NUMERICAL METHODS FOR SOLVING ITERATIVE PARTIAL DIFFERENTIAL EQUATIONS

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บทคัดย่อ

เนื้อหาของวิทยานิพนธ์ฉบับนี้กล่าวถึงการหาผลเฉลยเชิงตัวเลขของสมการเชิงอนุพันธ์ย่อยซ้ำที่อยู่ในรูปแบบ

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} = f\left(x, u(x), u^2(x), \cdots, u^m(x)\right), \quad x \in Z \qquad (1)$$

เมื่อ $Z \in [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ โดยมีเงื่อนไขเริ่มต้นดังนี้

$$u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) = g_{1,i}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \quad i = 1, 2, \cdots, n$$

$$u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n) = g_{2,i,j}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \quad i \neq j, i, j = 1, 2, 3, \cdots, n$$

$$\vdots \qquad (2)$$

$$u(x_1, 0, 0, \cdots, 0) = g_{n-1,2,3,\cdots,n}(x_1)$$

$$u(0, x_2, 0, \cdots, 0) = g_{n-1,1,3,\cdots,n}(x_2)$$

$$\vdots$$

$$u(0, 0, \cdots, 0, x_n) = g_{n-1,1,2,\cdots,n-1}(x_n)$$

$$u(0, 0, 0, \cdots, 0) = c = [c_1, c_2, \cdots, c_n]^T$$

หรือจะกล่าวได้ว่าเงื่อนไขเริ่มต้น คือ

$$g(x)=\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix} \tag{3}$$

$$\begin{aligned} &= (-1)^2\left(g_{1,1}(x_2,x_3,\cdots,x_n)+\cdots+g_{1,n}(x_1,x_2,\cdots,x_{n-1})\right) \\ &\quad +(-1)^3\left(g_{2,1,2}(x_3,x_4,\cdots x_n)+\cdots+g_{2,n-1,n}(x_1,x_2,\cdots,x_{n-2})\right) \\ &\quad \vdots \\ &\quad +(-1)^n\left(g_{n-1,2,3,\cdots n}(x_1)+\cdots+g_{n-1,1,2,\cdots n-1}(x_n)\right) \\ &\quad +(-1)^{n+1}c \end{aligned}$$

เมื่อ m เป็นจำนวนเต็มบวกที่มีค่ามากกว่า 1 และ

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u(x)=\begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix},$$

$$f(x)=\begin{bmatrix} f_1(x,u(x),u^2(x),\cdots,u^m(x)) \\ f_2(x,u(x),u^2(x),\cdots,u^m(x)) \\ \vdots \\ f_n(x,u(x),u^2(x),\cdots,u^m(x)) \end{bmatrix}$$

และ

$$u^2(x)=u(u(x))=\begin{bmatrix} u_1(u_1(x),u_2(x),\cdots,u_n(x)) \\ u_2(u_1(x),u_2(x),\cdots,u_n(x)) \\ \vdots \\ u_n(u_1(x),u_2(x),\cdots,u_n(x)) \end{bmatrix}$$

$$u^3(x)=u(u^2(x))$$

$$\vdots$$

$$u^m(x)=u(u^{m-1}(x))$$

โดยในวิทยานิพนธ์ฉบับนี้ประกอบด้วยระเบียบวิธีเชิงตัวเลข 3 วิธี คือ

1. ระเบียบวิธีที่ 1 ใช้วิธีผลต่างจำกัดถอยหลังโดยประมาณกับการประมาณค่าในช่วงด้วยวิธี ผลหารผลต่างสืบเนื่องของนิวตันแบบสองจุด
2. ระเบียบวิธีที่ 2 ใช้วิธีผลต่างจำกัดถอยหลังโดยประมาณกับการประมาณค่าในช่วงด้วยระเบียบวิธีผลหารผลต่างสืบเนื่องของนิวตันแบบ k จุด
3. ระเบียบวิธีที่ 3 ใช้วิธีผลต่างจำกัดถอยหลังโดยประมาณกับการประมาณค่าในช่วงด้วยวิธีลากรองจ์

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ABSTRACT

The objectives of this thesis is to find the numerical methods for solving the iterative partial differential equations of this form .

