bmatrix} f(x^k,y^k,z^k) \\ g(x^k,y^k,z^k) \\ h(x^k,y^k,z^k) \end{bmatrix}$$

and the matrix H is

$$H(x^k,y^k,z^k) = \begin{bmatrix} f_x(x^k,y^k,z^k) & 0 & 0 \\ 0 & g_y(x^k,y^k,z^k) & 0 \\ 0 & 0 & h_z(x^k,y^k,z^k) \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{f(x^{k+1},y^k,z^k) - f(x^k,y^k,z^k)}{x^{k+1} - x^k}$$

$$g_y(x^k,y^k,z^k) \approx \frac{g(x^k,y^{k+1},z^k) - g(x^k,y^k,z^k)}{y^{k+1} - y^k}$$

and $$h_z(x^k,y^k,z^k) \approx \frac{h(x^k,y^k,z^{k+1}) - h(x^k,y^k,z^k)}{z^{k+1} - z^k}$$

2-point backward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{f(x^k, y^k, z^k) - f(x^{k-1}, y^k, z^k)}{x^k - x^{k-1}}$$

$$g_y(x^k, y^k, z^k) \approx \frac{g(x^k, y^k, z^k) - g(x^k, y^{k-1}, z^k)}{y^k - y^{k-1}}$$

and

$$h_z(x^k, y^k, z^k) \approx \frac{h(x^k, y^k, z^k) - h(x^k, y^k, z^{k-1})}{z^k - z^{k-1}}$$

3-point central method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{f(x^{k+1}, y^k, z^k) - f(x^{k-1}, y^k, z^k)}{x^{k+1} - x^{k-1}}$$

$$g_y(x^k, y^k, z^k) \approx \frac{g(x^k, y^{k+1}, z^k) - g(x^k, y^{k-1}, z^k)}{y^{k+1} - y^{k-1}}$$

and

$$h_z(x^k, y^k, z^k) \approx \frac{h(x^k, y^k, z^{k+1}) - h(x^k, y^k, z^{k-1})}{z^{k+1} - z^{k-1}}$$

3-point forward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{-3f(x^k, y^k, z^k) + 4f(x^{k+1}, y^k, z^k) - f(x^{k+2}, y^k, z^k)}{x^{k+2} - x^k}$$

$$g_y(x^k, y^k, z^k) \approx \frac{-3g(x^k, y^k, z^k) + 4g(x^k, y^{k+1}, z^k) - g(x^k, y^{k+2}, z^k)}{y^{k+2} - y^k}$$

and

$$h_z(x^k, y^k, z^k) \approx \frac{-3h(x^k, y^k, z^k) + 4h(x^k, y^k, z^{k+1}) - h(x^k, y^k, z^{k+2})}{z^{k+2} - z^k}$$

3-point backward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{f(x^{k-2},y^k,z^k) - 4f(x^{k-1},y^k,z^k) + 3f(x^k,y^k,z^k)}{x^k - x^{k-2}}$$

$$g_y(x^k,y^k,z^k) \approx \frac{g(x^k,y^{k-2},z^k) - 4g(x^k,y^{k-1},z^k) + 3g(x^k,y^k,z^k)}{y^k - y^{k-2}}$$

and $$h_z(x^k,y^k,z^k) \approx \frac{h(x^k,y^k,z^{k-2}) - 4h(x^k,y^k,z^{k-1}) + 3h(x^k,y^k,z^k)}{z^k - z^{k-2}}$$

5-point central method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{1}{12(x^{k+2} - x^{k+1})}[f(x^{k-2},y^k,z^k) - 8f(x^{k-1},y^k,z^k) + 8f(x^{k+1},y^k,z^k) - f(x^{k+2},y^k,z^k)]$$

$$g_y(x^k,y^k,z^k) \approx \frac{1}{12(y^{k+2} - y^{k+1})}[g(x^k,y^{k-2},z^k) - 8g(x^k,y^{k-1},z^k) + 8g(x^k,y^{k+1},z^k) - g(x^k,y^{k+2},z^k)]$$

and $$h_z(x^k,y^k,z^k) \approx \frac{1}{12(z^{k+2} - z^{k+1})}[h(x^k,y^k,z^{k-2}) - 8h(x^k,y^k,z^{k-1}) + 8h(x^k,y^k,z^{k+1}) - h(x^k,y^k,z^{k+2})]$$

5-point forward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{1}{12(x^{k+1} - x^k)}[-25f(x^k,y^k,z^k) + 48f(x^{k+1},y^k,z^k) - 36f(x^{k+2},y^k,z^k) + 16f(x^{k+3},y^k,z^k) - 3f(x^{k+4},y^k,z^k)]$$

$$g_y(x^k,y^k,z^k) \approx \frac{1}{12(y^{k+1} - y^k)}[-25g(x^k,y^k,z^k) + 48g(x^k,y^{k+1},z^k) - 36g(x^k,y^{k+2},z^k) + 16g(x^k,y^{k+3},z^k) - 3g(x^k,y^{k+4},z^k)]$$

and $$h_z(x^k,y^k,z^k) \approx \frac{1}{12(z^{k+1}-z^k)}[-25h(x^k,y^k,z^k)+48h(x^k,y^k,z^{k+1}) - 36h(x^k,y^k,z^{k+2})+16h(x^k,y^k,z^{k+3})-3h(x^k,y^k,z^{k+4})]$$

5-point backward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{1}{12(x^k-x^{k-1})}[3f(x^{k-4},y^k,z^k)-16f(x^{k-3},y^k,z^k) + 36f(x^{k-2},y^k,z^k)-48f(x^{k-1},y^k,z^k)+25f(x^k,y^k,z^k)]$$

$$g_y(x^k,y^k,z^k) \approx \frac{1}{12(y^k-y^{k-1})}[3g(x^k,y^{k-4},z^k)-16g(x^k,y^{k-3},z^k) + 36g(x^k,y^{k-2},z^k)-48g(x^k,y^{k-1},z^k)+25g(x^k,y^k,z^k)]$$

and $$h_z(x^k,y^k,z^k) \approx \frac{1}{12(z^k-z^{k-1})}[3h(x^k,y^k,z^{k-4})-16h(x^k,y^k,z^{k-3}) + 36h(x^k,y^k,z^{k-2})-48h(x^k,y^k,z^{k-1})+25h(x^k,y^k,z^k)]$$

Table 4.4 The result of example 4.4

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	9	x = 2.9950562 y = 1.1790406 z = 1.0952150
2-point backward	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	11	x = 2.9950562 y = 1.1790406 z = 1.0952150
3-point central	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	10	x = 2.9950562 y = 1.1790406 z = 1.0952150

Table 4.4 (cont.)

Method	Initial vectors	Number of iterations	Solutions
3-point forward	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	10	x = 2.9950562 y = 1.1790406 z = 1.0952150
3-point backward	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	10	x = 2.9950562 y = 1.1790406 z = 1.0952150
5-point central	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	10	x = 2.9950562 y = 1.1790406 z = 1.0952150
5-point forward	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	10	x = 2.9950562 y = 1.1790406 z = 1.0952150
5-point backward	(2.5,2.5,2.5) and $\Delta x = 1.0,\ \Delta y = 1.0,$ $\Delta z = 1.0$	10	x = 2.9950562 y = 1.1790406 z = 1.0952150

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.4 in appendix.

Example 4.5 Find the solution of system

$$\begin{aligned} f(x,y,z) &= \sin x + \ln x + \ln y - e^{z} - 1.87073 = 0 \\ g(x,y,z) &= \cos x + \sin y + e^{-z} - 39.93815 = 0 \\ h(x,y,z) &= \ln x - e^{y} + \sin z + 13.09764 = 0. \end{aligned} \tag{4.5}$$

Iterative equation is

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} - [H(x^k, y^k, z^k)]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix}$$

and the matrix H is

$$H(x^k, y^k, z^k) = \begin{bmatrix} f_x(x^k, y^k, z^k) & 0 & 0 \\ 0 & g_y(x^k, y^k, z^k) & 0 \\ 0 & 0 & h_z(x^k, y^k, z^k) \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{f(x^{k+1}, y^k, z^k) - f(x^k, y^k, z^k)}{x^{k+1} - x^k}$$

$$g_y(x^k, y^k, z^k) \approx \frac{g(x^k, y^{k+1}, z^k) - g(x^k, y^k, z^k)}{y^{k+1} - y^k}$$

and $$h_z(x^k, y^k, z^k) \approx \frac{h(x^k, y^k, z^{k+1}) - h(x^k, y^k, z^k)}{z^{k+1} - z^k}$$

2-point backward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{f(x^k, y^k, z^k) - f(x^{k-1}, y^k, z^k)}{x^k - x^{k-1}}$$

$$g_y(x^k, y^k, z^k) \approx \frac{g(x^k, y^k, z^k) - g(x^k, y^{k-1}, z^k)}{y^k - y^{k-1}}$$

and

$$h_z(x^k, y^k, z^k) \approx \frac{h(x^k, y^k, z^k) - h(x^k, y^k, z^{k-1})}{z^k - z^{k-1}}$$

3-point central method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{f(x^{k+1}, y^k, z^k) - f(x^{k-1}, y^k, z^k)}{x^{k+1} - x^{k-1}}$$

$$g_y(x^k, y^k, z^k) \approx \frac{g(x^k, y^{k+1}, z^k) - g(x^k, y^{k-1}, z^k)}{y^{k+1} - y^{k-1}}$$

and

$$h_z(x^k, y^k, z^k) \approx \frac{h(x^k, y^k, z^{k+1}) - h(x^k, y^k, z^{k-1})}{z^{k+1} - z^{k-1}}$$

3-point forward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{-3f(x^k, y^k, z^k) + 4f(x^{k+1}, y^k, z^k) - f(x^{k+2}, y^k, z^k)}{x^{k+2} - x^k}$$

$$g_y(x^k, y^k, z^k) \approx \frac{-3g(x^k, y^k, z^k) + 4g(x^k, y^{k+1}, z^k) - g(x^k, y^{k+2}, z^k)}{y^{k+2} - y^k}$$

and

$$h_z(x^k, y^k, z^k) \approx \frac{-3h(x^k, y^k, z^k) + 4h(x^k, y^k, z^{k+1}) - h(x^k, y^k, z^{k+2})}{z^{k+2} - z^k}$$

3-point backward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{f(x^{k-2}, y^k, z^k) - 4f(x^{k-1}, y^k, z^k) + 3f(x^k, y^k, z^k)}{x^k - x^{k-2}}$$

$$g_y(x^k, y^k, z^k) \approx \frac{g(x^k, y^{k-2}, z^k) - 4g(x^k, y^{k-1}, z^k) + 3g(x^k, y^k, z^k)}{y^k - y^{k-2}}$$

and $$h_z(x^k, y^k, z^k) \approx \frac{h(x^k, y^k, z^{k-2}) - 4h(x^k, y^k, z^{k-1}) + 3h(x^k, y^k, z^k)}{z^k - z^{k-2}}$$

5-point central method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{1}{12(x^{k+2} - x^{k+1})}[f(x^{k-2}, y^k, z^k) - 8f(x^{k-1}, y^k, z^k) + 8f(x^{k+1}, y^k, z^k) - f(x^{k+2}, y^k, z^k)]$$

$$g_y(x^k, y^k, z^k) \approx \frac{1}{12(y^{k+2} - y^{k+1})}[g(x^k, y^{k-2}, z^k) - 8g(x^k, y^{k-1}, z^k) + 8g(x^k, y^{k+1}, z^k) - g(x^k, y^{k+2}, z^k)]$$

and $$h_z(x^k, y^k, z^k) \approx \frac{1}{12(z^{k+2} - z^{k+1})}[h(x^k, y^k, z^{k-2}) - 8h(x^k, y^k, z^{k-1}) + 8h(x^k, y^k, z^{k+1}) - h(x^k, y^k, z^{k+2})]$$

5-point forward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{1}{12(x^{k+1} - x^k)}[-25f(x^k, y^k, z^k) + 48f(x^{k+1}, y^k, z^k) - 36f(x^{k+2}, y^k, z^k) + 16f(x^{k+3}, y^k, z^k) - 3f(x^{k+4}, y^k, z^k)]$$

$$g_y(x^k, y^k, z^k) \approx \frac{1}{12(y^{k+1} - y^k)}[-25g(x^k, y^k, z^k) + 48g(x^k, y^{k+1}, z^k) - 36g(x^k, y^{k+2}, z^k) + 16g(x^k, y^{k+3}, z^k) - 3g(x^k, y^{k+4}, z^k)]$$

and $$h_z(x^k, y^k, z^k) \approx \frac{1}{12(z^{k+1} - z^k)}[-25h(x^k, y^k, z^k) + 48h(x^k, y^k, z^{k+1}) - 36h(x^k, y^k, z^{k+2}) + 16h(x^k, y^k, z^{k+3}) - 3h(x^k, y^k, z^{k+4})]$$

5-point backward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{1}{12(x^k - x^{k-1})}[3f(x^{k-4}, y^k, z^k) - 16f(x^{k-3}, y^k, z^k) + 36f(x^{k-2}, y^k, z^k) - 48f(x^{k-1}, y^k, z^k) + 25f(x^k, y^k, z^k)]$$

$$g_y(x^k, y^k, z^k) \approx \frac{1}{12(y^k - y^{k-1})}[3g(x^k, y^{k-4}, z^k) - 16g(x^k, y^{k-3}, z^k) + 36g(x^k, y^{k-2}, z^k) - 48g(x^k, y^{k-1}, z^k) + 25g(x^k, y^k, z^k)]$$

and $$h_z(x^k, y^k, z^k) \approx \frac{1}{12(z^k - z^{k-1})}[3h(x^k, y^k, z^{k-4}) - 16h(x^k, y^k, z^{k-3}) + 36h(x^k, y^k, z^{k-2}) - 48h(x^k, y^k, z^{k-1}) + 25h(x^k, y^k, z^k)]$$

Table 4.5 The result of example 4.5

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(3.0,3.0,-3.5) and $\Delta x = 0.5$, $\Delta y = 0.5$, $\Delta z = 0.5$	8	x = 3.4996749 y = 2.7000279 z = -3.7000015
2-point backward	(3.0,3.0,-3.5) and $\Delta x = 0.5$, $\Delta y = 0.5$, $\Delta z = 0.5$	7	x = 3.4996746 y = 2.7000279 z = -3.7000015
3-point central	(3.0,3.0,-3.5) and $\Delta x = 0.5$, $\Delta y = 0.5$, $\Delta z = 0.5$	9	x = 3.4996748 y = 2.7000279 z = -3.7000015

Table 4.5 (cont.)

Method	Initial vectors	Number of iterations	Solutions
3-point forward	(3.0,3.0,-3.5) and $\Delta x = 0.5,\ \Delta y = 0.5,$ $\Delta z = 0.5$	8	x = 3.4996748 y = 2.7000280 z = -3.7000015
3-point backward	(3.0,3.0,-3.5) and $\Delta x = 0.5,\ \Delta y = 0.5,$ $\Delta z = 0.5$	7	x = 3.4996751 y = 2.7000280 z = -3.7000015
5-point central	(3.0,3.0,-3.5) and $\Delta x = 0.5,\ \Delta y = 0.5,$ $\Delta z = 0.5$	8	x = 3.4996748 y = 2.7000279 z = -3.7000015
5-point forward	(3.0,3.0,-3.5) and $\Delta x = 0.5,\ \Delta y = 0.5,$ $\Delta z = 0.5$	8	x = 3.4996748 y = 2.7000280 z = -3.7000015
5-point backward	(3.0,3.0,-3.5) and $\Delta x = 0.5,\ \Delta y = 0.5,$ $\Delta z = 0.5$	8	x = 3.4996748 y = 2.7000280 z = -3.7000015

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.5 in appendix.

Example 4.6 Find the solution of system

$$f(x,y,z) = 3x - \cos(yz) - 0.5 = 0$$
$$g(x,y,z) = x^2 - 81(y+0.1)^2 + \sin z + 1.06 = 0$$
$$h(x,y,z) = e^{-xy} + 20z + \frac{10\pi - 3}{3} = 0. \tag{4.6}$$

Iterative equation is

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} - [H(x^k, y^k, z^k)]^{-1} \begin{bmatrix} f(x^k, y^k, z^k) \\ g(x^k, y^k, z^k) \\ h(x^k, y^k, z^k) \end{bmatrix}$$

and the matrix H is

$$H(x^k, y^k, z^k) = \begin{bmatrix} f_x(x^k, y^k, z^k) & 0 & 0 \\ 0 & g_y(x^k, y^k, z^k) & 0 \\ 0 & 0 & h_z(x^k, y^k, z^k) \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_x(x^k, y^k, z^k) \approx \frac{f(x^{k+1}, y^k, z^k) - f(x^k, y^k, z^k)}{x^{k+1} - x^k}$$

$$g_y(x^k, y^k, z^k) \approx \frac{g(x^k, y^{k+1}, z^k) - g(x^k, y^k, z^k)}{y^{k+1} - y^k}$$

and $$h_z(x^k, y^k, z^k) \approx \frac{h(x^k, y^k, z^{k+1}) - h(x^k, y^k, z^k)}{z^{k+1} - z^k}.$$

2-point backward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{f(x^k,y^k,z^k) - f(x^{k-1},y^k,z^k)}{x^k - x^{k-1}}$$

$$g_y(x^k,y^k,z^k) \approx \frac{g(x^k,y^k,z^k) - g(x^k,y^{k-1},z^k)}{y^k - y^{k-1}}$$

and

$$h_z(x^k,y^k,z^k) \approx \frac{h(x^k,y^k,z^k) - h(x^k,y^k,z^{k-1})}{z^k - z^{k-1}}$$

3-point central method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{f(x^{k+1},y^k,z^k) - f(x^{k-1},y^k,z^k)}{x^{k+1} - x^{k-1}}$$

$$g_y(x^k,y^k,z^k) \approx \frac{g(x^k,y^{k+1},z^k) - g(x^k,y^{k-1},z^k)}{y^{k+1} - y^{k-1}}$$

and

$$h_z(x^k,y^k,z^k) \approx \frac{h(x^k,y^k,z^{k+1}) - h(x^k,y^k,z^{k-1})}{z^{k+1} - z^{k-1}}$$

3-point forward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{-3f(x^k,y^k,z^k) + 4f(x^{k+1},y^k,z^k) - f(x^{k+2},y^k,z^k)}{x^{k+2} - x^k}$$

$$g_y(x^k,y^k,z^k) \approx \frac{-3g(x^k,y^k,z^k) + 4g(x^k,y^{k+1},z^k) - g(x^k,y^{k+2},z^k)}{y^{k+2} - y^k}$$

and

$$h_z(x^k,y^k,z^k) \approx \frac{-3h(x^k,y^k,z^k) + 4h(x^k,y^k,z^{k+1}) - h(x^k,y^k,z^{k+2})}{z^{k+2} - z^k}$$

3-point backward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{f(x^{k-2},y^k,z^k)-4f(x^{k-1},y^k,z^k)+3f(x^k,y^k,z^k)}{x^k-x^{k-2}}$$

$$g_y(x^k,y^k,z^k) \approx \frac{g(x^k,y^{k-2},z^k)-4g(x^k,y^{k-1},z^k)+3g(x^k,y^k,z^k)}{y^k-y^{k-2}}$$

and $$h_z(x^k,y^k,z^k) \approx \frac{h(x^k,y^k,z^{k-2})-4h(x^k,y^k,z^{k-1})+3h(x^k,y^k,z^k)}{z^k-z^{k-2}}$$

5-point central method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{1}{12(x^{k+2}-x^{k+1})}[f(x^{k-2},y^k,z^k)-8f(x^{k-1},y^k,z^k) +8f(x^{k+1},y^k,z^k)-f(x^{k+2},y^k,z^k)]$$

$$g_y(x^k,y^k,z^k) \approx \frac{1}{12(y^{k+2}-y^{k+1})}[g(x^k,y^{k-2},z^k)-8g(x^k,y^{k-1},z^k) +8g(x^k,y^{k+1},z^k)-g(x^k,y^{k+2},z^k)]$$

and $$h_z(x^k,y^k,z^k) \approx \frac{1}{12(z^{k+2}-z^{k+1})}[h(x^k,y^k,z^{k-2})-8h(x^k,y^k,z^{k-1}) +8h(x^k,y^k,z^{k+1})-h(x^k,y^k,z^{k+2})]$$

5-point forward method, we have

$$f_x(x^k,y^k,z^k) \approx \frac{1}{12(x^{k+1}-x^k)}[-25f(x^k,y^k,z^k)+48f(x^{k+1},y^k,z^k) -36f(x^{k+2},y^k,z^k)+16f(x^{k+3},y^k,z^k)-3f(x^{k+4},y^k,z^k)]$$

$$g_y(x^k,y^k,z^k) \approx \frac{1}{12(y^{k+1}-y^k)}[-25g(x^k,y^k,z^k)+48g(x^k,y^{k+1},z^k) -36g(x^k,y^{k+2},z^k)+16g(x^k,y^{k+3},z^k)-3g(x^k,y^{k+4},z^k)]$$

and $$h_z(x^k,y^k,z^k)\approx\frac{1}{12(z^{k+1}-z^k)}[-25h(x^k,y^k,z^k)+48h(x^k,y^k,z^{k+1}) -36h(x^k,y^k,z^{k+2})+16h(x^k,y^k,z^{k+3})-3h(x^k,y^k,z^{k+4})]$$

5-point backward method, we have

$$f_x(x^k,y^k,z^k)\approx\frac{1}{12(x^k-x^{k-1})}[3f(x^{k-4},y^k,z^k)-16f(x^{k-3},y^k,z^k) +36f(x^{k-2},y^k,z^k)-48f(x^{k-1},y^k,z^k)+25f(x^k,y^k,z^k)]$$

$$g_y(x^k,y^k,z^k)\approx\frac{1}{12(y^k-y^{k-1})}[3g(x^k,y^{k-4},z^k)-16g(x^k,y^{k-3},z^k) +36g(x^k,y^{k-2},z^k)-48g(x^k,y^{k-1},z^k)+25g(x^k,y^k,z^k)]$$

and $$h_z(x^k,y^k,z^k)\approx\frac{1}{12(z^k-z^{k-1})}[3h(x^k,y^k,z^{k-4})-16h(x^k,y^k,z^{k-3}) +36h(x^k,y^k,z^{k-2})-48h(x^k,y^k,z^{k-1})+25h(x^k,y^k,z^k)]$$

Table 4.6 The result of example 4.6

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(1.0,1.0,2.0) and $\Delta x=0.5,\ \Delta y=0.2,$ $\Delta z=0.5$	8	x = 0.5000000 y = 0.0000000 z = -0.5235988
2-point backward	(1.0,1.0,2.0) and $\Delta x=0.5,\ \Delta y=0.2,$ $\Delta z=0.5$	10	x = 0.5000000 y = 0.0000000 z = -0.5235988
3-point central	(1.0,1.0,2.0) and $\Delta x=0.5,\ \Delta y=0.2,$ $\Delta z=0.5$	9	x = 0.5000000 y = 0.0000000 z = -0.5235988

Table 4.6 (cont.)

Method	Initial vectors	Number of iterations	Solutions
3-point forward	(1.0,1.0,2.0) and $\Delta x = 0.5$, $\Delta y = 0.2$, $\Delta z = 0.5$	9	x = 0.5000000 y = 0.0000000 z = -0.5235988
3-point backward	(1.0,1.0,2.0) and $\Delta x = 0.5$, $\Delta y = 0.2$, $\Delta z = 0.5$	9	x = 0.5000000 y = 0.0000000 z = -0.5235988
5-point central	(1.0,1.0,2.0) and $\Delta x = 0.5$, $\Delta y = 0.2$, $\Delta z = 0.5$	9	x = 0.5000000 y = 0.0000000 z = -0.5235988
5-point forward	(1.0,1.0,2.0) and $\Delta x = 0.5$, $\Delta y = 0.2$, $\Delta z = 0.5$	9	x = 0.5000000 y = 0.0000000 z = -0.5235988
5-point backward	(1.0,1.0,2.0) and $\Delta x = 0.5$, $\Delta y = 0.2$, $\Delta z = 0.5$	9	x = 0.5000000 y = 0.0000000 z = -0.5235988

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.6 in appendix.

Example 4.7 Find the solution of system

$$
\begin{aligned}
f_1(x_1,x_2,x_3,x_4) &= 137x_1 + 55x_2 + 20x_3 - 76 = 0 \\
f_2(x_1,x_2,x_3,x_4) &= 76x_2 - 21x_2x_4 - 19 = 0 \\
f_3(x_1,x_2,x_3,x_4) &= x_1 + x_2 + x_3 - 1 = 0 \\
f_4(x_1,x_2,x_3,x_4) &= 76x_1 + 61x_1x_4 - 38 = 0.
\end{aligned} \tag{4.7}
$$

Iterative equation is

$$
\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \\ x_4^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^{k} \\ x_2^{k} \\ x_3^{k} \\ x_4^{k} \end{bmatrix} - [H(x_1^k,x_2^k,x_3^k,x_4^k)]^{-1} \begin{bmatrix} f_1(x_1^k,x_2^k,x_3^k,x_4^k) \\ f_2(x_1^k,x_2^k,x_3^k,x_4^k) \\ f_3(x_1^k,x_2^k,x_3^k,x_4^k) \\ f_4(x_1^k,x_2^k,x_3^k,x_4^k) \end{bmatrix}
$$

and the matrix H is

$$
H(x_1,x_2,x_3,x_4) = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ 0 & f_{22} & 0 & 0 \\ 0 & 0 & f_{33} & 0 \\ 0 & 0 & 0 & f_{44} \end{bmatrix}.
$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - f_1(x_1^k, x_2^k, x_3^k, x_4^k)}{x_1^{k+1} - x_1^k}$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - f_2(x_1^k, x_2^k, x_3^k, x_4^k)}{x_2^{k+1} - x_2^k}$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_3(x_1^k, x_2^k, x_3^{k+1}, x_4^k) - f_3(x_1^k, x_2^k, x_3^k, x_4^k)}{x_3^{k+1} - x_3^k}$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_4(x_1^k, x_2^k, x_3^k, x_4^{k+1}) - f_4(x_1^k, x_2^k, x_3^k, x_4^k)}{x_4^{k+1} - x_4^k}$$

2-point backward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_1(x_1^k, x_2^k, x_3^k, x_4^k) - f_1(x_1^{k-1}, x_2^k, x_3^k, x_4^k)}{x_1^k - x_1^{k-1}}$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_2(x_1^k, x_2^k, x_3^k, x_4^k) - f_2(x_1^k, x_2^{k-1}, x_3^k, x_4^k)}{x_2^k - x_2^{k-1}}$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_3(x_1^k, x_2^k, x_3^k, x_4^k) - f_3(x_1^k, x_2^k, x_3^{k-1}, x_4^k)}{x_3^k - x_3^{k-1}}$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_4(x_1^k, x_2^k, x_3^k, x_4^k) - f_4(x_1^k, x_2^k, x_3^k, x_4^{k-1})}{x_4^k - x_4^{k-1}}$$

3-point central method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - f_1(x_1^{k-1}, x_2^k, x_3^k, x_4^k)}{x_1^{k+1} - x_1^{k-1}}$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - f_2(x_1^k, x_2^{k-1}, x_3^k, x_4^k)}{x_2^{k+1} - x_2^{k-1}}$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_3(x_1^k, x_2^k, x_3^{k+1}, x_4^k) - f_3(x_1^k, x_2^k, x_3^{k-1}, x_4^k)}{x_3^{k+1} - x_3^{k-1}}$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_4(x_1^k, x_2^k, x_3^k, x_4^{k+1}) - f_4(x_1^k, x_2^k, x_3^k, x_4^{k-1})}{x_4^{k+1} - x_4^{k-1}}$$

3-point forward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_1^{k+2} - x_1^k}[-3f_1(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - f_1(x_1^{k+2}, x_2^k, x_3^k, x_4^k)]$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_2^{k+2} - x_2^k}[-3f_2(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - f_2(x_1^k, x_2^{k+2}, x_3^k, x_4^k)]$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_3^{k+2} - x_3^k}[-3f_3(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_3(x_1^k, x_2^k, x_3^{k+1}, x_4^k) - f_3(x_1^k, x_2^k, x_3^{k+2}, x_4^k)]$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_4^{k+2} - x_4^k}[-3f_4(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_4(x_1^k, x_2^k, x_3^k, x_4^{k+1}) - f_4(x_1^k, x_2^k, x_3^k, x_4^{k+2})]$$

3-point backward method, we have

$$f_{11}(x_1^k,x_2^k,x_3^k,x_4^k)\approx\frac{1}{x_1^k-x_1^{k-2}}[f_1(x_1^{k-2},x_2^k,x_3^k,x_4^k)-4f_1(x_1^{k-1},x_2^k,x_3^k,x_4^k)$$
$$+3f_1(x_1^k,x_2^k,x_3^k,x_4^k)]$$

$$f_{22}(x_1^k,x_2^k,x_3^k,x_4^k)\approx\frac{1}{x_2^k-x_2^{k-2}}[f_2(x_1^k,x_2^{k-2},x_3^k,x_4^k)-4f_2(x_1^k,x_2^{k-1},x_3^k,x_4^k)$$
$$+3f_2(x_1^k,x_2^k,x_3^k,x_4^k)]$$

$$f_{33}(x_1^k,x_2^k,x_3^k,x_4^k)\approx\frac{1}{x_3^k-x_3^{k-2}}[f_3(x_1^k,x_2^k,x_3^{k-2},x_4^k)-4f_3(x_1^k,x_2^k,x_3^{k-1},x_4^k)$$
$$+3f_3(x_1^k,x_2^k,x_3^k,x_4^k)]$$

and $$f_{44}(x_1^k,x_2^k,x_3^k,x_4^k)\approx\frac{1}{x_4^k-x_4^{k-2}}[f_4(x_1^k,x_2^k,x_3^k,x_4^{k-2})-4f_4(x_1^k,x_2^k,x_3^k,x_4^{k-1})$$
$$+3f_4(x_1^k,x_2^k,x_3^k,x_4^k)]$$

5-point central method, we have

$$f_{11}(x_1^k,x_2^k,x_3^k,x_4^k)\approx\frac{1}{12(x_1^{k+2}-x_1^{k+1})}[f_1(x_1^{k-2},x_2^k,x_3^k,x_4^k)-8f_1(x_1^{k-1},x_2^k,x_3^k,x_4^k)$$
$$+8f_1(x_1^{k+1},x_2^k,x_3^k,x_4^k)-f_1(x_1^{k+2},x_2^k,x_3^k,x_4^k)]$$

$$f_{22}(x_1^k,x_2^k,x_3^k,x_4^k)\approx\frac{1}{12(x_2^{k+2}-x_2^{k+1})}[f_2(x_1^k,x_2^{k-2},x_3^k,x_4^k)-8f_2(x_1^k,x_2^{k-1},x_3^k,x_4^k)$$
$$+8f_2(x_1^k,x_2^{k+1},x_3^k,x_4^k)-f_2(x_1^k,x_2^{k+2},x_3^k,x_4^k)]$$

$$f_{33}(x_1^k,x_2^k,x_3^k,x_4^k)\approx\frac{1}{12(x_3^{k+2}-x_3^{k+1})}[f_3(x_1^k,x_2^k,x_3^{k-2},x_4^k)-8f_3(x_1^k,x_2^k,x_3^{k-1},x_4^k)$$
$$+8f_3(x_1^k,x_2^k,x_3^{k+1},x_4^k)-f_3(x_1^k,x_2^k,x_3^{k+2},x_4^k)]$$

$$f_{44}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_4^{k+2}-x_4^{k+1})}[f_4(x_1^k,x_2^k,x_3^k,x_4^{k-2}) - 8f_4(x_1^k,x_2^k,x_3^k,x_4^{k-1}) + 8f_4(x_1^k,x_2^k,x_3^k,x_4^{k+1}) - f_4(x_1^k,x_2^k,x_3^k,x_4^{k+2})]$$

5-point forward method, we have

$$f_{11}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_1^{k+1}-x_1^k)}[-25f_1(x_1^k,x_2^k,x_3^k,x_4^k) + 48f_1(x_1^{k+1},x_2^k,x_3^k,x_4^k) - 36f_1(x_1^{k+2},x_2^k,x_3^k,x_4^k) + 16f_1(x_1^{k+3},x_2^k,x_3^k,x_4^k) - 3f_1(x_1^{k+4},x_2^k,x_3^k,x_4^k)]$$

$$f_{22}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_2^{k+1}-x_2^k)}[-25f_2(x_1^k,x_2^k,x_3^k,x_4^k) + 48f_2(x_1^k,x_2^{k+1},x_3^k,x_4^k) - 36f_2(x_1^k,x_2^{k+2},x_3^k,x_4^k) + 16f_2(x_1^k,x_2^{k+3},x_3^k,x_4^k) - 3f_2(x_1^k,x_2^{k+4},x_3^k,x_4^k)]$$

$$f_{33}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_3^{k+1}-x_3^k)}[-25f_3(x_1^k,x_2^k,x_3^k,x_4^k) + 48f_3(x_1^k,x_2^k,x_3^{k+1},x_4^k) - 36f_3(x_1^k,x_2^k,x_3^{k+2},x_4^k) + 16f_3(x_1^k,x_2^k,x_3^{k+3},x_4^k) - 3f_3(x_1^k,x_2^k,x_3^{k+4},x_4^k)]$$

$$f_{44}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_4^{k+1}-x_4^k)}[-25f_4(x_1^k,x_2^k,x_3^k,x_4^k) + 48f_4(x_1^k,x_2^k,x_3^k,x_4^{k+1}) - 36f_4(x_1^k,x_2^k,x_3^k,x_4^{k+2}) + 16f_4(x_1^k,x_2^k,x_3^k,x_4^{k+3}) - 3f_4(x_1^k,x_2^k,x_3^k,x_4^{k+4})]$$

5-point backward method, we have

$$f_{11}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_1^k-x_1^{k-1})}[3f_1(x_1^{k-4},x_2^k,x_3^k,x_4^k) - 16f_1(x_1^{k-3},x_2^k,x_3^k,x_4^k) + 36f_1(x_1^{k-2},x_2^k,x_3^k,x_4^k) - 48f_1(x_1^{k-1},x_2^k,x_3^k,x_4^k) + 25f_1(x_1^k,x_2^k,x_3^k,x_4^k)]$$

$$f_{22}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_2^k-x_2^{k-1})}[3f_2(x_1^k,x_2^{k-4},x_3^k,x_4^k) - 16f_2(x_1^k,x_2^{k-3},x_3^k,x_4^k) + 36f_2(x_1^k,x_2^{k-2},x_3^k,x_4^k) - 48f_2(x_1^k,x_2^{k-1},x_3^k,x_4^k) + 25f_2(x_1^k,x_2^k,x_3^k,x_4^k)]$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_3^k - x_3^{k-1})}[3f_3(x_1^k, x_2^k, x_3^{k-4}, x_4^k) - 16f_3(x_1^k, x_2^k, x_3^{k-3}, x_4^k) + 36f_3(x_1^k, x_2^k, x_3^{k-2}, x_4^k) - 48f_3(x_1^k, x_2^k, x_3^{k-1}, x_4^k) + 25f_3(x_1^k, x_2^k, x_3^k, x_4^k)]$$

$$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_4^k - x_4^{k-1})}[3f_4(x_1^k, x_2^k, x_3^k, x_4^{k-4}) - 16f_4(x_1^k, x_2^k, x_3^k, x_4^{k-3}) + 36f_4(x_1^k, x_2^k, x_3^k, x_4^{k-2}) - 48f_4(x_1^k, x_2^k, x_3^k, x_4^{k-1}) + 25f_4(x_1^k, x_2^k, x_3^k, x_4^k)]$$

Table 4.7 The result of example 4.7

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	31	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$
2-point backward	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	31	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$
3-point central	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	31	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$
3-point forward	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	31	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$

Table 4.7 (cont.)

Method	Initial vectors	Number of iterations	Solutions
3-point backward	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	31	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$
5-point central	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	29	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$
5-point forward	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	31	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$
5-point backward	(0.1,1.0,-0.2,2.5) and $\Delta x_1 = 0.1$, $\Delta x_2 = 0.2$, $\Delta x_3 = 0.1$, $\Delta x_4 = 0.1$	31	$x_1 = 0.1545897$ $x_2 = 1.0832286$ $x_3 = -0.2378183$ $x_4 = 2.7838020$

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.7 in appendix.

Example 4.8

The plane $x+y+z-12=0$ intersects the paraboloid $z=x^2+y^2$ in an ellipse (Figure 4.1). Find the highest and lowest point on this ellipse by Lagrange Multipliers with two constraints.

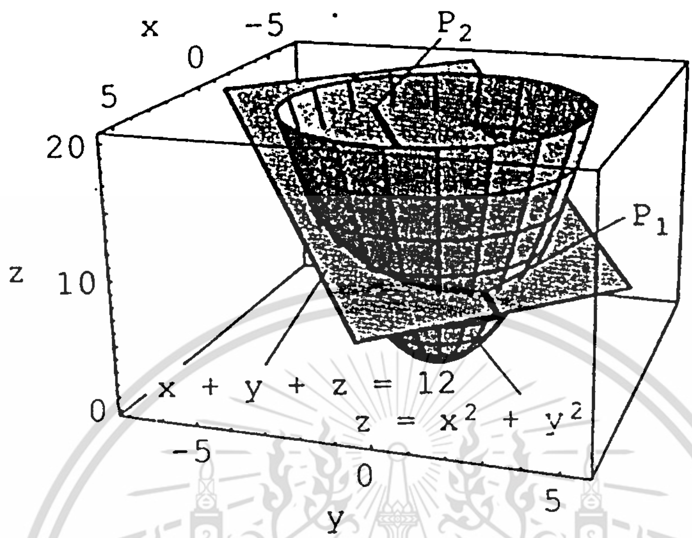

Figure 4.1 The plane and paraboloid intersecting in the ellipse

Solution The height of the point (x,y,z) is z, so we want to find the maximum and minimum values of

$$f(x,y,z)=z$$

subject to the two conditions

$$g(x,y,z)=x+y+z-12=0$$

and

$$h(x,y,z)=x^2+y^2-z=0.$$

By Lagrange Multipliers with two constraints, we have Lagrange function

$$L(x,y,z,\lambda,\mu) = z + \lambda(x+y+z-12) + \mu(x^2+y^2-z)$$

and a system of nonlinear equations

$$L_x(x,y,z,\lambda,\mu) = \lambda + 2\mu x = 0$$
$$L_y(x,y,z,\lambda,\mu) = \lambda + 2\mu y = 0$$
$$L_z(x,y,z,\lambda,\mu) = 1 + \lambda - \mu = 0$$
$$L_\lambda(x,y,z,\lambda,\mu) = x + y + z - 12 = 0$$
$$L_\mu(x,y,z,\lambda,\mu) = x^2 + y^2 - z = 0.$$

Iterative equation is

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \\ \lambda^{k+1} \\ \mu^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ z^k \\ \lambda^k \\ \mu^k \end{bmatrix} - [H(x^k,y^k,z^k,\lambda^k,\mu^k)]^{-1} \begin{bmatrix} L_x(x^k,y^k,z^k,\lambda^k,\mu^k) \\ L_y(x^k,y^k,z^k,\lambda^k,\mu^k) \\ L_z(x^k,y^k,z^k,\lambda^k,\mu^k) \\ L_\lambda(x^k,y^k,z^k,\lambda^k,\mu^k) \\ L_\mu(x^k,y^k,z^k,\lambda^k,\mu^k) \end{bmatrix}$$

and the matrix H is

$$H(x,y,z,\lambda,\mu) = \begin{bmatrix} L_{xx} & 0 & 0 & 0 & 0 \\ 0 & L_{yy} & 0 & 0 & 0 \\ 0 & 0 & L_{zz} & 0 & 0 \\ 0 & 0 & 0 & L_{\lambda\lambda} & 0 \\ 0 & 0 & 0 & 0 & L_{\mu\mu} \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$L_{xx}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_x(x^{k+1},y^k,z^k,\lambda^k,\mu^k) - L_x(x^k,y^k,z^k,\lambda^k,\mu^k)}{x^{k+1}-x^k}$$

$$L_{yy}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_y(x^k,y^{k+1},z^k,\lambda^k,\mu^k) - L_y(x^k,y^k,z^k,\lambda^k,\mu^k)}{y^{k+1}-y^k}$$

$$L_{zz}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_z(x^k,y^k,z^{k+1},\lambda^k,\mu^k) - L_z(x^k,y^k,z^k,\lambda^k,\mu^k)}{z^{k+1}-z^k}$$

$$L_{\lambda\lambda}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_\lambda(x^k,y^k,z^k,\lambda^{k+1},\mu^k) - L_\lambda(x^k,y^k,z^k,\lambda^k,\mu^k)}{\lambda^{k+1}-\lambda^k}$$

and $$L_{\mu\mu}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+1}) - L_\mu(x^k,y^k,z^k,\lambda^k,\mu^k)}{\mu^{k+1}-\mu^k}$$

2-point backward method, we have

$$L_{xx}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_x(x^k,y^k,z^k,\lambda^k,\mu^k) - L_x(x^{k-1},y^k,z^k,\lambda^k,\mu^k)}{x^k-x^{k-1}}$$

$$L_{yy}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_y(x^k,y^k,z^k,\lambda^k,\mu^k) - L_y(x^k,y^{k-1},z^k,\lambda^k,\mu^k)}{y^k-y^{k-1}}$$

$$L_{zz}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_z(x^k,y^k,z^k,\lambda^k,\mu^k) - L_z(x^k,y^k,z^{k-1},\lambda^k,\mu^k)}{z^k-z^{k-1}}$$

$$L_{\lambda\lambda}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_\lambda(x^k,y^k,z^k,\lambda^k,\mu^k) - L_\lambda(x^k,y^k,z^k,\lambda^{k-1},\mu^k)}{\lambda^k-\lambda^{k-1}}$$

and $$L_{\mu\mu}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{L_\mu(x^k,y^k,z^k,\lambda^k,\mu^k) - L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-1})}{\mu^k-\mu^{k-1}}$$