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} = f\left(x, u(x), u^2(x), \cdots, u^m(x)\right), \qquad x \in Z \qquad (1)$$

where $Z \in [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ with the initial conditions

$$u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) = g_{1,i}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \quad i = 1,2,\cdots,n$$

$$u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n) = g_{2,i,j}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \quad i \neq j, i, j = 1,2,3,\cdots,n$$

$$\vdots \qquad (2)$$

$$u(x_1, 0, 0, \cdots, 0) = g_{n-1,2,3,\cdots,n}(x_1)$$

$$u(0, x_2, 0, \cdots, 0) = g_{n-1,1,3,\cdots,n}(x_2)$$

$$\vdots$$

$$u(0, 0, \cdots, 0, x_n) = g_{n-1,1,2,\cdots,n-1}(x_n)$$

$$u(0,0,0,\cdots,0) = c = [c_1, c_2, \cdots, c_n]^T$$

or the compatibility initial condition

$$g(x)=\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix} \tag{3}$$

$$\begin{aligned} &= (-1)^2\big(g_{1,1}(x_2,x_3,\cdots,x_n)+\cdots+g_{1,n}(x_1,x_2,\cdots,x_{n-1})\big) \\ &\quad +(-1)^3\big(g_{2,1,2}(x_3,x_4,\cdots x_n)+\cdots+g_{2,n-1,n}(x_1,x_2,\cdots,x_{n-2})\big) \\ &\quad \vdots \\ &\quad +(-1)^n\big(g_{n-1,2,3,\cdots n}(x_1)+\cdots+g_{n-1,1,2,\cdots n-1}(x_n)\big) \\ &\quad +(-1)^{n+1}c \end{aligned}$$

where m is a positive integer greater than 1 and

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\quad u(x)=\begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix},\quad f(x)=\begin{bmatrix} f_1(x,u(x),u^2(x),\cdots,u^m(x)) \\ f_2(x,u(x),u^2(x),\cdots,u^m(x)) \\ \vdots \\ f_n(x,u(x),u^2(x),\cdots,u^m(x)) \end{bmatrix}$$

and

$$u^2(x)=u(u(x))=\begin{bmatrix} u_1(u_1(x),u_2(x),\cdots,u_n(x)) \\ u_2(u_1(x),u_2(x),\cdots,u_n(x)) \\ \vdots \\ u_n(u_1(x),u_2(x),\cdots,u_n(x)) \end{bmatrix}$$

$$u^3(x)=u(u^2(x))$$

$$\vdots$$

$$u^m(x)=u(u^{m-1}(x)).$$

By using the following three numerical methods.

1. The first method is the Backward Finite Difference Approximation and the Two Points Newton's Divided Difference Method.
2. The second method is the Backward Finite Difference Approximation and the k Points Newton's Divided Difference Method.
3. The third method is the Backward Finite Difference Approximation and the Lagrange Interpolation Method.

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Chinda Chaichuay

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CHAPTER 1

INTRODUCTION

Iterative Differential Equation is one type of the functional differential equations. The study of the functional differential equations of this type was started more than 90 years ago. There are many mathematicians around the world who gave their efforts in the study in this field. We may classify the study of the iterative differential equations into two main types.

1.Iterative Ordinary Differential Equations.

Iterative ordinary differential equation in the following form is used extensively as a reference in many studies.

$$y'(x) = G\left(x, y(x), y^2(x), \cdots, y^m(x)\right) \tag{1.1}$$

where

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y(y(x))) = y(y^2(x)) \\ &\vdots \\ y^m(x) &= y(y^{m-1}(x)). \end{aligned} \tag{1.2}$$

This type of differential equation is used as a basis in much research.

One such study is the work of G Barba [1], which was presented in 1930. The aim of her work is to find the solution of the ordinary differential equation of the form

$$f(x)f'(x) = f(f(x)). \tag{1.3}$$

In another study, presented in 1968 – 1971 , A.Pelczar [2–4] presented the proof of the uniqueness theorem of the solution of the problem in following form

$$y'(x) = f(x, y(x), y(y(x))) \tag{1.4}$$

with the initial condition

$$y(0) = c. \tag{1.5}$$

In 1992, M. Podisuk [5] studied the existence and uniqueness for the solution of the iterative ordinary differential equations of the form

$$y'(x) = f\left(x, y(x), y^2(x), \cdots, y^m(x)\right), \quad x \in [0, a] \tag{1.6}$$

with initial condition

$$y(0) = c \quad , c \in [0, a] \tag{1.7}$$

where $y^k(x)$ denotes k^{th} iteration of the (unknown) function $y(x)$ which satisfies equation (1.6) – (1.7).

In 1999, P.Pataranavik[6] discussed the existence and uniqueness for the solution of the simple iterative ordinary differential equations of the form

$$y'(x) = y^m(x), \quad x \in [0, a] \tag{1.8}$$

with the initial condition

$$y(0) = c \quad , c \in [0, a] \tag{1.9}$$

where a is positive real number,

c is positive real number in interval $[0, a]$,

m is positive integer greater than 1 and

$$y^2(x) = y(y(x))$$
$$y^3(x) = y(y(y(x))) = y(y^2(x))$$
$$y^4(x) = y(y^3(x))$$
$$\vdots$$
$$y^m(x) = y(y^{m-1}(x)) .$$