3-point central method, we have

$$L_{xx}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{L_x(x^{k+1},y^k,z^k,\lambda^k,\mu^k)-L_x(x^{k-1},y^k,z^k,\lambda^k,\mu^k)}{x^{k+1}-x^{k-1}}$$

$$L_{yy}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{L_y(x^k,y^{k+1},z^k,\lambda^k,\mu^k)-L_y(x^k,y^{k-1},z^k,\lambda^k,\mu^k)}{y^{k+1}-y^{k-1}}$$

$$L_{zz}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{L_z(x^k,y^k,z^{k+1},\lambda^k,\mu^k)-L_z(x^k,y^k,z^{k-1},\lambda^k,\mu^k)}{z^{k+1}-z^{k-1}}$$

$$L_{\lambda\lambda}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{L_\lambda(x^k,y^k,z^k,\lambda^{k+1},\mu^k)-L_\lambda(x^k,y^k,z^k,\lambda^{k-1},\mu^k)}{\lambda^{k+1}-\lambda^{k-1}}$$

and $$L_{\mu\mu}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+1})-L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-1})}{\mu^{k+1}-\mu^{k-1}}$$

3-point forward method, we have

$$L_{xx}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{x^{k+2}-x^k}[-3L_x(x^k,y^k,z^k,\lambda^k,\mu^k)+4L_x(x^{k+1},y^k,z^k,\lambda^k,\mu^k)-L_x(x^{k+2},y^k,z^k,\lambda^k,\mu^k)]$$

$$L_{yy}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{y^{k+2}-y^k}[-3L_y(x^k,y^k,z^k,\lambda^k,\mu^k)+4L_y(x^k,y^{k+1},z^k,\lambda^k,\mu^k)-L_y(x^k,y^{k+2},z^k,\lambda^k,\mu^k)]$$

$$L_{zz}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{z^{k+2}-z^k}[-3L_z(x^k,y^k,z^k,\lambda^k,\mu^k)+4L_z(x^k,y^k,z^{k+1},\lambda^k,\mu^k)-L_z(x^k,y^k,z^{k+2},\lambda^k,\mu^k)]$$

$$L_{\lambda\lambda}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{\lambda^{k+2}-\lambda^k}[-3L_\lambda(x^k,y^k,z^k,\lambda^k,\mu^k)+4L_\lambda(x^k,y^k,z^k,\lambda^{k+1},\mu^k)-L_\lambda(x^k,y^k,z^k,\lambda^{k+2},\mu^k)]$$

and $$L_{\mu\mu}(x^k, y^k, z^k, \lambda^k, \mu^k) \approx \frac{1}{\mu^{k+2} - \mu^k}[-3L_\mu(x^k, y^k, z^k, \lambda^k, \mu^k) + 4L_\mu(x^k, y^k, z^k, \lambda^k, \mu^{k+1}) - L_\mu(x^k, y^k, z^k, \lambda^k, \mu^{k+2})]$$

3-point backward method, we have

$$L_{xx}(x^k, y^k, z^k, \lambda^k, \mu^k) \approx \frac{1}{x^k - x^{k-2}}[L_x(x^{k-2}, y^k, z^k, \lambda^k, \mu^k) - 4L_x(x^{k-1}, y^k, z^k, \lambda^k, \mu^k) + 3L_x(x^k, y^k, z^k, \lambda^k, \mu^k)]$$

$$L_{yy}(x^k, y^k, z^k, \lambda^k, \mu^k) \approx \frac{1}{y^k - y^{k-2}}[L_y(x^k, y^{k-2}, z^k, \lambda^k, \mu^k) - 4L_y(x^k, y^{k-1}, z^k, \lambda^k, \mu^k) + 3L_y(x^k, y^k, z^k, \lambda^k, \mu^k)]$$

$$L_{zz}(x^k, y^k, z^k, \lambda^k, \mu^k) \approx \frac{1}{z^k - z^{k-2}}[L_z(x^k, y^k, z^{k-2}, \lambda^k, \mu^k) - 4L_z(x^k, y^k, z^{k-1}, \lambda^k, \mu^k) + 3L_z(x^k, y^k, z^k, \lambda^k, \mu^k)]$$

$$L_{\lambda\lambda}(x^k, y^k, z^k, \lambda^k, \mu^k) \approx \frac{1}{\lambda^k - \lambda^{k-2}}[L_\lambda(x^k, y^k, z^k, \lambda^{k-2}, \mu^k) - 4L_\lambda(x^k, y^k, z^k, \lambda^{k-1}, \mu^k) + 3L_\lambda(x^k, y^k, z^k, \lambda^k, \mu^k)]$$

and $$L_{\mu\mu}(x^k, y^k, z^k, \lambda^k, \mu^k) \approx \frac{1}{\mu^k - \mu^{k-2}}[L_\mu(x^k, y^k, z^k, \lambda^k, \mu^{k-2}) - 4L_\mu(x^k, y^k, z^k, \lambda^k, \mu^{k-1}) + 3L_\mu(x^k, y^k, z^k, \lambda^k, \mu^k)]$$

5-point central method, we have

$$L_{xx}(x^k, y^k, z^k, \lambda^k, \mu^k) \approx \frac{1}{12(x^{k+2} - x^{k+1})}[L_x(x^{k-2}, y^k, z^k, \lambda^k, \mu^k) - 8L_x(x^{k-1}, y^k, z^k, \lambda^k, \mu^k) + 8L_x(x^{k+1}, y^k, z^k, \lambda^k, \mu^k) - L_x(x^{k+2}, y^k, z^k, \lambda^k, \mu^k)]$$

$$L_{yy}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{12(y^{k+2}-y^{k+1})}[L_y(x^k,y^{k-2},z^k,\lambda^k,\mu^k)$$
$$-8L_y(x^k,y^{k-1},z^k,\lambda^k,\mu^k)+8L_y(x^k,y^{k+1},z^k,\lambda^k,\mu^k)$$
$$-L_y(x^k,y^{k+2},z^k,\lambda^k,\mu^k)]$$

$$L_{zz}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{12(z^{k+2}-z^{k+1})}[L_z(x^k,y^k,z^{k-2},\lambda^k,\mu^k)$$
$$-8L_z(x^k,y^k,z^{k-1},\lambda^k,\mu^k)+8L_z(x^k,y^k,z^{k+1},\lambda^k,\mu^k)$$
$$-L_z(x^k,y^k,z^{k+2},\lambda^k,\mu^k)]$$

$$L_{\lambda\lambda}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{12(\lambda^{k+2}-\lambda^{k+1})}[L_\lambda(x^k,y^k,z^k,\lambda^{k-2},\mu^k)$$
$$-8L_\lambda(x^k,y^k,z^k,\lambda^{k-1},\mu^k)+8L_\lambda(x^k,y^k,z^k,\lambda^{k+1},\mu^k)$$
$$-L_\lambda(x^k,y^k,z^k,\lambda^{k+2},\mu^k)]$$

and $L_{\mu\mu}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{12(\mu^{k+2}-\mu^{k+1})}[L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-2})$

$$-8L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-1})+8L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+1})$$
$$-L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+2})]$$

5-point forward method, we have

$$L_{xx}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{12(x^{k+1}-x^k)}[-25L_x(x^k,y^k,z^k,\lambda^k,\mu^k)$$
$$+48L_x(x^{k+1},y^k,z^k,\lambda^k,\mu^k)-36L_x(x^{k+2},y^k,z^k,\lambda^k,\mu^k)$$
$$+16L_x(x^{k+3},y^k,z^k,\lambda^k,\mu^k)-3L_x(x^{k+4},y^k,z^k,\lambda^k,\mu^k)]$$

$$L_{yy}(x^k,y^k,z^k,\lambda^k,\mu^k)\approx\frac{1}{12(y^{k+1}-y^k)}[-25L_y(x^k,y^k,z^k,\lambda^k,\mu^k)$$
$$+48L_y(x^k,y^{k+1},z^k,\lambda^k,\mu^k)-36L_y(x^k,y^{k+2},z^k,\lambda^k,\mu^k)$$
$$+16L_y(x^k,y^{k+3},z^k,\lambda^k,\mu^k)-3L_y(x^k,y^{k+4},z^k,\lambda^k,\mu^k)]$$

$$L_{zz}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(z^{k+1}-z^k)}[-25L_z(x^k,y^k,z^k,\lambda^k,\mu^k)$$
$$+48L_z(x^k,y^k,z^{k+1},\lambda^k,\mu^k)-36L_z(x^k,y^k,z^{k+2},\lambda^k,\mu^k)$$
$$+16L_z(x^k,y^k,z^{k+3},\lambda^k,\mu^k)-3L_z(x^k,y^k,z^{k+4},\lambda^k,\mu^k)]$$

$$L_{\lambda\lambda}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(\lambda^{k+1}-\lambda^k)}[-25L_\lambda(x^k,y^k,z^k,\lambda^k,\mu^k)$$
$$+48L_\lambda(x^k,y^k,z^k,\lambda^{k+1},\mu^k)-36L_\lambda(x^k,y^k,z^k,\lambda^{k+2},\mu^k)$$
$$+16L_\lambda(x^k,y^k,z^k,\lambda^{k+3},\mu^k)-3L_\lambda(x^k,y^k,z^k,\lambda^{k+4},\mu^k)]$$

$$L_{\mu\mu}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(\mu^{k+1}-\mu^k)}[-25L_\mu(x^k,y^k,z^k,\lambda^k,\mu^k)$$
$$+48L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+1})-36L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+2})$$
$$+16L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+3})-3L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k+4})]$$

5-point backward method, we have

$$L_{xx}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(x^k-x^{k-1})}[3L_x(x^{k-4},y^k,z^k,\lambda^k,\mu^k)$$
$$-16L_x(x^{k-3},y^k,z^k,\lambda^k,\mu^k)+36L_x(x^{k-2},y^k,z^k,\lambda^k,\mu^k)$$
$$-48L_x(x^{k-1},y^k,z^k,\lambda^k,\mu^k)+25L_x(x^k,y^k,z^k,\lambda^k,\mu^k)]$$

$$L_{yy}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(y^k-y^{k-1})}[3L_y(x^k,y^{k-4},z^k,\lambda^k,\mu^k)$$
$$-16L_y(x^k,y^{k-3},z^k,\lambda^k,\mu^k)+36L_y(x^k,y^{k-2},z^k,\lambda^k,\mu^k)$$
$$-48L_y(x^k,y^{k-1},z^k,\lambda^k,\mu^k)+25L_y(x^k,y^k,z^k,\lambda^k,\mu^k)]$$

$$L_{zz}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(z^k-z^{k-1})}[3L_z(x^k,y^k,z^{k-4},\lambda^k,\mu^k)$$
$$-16L_z(x^k,y^k,z^{k-3},\lambda^k,\mu^k)+36L_z(x^k,y^k,z^{k-2},\lambda^k,\mu^k)$$
$$-48L_z(x^k,y^k,z^{k-1},\lambda^k,\mu^k)+25L_z(x^k,y^k,z^k,\lambda^k,\mu^k)]$$

$$L_{\lambda\lambda}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(\lambda^k-\lambda^{k-1})}[3L_\lambda(x^k,y^k,z^k,\lambda^{k-4},\mu^k)$$
$$-16L_\lambda(x^k,y^k,z^k,\lambda^{k-3},\mu^k)+36L_\lambda(x^k,y^k,z^k,\lambda^{k-2},\mu^k)$$
$$-48L_\lambda(x^k,y^k,z^k,\lambda^{k-1},\mu^k)+25L_\lambda(x^k,y^k,z^k,\lambda^k,\mu^k)]$$

$$L_{\mu\mu}(x^k,y^k,z^k,\lambda^k,\mu^k) \approx \frac{1}{12(\mu^k-\mu^{k-1})}[3L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-4})$$
$$-16L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-3})+36L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-2})$$
$$-48L_\mu(x^k,y^k,z^k,\lambda^k,\mu^{k-1})+25L_\mu(x^k,y^k,z^k,\lambda^k,\mu^k)]$$

Table 4.8 The result of example 4.8

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x=0.1$, $\Delta y=0.1$, $\Delta z=0.2$, $\Delta\lambda=0.1, \Delta\mu=0.32$	19	x = -3.0000000 y = -3.0000000 z = 17.9999999 λ = -1.2000000 μ = -0.2000000
2-point backward	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x=0.1$, $\Delta y=0.1$, $\Delta z=0.2$, $\Delta\lambda=0.1, \Delta\mu=0.32$	18	x = -3.0000000 y = -3.0000000 z = 17.9999998 λ = -1.2000000 μ = -0.2000000
3-point central	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x=0.1$, $\Delta y=0.1$, $\Delta z=0.2$, $\Delta\lambda=0.1, \Delta\mu=0.32$	19	x = -3.0000000 y = -3.0000000 z = 17.9999999 λ = -1.2000000 μ = -0.2000000

Table 4.8 (cont.)

Method	Initial vectors	Number of iterations	Solutions
3-point forward	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta z = 0.2$, $\Delta \lambda = 0.1, \Delta \mu = 0.32$	19	x = -3.0000000 y = -3.0000000 z = 17.9999999 λ = -1.2000000 μ = -0.2000000
3-point backward	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta z = 0.2$, $\Delta \lambda = 0.1, \Delta \mu = 0.32$	19	x = -3.0000000 y = -3.0000000 z = 17.9999999 λ = -1.2000000 μ = -0.2000000
5-point central	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta z = 0.2$, $\Delta \lambda = 0.1, \Delta \mu = 0.32$	19	x = -3.0000000 y = -3.0000000 z = 17.9999999 λ = -1.2000000 μ = -0.2000000
5-point forward	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta z = 0.2$, $\Delta \lambda = 0.1, \Delta \mu = 0.32$	19	x = -3.0000000 y = -3.0000000 z = 17.9999999 λ = -1.2000000 μ = -0.2000000

Table 4.8 (cont.)

Method	Initial vectors	Number of iterations	Solutions
5-point backward	(-2.5,-2.5,16.0,1.0,5.0) and $\Delta x = 0.1$, $\Delta y = 0.1$, $\Delta z = 0.2$, $\Delta\lambda = 0.1, \Delta\mu = 0.32$	19	x = -3.0000000 y = -3.0000000 z = 17.9999999 λ = -1.2000000 μ = -0.2000000

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.8 in appendix.

Example 4.9

The application is taken from the Bohr theory of the hydorgen spectrum in atomic physics. A certain constant known as the Rydberg constant, usually denoted by R_∞, plays an important part in that theory, but it cannot be measured directly The value must be inferred from measurements of other constants. Approximate values R_H and R_{He} can be determined from measurements of the spectrum of hydrogen and helium, respectively. If we knew the exact mass M_H of the nucleus of a hydrogen atom and the exact mass m of an electron, then theoretically. We could determine R_∞ from the equation

$$R_\infty = \left(1 + \frac{m}{M_H}\right) R_H$$

Suppose, for the moment, that m and M_H are unknown (because even our best values are inferred from other measured variables). The atomic weight A_H of hydrogen can be measured. Also, $A_H = M_H + m$. Similar equations hold for the helium atom. So we have a system of four equations:

$$R_\infty = \left(1 + \frac{m}{M_H}\right) R_H$$

$$R_\infty = \left(1 + \frac{m}{M_{He}}\right) R_{He}$$

$$A_H = M_H + m$$

$$A_{He} = M_{He} + m$$

In this system, the numbers A_H, A_{He}, R_H and R_{He} can be taken as known because their values can be determined from laboratory measurements. The number R_∞ and masses m, M_H and M_{He} will be regarded as unknown. Their values can be determined by solving the system of four equations.

The work is simplified if we first make some algebraic changes in the system. Let $x_1 = R_\infty, x_2 = m, x_3 = M_H$ and $x_4 = M_{He}$. Then our system becomes

$$x_1 = \left(1 + \frac{x_2}{x_3}\right) R_H$$

$$x_1 = \left(1 + \frac{x_2}{x_4}\right) R_{He}$$

$$x_2 + x_3 = A_H$$

$$2x_2 + x_4 = A_{He}$$

Since $x_1 = R_\infty$ and R_∞ is approximated by the measured number R_H, we have an initial estimate of x_1. Similarly, since x_2 is the mass of the electron and we know that the value is somewhere around $\frac{1}{2000}$ of a unit of atomic mass, an initial estimate of 0.0005 for x_2 seems reasonable. We need values of A_H, A_{He}, R_H and R_{He} to perform the calculations. Suppose laboratory measurements produced the following values: A_H = 1.00812, A_{He} = 4.00388, R_H = 109677.68, and R_{He} = 109722.34. What are the corresponding values for Rydberg constant R_∞ and masses m, M_H and M_{He}?. A machine implementation of the iterations suggested here produced the following results.

Iterative equation is

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \\ x_4^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^{k} \\ x_2^{k} \\ x_3^{k} \\ x_4^{k} \end{bmatrix} - [H(x_1^k, x_2^k, x_3^k, x_4^k)]^{-1} \begin{bmatrix} f_1(x_1^k, x_2^k, x_3^k, x_4^k) \\ f_2(x_1^k, x_2^k, x_3^k, x_4^k) \\ f_3(x_1^k, x_2^k, x_3^k, x_4^k) \\ f_4(x_1^k, x_2^k, x_3^k, x_4^k) \end{bmatrix}$$

and the matrix H is

$$H(x_1, x_2, x_3, x_4) = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ 0 & f_{22} & 0 & 0 \\ 0 & 0 & f_{33} & 0 \\ 0 & 0 & 0 & f_{44} \end{bmatrix}.$$

We can find the matrix H from theorem 3.1-3.8

2-point forward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - f_1(x_1^k, x_2^k, x_3^k, x_4^k)}{x_1^{k+1} - x_1^k}$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - f_2(x_1^k, x_2^k, x_3^k, x_4^k)}{x_2^{k+1} - x_2^k}$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_3(x_1^k, x_2^k, x_3^{k+1}, x_4^k) - f_3(x_1^k, x_2^k, x_3^k, x_4^k)}{x_3^{k+1} - x_3^k}$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_4(x_1^k, x_2^k, x_3^k, x_4^{k+1}) - f_4(x_1^k, x_2^k, x_3^k, x_4^k)}{x_4^{k+1} - x_4^k}$$

2-point backward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_1(x_1^k, x_2^k, x_3^k, x_4^k) - f_1(x_1^{k-1}, x_2^k, x_3^k, x_4^k)}{x_1^k - x_1^{k-1}}$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_2(x_1^k, x_2^k, x_3^k, x_4^k) - f_2(x_1^k, x_2^{k-1}, x_3^k, x_4^k)}{x_2^k - x_2^{k-1}}$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_3(x_1^k, x_2^k, x_3^k, x_4^k) - f_3(x_1^k, x_2^k, x_3^{k-1}, x_4^k)}{x_3^k - x_3^{k-1}}$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_4(x_1^k, x_2^k, x_3^k, x_4^k) - f_4(x_1^k, x_2^k, x_3^k, x_4^{k-1})}{x_4^k - x_4^{k-1}}$$

3-point central method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - f_1(x_1^{k-1}, x_2^k, x_3^k, x_4^k)}{x_1^{k+1} - x_1^{k-1}}$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - f_2(x_1^k, x_2^{k-1}, x_3^k, x_4^k)}{x_2^{k+1} - x_2^{k-1}}$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_3(x_1^k, x_2^k, x_3^{k+1}, x_4^k) - f_3(x_1^k, x_2^k, x_3^{k-1}, x_4^k)}{x_3^{k+1} - x_3^{k-1}}$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{f_4(x_1^k, x_2^k, x_3^k, x_4^{k+1}) - f_4(x_1^k, x_2^k, x_3^k, x_4^{k-1})}{x_4^{k+1} - x_4^{k-1}}$$

3-point forward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_1^{k+2} - x_1^k}[-3f_1(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - f_1(x_1^{k+2}, x_2^k, x_3^k, x_4^k)]$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_2^{k+2} - x_2^k}[-3f_2(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - f_2(x_1^k, x_2^{k+2}, x_3^k, x_4^k)]$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_3^{k+2} - x_3^k}[-3f_3(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_3(x_1^k, x_2^k, x_3^{k+1}, x_4^k) - f_3(x_1^k, x_2^k, x_3^{k+2}, x_4^k)]$$

and $f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_4^{k+2} - x_4^k}[-3f_4(x_1^k, x_2^k, x_3^k, x_4^k) + 4f_4(x_1^k, x_2^k, x_3^k, x_4^{k+1}) - f_4(x_1^k, x_2^k, x_3^k, x_4^{k+2})]$

3-point backward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_1^k - x_1^{k-2}}[f_1(x_1^{k-2}, x_2^k, x_3^k, x_4^k) - 4f_1(x_1^{k-1}, x_2^k, x_3^k, x_4^k) + 3f_1(x_1^k, x_2^k, x_3^k, x_4^k)]$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_2^k - x_2^{k-2}}[f_2(x_1^k, x_2^{k-2}, x_3^k, x_4^k) - 4f_2(x_1^k, x_2^{k-1}, x_3^k, x_4^k) + 3f_2(x_1^k, x_2^k, x_3^k, x_4^k)]$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_3^k - x_3^{k-2}}[f_3(x_1^k, x_2^k, x_3^{k-2}, x_4^k) - 4f_3(x_1^k, x_2^k, x_3^{k-1}, x_4^k) + 3f_3(x_1^k, x_2^k, x_3^k, x_4^k)]$$

and $$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{x_4^k - x_4^{k-2}}[f_4(x_1^k, x_2^k, x_3^k, x_4^{k-2}) - 4f_4(x_1^k, x_2^k, x_3^k, x_4^{k-1}) + 3f_4(x_1^k, x_2^k, x_3^k, x_4^k)]$$

5-point central method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_1^{k+2} - x_1^{k+1})}[f_1(x_1^{k-2}, x_2^k, x_3^k, x_4^k) - 8f_1(x_1^{k-1}, x_2^k, x_3^k, x_4^k) + 8f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - f_1(x_1^{k+2}, x_2^k, x_3^k, x_4^k)]$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_2^{k+2} - x_2^{k+1})}[f_2(x_1^k, x_2^{k-2}, x_3^k, x_4^k) - 8f_2(x_1^k, x_2^{k-1}, x_3^k, x_4^k) + 8f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - f_2(x_1^k, x_2^{k+2}, x_3^k, x_4^k)]$$

$$f_{33}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_3^{k+2} - x_3^{k+1})}[f_3(x_1^k, x_2^k, x_3^{k-2}, x_4^k) - 8f_3(x_1^k, x_2^k, x_3^{k-1}, x_4^k) + 8f_3(x_1^k, x_2^k, x_3^{k+1}, x_4^k) - f_3(x_1^k, x_2^k, x_3^{k+2}, x_4^k)]$$

$$f_{44}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_4^{k+2} - x_4^{k+1})}[f_4(x_1^k, x_2^k, x_3^k, x_4^{k-2}) - 8f_4(x_1^k, x_2^k, x_3^k, x_4^{k-1}) + 8f_4(x_1^k, x_2^k, x_3^k, x_4^{k+1}) - f_4(x_1^k, x_2^k, x_3^k, x_4^{k+2})]$$

5-point forward method, we have

$$f_{11}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_1^{k+1} - x_1^k)}[-25f_1(x_1^k, x_2^k, x_3^k, x_4^k) + 48f_1(x_1^{k+1}, x_2^k, x_3^k, x_4^k) - 36f_1(x_1^{k+2}, x_2^k, x_3^k, x_4^k) + 16f_1(x_1^{k+3}, x_2^k, x_3^k, x_4^k) - 3f_1(x_1^{k+4}, x_2^k, x_3^k, x_4^k)]$$

$$f_{22}(x_1^k, x_2^k, x_3^k, x_4^k) \approx \frac{1}{12(x_2^{k+1} - x_2^k)}[-25f_2(x_1^k, x_2^k, x_3^k, x_4^k) + 48f_2(x_1^k, x_2^{k+1}, x_3^k, x_4^k) - 36f_2(x_1^k, x_2^{k+2}, x_3^k, x_4^k) + 16f_2(x_1^k, x_2^{k+3}, x_3^k, x_4^k) - 3f_2(x_1^k, x_2^{k+4}, x_3^k, x_4^k)]$$

$$f_{33}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_3^{k+1}-x_3^k)}[-25f_3(x_1^k,x_2^k,x_3^k,x_4^k)+48f_3(x_1^k,x_2^k,x_3^{k+1},x_4^k)$$
$$-36f_3(x_1^k,x_2^k,x_3^{k+2},x_4^k)+16f_3(x_1^k,x_2^k,x_3^{k+3},x_4^k)-3f_3(x_1^k,x_2^k,x_3^{k+4},x_4^k)]$$

$$f_{44}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_4^{k+1}-x_4^k)}[-25f_4(x_1^k,x_2^k,x_3^k,x_4^k)+48f_4(x_1^k,x_2^k,x_3^k,x_4^{k+1})$$
$$-36f_4(x_1^k,x_2^k,x_3^k,x_4^{k+2})+16f_4(x_1^k,x_2^k,x_3^k,x_4^{k+3})-3f_4(x_1^k,x_2^k,x_3^k,x_4^{k+4})]$$

5-point backward method, we have

$$f_{11}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_1^k-x_1^{k-1})}[3f_1(x_1^{k-4},x_2^k,x_3^k,x_4^k)-16f_1(x_1^{k-3},x_2^k,x_3^k,x_4^k)$$
$$+36f_1(x_1^{k-2},x_2^k,x_3^k,x_4^k)-48f_1(x_1^{k-1},x_2^k,x_3^k,x_4^k)+25f_1(x_1^k,x_2^k,x_3^k,x_4^k)]$$

$$f_{22}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_2^k-x_2^{k-1})}[3f_2(x_1^k,x_2^{k-4},x_3^k,x_4^k)-16f_2(x_1^k,x_2^{k-3},x_3^k,x_4^k)$$
$$+36f_2(x_1^k,x_2^{k-2},x_3^k,x_4^k)-48f_2(x_1^k,x_2^{k-1},x_3^k,x_4^k)+25f_2(x_1^k,x_2^k,x_3^k,x_4^k)]$$

$$f_{33}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_3^k-x_3^{k-1})}[3f_3(x_1^k,x_2^k,x_3^{k-4},x_4^k)-16f_3(x_1^k,x_2^k,x_3^{k-3},x_4^k)$$
$$+36f_3(x_1^k,x_2^k,x_3^{k-2},x_4^k)-48f_3(x_1^k,x_2^k,x_3^{k-1},x_4^k)+25f_3(x_1^k,x_2^k,x_3^k,x_4^k)]$$

$$f_{44}(x_1^k,x_2^k,x_3^k,x_4^k) \approx \frac{1}{12(x_4^k-x_4^{k-1})}[3f_4(x_1^k,x_2^k,x_3^k,x_4^{k-4})-16f_4(x_1^k,x_2^k,x_3^k,x_4^{k-3})$$
$$+36f_4(x_1^k,x_2^k,x_3^k,x_4^{k-2})-48f_4(x_1^k,x_2^k,x_3^k,x_4^{k-1})+25f_4(x_1^k,x_2^k,x_3^k,x_4^k)]$$

Table 4.9 The result of example 4.9

Method	Initial vectors	Number of iterations	Solutions
2-point forward	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5,\ \Delta x_2 = 0.5,$ $\Delta x_3 = 0.5,\ \Delta x_4 = 0.5$	41	x_1 = 109737.3715400 x_2 = 0.0005484 x_3 = 1.0075716 x_4 = 4.0027833
2-point backward	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5,\ \Delta x_2 = 0.5,$ $\Delta x_3 = 0.5,\ \Delta x_4 = 0.5$	40	x_1 = 109737.3715400 x_2 = 0.0005484 x_3 = 1.0075716 x_4 = 4.0027833
3-point central	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5,\ \Delta x_2 = 0.5,$ $\Delta x_3 = 0.5,\ \Delta x_4 = 0.5$	40	x_1 = 109737.3715400 x_2 = 0.0005484 x_3 = 1.0075716 x_4 = 4.0027833
3-point forward	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5,\ \Delta x_2 = 0.5,$ $\Delta x_3 = 0.5,\ \Delta x_4 = 0.5$	42	x_1 = 109737.3715400 x_2 = 0.0005484 x_3 = 1.0075716 x_4 = 4.0027833
3-point backward	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5,\ \Delta x_2 = 0.5,$ $\Delta x_3 = 0.5,\ \Delta x_4 = 0.5$	40	x_1 = 109737.3715400 x_2 = 0.0005484 x_3 = 1.0075716 x_4 = 4.0027833
5-point central	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5,\ \Delta x_2 = 0.5,$ $\Delta x_3 = 0.5,\ \Delta x_4 = 0.5$	40	x_1 = 109737.3715400 x_2 = 0.0005484 x_3 = 1.0075716 x_4 = 4.0027833

Table 4.9 (cont.)

Method	Initial vectors	Number of iterations	Solutions
5-point forward	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5$, $\Delta x_2 = 0.5$, $\Delta x_3 = 0.5$, $\Delta x_4 = 0.5$	42	$x_1 = 109737.3715400$ $x_2 = 0.0005484$ $x_3 = 1.0075716$ $x_4 = 4.0027833$
5-point backward	(0.5,0.5,0.5,0.5) and $\Delta x_1 = 0.5$, $\Delta x_2 = 0.5$, $\Delta x_3 = 0.5$, $\Delta x_4 = 0.5$	42	$x_1 = 109737.3715400$ $x_2 = 0.0005484$ $x_3 = 1.0075716$ $x_4 = 4.0027833$

Repeat the method until the error less than 0.0000005. Showing the iterations of table 4.9 in appendix.

CHAPTER 5

CONCLUSION AND SUGGESTIONS

5.1 Conclusion

In this thesis, we found the new iterative methods that do not use any derivative that can solve systems of nonlinear equations. The iterative equation is

$$x^{k+1} = x^k - [H(x^k)]^{-1} f(x^k), \qquad k = 0,1,2,\ldots \tag{5.1}$$

where $H(x)$ is the diagonal matrix with values (from upper left to lower right) of $f_{ii}(x)$, $i = 1,2,\ldots,n$ which approximate from The Numerical Analysis. We have new eight iterative methods for solving systems of nonlinear equations are

2-point forward method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^k)}{x_i^{k+1} - x_i^k} \tag{5.2}$$

2-point backward method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{f_i(x^k) - f_i(x^{k-1})}{x_i^k - x_i^{k-1}} \tag{5.3}$$

3-point central method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_i^{k+1} - x_i^{k-1}} \tag{5.4}$$

3-point forward method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{-3f_i(x^k) + 4f_i(x^{k+1}) - f_i(x^{k+2})}{x_i^{k+2} - x_i^k} \tag{5.5}$$

3-point backward method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2}) - 4f_i(x^{k-1}) + 3f_i(x^k)}{x_i^k - x_i^{k-2}} \tag{5.6}$$

5-point central method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{f_i(x^{k-2}) - 8f_i(x^{k-1}) + 8f_i(x^{k+1}) - f_i(x^{k+2})}{12(x_i^{k+2} - x_i^{k+1})} \tag{5.7}$$

5-point forward method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{-25f_i(x^k) + 48f_i(x^{k+1}) - 36f_i(x^{k+2}) + 16f_i(x^{k+3}) - 3f_i(x^{k+4})}{12(x_i^{k+1} - x_i^k)} \tag{5.8}$$

and 5-point backward method with approximate derivative from

$$f_{ii}(x^k) \approx \frac{3f_i(x^{k-4}) - 16f_i(x^{k-3}) + 36f_i(x^{k-2}) - 48f_i(x^{k-1}) + 25f_i(x^k)}{12(x_i^k - x_i^{k-1})} \tag{5.9}$$

The new eight iterative methods can use to solve systems of nonlinear equations that is difficult to find derivative or are difficult to calculate. However, the number of iteration of the new iterative method may be more than the number of iteration of the other methods because we do not use any derivative.

For all new eight methods, I thought that the 2-point method is probably the best method because it has the number of iteration equal to the other methods or it may be less than the other methods in some example. and it is easy for writing algorithms.

The important problem is technique to choose the initial value. In order to solve the systems of nonlinear equations, guessing an initial value is easier than using theorem to find an initial value. Thus, sometimes, we choose the initial value that can obtain the solution and sometimes, we choose the initial value that cannot obtain the solution, though in actually this system has a solution. However, the systems may have many solutions or have a unique solution or have no solution.

For our algorithms, when do we stop the iteration?. We set an absolute error ε and iterated until either $\sum_{i,j=1}^{n} \left| f_i(x_j) - f_i(x_{j-1}) \right| < \varepsilon$, or we reached a maximum number of iterations. It may happen that if we set an absolute error too small then we may need initial value that closed to its solution. In this thesis, we will consider an absolute error is $\varepsilon = 0.0000005$ and the number of iterations of our methods from assuming for all examples.

5.2 Suggestions

For solving systems of nonlinear equations, in order to solve complicated functions or the functions that are difficult to find derivative or are difficult to calculate its derivative. We can modify by using

$$f_{ij}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^k)}{x_j^{k+1} - x_j^k} \tag{5.10}$$

$$f_{ij}(x^k) \approx \frac{f_i(x^k) - f_i(x^{k-1})}{x_j^k - x_j^{k-1}} \tag{5.11}$$

$$f_{ij}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_j^{k+1} - x_j^{k-1}} \tag{5.12}$$

$$f_{ij}(x^k) \approx \frac{-3f_i(x^k) + 4f_i(x^{k+1}) - f_i(x^{k+2})}{x_j^{k+2} - x_j^k} \tag{5.13}$$

$$f_{ij}(x^k) \approx \frac{f_i(x^{k-2}) - 4f_i(x^{k-1}) + 3f_i(x^k)}{x_j^k - x_j^{k-2}} \tag{5.14}$$

$$f_{ij}(x^k) \approx \frac{f_i(x^{k-2}) - 8f_i(x^{k-1}) + 8f_i(x^{k+1}) - f_i(x^{k+2})}{12(x_j^{k+2} - x_j^{k+1})} \tag{5.15}$$

$$f_{ij}(x^k) \approx \frac{-25f_i(x^k) + 48f_i(x^{k+1}) - 36f_i(x^{k+2}) + 16f_i(x^{k+3}) - 3f_i(x^{k+4})}{12(x_j^{k+1} - x_j^k)} \tag{5.16}$$

$$f_{ij}(x^k) \approx \frac{3f_i(x^{k-4}) - 16f_i(x^{k-3}) + 36f_i(x^{k-2}) - 48f_i(x^{k-1}) + 25f_i(x^k)}{12(x_j^k - x_j^{k-1})} \tag{5.17}$$

for the Jacobian matrix $J(x) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$ in Newton method.

REFERENCES

[1] Gerald, C.F and Wheatley P.O. **"Applied Numerical Analysis."** Fifth Edition. Addison-Wesley Publishing Company, 1994. pp.209.

[2] Dennis, S.**"RF 21 Feature Nonlinear function minimizer."**[Online].Available: http://members.aol.com/rf21exe/solver.htm., 1983.

[3] Podisuk, M. **" Modified Newton Iterative Method for Solving Systems of Nonlinear Equations."** SEA Bull. Math., Vol.15, No.2, 1991. pp.123-130.

[4] Ortega, J.M. **" Numerical Analysis, A Second Course."** Society for Industrial and Applied Mathematics (SIAM), 1990.

[5] Kantrorovich, L.V. and Akilov, G.P.**"Functional Analysis in Normed Spaces."** MaxMillan, New York, N.Y, 1964.

[6] Dennis, J.E., Jr. **"On The Convergence of Newton-like Methods."** Numerical Methods for Nonlinear Algebraic Equations. Philip Rabinowitz, Gordon and Breach Science Publishers, London, 1970. pp.163-181.

[7] Rattanametawee, W. **" Modified Newton Method for Solving Systems of Nonlinear Equations."** Master degree Thesis, Dep. Of Math and Com, KMITL, 2001.

[8] Carnahan, B., Luther H.A. and Wilkes, J.O. **" Applied Numerical Methods."** John Wiley and Sons, Inc., 1969. pp.319.

[9] Fausett, V.L. Fausett. **"Applied Numerical Analysis Using Matlab."** Prentice Hall, Inc., 1999. pp.142-145.

[10] Plybon, F.B. **"An Introduction to Applied Numerical Analysis."** PWS-KENT Publishing Company., 1992. pp.210-212.

[11] Pozrikidis, C. **"Numerical Computation in Science and Engineering."** Oxford University Press, Inc., 1998. pp.184-185.

[12] Traub, J.F. " **Iterative Methods for the Solution of Equations.**" Frentice-Hell, Inc., 1964

[13] Rabinowitz, P. "**Numerical Methods for Nonlinear Algebraic Equations.**" Gordon and Breach, Science Publishers Ltd, 1970.

[14] Chapra, C.S. and Canale, P.R. "**Numerical Methods for Engineers.**" The McGraw-Hill Companies, Inc., 1998.

[15] ไมตรี โพธิ์สุข. "**การวิเคราะห์เชิงตัวเลขพื้นฐาน**" สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง. 2529.

APPENDIX

The iterations of table 4.1

The result from 2-point forward method

k	x^k	y^k	error
1	1.1350784	-1.2903270	0.3576566
2	1.8136535	-1.2841867	0.2802647
3	1.7866196	-2.1534242	0.0567633
4	1.8411997	-1.9960591	0.0400987
5	1.8694665	-2.0759363	0.0106707
6	1.8583161	-2.0939485	0.0036941
7	1.8546380	-2.0873661	0.0012697
8	1.8559954	-2.0852296	0.0004417
9	1.8564200	-2.0860265	0.0001484
10	1.8562617	-2.0862760	0.0000516
11	1.8562119	-2.0861831	0.0000174
12	1.8562304	-2.0861539	0.0000060
13	1.8562363	-2.0861648	0.0000020
14	1.8562341	-2.0861682	0.0000007
15	1.8562334	-2.0861669	0.0000002

Estimates values are $x = 1.8562334$ and $y = -2.0861665$ after 15 iterations.

The result from 2-point backward method

k	x^k	y^k	error
1	1.1847591	-0.8591939	0.4950244
2	1.9941400	-1.9700451	0.1197998
3	1.8399032	-2.2012911	0.0472889
4	1.8249952	-2.0780470	0.0168118
5	1.8579660	-2.0669426	0.0076799
6	1.8598639	-2.0871607	0.0019160
7	1.8560367	-2.0882854	0.0008524
8	1.8558087	-2.0860508	0.0002248
9	1.8562569	-2.0859171	0.0001003
10	1.8562835	-2.0861803	0.0000263
11	1.8562310	-2.0861959	0.0000117
12	1.8562279	-2.0861651	0.0000031
13	1.8562340	-2.0861633	0.0000014
14	1.8562344	-2.0861669	0.0000004

Estimates values are $x = 1.8562344$ and $y = -2.0861664$ after 14 iterations.

The result from 3-point central method

k	x^k	y^k	error
1	1.1588167	-1.0470138	0.4452414
2	1.8965719	-1.5086959	0.2421013
3	1.8313832	-2.1679140	0.0339988
4	1.8355850	-2.0621795	0.0141431
5	1.8608370	-2.0737617	0.0060540
6	1.8586163	-2.0888289	0.0015971
7	1.8557006	-2.0875609	0.0006831
8	1.8559545	-2.0858533	0.0001876
9	1.8562962	-2.0860028	0.0000803
10	1.8562664	-2.0862033	0.0000219
11	1.8562264	-2.0861859	0.0000094
12	1.8562299	-2.0861624	0.0000026
13	1.8562346	-2.0861644	0.0000011
14	1.8562342	-2.0861672	0.0000003

Estimates values are x= 1.8562342 and y= -2.0861672 after 14 iterations.

The result from 3-point forward method

k	x^k	y^k	error
1	1.1559163	-1.0398758	0.4472917
2	1.7723678	-1.2504848	0.2676603
3	1.7767795	-2.1081442	0.0531315
4	1.8531840	-2.0433077	0.0184218
5	1.8637136	-2.0841391	0.0050188
6	1.8566482	-2.0905101	0.0019116
7	1.8553565	-2.0864079	0.0005893
8	1.8561858	-2.0856513	0.0002263
9	1.8563363	-2.0861385	0.0000689
10	1.8562393	-2.0862269	0.0000265
11	1.8562217	-2.0861700	0.0000081
12	1.8562331	-2.0861596	0.0000031
13	1.8562351	-2.0861663	0.0000009
14	1.8562338	-2.0861675	0.0000004

Estimates values are x= 1.8562338 and y= -2.0861675 after 14 iterations.

The result from 3-point backward method

k	x^k	y^k	error
1	1.1561615	-1.0377786	0.4479472
2	1.8239301	-1.2798129	0.2910131
3	1.7889883	-2.1507731	0.0548461
4	1.8414309	-2.0631523	0.0125931
5	1.8605837	-2.0772704	0.0049630
6	1.8579623	-2.0886913	0.0014080
7	1.8557276	-2.0871789	0.0005674
8	1.8560313	-2.0858693	0.0001655
9	1.8562930	-2.0860478	0.0000666
10	1.8562574	-2.0862015	0.0000194
11	1.8562268	-2.0861806	0.0000078
12	1.8562309	-2.0861626	0.0000023
13	1.8562345	-2.0861651	0.0000009
14	1.8562340	-2.0861672	0.0000003

Estimates values are $x = 1.8562340$ and $y = -2.0861672$ after 14 iterations.

The result from 5-point central method

k	x^k	y^k	error
1	1.1578877	-1.0442756	0.4460387
2	1.8601147	-1.3852780	0.2706078
3	1.8106612	-2.1541819	0.0429619
4	1.8399308	-2.0554116	0.0161219
5	1.8619298	-2.0763669	0.0060158
6	1.8581339	-2.0894683	0.0017694
7	1.8555703	-2.0872788	0.0006878
8	1.8560112	-2.0857768	0.0002085
9	1.8563114	-2.0860361	0.0000807
10	1.8562598	-2.0862123	0.0000244
11	1.8562246	-2.0861820	0.0000094
12	1.8562307	-2.0861614	0.0000029
13	1.8562348	-2.0861649	0.0000011
14	1.8562341	-2.0861673	0.0000003

Estimates values are $x = 1.8562341$ and $y = -2.0861673$ after 14 iterations.

The result from 5-point forward method

k	x^k	y^k	error
1	1.1578985	-1.0442885	0.4460353
2	1.8775865	-1.1991576	0.3748078
3	1.7618270	-2.2866729	0.0924283
4	1.7870394	-2.0511454	0.0335943
5	1.8640776	-2.0422279	0.0167138
6	1.8638691	-2.0904973	0.0034350
7	1.8553766	-2.0906079	0.0017001
8	1.8553360	-2.0856603	0.0004066
9	1.8563348	-2.0856393	0.0002015
10	1.8563387	-2.0862260	0.0000475
11	1.8562219	-2.0862283	0.0000235
12	1.8562214	-2.0861598	0.0000056
13	1.8562351	-2.0861595	0.0000028
14	1.8562352	-2.0861675	0.0000007
15	1.8562336	-2.0861675	0.0000003

Estimates values are $x = 1.8562336$ and $y = -2.0861675$ after 15 iterations.

The result from 5-point backward method

k	x^k	y^k	error
1	1.1578952	-1.0443166	0.4460265
2	1.8543025	-1.1919008	0.3591753
3	1.7615338	-2.2619611	0.0892884
4	1.7995857	-1.9916868	0.0503107
5	1.8713760	-2.0503824	0.0188939
6	1.8626355	-2.0947031	0.0048779
7	1.8545034	-2.0898930	0.0020055
8	1.8554824	-2.0851465	0.0005792
9	1.8564368	-2.0857253	0.0002371
10	1.8563216	-2.0862858	0.0000677
11	1.8562100	-2.0862183	0.0000277
12	1.8562234	-2.0861528	0.0000079
13	1.8562365	-2.0861607	0.0000032
14	1.8562349	-2.0861683	0.0000009
15	1.8562334	-2.0861674	0.0000004

Estimates values are $x = 1.8562334$ and $y = -2.0861674$ after 15 iterations.

The iterations of table 4.2

The result from 2-point forward method

k	x^k	y^k	error
1	0.6705810	24.2597321	40.6224863
2	-0.0025097	1.1844212	0.1844181
3	0.0003583	1.0050322	0.0064692
4	0.0000003	0.9992836	0.0007164
5	-0.0000000	0.9999994	0.0000006
6	-0.0000000	1.0000000	0.0000000

Estimates values are $x = 0.0000000$ and $y = 1.0000000$ after 6 iterations.

The result from 2-point backward method

k	x^k	y^k	error
1	0.3272288	-16.3889964	21.2418564
2	-0.0346417	0.0441515	1.0275977
3	-0.0051900	1.0698291	0.0698157
4	0.0000433	1.0104342	0.0106081
5	-0.0000001	0.9999135	0.0000868
6	-0.0000000	1.0000001	0.0000001

Estimates values are $x = 0.0000000$ and $y = 1.0000001$ after 6 iterations.

The result from 3-point central method

k	x^k	y^k	error
1	0.5283453	-209.5019847	319.2973236
2	-0.0016761	-0.9128542	1.9131449
3	-0.0005874	1.0032927	0.0032925
4	0.0000001	1.0011755	0.0011758
5	0.0000000	0.9999998	0.0000002

Estimates values are $x = 0.0000000$ and $y = 0.9999998$ after 5 iterations.

The result from 3-point forward method

k	x^k	y^k	error
1	0.4067200	4.5716032	7.2982219
2	-0.0011912	-1.8931742	2.8931749
3	-0.0000533	1.0023852	0.0023852
4	0.0000000	1.0001066	0.0001066
5	-0.0000000	1.0000000	0.0000000

Estimates values are $x = 0.0000000$ and $y = 1.0000000$ after 5 iterations.

The result from 3-point backward method

k	x^k	y^k	error
1	0.4580386	5.3717432	8.5008863
2	-0.0849069	-2.1851027	3.1896700
3	-0.0107933	1.2216515	0.2215938
4	0.0000346	1.0218210	0.0219607
5	-0.0000000	0.9999309	0.0000691
6	0.0000000	1.0000000	0.0000000

Estimates values are $x = 0.0000000$ and $y = 1.0000000$ after 6 iterations.

The result from 5-point central method

k	x^k	y^k	error
1	0.4985138	11.0018098	15.9147485
2	0.0113947	0.0534873	0.9465778
3	0.0000406	0.9774675	0.0225325
4	0.0000000	0.9999188	0.0000812
5	0.0000000	1.0000000	0.0000000

Estimates values are $x = 0.0000000$ and $y = 1.0000000$ after 5 iterations.

The result from 5-point forward method

k	x^k	y^k	error
1	0.4788573	4.1679854	7.1679502
2	0.0215243	-1.0059536	2.0061852
3	0.0123744	0.9578677	0.0422095
4	0.0000392	0.9755540	0.0244460
5	0.0000000	0.9999216	0.0000784
6	0.0000000	1.0000000	0.0000000

Estimates values are $x = 0.0000000$ and $y = 1.0000000$ after 6 iterations.

The result from 5-point backward method

k	x^k	y^k	error
1	0.4960794	3.8848091	6.8874989
2	-0.2526204	-1.7558100	2.7995948
3	-0.0635452	1.8203181	0.8185990
4	0.0007057	1.1357732	0.1387878
5	0.0000001	0.9985896	0.0014104
6	0.0000000	0.9999998	0.0000002

Estimates values are $x = 0.0000000$ and $y = 0.9999998$ after 6 iterations.

The iterations of table 4.3

The result from 2-point forward method

k	x^k	y^k	error
1	0.5300391	3.1415927	0.0256330
2	0.5061394	3.1512877	0.0099095
3	0.4978111	3.1444435	0.0042978
4	0.4996379	3.1404705	0.0010832
5	0.5001752	3.1414109	0.0003061
6	0.5000294	3.1416801	0.0000846
7	0.4999861	3.1416074	0.0000246
8	0.4999977	3.1415857	0.0000067
9	0.5000011	3.1415915	0.0000020
10	0.5000002	3.1415932	0.0000005
11	0.4999999	3.1415927	0.0000002

Estimates values are $x = 0.4999999$ and $y = 3.1415927$ after 11 iterations.

The result from 2-point backward method

k	x^k	y^k	error
1	0.4430969	3.1415927	0.0607064
2	0.5000000	3.0951059	0.0437871
3	0.5108557	3.1415927	0.0098211
4	0.5000000	3.1463340	0.0044812
5	0.4992862	3.1415927	0.0006670
6	0.5000000	3.1412328	0.0003400
7	0.5000575	3.1415927	0.0000536
8	0.5000000	3.1416214	0.0000271
9	0.4999954	3.1415927	0.0000043
10	0.5000000	3.1415904	0.0000022
11	0.5000004	3.1415927	0.0000003

Estimates values are $x = 0.5000004$ and $y = 3.1415927$ after 11 iterations.

The result from 3-point central method

k	x^k	y^k	error
1	0.6272571	3.1415927	0.1363733
2	0.5331612	3.1030882	0.0573011
3	0.5088465	3.1516707	0.0107829
4	0.4988864	3.1455606	0.0041617
5	0.4993696	3.1410287	0.0005721
6	0.5000923	3.1412752	0.0003355
7	0.5000505	3.1416388	0.0000470
8	0.4999927	3.1416179	0.0000267
9	0.4999960	3.1415890	0.0000037
10	0.5000006	3.1415906	0.0000021
11	0.5000003	3.1415929	0.0000003

Estimates values are $x = 0.5000003$ and $y = 3.1415929$ after 11 iterations.

The result from 3-point forward method

k	x^k	y^k	error
1	0.5579660	3.1415927	0.0448459
2	0.5108072	3.1503706	0.0099798
3	0.4992446	3.1463159	0.0044183
4	0.4992483	3.1412117	0.0004062
5	0.5000639	3.1412135	0.0003575
6	0.5000603	3.1416246	0.0000343
7	0.4999949	3.1416228	0.0000284
8	0.4999952	3.1415901	0.0000027
9	0.5000004	3.1415903	0.0000023
10	0.5000004	3.1415929	0.0000002

Estimates values are $x = 0.5000004$ and $y = 3.1415929$ after 10 iterations.

The result from 3-point backward method

k	x^k	y^k	error
1	0.4430969	3.1415927	0.0607064
2	0.5362283	3.0951059	0.0655559
3	0.5102954	3.1519297	0.0113483
4	0.4989948	3.1461231	0.0045019
5	0.4992793	3.1410842	0.0005248
6	0.5000841	3.1412293	0.0003639
7	0.5000578	3.1416347	0.0000436
8	0.4999933	3.1416215	0.0000289
9	0.4999954	3.1415893	0.0000035
10	0.5000005	3.1415904	0.0000023
11	0.5000004	3.1415929	0.0000003

Estimates values are $x = 0.5000004$ and $y = 3.1415929$ after 11 iterations.

The result from 5-point central method

k	x^k	y^k	error
1	0.6928628	3.1415927	0.3145533
2	0.5595251	2.9923675	0.1569787
3	0.5281044	3.1500261	0.0177064
4	0.5019810	3.1509902	0.0090621
5	0.4985623	3.1425604	0.0021074
6	0.4998542	3.1408619	0.0007000
7	0.5001166	3.1415196	0.0001661
8	0.5000117	3.1416509	0.0000558
9	0.4999907	3.1415985	0.0000132
10	0.4999991	3.1415880	0.0000044
11	0.5000007	3.1415922	0.0000011
12	0.5000001	3.1415930	0.0000004

Estimates values are $x = 0.5000001$ and $y = 3.1415930$ after 12 iterations.

The result from 5-point forward method

k	x^k	y^k	error
1	0.4292772	3.1415927	0.0775563
2	0.5788107	3.0786223	0.1013000
3	0.5227450	3.1431139	0.0186935
4	0.5018848	3.1499274	0.0080453
5	0.4987231	3.1425145	0.0019202
6	0.4998597	3.1409448	0.0006214
7	0.5001034	3.1415224	0.0001516
8	0.5000112	3.1416443	0.0000496
9	0.4999918	3.1415983	0.0000121
10	0.4999991	3.1415885	0.0000039
11	0.5000007	3.1415922	0.0000010
12	0.5000001	3.1415930	0.0000003

Estimates values are $x = 0.5000001$ and $y = 3.1415930$ after 12 iterations.

The result from 5-point backward method

k	x^k	y^k	error
1	0.5595471	3.1415927	0.0457897
2	0.5109513	3.1500210	0.0096803
3	0.4993255	3.1463696	0.0044703
4	0.4992398	3.1412528	0.0004433
5	0.5000574	3.1412092	0.0003584
6	0.5000610	3.1416213	0.0000344
7	0.4999955	3.1416231	0.0000285
8	0.4999951	3.1415904	0.0000027
9	0.5000004	3.1415902	0.0000023
10	0.5000004	3.1415928	0.0000002

Estimates values are $x = 0.5000004$ and $y = 3.1415928$ after 10 iterations.

The iterations of table 4.4

The result from 2-point forward method

k	x^k	y^k	z^k	error
1	2.8695652	1.4843750	1.3399342	7.6486334
2	2.9762198	1.1808460	1.0967788	0.4032192
3	2.9941884	1.1761305	1.0932612	0.0599036
4	2.9951044	1.1785853	1.0947999	0.0090313
5	2.9950585	1.1789811	1.0951702	0.0010260
6	2.9950570	1.1790337	1.0952087	0.0001391
7	2.9950562	1.1790397	1.0952144	0.0000151
8	2.9950562	1.1790405	1.0952149	0.0000021
9	2.9950562	1.1790406	1.0952150	0.0000002

Estimates values are $x=$ 2.9950562, $y=$ 1.1790406 and z = 1.0952150 after 9 iterations.

The result from 2-point backward method

k	x^k	y^k	z^k	error
1	5.3333333	1.3392857	1.2217080	105.7463965
2	2.6645487	1.5505350	1.3484336	11.7276076
3	2.7783847	1.1716667	1.1027437	3.8871039
4	3.0457140	1.1459469	1.0713774	1.5657408
5	2.9893928	1.1834152	1.0971340	0.1733242
6	2.9947648	1.1784214	1.0950684	0.0125959
7	2.9950851	1.1789695	1.0951148	0.0021223
8	2.9950544	1.1790290	1.0952104	0.0001949
9	2.9950566	1.1790395	1.0952136	0.0000306
10	2.9950561	1.1790404	1.0952149	0.0000029
11	2.9950562	1.1790406	1.0952150	0.0000005

Estimates values are $x=$ 2.9950562, $y=$ 1.1790406 and z = 1.0952150 after 11 iterations.

The result from 3-point central method

k	x^k	y^k	z^k	error
1	3.1538462	1.4166667	1.2836872	6.6117939
2	3.0075535	1.2375593	1.1427082	1.1928444
3	2.9945112	1.1889327	1.1031936	0.1779681
4	2.9949431	1.1802496	1.0962488	0.0231337
5	2.9950458	1.1791897	1.0953357	0.0027265
6	2.9950545	1.1790584	1.0952303	0.0003419
7	2.9950560	1.1790428	1.0952168	0.0000402
8	2.9950561	1.1790408	1.0952152	0.0000050
9	2.9950562	1.1790406	1.0952150	0.0000006
10	2.9950562	1.1790406	1.0952150	0.0000001

Estimates values are x= 2.9950562, y= 1.1790406 and z = 1.0952150 after 10 iterations.

The result from 3-point forward method

k	x^k	y^k	z^k	error
1	3.4444444	1.4166667	1.3705479	12.8952036
2	3.0468055	1.2963841	1.1595272	2.6030109
3	2.9924863	1.1985671	1.1139227	0.4052982
4	2.9950120	1.1816748	1.0971079	0.0434774
5	2.9950123	1.1793402	1.0955004	0.0062834
6	2.9950553	1.1790797	1.0952432	0.0006494
7	2.9950555	1.1790450	1.0952192	0.0000927
8	2.9950561	1.1790411	1.0952154	0.0000096
9	2.9950561	1.1790406	1.0952151	0.0000014
10	2.9950562	1.1790406	1.0952150	0.0000001

Estimates values are x= 2.9950562, y= 1.1790406 and z = 1.0952150 after 10 iterations.

The result from 3-point backward method

k	x^k	y^k	z^k	error
1	3.4444444	1.4166667	1.3217761	12.5755265
2	3.0447570	1.2887833	1.1799700	2.6041125
3	2.9940027	1.2013784	1.1131574	0.3981787
4	2.9947968	1.1818042	1.0975793	0.0529163
5	2.9950326	1.1793817	1.0954911	0.0062350
6	2.9950523	1.1790814	1.0952500	0.0007825
7	2.9950558	1.1790456	1.0952191	0.0000921
8	2.9950561	1.1790412	1.0952155	0.0000115
9	2.9950561	1.1790406	1.0952151	0.0000014
10	2.9950562	1.1790406	1.0952150	0.0000002

Estimates values are x= 2.9950562, y= 1.1790406 and z = 1.0952150 after 10 iterations.

The result from 5-point central method

k	x^k	y^k	z^k	error
1	3.2285714	1.4166667	1.3054348	8.0638049
2	3.0179989	1.2525940	1.1498289	1.5732529
3	2.9942373	1.1918771	1.1060175	0.2368588
4	2.9949361	1.1806663	1.0965466	0.0299977
5	2.9950388	1.1792368	1.0953809	0.0037198
6	2.9950544	1.1790646	1.0952347	0.0004439
7	2.9950559	1.1790435	1.0952175	0.0000549
8	2.9950561	1.1790409	1.0952153	0.0000065
9	2.9950561	1.1790406	1.0952151	0.0000008
10	2.9950562	1.1790406	1.0952150	0.0000001

Estimates values are x= 2.9950562, y= 1.1790406 and z = 1.0952150 after 10 iterations.

The result from 5-point forward method

k	x^k	y^k	z^k	error
1	3.2285714	1.4166667	1.2917197	7.9751744
2	3.0173542	1.2504566	1.1562391	1.5769658
3	2.9947152	1.1927791	1.1057752	0.2347517
4	2.9948682	1.1807060	1.0966977	0.0329839
5	2.9950454	1.1792503	1.0953778	0.0037032
6	2.9950534	1.1790652	1.0952369	0.0004871
7	2.9950560	1.1790437	1.0952174	0.0000547
8	2.9950561	1.1790409	1.0952153	0.0000072
9	2.9950562	1.1790406	1.0952151	0.0000008
10	2.9950562	1.1790406	1.0952150	0.0000001

Estimates values are $x=$ 2.9950562, $y=$ 1.1790406 and z = 1.0952150 after 10 iterations.

The result from 5-point backward method

k	x^k	y^k	z^k	error
1	3.2285714	1.4166667	1.3332211	8.2444813
2	3.0193052	1.2569244	1.1381998	1.5743753
3	2.9933507	1.1902708	1.1065451	0.2420910
4	2.9950636	1.1806056	1.0962738	0.0258892
5	2.9950262	1.1792133	1.0953882	0.0037850
6	2.9950562	1.1790637	1.0952307	0.0003825
7	2.9950557	1.1790431	1.0952176	0.0000558
8	2.9950562	1.1790409	1.0952152	0.0000056
9	2.9950561	1.1790406	1.0952151	0.0000008
10	2.9950562	1.1790406	1.0952150	0.0000001

Estimates values are $x=$ 2.9950562, $y=$ 1.1790406 and z = 1.0952150 after 10 iterations.

The iterations of table 4.5

The result from 2-point forward method

k	x^k	y^k	z^k	error
1	3.6475409	2.7874698	-3.7943830	5.2841587
2	3.5047705	2.7026602	-3.7172882	0.7297952
3	3.5018938	2.7010394	-3.6994957	0.0365244
4	3.5002316	2.7000414	-3.7000095	0.0008983
5	3.4996829	2.7000391	-3.6999970	0.0003588
6	3.4996810	2.7000278	-3.7000017	0.0000170
7	3.4996748	2.7000281	-3.7000015	0.0000044
8	3.4996749	2.7000279	-3.7000015	0.0000003

Estimates values are x= 3.4996749, y= 2.7000279 and z = -3.7000015 after 8 iterations.

The result from 2-point backward method

k	x^k	y^k	z^k	error
1	3.7952156	2.6495969	-3.6785523	1.6610905
2	3.4921589	2.6964912	-3.6970489	0.1700580
3	3.4980311	2.6997965	-3.7000186	0.0042995
4	3.4995415	2.6999977	-3.7000106	0.0008411
5	3.4996579	2.7000259	-3.7000020	0.0000515
6	3.4996737	2.7000276	-3.7000016	0.0000088
7	3.4996746	2.7000279	-3.7000015	0.0000004

Estimates values are x= 3.4996746, y= 2.7000279 and z = -3.7000015 after 7 iterations.

The result from 3-point central method

k	x^k	y^k	z^k	error
1	3.7138206	2.7354172	-3.7222831	1.5509053
2	3.4769430	2.7060602	-3.6989027	0.1721525
3	3.5033033	2.6995495	-3.7003284	0.0258926
4	3.4994111	2.7001164	-3.6999593	0.0035103
5	3.4997235	2.7000205	-3.7000058	0.0003592
6	3.4996707	2.7000291	-3.7000009	0.0000486
7	3.4996755	2.7000278	-3.7000016	0.0000052
8	3.4996748	2.7000280	-3.7000015	0.0000007
9	3.4996748	2.7000279	-3.7000015	0.0000001

Estimates values are x= 3.4996748, y= 2.7000279 and z = -3.7000015 after 9 iterations.

The result from 3-point forward method

k	x^k	y^k	z^k	error
1	3.5904901	2.6854384	-3.7459885	2.2940210
2	3.5013194	2.7050157	-3.6987400	0.1306812
3	3.5024593	2.6999996	-3.7001008	0.0081403
4	3.4996608	2.7000870	-3.6999767	0.0019983
5	3.4997075	2.7000263	-3.7000030	0.0001288
6	3.4996739	2.7000287	-3.7000012	0.0000259
7	3.4996752	2.7000279	-3.7000015	0.0000021
8	3.4996748	2.7000280	-3.7000015	0.0000003

Estimates values are x= 3.4996748, y= 2.7000280 and z = -3.7000015 after 8 iterations.

The result from 3-point backward method

k	x^k	y^k	z^k	error
1	3.5931805	2.7072007	-3.7642716	2.7979589
2	3.5187450	2.7053065	-3.7000890	0.0892659
3	3.5026683	2.7004101	-3.6999500	0.0080582
4	3.4998891	2.7000825	-3.6999840	0.0015674
5	3.4997052	2.7000311	-3.7000009	0.0000748
6	3.4996766	2.7000285	-3.7000013	0.0000165
7	3.4996751	2.7000280	-3.7000015	0.0000005

Estimates values are x= 3.4996751, y= 2.7000280 and z = -3.7000015 after 7 iterations.

The result from 5-point central method

k	x^k	y^k	z^k	error
1	3.6679294	2.7236612	-3.7321596	1.7496229
2	3.5053057	2.7051551	-3.6993025	0.1081737
3	3.5025633	2.7001089	-3.7000674	0.0057786
4	3.4997217	2.7000871	-3.6999782	0.0018764
5	3.4997076	2.7000275	-3.7000024	0.0000870
6	3.4996746	2.7000286	-3.7000012	0.0000234
7	3.4996752	2.7000279	-3.7000015	0.0000015
8	3.4996748	2.7000279	-3.7000015	0.0000003

Estimates values are x= 3.4996748, y= 2.7000279 and z = -3.7000015 after 8 iterations.

The result from 5-point forward method

k	x^k	y^k	z^k	error
1	3.6752399	2.7157422	-3.7329772	1.6790701
2	3.5020403	2.7052111	-3.6990419	0.1204765
3	3.5025869	2.7000321	-3.7000975	0.0076450
4	3.4996791	2.7000893	-3.6999763	0.0020272
5	3.4997088	2.7000266	-3.7000029	0.0001207
6	3.4996741	2.7000287	-3.7000012	0.0000261
7	3.4996752	2.7000279	-3.7000015	0.0000020
8	3.4996748	2.7000280	-3.7000015	0.0000003

Estimates values are x= 3.4996748, y= 2.7000280 and z = -3.7000015 after 8 iterations.

The result from 5-point backward method

k	x^k	y^k	z^k	error
1	3.6643886	2.7226880	-3.7388125	2.0062827
2	3.5019702	2.7054072	-3.6995493	0.1028227
3	3.5027177	2.7000606	-3.7001021	0.0075434
4	3.4996953	2.7000921	-3.6999758	0.0020719
5	3.4997103	2.7000269	-3.7000028	0.0001149
6	3.4996743	2.7000287	-3.7000012	0.0000266
7	3.4996752	2.7000279	-3.7000015	0.0000019
8	3.4996748	2.7000280	-3.7000015	0.0000003

Estimates values are x= 3.4996748, y= 2.7000280 and z = -3.7000015 after 8 iterations.

The iterations of table 4.6

The result from 2-point forward method

k	x^k	y^k	z^k	error
1	0.0279511	0.5111075	-0.4919927	31.6638770
2	0.4895166	0.0117526	-0.5228896	0.2512178
3	0.4999937	0.0212111	-0.5233119	0.3846750
4	0.4999795	0.0025946	-0.5230713	0.0514546
5	0.4999997	-0.0001939	-0.5235340	0.0045888
6	0.5000000	0.0000064	-0.5236036	0.0002087
7	0.5000000	-0.0000003	-0.5235986	0.0000076
8	0.5000000	0.0000000	-0.5235988	0.0000003

Estimates values are x= 0.5000000, y= 0.0000000 and z = -0.5235988 after 8 iterations.

The result from 2-point backward method

k	x^k	y^k	z^k	error
1	0.0279511	0.4133290	-0.4919927	22.7718168
2	0.4931315	0.2545006	-0.5230245	9.4938667
3	0.4970513	0.1211233	-0.5177015	3.2147603
4	0.4993449	0.0535989	-0.5206774	1.1327200
5	0.4998702	0.0173869	-0.5222783	0.3232442
6	0.4999863	0.0034852	-0.5231661	0.0640352
7	0.4999994	0.0002945	-0.5235117	0.0062990
8	0.5000000	0.0000096	-0.5235914	0.0002912
9	0.5000000	0.0000004	-0.5235985	0.0000110
10	0.5000000	0.0000000	-0.5235988	0.0000004

Estimates values are x= 0.5000000, y= 0.0000000 and z = -0.5235988 after 10 iterations.

The result from 3-point central method

k	x^k	y^k	z^k	error
1	0.0279511	0.4666627	-0.4919927	27.4303071
2	0.4912529	0.1897410	-0.5229508	6.0954076
3	0.4983604	0.0619545	-0.5191488	1.3753448
4	0.4998276	0.0119348	-0.5220786	0.2286908
5	0.4999935	0.0006994	-0.5233014	0.0167359
6	0.5000000	0.0000178	-0.5235813	0.0006144
7	0.5000000	0.0000009	-0.5235983	0.0000232
8	0.5000000	0.0000000	-0.5235988	0.0000008
9	0.5000000	0.0000000	-0.5235988	0.0000000

Estimates values are x= 0.5000000, y= 0.0000000 and z = -0.5235988 after 9 iterations.

The result from 3-point forward method

k	x^k	y^k	z^k	error
1	0.0279511	0.4666627	-0.4919927	27.4303071
2	0.4912529	0.1897410	-0.5229508	6.0954076
3	0.4983604	0.0619545	-0.5191488	1.3753448
4	0.4998276	0.0119348	-0.5220786	0.2286908
5	0.4999935	0.0006994	-0.5233014	0.0167359
6	0.5000000	0.0000178	-0.5235813	0.0006144
7	0.5000000	0.0000009	-0.5235983	0.0000232
8	0.5000000	0.0000000	-0.5235988	0.0000008
9	0.5000000	0.0000000	-0.5235988	0.0000000

Estimates values are x= 0.5000000, y= 0.0000000 and z = -0.5235988 after 9 iterations.

The result from 3-point backward method

k	x^k	y^k	z^k	error
1	0.0279511	0.4666627	-0.4919927	27.4303071
2	0.4912529	0.1897410	-0.5229508	6.0954076
3	0.4983604	0.0619545	-0.5191488	1.3753448
4	0.4998276	0.0119348	-0.5220786	0.2286908
5	0.4999935	0.0006994	-0.5233014	0.0167359
6	0.5000000	0.0000178	-0.5235813	0.0006144
7	0.5000000	0.0000009	-0.5235983	0.0000232
8	0.5000000	0.0000000	-0.5235988	0.0000008
9	0.5000000	0.0000000	-0.5235988	0.0000000

Estimates values are $x=$ 0.5000000, $y=$ 0.0000000 and z = -0.5235988 after 9 iterations.

The result from 5-point central method

k	x^k	y^k	z^k	error
1	0.0279511	0.4666627	-0.4919927	27.4303071
2	0.4912529	0.1897410	-0.5229508	6.0954076
3	0.4983604	0.0619545	-0.5191488	1.3753448
4	0.4998276	0.0119348	-0.5220786	0.2286908
5	0.4999935	0.0006994	-0.5233014	0.0167359
6	0.5000000	0.0000178	-0.5235813	0.0006144
7	0.5000000	0.0000009	-0.5235983	0.0000232
8	0.5000000	0.0000000	-0.5235988	0.0000008
9	0.5000000	0.0000000	-0.5235988	0.0000000

Estimates values are $x=$ 0.5000000, $y=$ 0.0000000 and z = -0.5235988 after 9 iterations.

The result from 5-point forward method

k	x^k	y^k	z^k	error
1	0.0279511	0.4666627	-0.4919927	27.4303071
2	0.4912529	0.1897410	-0.5229508	6.0954076
3	0.4983604	0.0619545	-0.5191488	1.3753448
4	0.4998276	0.0119348	-0.5220786	0.2286908
5	0.4999935	0.0006994	-0.5233014	0.0167359
6	0.5000000	0.0000178	-0.5235813	0.0006144
7	0.5000000	0.0000009	-0.5235983	0.0000232
8	0.5000000	0.0000000	-0.5235988	0.0000008
9	0.5000000	0.0000000	-0.5235988	0.0000000

Estimates values are x= 0.5000000, y= 0.0000000 and z = -0.5235988 after 9 iterations.

The result from 5-point backward method

k	x^k	y^k	z^k	error
1	0.0279511	0.4666627	-0.4919927	27.4303071
2	0.4912529	0.1897410	-0.5229508	6.0954076
3	0.4983604	0.0619545	-0.5191488	1.3753448
4	0.4998276	0.0119348	-0.5220786	0.2286908
5	0.4999935	0.0006994	-0.5233014	0.0167359
6	0.5000000	0.0000178	-0.5235813	0.0006144
7	0.5000000	0.0000009	-0.5235983	0.0000232
8	0.5000000	0.0000000	-0.5235988	0.0000008
9	0.5000000	0.0000000	-0.5235988	0.0000000

Estimates values are x= 0.5000000, y= 0.0000000 and z = -0.5235988 after 9 iterations.

The iterations of table 4.7

The result from 2-point forward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7142857	17.4325670
2	0.1573034	1.0039387	-0.3717565	2.8684910	11.5343599
3	0.1514077	1.1251740	-0.1612421	2.7178353	7.2037926
4	0.1571625	1.0633088	-0.2765817	2.8149392	3.5500960
5	0.1534044	1.0909158	-0.2204713	2.7681547	1.5622926
6	0.1551923	1.0798732	-0.2443202	2.7896877	0.6374561
7	0.1543643	1.0840919	-0.2350655	2.7812067	0.2291249
8	0.1546894	1.0827892	-0.2384562	2.7844672	0.0778509
9	0.1545642	1.0832124	-0.2374785	2.7834631	0.0195995
10	0.1546027	1.0831686	-0.2377766	2.7837895	0.0045367
11	0.1545902	1.0831810	-0.2377713	2.7837558	0.0021473
12	0.1545915	1.0832103	-0.2377712	2.7837654	0.0008137
13	0.1545911	1.0832071	-0.2378018	2.7837879	0.0009356
14	0.1545903	1.0832191	-0.2377982	2.7837854	0.0002921
15	0.1545904	1.0832199	-0.2378094	2.7837947	0.0003134
16	0.1545900	1.0832238	-0.2378103	2.7837954	0.0001441
17	0.1545900	1.0832250	-0.2378138	2.7837983	0.0001232
18	0.1545899	1.0832263	-0.2378150	2.7837992	0.0000729
19	0.1545898	1.0832270	-0.2378162	2.7838003	0.0000521
20	0.1545898	1.0832276	-0.2378169	2.7838008	0.0000340
21	0.1545898	1.0832279	-0.2378174	2.7838012	0.0000230
22	0.1545898	1.0832281	-0.2378177	2.7838015	0.0000154
23	0.1545898	1.0832283	-0.2378179	2.7838017	0.0000103
24	0.1545897	1.0832284	-0.2378180	2.7838018	0.0000069
25	0.1545897	1.0832285	-0.2378181	2.7838019	0.0000046
26	0.1545897	1.0832285	-0.2378182	2.7838019	0.0000031
27	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000021
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000014
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000009
30	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006
31	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000004

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 = -0.2378183, x_4 = 2.7838020$ after 31 iterations.

The result from 2-point backward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7142857	17.4325670
2	0.1573034	1.0039387	-0.3717565	2.8684910	11.5343599
3	0.1514077	1.1251740	-0.1612421	2.7178353	7.2037926
4	0.1571625	1.0633088	-0.2765817	2.8149392	3.5500960
5	0.1534044	1.0909158	-0.2204713	2.7681547	1.5622926
6	0.1551923	1.0798732	-0.2443202	2.7896877	0.6374561
7	0.1543643	1.0840919	-0.2350655	2.7812067	0.2291249
8	0.1546894	1.0827892	-0.2384562	2.7844672	0.0778509
9	0.1545642	1.0832124	-0.2374785	2.7834631	0.0195995
10	0.1546027	1.0831686	-0.2377766	2.7837895	0.0045367
11	0.1545902	1.0831810	-0.2377713	2.7837558	0.0021473
12	0.1545915	1.0832103	-0.2377712	2.7837654	0.0008137
13	0.1545911	1.0832071	-0.2378018	2.7837879	0.0009356
14	0.1545903	1.0832191	-0.2377982	2.7837854	0.0002921
15	0.1545904	1.0832199	-0.2378094	2.7837947	0.0003134
16	0.1545900	1.0832238	-0.2378103	2.7837954	0.0001441
17	0.1545900	1.0832250	-0.2378138	2.7837983	0.0001232
18	0.1545899	1.0832263	-0.2378150	2.7837992	0.0000729
19	0.1545898	1.0832270	-0.2378162	2.7838003	0.0000521
20	0.1545898	1.0832276	-0.2378169	2.7838008	0.0000340
21	0.1545898	1.0832279	-0.2378174	2.7838012	0.0000230
22	0.1545898	1.0832281	-0.2378177	2.7838015	0.0000154
23	0.1545898	1.0832283	-0.2378179	2.7838017	0.0000103
24	0.1545897	1.0832284	-0.2378180	2.7838018	0.0000069
25	0.1545897	1.0832285	-0.2378181	2.7838019	0.0000046
26	0.1545897	1.0832285	-0.2378182	2.7838019	0.0000031
27	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000021
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000014
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000009
30	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006

The result from 2-point backward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
31	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000004

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 = -0.2378183, x_4 = 2.7838020$ after 31 iterations.

The result from 3-point central method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7142857	17.4325670
2	0.1573034	1.0039387	-0.3717565	2.8684910	11.5343599
3	0.1514077	1.1251740	-0.1612421	2.7178353	7.2037926
4	0.1571625	1.0633088	-0.2765817	2.8149392	3.5500960
5	0.1534044	1.0909158	-0.2204713	2.7681547	1.5622926
6	0.1551923	1.0798732	-0.2443202	2.7896877	0.6374561
7	0.1543643	1.0840919	-0.2350655	2.7812067	0.2291249
8	0.1546894	1.0827892	-0.2384562	2.7844672	0.0778509
9	0.1545642	1.0832124	-0.2374785	2.7834631	0.0195995
10	0.1546027	1.0831686	-0.2377766	2.7837895	0.0045367
11	0.1545902	1.0831810	-0.2377713	2.7837558	0.0021473
12	0.1545915	1.0832103	-0.2377712	2.7837654	0.0008137
13	0.1545911	1.0832071	-0.2378018	2.7837879	0.0009356
14	0.1545903	1.0832191	-0.2377982	2.7837854	0.0002921
15	0.1545904	1.0832199	-0.2378094	2.7837947	0.0003134
16	0.1545900	1.0832238	-0.2378103	2.7837954	0.0001441
17	0.1545900	1.0832250	-0.2378138	2.7837983	0.0001232
18	0.1545899	1.0832263	-0.2378150	2.7837992	0.0000729
19	0.1545898	1.0832270	-0.2378162	2.7838003	0.0000521
20	0.1545898	1.0832276	-0.2378169	2.7838008	0.0000340
21	0.1545898	1.0832279	-0.2378174	2.7838012	0.0000230
22	0.1545898	1.0832281	-0.2378177	2.7838015	0.0000154
23	0.1545898	1.0832283	-0.2378179	2.7838017	0.0000103
24	0.1545897	1.0832284	-0.2378180	2.7838018	0.0000069
25	0.1545897	1.0832285	-0.2378181	2.7838019	0.0000046

The result from 3-point central method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
26	0.1545897	1.0832285	-0.2378182	2.7838019	0.0000031
27	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000021
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000014
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000009
30	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006
31	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000004

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 =-0.2378183, x_4 = 2.7838020$ after 31 iterations.

The result from 3-point forward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7142857	17.4325670
2	0.1573034	1.0039387	-0.3717565	2.8684910	11.5343599
3	0.1514077	1.1251740	-0.1612421	2.7178353	7.2037926
4	0.1571625	1.0633088	-0.2765817	2.8149392	3.5500960
5	0.1534044	1.0909158	-0.2204713	2.7681547	1.5622926
6	0.1551923	1.0798732	-0.2443202	2.7896877	0.6374561
7	0.1543643	1.0840919	-0.2350655	2.7812067	0.2291249
8	0.1546894	1.0827892	-0.2384562	2.7844672	0.0778509
9	0.1545642	1.0832124	-0.2374785	2.7834631	0.0195995
10	0.1546027	1.0831686	-0.2377766	2.7837895	0.0045367
11	0.1545902	1.0831810	-0.2377713	2.7837558	0.0021473
12	0.1545915	1.0832103	-0.2377712	2.7837654	0.0008137
13	0.1545911	1.0832071	-0.2378018	2.7837879	0.0009356
14	0.1545903	1.0832191	-0.2377982	2.7837854	0.0002921
15	0.1545904	1.0832199	-0.2378094	2.7837947	0.0003134
16	0.1545900	1.0832238	-0.2378103	2.7837954	0.0001441
17	0.1545900	1.0832250	-0.2378138	2.7837983	0.0001232
18	0.1545899	1.0832263	-0.2378150	2.7837992	0.0000729
19	0.1545898	1.0832270	-0.2378162	2.7838003	0.0000521
20	0.1545898	1.0832276	-0.2378169	2.7838008	0.0000340

The result from 3-point forward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
21	0.1545898	1.0832279	-0.2378174	2.7838012	0.0000230
22	0.1545898	1.0832281	-0.2378177	2.7838015	0.0000154
23	0.1545898	1.0832283	-0.2378179	2.7838017	0.0000103
24	0.1545897	1.0832284	-0.2378180	2.7838018	0.0000069
25	0.1545897	1.0832285	-0.2378181	2.7838019	0.0000046
26	0.1545897	1.0832285	-0.2378182	2.7838019	0.0000031
27	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000021
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000014
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000009
30	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006
31	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000004

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 = -0.2378183, x_4 = 2.7838020$ after 31 iterations.

The result from 3-point backward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7142857	17.4325670
2	0.1573034	1.0039387	-0.3717565	2.8684910	11.5343599
3	0.1514077	1.1251740	-0.1612421	2.7178353	7.2037926
4	0.1571625	1.0633088	-0.2765817	2.8149392	3.5500960
5	0.1534044	1.0909158	-0.2204713	2.7681547	1.5622926
6	0.1551923	1.0798732	-0.2443202	2.7896877	0.6374561
7	0.1543643	1.0840919	-0.2350655	2.7812067	0.2291249
8	0.1546894	1.0827892	-0.2384562	2.7844672	0.0778509
9	0.1545642	1.0832124	-0.2374785	2.7834631	0.0195995
10	0.1546027	1.0831686	-0.2377766	2.7837895	0.0045367
11	0.1545902	1.0831810	-0.2377713	2.7837558	0.0021473
12	0.1545915	1.0832103	-0.2377712	2.7837654	0.0008137
13	0.1545911	1.0832071	-0.2378018	2.7837879	0.0009356
14	0.1545903	1.0832191	-0.2377982	2.7837854	0.0002921
15	0.1545904	1.0832199	-0.2378094	2.7837947	0.0003134

The result from 3-point backward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
16	0.1545900	1.0832238	-0.2378103	2.7837954	0.0001441
17	0.1545900	1.0832250	-0.2378138	2.7837983	0.0001232
18	0.1545899	1.0832263	-0.2378150	2.7837992	0.0000729
19	0.1545898	1.0832270	-0.2378162	2.7838003	0.0000521
20	0.1545898	1.0832276	-0.2378169	2.7838008	0.0000340
21	0.1545898	1.0832279	-0.2378174	2.7838012	0.0000230
22	0.1545898	1.0832281	-0.2378177	2.7838015	0.0000154
23	0.1545898	1.0832283	-0.2378179	2.7838017	0.0000103
24	0.1545897	1.0832284	-0.2378180	2.7838018	0.0000069
25	0.1545897	1.0832285	-0.2378181	2.7838019	0.0000046
26	0.1545897	1.0832285	-0.2378182	2.7838019	0.0000031
27	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000021
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000014
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000009
30	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006
31	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000004

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 = -0.2378183, x_4 = 2.7838020$ after 31 iterations.

The result from 5-point central method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7177419	17.3801357
2	0.1571662	1.0039387	-0.3717565	2.8531760	11.0491535
3	0.1519734	1.1255156	-0.1611049	2.7443633	6.3006268
4	0.1561177	1.0618498	-0.2774891	2.8025492	3.1665845
5	0.1538739	1.0938484	-0.2179675	2.7732493	1.6164840
6	0.1549956	1.0777933	-0.2477223	2.7885955	0.8052723
7	0.1544061	1.0858190	-0.2327889	2.7811650	0.4008851
8	0.1546910	1.0818572	-0.2402251	2.7849756	0.2000438
9	0.1545447	1.0838516	-0.2365482	2.7831337	0.0988565
10	0.1546154	1.0828788	-0.2383963	2.7840790	0.0495338

The result from 5-point central method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
11	0.1545791	1.0833749	-0.2374942	2.7836276	0.0243230
12	0.1545964	1.0831372	-0.2379540	2.7838638	0.0122805
13	0.1545874	1.0832613	-0.2377336	2.7837545	0.0059669
14	0.1545916	1.0832037	-0.2378486	2.7838142	0.0030555
15	0.1545893	1.0832351	-0.2377953	2.7837882	0.0014559
16	0.1545903	1.0832214	-0.2378243	2.7838036	0.0007654
17	0.1545897	1.0832295	-0.2378116	2.7837977	0.0003515
18	0.1545899	1.0832263	-0.2378191	2.7838018	0.0001940
19	0.1545897	1.0832285	-0.2378162	2.7838005	0.0000831
20	0.1545898	1.0832278	-0.2378182	2.7838016	0.0000501
21	0.1545897	1.0832284	-0.2378176	2.7838015	0.0000188
22	0.1545898	1.0832283	-0.2378182	2.7838018	0.0000140
23	0.1545897	1.0832285	-0.2378181	2.7838018	0.0000042
24	0.1545897	1.0832285	-0.2378182	2.7838021	0.0000100
25	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000032
26	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000011
27	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000011
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000005

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 = -0.2378183, x_4 = 2.7838020$ after 29 iterations.

The result from 5-point forward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7142857	17.4325670
2	0.1573034	1.0039387	-0.3717565	2.8684910	11.5343599
3	0.1514077	1.1251740	-0.1612421	2.7178353	7.2037926
4	0.1571625	1.0633088	-0.2765817	2.8149392	3.5500960
5	0.1534044	1.0909158	-0.2204713	2.7681547	1.5622926
6	0.1551923	1.0798732	-0.2443202	2.7896877	0.6374561
7	0.1543643	1.0840919	-0.2350655	2.7812067	0.2291249

The result from 5-point forward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
8	0.1546894	1.0827892	-0.2384562	2.7844672	0.0778509
9	0.1545642	1.0832124	-0.2374785	2.7834631	0.0195995
10	0.1546027	1.0831686	-0.2377766	2.7837895	0.0045367
11	0.1545902	1.0831810	-0.2377713	2.7837558	0.0021473
12	0.1545915	1.0832103	-0.2377712	2.7837654	0.0008137
13	0.1545911	1.0832071	-0.2378018	2.7837879	0.0009356
14	0.1545903	1.0832191	-0.2377982	2.7837854	0.0002921
15	0.1545904	1.0832199	-0.2378094	2.7837947	0.0003134
16	0.1545900	1.0832238	-0.2378103	2.7837954	0.0001441
17	0.1545900	1.0832250	-0.2378138	2.7837983	0.0001232
18	0.1545899	1.0832263	-0.2378150	2.7837992	0.0000729
19	0.1545898	1.0832270	-0.2378162	2.7838003	0.0000521
20	0.1545898	1.0832276	-0.2378169	2.7838008	0.0000340
21	0.1545898	1.0832279	-0.2378174	2.7838012	0.0000230
22	0.1545898	1.0832281	-0.2378177	2.7838015	0.0000154
23	0.1545898	1.0832283	-0.2378179	2.7838017	0.0000103
24	0.1545897	1.0832284	-0.2378180	2.7838018	0.0000069
25	0.1545897	1.0832285	-0.2378181	2.7838019	0.0000046
26	0.1545897	1.0832285	-0.2378182	2.7838019	0.0000031
27	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000021
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000014
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000009
30	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006
31	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000004

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 = -0.2378183, x_4 = 2.7838020$ after 31 iterations.

The result from 5-point backward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	0.1663020	1.2054545	-0.1000000	2.7142857	17.4325670
2	0.1573034	1.0039387	-0.3717565	2.8684910	11.5343599
3	0.1514077	1.1251740	-0.1612421	2.7178353	7.2037926
4	0.1571625	1.0633088	-0.2765817	2.8149392	3.5500960
5	0.1534044	1.0909158	-0.2204713	2.7681547	1.5622926
6	0.1551923	1.0798732	-0.2443202	2.7896877	0.6374561
7	0.1543643	1.0840919	-0.2350655	2.7812067	0.2291249
8	0.1546894	1.0827892	-0.2384562	2.7844672	0.0778509
9	0.1545642	1.0832124	-0.2374785	2.7834631	0.0195995
10	0.1546027	1.0831686	-0.2377766	2.7837895	0.0045367
11	0.1545902	1.0831810	-0.2377713	2.7837558	0.0021473
12	0.1545915	1.0832103	-0.2377712	2.7837654	0.0008137
13	0.1545911	1.0832071	-0.2378018	2.7837879	0.0009356
14	0.1545903	1.0832191	-0.2377982	2.7837854	0.0002921
15	0.1545904	1.0832199	-0.2378094	2.7837947	0.0003134
16	0.1545900	1.0832238	-0.2378103	2.7837954	0.0001441
17	0.1545900	1.0832250	-0.2378138	2.7837983	0.0001232
18	0.1545899	1.0832263	-0.2378150	2.7837992	0.0000729
19	0.1545898	1.0832270	-0.2378162	2.7838003	0.0000521
20	0.1545898	1.0832276	-0.2378169	2.7838008	0.0000340
21	0.1545898	1.0832279	-0.2378174	2.7838012	0.0000230
22	0.1545898	1.0832281	-0.2378177	2.7838015	0.0000154
23	0.1545898	1.0832283	-0.2378179	2.7838017	0.0000103
24	0.1545897	1.0832284	-0.2378180	2.7838018	0.0000069
25	0.1545897	1.0832285	-0.2378181	2.7838019	0.0000046
26	0.1545897	1.0832285	-0.2378182	2.7838019	0.0000031
27	0.1545897	1.0832285	-0.2378182	2.7838020	0.0000021
28	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000014
29	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000009
30	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000006
31	0.1545897	1.0832286	-0.2378183	2.7838020	0.0000004

Estimates values are $x_1 = 0.1545897, x_2 = 1.0832286, x_3 = -0.2378183, x_4 = 2.7838020$ after 31 iterations.

The iterations of table 4.8

The result from 2-point forward method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8571429	-2.8571429	17.0000000	-1.4285714	-0.2500000	1.5663265
2	-2.9126050	-2.9126050	17.7142857	-1.2356506	-0.2121212	0.8822032
3	-2.9761819	-2.9761819	17.8252101	-1.2335968	-0.2072449	0.2633983
4	-2.9853154	-2.9853154	17.9523638	-1.2056123	-0.2019238	0.1501037
5	-2.9960305	-2.9960305	17.9706307	-1.2054931	-0.2011817	0.0439747
6	-2.9975492	-2.9975492	17.9920610	-1.2009265	-0.2003181	0.0251041
7	-2.9993384	-2.9993384	17.9950985	-1.2009126	-0.2001963	0.0073312
8	-2.9995914	-2.9995914	17.9986769	-1.2001542	-0.2000529	0.0041864
9	-2.9998897	-2.9998897	17.9991829	-1.2001520	-0.2000327	0.0012219
10	-2.9999319	-2.9999319	17.9997795	-1.2000257	-0.2000088	0.0006978
11	-2.9999816	-2.9999816	17.9998638	-1.2000253	-0.2000054	0.0002037
12	-2.9999887	-2.9999887	17.9999632	-1.2000043	-0.2000015	0.0001163
13	-2.9999969	-2.9999969	17.9999773	-1.2000042	-0.2000009	0.0000339
14	-2.9999981	-2.9999981	17.9999939	-1.2000007	-0.2000002	0.0000194
15	-2.9999995	-2.9999995	17.9999962	-1.2000007	-0.2000002	0.0000057
16	-2.9999997	-2.9999997	17.9999990	-1.2000001	-0.2000000	0.0000032
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000009
18	-2.9999999	-2.9999999	17.9999998	-1.2000000	-0.2000000	0.0000005
19	-3.0000000	-3.0000000	17.9999999	-1.2000000	-0.2000000	0.0000002

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999999, λ=-1.2000000 and μ = -0.2000000 after 19 iterations.

The result from 2-point backward method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8431373	-2.8431373	17.0000000	-1.4215686	-0.2500000	1.6909842
2	-2.9211009	-2.9211009	17.6862745	-1.2466623	-0.2133891	0.8098139
3	-2.9749340	-2.9749340	17.8422018	-1.2287526	-0.2065176	0.2716384
4	-2.9869537	-2.9869537	17.9498680	-1.2068822	-0.2020256	0.1349791
5	-2.9958505	-2.9958505	17.9739074	-1.2046266	-0.2010492	0.0450382
6	-2.9978284	-2.9978284	17.9917010	-1.2011250	-0.2003325	0.0224989
7	-2.9993092	-2.9993092	17.9956568	-1.2007667	-0.2001739	0.0075000
8	-2.9996381	-2.9996381	17.9986184	-1.2001869	-0.2000553	0.0037499
9	-2.9998849	-2.9998849	17.9992763	-1.2001277	-0.2000290	0.0012498
10	-2.9999397	-2.9999397	17.9997698	-1.2000311	-0.2000092	0.0006250
11	-2.9999808	-2.9999808	17.9998794	-1.2000213	-0.2000048	0.0002083
12	-2.9999899	-2.9999899	17.9999616	-1.2000052	-0.2000015	0.0001042
13	-2.9999968	-2.9999968	17.9999799	-1.2000035	-0.2000008	0.0000347
14	-2.9999983	-2.9999983	17.9999936	-1.2000009	-0.2000003	0.0000174
15	-2.9999995	-2.9999995	17.9999966	-1.2000006	-0.2000001	0.0000058
16	-2.9999997	-2.9999997	17.9999989	-1.2000001	-0.2000000	0.0000029
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000010
18	-3.0000000	-3.0000000	17.9999998	-1.2000000	-0.2000000	0.0000005

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999998, λ=-1.2000000 and μ= -0.2000000 after 18 iterations.

The result from 3-point central method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8500000	-2.8500000	17.0000000	-1.4250000	-0.2500000	1.6300000
2	-2.9162281	-2.9162281	17.7000000	-1.2409481	-0.2127660	0.8518660
3	-2.9754851	-2.9754851	17.8324561	-1.2314587	-0.2069341	0.2684719
4	-2.9860240	-2.9860240	17.9509701	-1.2062379	-0.2019806	0.1436270
5	-2.9959278	-2.9959278	17.9720479	-1.2051082	-0.2011244	0.0446730
6	-2.9976703	-2.9976703	17.9918555	-1.2010244	-0.2003263	0.0239847
7	-2.9993217	-2.9993217	17.9953405	-1.2008477	-0.2001866	0.0074435
8	-2.9996117	-2.9996117	17.9986433	-1.2001703	-0.2000543	0.0039987
9	-2.9998870	-2.9998870	17.9992234	-1.2001412	-0.2000311	0.0012405
10	-2.9999353	-2.9999353	17.9997739	-1.2000284	-0.2000090	0.0006665
11	-2.9999812	-2.9999812	17.9998706	-1.2000235	-0.2000052	0.0002068
12	-2.9999892	-2.9999892	17.9999623	-1.2000047	-0.2000015	0.0001111
13	-2.9999969	-2.9999969	17.9999784	-1.2000039	-0.2000009	0.0000345
14	-2.9999982	-2.9999982	17.9999937	-1.2000008	-0.2000003	0.0000185
15	-2.9999995	-2.9999995	17.9999964	-1.2000007	-0.2000001	0.0000057
16	-2.9999997	-2.9999997	17.9999990	-1.2000001	-0.2000000	0.0000031
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000010
18	-3.0000000	-3.0000000	17.9999998	-1.2000000	-0.2000000	0.0000005
19	-3.0000000	-3.0000000	17.9999999	-1.2000000	-0.2000000	0.0000002

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999999, λ=-1.2000000 and μ= -0.2000000 after 19 iterations.

The result from 3-point forward method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8500000	-2.8500000	17.0000000	-1.4250000	-0.2500000	1.6300000
2	-2.9162281	-2.9162281	17.7000000	-1.2409481	-0.2127660	0.8518660
3	-2.9754851	-2.9754851	17.8324561	-1.2314587	-0.2069341	0.2684719
4	-2.9860240	-2.9860240	17.9509701	-1.2062379	-0.2019806	0.1436270
5	-2.9959278	-2.9959278	17.9720479	-1.2051082	-0.2011244	0.0446730
6	-2.9976703	-2.9976703	17.9918555	-1.2010244	-0.2003263	0.0239847
7	-2.9993217	-2.9993217	17.9953405	-1.2008477	-0.2001866	0.0074435
8	-2.9996117	-2.9996117	17.9986433	-1.2001703	-0.2000543	0.0039987
9	-2.9998870	-2.9998870	17.9992234	-1.2001412	-0.2000311	0.0012405
10	-2.9999353	-2.9999353	17.9997739	-1.2000284	-0.2000090	0.0006665
11	-2.9999812	-2.9999812	17.9998706	-1.2000235	-0.2000052	0.0002068
12	-2.9999892	-2.9999892	17.9999623	-1.2000047	-0.2000015	0.0001111
13	-2.9999969	-2.9999969	17.9999784	-1.2000039	-0.2000009	0.0000345
14	-2.9999982	-2.9999982	17.9999937	-1.2000008	-0.2000003	0.0000185
15	-2.9999995	-2.9999995	17.9999964	-1.2000007	-0.2000001	0.0000057
16	-2.9999997	-2.9999997	17.9999990	-1.2000001	-0.2000000	0.0000031
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000010
18	-3.0000000	-3.0000000	17.9999998	-1.2000000	-0.2000000	0.0000005
19	-3.0000000	-3.0000000	17.9999999	-1.2000000	-0.2000000	0.0000002

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999999, λ=-1.2000000 and μ= -0.2000000 after 19 iterations.

The result from 3-point backward method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8500000	-2.8500000	17.0000000	-1.4250000	-0.2500000	1.6300000
2	-2.9162281	-2.9162281	17.7000000	-1.2409481	-0.2127660	0.8518660
3	-2.9754851	-2.9754851	17.8324561	-1.2314587	-0.2069341	0.2684719
4	-2.9860240	-2.9860240	17.9509701	-1.2062379	-0.2019806	0.1436270
5	-2.9959278	-2.9959278	17.9720479	-1.2051082	-0.2011244	0.0446730
6	-2.9976703	-2.9976703	17.9918555	-1.2010244	-0.2003263	0.0239847
7	-2.9993217	-2.9993217	17.9953405	-1.2008477	-0.2001866	0.0074435
8	-2.9996117	-2.9996117	17.9986433	-1.2001703	-0.2000543	0.0039987
9	-2.9998870	-2.9998870	17.9992234	-1.2001412	-0.2000311	0.0012405
10	-2.9999353	-2.9999353	17.9997739	-1.2000284	-0.2000090	0.0006665
11	-2.9999812	-2.9999812	17.9998706	-1.2000235	-0.2000052	0.0002068
12	-2.9999892	-2.9999892	17.9999623	-1.2000047	-0.2000015	0.0001111
13	-2.9999969	-2.9999969	17.9999784	-1.2000039	-0.2000009	0.0000345
14	-2.9999982	-2.9999982	17.9999937	-1.2000008	-0.2000003	0.0000185
15	-2.9999995	-2.9999995	17.9999964	-1.2000007	-0.2000001	0.0000057
16	-2.9999997	-2.9999997	17.9999990	-1.2000001	-0.2000000	0.0000031
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000010
18	-3.0000000	-3.0000000	17.9999998	-1.2000000	-0.2000000	0.0000005
19	-3.0000000	-3.0000000	17.9999999	-1.2000000	-0.2000000	0.0000002

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999999, λ=-1.2000000 and μ = -0.2000000 after 19 iterations.

The result from 5-point central method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8500000	-2.8500000	17.0000000	-1.4250000	-0.2500000	1.6300000
2	-2.9162281	-2.9162281	17.7000000	-1.2409481	-0.2127660	0.8518660
3	-2.9754851	-2.9754851	17.8324561	-1.2314587	-0.2069341	0.2684719
4	-2.9860240	-2.9860240	17.9509701	-1.2062379	-0.2019806	0.1436270
5	-2.9959278	-2.9959278	17.9720479	-1.2051082	-0.2011244	0.0446730
6	-2.9976703	-2.9976703	17.9918555	-1.2010244	-0.2003263	0.0239847
7	-2.9993217	-2.9993217	17.9953405	-1.2008477	-0.2001866	0.0074435
8	-2.9996117	-2.9996117	17.9986433	-1.2001703	-0.2000543	0.0039987
9	-2.9998870	-2.9998870	17.9992234	-1.2001412	-0.2000311	0.0012405
10	-2.9999353	-2.9999353	17.9997739	-1.2000284	-0.2000090	0.0006665
11	-2.9999812	-2.9999812	17.9998706	-1.2000235	-0.2000052	0.0002068
12	-2.9999892	-2.9999892	17.9999623	-1.2000047	-0.2000015	0.0001111
13	-2.9999969	-2.9999969	17.9999784	-1.2000039	-0.2000009	0.0000345
14	-2.9999982	-2.9999982	17.9999937	-1.2000008	-0.2000003	0.0000185
15	-2.9999995	-2.9999995	17.9999964	-1.2000007	-0.2000001	0.0000057
16	-2.9999997	-2.9999997	17.9999990	-1.2000001	-0.2000000	0.0000031
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000010
18	-3.0000000	-3.0000000	17.9999998	-1.2000000	-0.2000000	0.0000005
19	-3.0000000	-3.0000000	17.9999999	-1.2000000	-0.2000000	0.0000002

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999999, λ=-1.2000000 and μ = -0.2000000 after 19 iterations.

The result from 5-point forward method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8500000	-2.8500000	17.0000000	-1.4250000	-0.2500000	1.6300000
2	-2.9162281	-2.9162281	17.7000000	-1.2409481	-0.2127660	0.8518660
3	-2.9754851	-2.9754851	17.8324561	-1.2314587	-0.2069341	0.2684719
4	-2.9860240	-2.9860240	17.9509701	-1.2062379	-0.2019806	0.1436270
5	-2.9959278	-2.9959278	17.9720479	-1.2051082	-0.2011244	0.0446730
6	-2.9976703	-2.9976703	17.9918555	-1.2010244	-0.2003263	0.0239847
7	-2.9993217	-2.9993217	17.9953405	-1.2008477	-0.2001866	0.0074435
8	-2.9996117	-2.9996117	17.9986433	-1.2001703	-0.2000543	0.0039987
9	-2.9998870	-2.9998870	17.9992234	-1.2001412	-0.2000311	0.0012405
10	-2.9999353	-2.9999353	17.9997739	-1.2000284	-0.2000090	0.0006665
11	-2.9999812	-2.9999812	17.9998706	-1.2000235	-0.2000052	0.0002068
12	-2.9999892	-2.9999892	17.9999623	-1.2000047	-0.2000015	0.0001111
13	-2.9999969	-2.9999969	17.9999784	-1.2000039	-0.2000009	0.0000345
14	-2.9999982	-2.9999982	17.9999937	-1.2000008	-0.2000003	0.0000185
15	-2.9999995	-2.9999995	17.9999964	-1.2000007	-0.2000001	0.0000057
16	-2.9999997	-2.9999997	17.9999990	-1.2000001	-0.2000000	0.0000031
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000010
18	-3.0000000	-3.0000000	17.9999998	-1.2000000	-0.2000000	0.0000005
19	-3.0000000	-3.0000000	17.9999999	-1.2000000	-0.2000000	0.0000002

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999999, λ=-1.2000000 and μ = -0.2000000 after 19 iterations.

The result from 5-point backward method

k	x^k	y^k	z^k	λ^k	μ^k	error
1	-2.8500000	-2.8500000	17.0000000	-1.4250000	-0.2500000	1.6300000
2	-2.9162281	-2.9162281	17.7000000	-1.2409481	-0.2127660	0.8518660
3	-2.9754851	-2.9754851	17.8324561	-1.2314587	-0.2069341	0.2684719
4	-2.9860240	-2.9860240	17.9509701	-1.2062379	-0.2019806	0.1436270
5	-2.9959278	-2.9959278	17.9720479	-1.2051082	-0.2011244	0.0446730
6	-2.9976703	-2.9976703	17.9918555	-1.2010244	-0.2003263	0.0239847
7	-2.9993217	-2.9993217	17.9953405	-1.2008477	-0.2001866	0.0074435
8	-2.9996117	-2.9996117	17.9986433	-1.2001703	-0.2000543	0.0039987
9	-2.9998870	-2.9998870	17.9992234	-1.2001412	-0.2000311	0.0012405
10	-2.9999353	-2.9999353	17.9997739	-1.2000284	-0.2000090	0.0006665
11	-2.9999812	-2.9999812	17.9998706	-1.2000235	-0.2000052	0.0002068
12	-2.9999892	-2.9999892	17.9999623	-1.2000047	-0.2000015	0.0001111
13	-2.9999969	-2.9999969	17.9999784	-1.2000039	-0.2000009	0.0000345
14	-2.9999982	-2.9999982	17.9999937	-1.2000008	-0.2000003	0.0000185
15	-2.9999995	-2.9999995	17.9999964	-1.2000007	-0.2000001	0.0000057
16	-2.9999997	-2.9999997	17.9999990	-1.2000001	-0.2000000	0.0000031
17	-2.9999999	-2.9999999	17.9999994	-1.2000001	-0.2000000	0.0000010
18	-3.0000000	-3.0000000	17.9999998	-1.2000000	-0.2000000	0.0000005
19	-3.0000000	-3.0000000	17.9999999	-1.2000000	-0.2000000	0.0000002

Estimates values are x=-3.0000000, y=-3.0000000, z =17.9999999, λ=-1.2000000 and μ = -0.2000000 after 19 iterations.

The iterations of table 4.9

The result from 2-point forward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.6800000	-0.4999977	0.5081200	3.0038800	345680.1790300
2	91458.9873040	0.5085338	1.5081177	5.0038754	84618.9918470
3	120873.2009300	-0.2505153	0.4995862	2.9868124	86548.8703150
4	100519.5117900	0.0509960	1.2586353	4.5049105	24047.7723620
5	110964.4079900	-0.1050970	0.9571240	3.9018879	17527.7815860
6	106766.9790600	0.0112289	1.1132170	4.2140740	7265.0842873
7	110014.7078300	-0.0295433	0.9968911	3.9814222	4694.0363165
8	108908.1683400	0.0030633	1.0376633	4.0629666	1990.2933139
9	109805.0669600	-0.0072804	1.0050567	3.9977533	1204.4385757
10	109522.5229800	0.0011673	1.0154004	4.0184407	512.9625346
11	109754.2138700	-0.0014364	1.0069527	4.0015453	304.2616792
12	109682.9525500	0.0007027	1.0095564	4.0067529	129.6996375
13	109741.5818600	0.0000485	1.0074173	4.0024747	76.5314689
14	109723.6704500	0.0005870	1.0080715	4.0037829	32.6263975
15	109738.4253000	0.0004227	1.0075330	4.0027061	19.2287654
16	109733.9273000	0.0005580	1.0076973	4.0030346	8.1965633
17	109737.6353500	0.0005168	1.0075620	4.0027639	4.8299316
18	109736.5060400	0.0005508	1.0076032	4.0028464	2.0585013
19	109737.4375900	0.0005404	1.0075692	4.0027784	1.2131091
20	109737.1540700	0.0005490	1.0075796	4.0027991	0.5169330
21	109737.3880800	0.0005464	1.0075710	4.0027821	0.3046849
22	109737.3169000	0.0005485	1.0075736	4.0027873	0.1298100
23	109737.3756800	0.0005479	1.0075715	4.0027830	0.0765245
24	109737.3578100	0.0005484	1.0075721	4.0027843	0.0325974
25	109737.3725800	0.0005482	1.0075716	4.0027832	0.0192200
26	109737.3680900	0.0005484	1.0075718	4.0027835	0.0081858
27	109737.3718000	0.0005483	1.0075716	4.0027832	0.0048274
28	109737.3706700	0.0005484	1.0075717	4.0027833	0.0020556
29	109737.3716000	0.0005484	1.0075716	4.0027833	0.0012125
30	109737.3713200	0.0005484	1.0075716	4.0027833	0.0005162
31	109737.3715500	0.0005484	1.0075716	4.0027833	0.0003043

The result from 2-point forward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
32	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001292
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000763
34	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000327
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000191
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000076
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000044
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000019
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000012
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000007
41	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000004

Estimates values are $x_1 = 109737.3715400$, $x_2 = 0.0005484$, $x_3 = 1.0075716$, $x_4 = 4.0027833$ after 41 iterations.

The result from 2-point backward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.6800000	-0.4999977	0.5081200	3.0038800	345680.1790300
2	91458.9873030	0.5085338	1.5081177	5.0038754	84618.9918470
3	120873.2009300	-0.2505153	0.4995862	2.9868124	86548.8703160
4	100519.5117900	0.0509960	1.2586353	4.5049105	24047.7723620
5	110964.4079900	-0.1050970	0.9571240	3.9018879	17527.7815860
6	106766.9790600	0.0112289	1.1132170	4.2140740	7265.0842871
7	110014.7078300	-0.0295433	0.9968911	3.9814222	4694.0363160
8	108908.1683400	0.0030633	1.0376633	4.0629666	1990.2933136
9	109805.0669600	-0.0072804	1.0050567	3.9977533	1204.4385756
10	109522.5229800	0.0011673	1.0154004	4.0184407	512.9625344
11	109754.2138700	-0.0014364	1.0069527	4.0015453	304.2616791
12	109682.9525500	0.0007027	1.0095564	4.0067529	129.6996375
13	109741.5818600	0.0000485	1.0074173	4.0024747	76.5314689
14	109723.6704500	0.0005870	1.0080715	4.0037829	32.6263975
15	109738.4253000	0.0004227	1.0075330	4.0027061	19.2287654
16	109733.9273000	0.0005580	1.0076973	4.0030346	8.1965632

The result from 2-point backward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
17	109737.6353500	0.0005168	1.0075620	4.0027639	4.8299316
18	109736.5060400	0.0005508	1.0076032	4.0028464	2.0585013
19	109737.4375900	0.0005404	1.0075692	4.0027784	1.2131093
20	109737.1540700	0.0005490	1.0075796	4.0027991	0.5169333
21	109737.3880800	0.0005464	1.0075710	4.0027821	0.3046852
22	109737.3169000	0.0005485	1.0075736	4.0027873	0.1298101
23	109737.3756800	0.0005479	1.0075715	4.0027830	0.0765243
24	109737.3578100	0.0005484	1.0075721	4.0027843	0.0325974
25	109737.3725800	0.0005482	1.0075716	4.0027832	0.0192200
26	109737.3680900	0.0005484	1.0075718	4.0027835	0.0081858
27	109737.3718000	0.0005483	1.0075716	4.0027832	0.0048274
28	109737.3706700	0.0005484	1.0075717	4.0027833	0.0020556
29	109737.3716000	0.0005484	1.0075716	4.0027833	0.0012124
30	109737.3713200	0.0005484	1.0075716	4.0027833	0.0005164
31	109737.3715500	0.0005484	1.0075716	4.0027833	0.0003048
32	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001298
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000768
34	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000327
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000191
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000079
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000046
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000017
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000010
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000005

Estimates values are x_1 = 109737.3715400, x_2 = 0.0005484, x_3 = 1.0075716, x_4 = 4.0027833 after 40 iterations.

The result from 3-point central method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.6800000	-0.4999977	0.5081200	3.0038800	345680.1790300
2	91458.9873040	0.5085338	1.5081177	5.0038754	84618.9918480
3	120873.2009300	-0.2505153	0.4995862	2.9868124	86548.8703130
4	100519.5117900	0.0509960	1.2586353	4.5049105	24047.7723610
5	110964.4079900	-0.1050970	0.9571240	3.9018879	17527.7815860
6	106766.9790600	0.0112289	1.1132170	4.2140740	7265.0842869
7	110014.7078300	-0.0295433	0.9968911	3.9814222	4694.0363159
8	108908.1683400	0.0030633	1.0376633	4.0629666	1990.2933136
9	109805.0669600	-0.0072804	1.0050567	3.9977533	1204.4385757
10	109522.5229800	0.0011673	1.0154004	4.0184407	512.9625349
11	109754.2138700	-0.0014364	1.0069527	4.0015453	304.2616794
12	109682.9525500	0.0007027	1.0095564	4.0067529	129.6996375
13	109741.5818600	0.0000485	1.0074173	4.0024747	76.5314689
14	109723.6704500	0.0005870	1.0080715	4.0037829	32.6263975
15	109738.4253000	0.0004227	1.0075330	4.0027061	19.2287654
16	109733.9273000	0.0005580	1.0076973	4.0030346	8.1965632
17	109737.6353500	0.0005168	1.0075620	4.0027639	4.8299316
18	109736.5060400	0.0005508	1.0076032	4.0028464	2.0585013
19	109737.4375900	0.0005404	1.0075692	4.0027784	1.2131091
20	109737.1540700	0.0005490	1.0075796	4.0027991	0.5169331
21	109737.3880800	0.0005464	1.0075710	4.0027821	0.3046849
22	109737.3169000	0.0005485	1.0075736	4.0027873	0.1298101
23	109737.3756800	0.0005479	1.0075715	4.0027830	0.0765243
24	109737.3578100	0.0005484	1.0075721	4.0027843	0.0325971
25	109737.3725800	0.0005482	1.0075716	4.0027832	0.0192200
26	109737.3680900	0.0005484	1.0075718	4.0027835	0.0081856
27	109737.3718000	0.0005483	1.0075716	4.0027832	0.0048269
28	109737.3706700	0.0005484	1.0075717	4.0027833	0.0020550
29	109737.3716000	0.0005484	1.0075716	4.0027833	0.0012122
30	109737.3713200	0.0005484	1.0075716	4.0027833	0.0005162
31	109737.3715500	0.0005484	1.0075716	4.0027833	0.0003046
32	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001298
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000768
34	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000331

The result from 3-point central method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000198
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000085
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000051
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000023
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000012
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000005

Estimates values are x_1= 109737.3715400, x_2= 0.0005484, x_3= 1.0075716, x_4 = 4.0027833 after 40 iterations.

The result from 3-point forward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.7323200	-0.4999977	0.5081200	3.0038800	345680.2836700
2	91458.9873040	0.5085340	1.5081177	5.0038754	84619.0147910
3	120873.2062500	-0.2505153	0.4995860	2.9868119	86548.9091200
4	100519.5102900	0.0509960	1.2586353	4.5049105	24047.7752880
5	110964.4079800	-0.1050970	0.9571240	3.9018879	17527.7839980
6	106766.9785800	0.0112289	1.1132170	4.2140740	7265.0852167
7	110014.7078300	-0.0295433	0.9968911	3.9814222	4694.0370400
8	108908.1682000	0.0030633	1.0376633	4.0629666	1990.2936001
9	109805.0669500	-0.0072804	1.0050567	3.9977533	1204.4387683
10	109522.5229400	0.0011673	1.0154004	4.0184407	512.9626113
11	109754.2138700	-0.0014365	1.0069527	4.0015453	304.2617283
12	109682.9525400	0.0007027	1.0095565	4.0067529	129.6996569
13	109741.5818600	0.0000485	1.0074173	4.0024747	76.5314812
14	109723.6704500	0.0005870	1.0080715	4.0037829	32.6264025
15	109738.4253000	0.0004227	1.0075330	4.0027061	19.2287685
16	109733.9273000	0.0005580	1.0076973	4.0030346	8.1965641
17	109737.6353500	0.0005168	1.0075620	4.0027639	4.8299324
18	109736.5060400	0.0005508	1.0076032	4.0028464	2.0585021
19	109737.4375900	0.0005404	1.0075692	4.0027784	1.2131096
20	109737.1540700	0.0005490	1.0075796	4.0027991	0.5169330

The result from 3-point forward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
21	109737.3880800	0.0005464	1.0075710	4.0027821	0.3046849
22	109737.3169000	0.0005485	1.0075736	4.0027873	0.1298100
23	109737.3756800	0.0005479	1.0075715	4.0027830	0.0765245
24	109737.3578100	0.0005484	1.0075721	4.0027843	0.0325971
25	109737.3725800	0.0005482	1.0075716	4.0027832	0.0192200
26	109737.3680900	0.0005484	1.0075718	4.0027835	0.0081858
27	109737.3718000	0.0005483	1.0075716	4.0027832	0.0048274
28	109737.3706700	0.0005484	1.0075717	4.0027833	0.0020554
29	109737.3716000	0.0005484	1.0075716	4.0027833	0.0012124
30	109737.3713200	0.0005484	1.0075716	4.0027833	0.0005164
31	109737.3715500	0.0005484	1.0075716	4.0027833	0.0003046
32	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001298
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000768
34	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000327
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000193
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000079
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000044
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000019
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000012
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000011
41	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000006
42	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000000

Estimates values are x_1= 109737.3715400, x_2= 0.0005484, x_3= 1.0075716, x_4 = 4.0027833 after 42 iterations.

The result from 3-point backward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.6800000	-0.4999977	0.5081200	3.0038800	345680.1790300
2	91458.9873030	0.5085338	1.5081177	5.0038754	84618.9918530
3	120873.2009300	-0.2505153	0.4995862	2.9868124	86548.8703080
4	100519.5117900	0.0509960	1.2586353	4.5049105	24047.7723590
5	110964.4079900	-0.1050970	0.9571240	3.9018879	17527.7815840
6	106766.9790600	0.0112289	1.1132170	4.2140740	7265.0842866
7	110014.7078300	-0.0295433	0.9968911	3.9814222	4694.0363157
8	108908.1683400	0.0030633	1.0376633	4.0629666	1990.2933132
9	109805.0669600	-0.0072804	1.0050567	3.9977533	1204.4385754
10	109522.5229800	0.0011673	1.0154004	4.0184407	512.9625346
11	109754.2138700	-0.0014364	1.0069527	4.0015453	304.2616792
12	109682.9525500	0.0007027	1.0095564	4.0067529	129.6996375
13	109741.5818600	0.0000485	1.0074173	4.0024747	76.5314689
14	109723.6704500	0.0005870	1.0080715	4.0037829	32.6263977
15	109738.4253000	0.0004227	1.0075330	4.0027061	19.2287654
16	109733.9273000	0.0005580	1.0076973	4.0030346	8.1965633
17	109737.6353500	0.0005168	1.0075620	4.0027639	4.8299316
18	109736.5060400	0.0005508	1.0076032	4.0028464	2.0585013
19	109737.4375900	0.0005404	1.0075692	4.0027784	1.2131091
20	109737.1540700	0.0005490	1.0075796	4.0027991	0.5169333
21	109737.3880800	0.0005464	1.0075710	4.0027821	0.3046849
22	109737.3169000	0.0005485	1.0075736	4.0027873	0.1298100
23	109737.3756800	0.0005479	1.0075715	4.0027830	0.0765243
24	109737.3578100	0.0005484	1.0075721	4.0027843	0.0325971
25	109737.3725800	0.0005482	1.0075716	4.0027832	0.0192200
26	109737.3680900	0.0005484	1.0075718	4.0027835	0.0081856
27	109737.3718000	0.0005483	1.0075716	4.0027832	0.0048271
28	109737.3706700	0.0005484	1.0075717	4.0027833	0.0020552
29	109737.3716000	0.0005484	1.0075716	4.0027833	0.0012124
30	109737.3713200	0.0005484	1.0075716	4.0027833	0.0005158
31	109737.3715500	0.0005484	1.0075716	4.0027833	0.0003041
32	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001292
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000763
34	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000324

The result from 3-point backward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000193
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000079
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000044
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000019
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000012
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000004

Estimates values are $x_1 =$ 109737.3715400, $x_2 =$ 0.0005484, $x_3 =$ 1.0075716, $x_4 = 4.0027833$ after 40 iterations.

The result from 5-point central method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.6887200	-0.4999977	0.5081200	3.8353640	341719.9791900
2	95418.3730970	0.5085338	1.5081177	4.8405728	77076.2486340
3	121249.3896700	-0.1960719	0.4995862	3.3416609	72583.5305260
4	103284.3827000	0.0527096	1.2041919	4.2305847	18999.6698060
5	111089.3897900	-0.0701944	0.9554104	3.9478009	12788.0738620
6	107771.4077900	0.0122975	1.0783144	4.1131298	5436.2766741
7	110050.3898000	-0.0187418	0.9958225	4.0008958	3278.9993313
8	109208.3556600	0.0033840	1.0268618	4.0353756	1436.8250587
9	109814.3522400	-0.0043941	1.0047360	4.0036392	828.7831862
10	109601.9176900	0.0012520	1.0125141	4.0113566	366.0692016
11	109756.5866300	-0.0006994	1.0068680	4.0031317	208.5149081
12	109703.1696000	0.0007244	1.0088194	4.0049690	92.2843283
13	109742.1855700	0.0002345	1.0073956	4.0028841	52.3999549
14	109728.7665800	0.0005925	1.0078855	4.0033352	23.2007798
15	109738.5787300	0.0004695	1.0075275	4.0028099	13.1644228
16	109735.2085600	0.0005594	1.0076505	4.0029222	5.8287616
17	109737.6743300	0.0005285	1.0075606	4.0027901	3.3070631
18	109736.8279700	0.0005511	1.0075915	4.0028182	1.4641121
19	109737.4474900	0.0005434	1.0075689	4.0027850	0.8307601
20	109737.2349400	0.0005491	1.0075766	4.0027921	0.3677506

The result from 5-point central method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
21	109737.3905900	0.0005471	1.0075709	4.0027837	0.2086925
22	109737.3372100	0.0005485	1.0075729	4.0027855	0.0923692
23	109737.3763200	0.0005481	1.0075715	4.0027834	0.0524252
24	109737.3629100	0.0005484	1.0075719	4.0027838	0.0232007
25	109737.3727400	0.0005483	1.0075716	4.0027833	0.0131694
26	109737.3693700	0.0005484	1.0075717	4.0027834	0.0058277
27	109737.3718400	0.0005483	1.0075716	4.0027833	0.0033085
28	109737.3709900	0.0005484	1.0075717	4.0027833	0.0014636
29	109737.3716100	0.0005484	1.0075716	4.0027833	0.0008310
30	109737.3714000	0.0005484	1.0075716	4.0027833	0.0003680
31	109737.3715600	0.0005484	1.0075716	4.0027833	0.0002091
32	109737.3715000	0.0005484	1.0075716	4.0027833	0.0000924
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000523
34	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000230
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000132
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000061
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000035
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000015
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000008
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000002

Estimates values are $x_1 =$ 109737.3715400, $x_2 =$ 0.0005484, $x_3 =$ 1.0075716, $x_4 =$ 4.0027833 after 40 iterations.

The result from 5-point forward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.5753600	-0.4999977	0.5081200	3.0038800	345679.9697500
2	91458.9873040	0.5085333	1.5081177	5.0038754	84618.9459590
3	120873.1903000	-0.2505153	0.4995867	2.9868134	86548.7926960
4	100519.5147700	0.0509960	1.2586353	4.5049105	24047.7665050
5	110964.4080200	-0.1050970	0.9571240	3.9018879	17527.7767600
6	106766.9800200	0.0112289	1.1132170	4.2140739	7265.0824277
7	110014.7078400	-0.0295433	0.9968911	3.9814222	4694.0348684
8	108908.1686400	0.0030633	1.0376633	4.0629666	1990.2927414
9	109805.0669600	-0.0072804	1.0050567	3.9977533	1204.4381913
10	109522.5230600	0.0011673	1.0154004	4.0184407	512.9623821
11	109754.2138700	-0.0014364	1.0069527	4.0015453	304.2615812
12	109682.9525700	0.0007027	1.0095564	4.0067529	129.6995980
13	109741.5818600	0.0000485	1.0074173	4.0024747	76.5314441
14	109723.6704600	0.0005870	1.0080715	4.0037829	32.6263877
15	109738.4253000	0.0004227	1.0075330	4.0027061	19.2287592
16	109733.9273000	0.0005580	1.0076973	4.0030346	8.1965605
17	109737.6353500	0.0005168	1.0075620	4.0027639	4.8299300
18	109736.5060400	0.0005508	1.0076032	4.0028464	2.0585008
19	109737.4375900	0.0005404	1.0075692	4.0027784	1.2131088
20	109737.1540700	0.0005490	1.0075796	4.0027991	0.5169333
21	109737.3880800	0.0005464	1.0075710	4.0027821	0.3046852
22	109737.3169000	0.0005485	1.0075736	4.0027873	0.1298101
23	109737.3756800	0.0005479	1.0075715	4.0027830	0.0765243
24	109737.3578100	0.0005484	1.0075721	4.0027843	0.0325975
25	109737.3725800	0.0005482	1.0075716	4.0027832	0.0192194
26	109737.3680900	0.0005484	1.0075718	4.0027835	0.0081858
27	109737.3718000	0.0005483	1.0075716	4.0027832	0.0048274
28	109737.3706700	0.0005484	1.0075717	4.0027833	0.0020556
29	109737.3716000	0.0005484	1.0075716	4.0027833	0.0012124
30	109737.3713200	0.0005484	1.0075716	4.0027833	0.0005158
31	109737.3715500	0.0005484	1.0075716	4.0027833	0.0003043
32	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001298
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000765
34	109737.3715200	0.0005484	1.0075716	4.0027833	0.0000327

The result from 5-point forward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000193
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000089
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000055
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000024
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000012
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000014
41	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000006
42	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000000

Estimates values are $x_1 = 109737.3715400$, $x_2 = 0.0005484$, $x_3 = 1.0075716$, $x_4 = 4.0027833$ after 42 iterations.

The result from 5-point backward method

k	x_1^k	x_2^k	x_3^k	x_4^k	error
1	219444.9590400	-0.4999977	0.5081200	3.0038800	345680.7371100
2	91458.9873030	0.5085351	1.5081177	5.0038754	84619.1142010
3	120873.2292800	-0.2505153	0.4995849	2.9868098	86549.0773190
4	100519.5038200	0.0509960	1.2586353	4.5049105	24047.7879850
5	110964.4079200	-0.1050971	0.9571240	3.9018879	17527.7944630
6	106766.9764900	0.0112289	1.1132171	4.2140742	7265.0892492
7	110014.7078000	-0.0295433	0.9968911	3.9814222	4694.0401792
8	108908.1675600	0.0030633	1.0376633	4.0629667	1990.2948413
9	109805.0669500	-0.0072804	1.0050567	3.9977533	1204.4396014
10	109522.5227700	0.0011673	1.0154004	4.0184408	512.9629428
11	109754.2138600	-0.0014365	1.0069527	4.0015453	304.2619408
12	109682.9525000	0.0007027	1.0095565	4.0067529	129.6997411
13	109741.5818600	0.0000485	1.0074173	4.0024747	76.5315347
14	109723.6704400	0.0005870	1.0080715	4.0037829	32.6264239
15	109738.4253000	0.0004227	1.0075330	4.0027061	19.2287820
16	109733.9272900	0.0005580	1.0076973	4.0030346	8.1965698
17	109737.6353500	0.0005168	1.0075620	4.0027639	4.8299358
18	109736.5060400	0.0005508	1.0076032	4.0028464	2.0585025

The result from 5-point backward method (cont.)

k	x_1^k	x_2^k	x_3^k	x_4^k	error
19	109737.4375900	0.0005404	1.0075692	4.0027784	1.2131099
20	109737.1540700	0.0005490	1.0075796	4.0027991	0.5169332
21	109737.3880800	0.0005464	1.0075710	4.0027821	0.3046847
22	109737.3169000	0.0005485	1.0075736	4.0027873	0.1298097
23	109737.3756800	0.0005479	1.0075715	4.0027830	0.0765243
24	109737.3578100	0.0005484	1.0075721	4.0027843	0.0325974
25	109737.3725800	0.0005482	1.0075716	4.0027832	0.0192198
26	109737.3680900	0.0005484	1.0075718	4.0027835	0.0081853
27	109737.3718000	0.0005483	1.0075716	4.0027832	0.0048269
28	109737.3706700	0.0005484	1.0075717	4.0027833	0.0020554
29	109737.3716000	0.0005484	1.0075716	4.0027833	0.0012125
30	109737.3713200	0.0005484	1.0075716	4.0027833	0.0005164
31	109737.3715500	0.0005484	1.0075716	4.0027833	0.0003046
32	109737.3714800	0.0005484	1.0075716	4.0027833	0.0001292
33	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000758
34	109737.3715300	0.0005484	1.0075716	4.0027833	0.0000318
35	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000188
36	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000076
37	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000046
38	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000024
39	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000014
40	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000011
41	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000006
42	109737.3715400	0.0005484	1.0075716	4.0027833	0.0000004

Estimates values are $x_1 = 109737.3715400$, $x_2 = 0.0005484$, $x_3 = 1.0075716$, $x_4 = 4.0027833$ after 42 iterations.

AUTUOR BIOGRAPHY

Author	:	Somnuk Srisawat
Date of Birth	:	August 25, 1967
Bachelor Degree	:	B.S. (Mathematics)
Institution	:	Ramkhamhaeng University Bangkok, Thailand
Year of Graduation	:	1990