In 2001, C.Pornchai [7] studied the numerical methods for solving an iterative ordinary differential equations of form

$$y'(x) = f\left(x, y(x), y^2(x), \cdots, y^m(x)\right) , \quad x \in [a,b] \tag{1.10}$$

with the initial condition

$$y(a) = c \tag{1.11}$$

where m is positive integer greater than 1 and

$$y^2(x) = y(y(x))$$
$$y^3(x) = y(y(y(x))) = y(y^2(x))$$
$$y^4(x) = y(y^3(x))$$
$$\vdots$$
$$y^m(x) = y(y^{m-1}(x)) .$$

2. Iterative Partial differential Equations.

In 1992, M.Podisuk studied the existence and uniqueness for the solution of iterative partial differential equation of the form

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} = f\left(x, u(x), u^2(x), \cdots, u^m(x)\right) , \qquad x \in Z \tag{1.12}$$

where $Z \in [0,a_1]\times[0,a_2]\times\cdots\times[0,a_n]$ with the initial conditions

$$u(x_1,x_2,\cdots,x_{i-1},0,x_{i+1},\cdots,x_n) = g_{1,i}(x_1,x_2,\cdots,x_{i-1},x_{i+1},\cdots,x_n)\ ,\quad i=1,2,\cdots,n$$

$$u(x_1,x_2,\cdots,x_{i-1},0,x_{i+1},\cdots,x_{j-1},0,x_{j+1},\cdots,x_n)$$
$$= g_{2,i,j}(x_1,x_2,\cdots,x_{i-1},x_{i+1},\cdots,x_{j-1},x_{j+1},\cdots,x_n)\ ,\quad i\neq j\ ,i,j=1,2,3,\cdots,n$$

$$\vdots \tag{1.13}$$

$$u(x_1,0,0,\cdots,0) = g_{n-1,2,3,\cdots,n}(x_1)$$
$$u(0,x_2,0,\cdots,0) = g_{n-1,1,3,\cdots,n}(x_2)$$
$$\vdots$$
$$u(0,0,\cdots,0,x_n) = g_{n-1,1,2,\cdots,n-1}(x_n)$$
$$u(0,0,0,\cdots,0) = c = [c_1,c_2,\cdots,c_n]^T$$

or the compatibility initial condition

$$g(x)=\begin{bmatrix} g_1(x)\\ g_2(x)\\ \vdots \\ g_n(x)\end{bmatrix} \tag{1.14}$$

$$\begin{aligned} =\ & (-1)^2\left(g_{1,1}(x_2,x_3,\cdots,x_n)+\cdots+g_{1,n}(x_1,x_2,\cdots,x_{n-1})\right)\\ &+(-1)^3\left(g_{2,1,2}(x_3,x_4,\cdots x_n)+\cdots+g_{2,n-1,n}(x_1,x_2,\cdots,x_{n-2})\right)\\ &\vdots\\ &+(-1)^n\left(g_{n-1,2,3,\cdots n}(x_1)+\cdots+g_{n-1,1,2,\cdots n-1}(x_n)\right)\\ &+(-1)^{n+1}c \end{aligned}$$

where m is a positive integer greater than 1 and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix},$$

$$f(x) = \begin{bmatrix} f_1(x, u(x), u^2(x), \cdots, u^m(x)) \\ f_2(x, u(x), u^2(x), \cdots, u^m(x)) \\ \vdots \\ f_n(x, u(x), u^2(x), \cdots, u^m(x)) \end{bmatrix}$$

and

$$u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x), u_2(x), \cdots, u_n(x)) \\ u_2(u_1(x), u_2(x), \cdots, u_n(x)) \\ \vdots \\ u_n(u_1(x), u_2(x), \cdots, u_n(x)) \end{bmatrix}.$$

$$u^3(x) = u(u^2(x))$$

$$\vdots$$

$$u^m(x) = u(u^{m-1}(x))$$

and

$$Z = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$$

$$u_i : Z \to R, \qquad f_i : Z \times R^{mn} \to R, \quad i = 1, 2, \cdots, n$$

$$u : Z \to R^n, \qquad f : Z \times R^{mn} \to R^n.$$

When f and g are continuous then the problem (1.12)-(1.13) is equivalent to the problem of the continuous solution of the integral equation

$$u(x) = g(x) + \int_0^x f(t, u(t), u^2(t), u^3(t), \cdots u^m(t))dt. \tag{1.15}$$

In fact, this equation is very difficult to find the exact solutions. Many mathematicians according to research in this field. One of many methods that they use, are numerical methods.

In this thesis, we shall study the numerical methods for solving an iterative partial differential equation of the form

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} = f\left(x, u(x), u^2(x), \cdots, u^m(x)\right), \qquad x \in Z \tag{1.16}$$

where $Z \in [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ with the initial conditions

$$\begin{aligned}
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) &= g_{1,i}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \\
&\qquad i = 1, 2, \cdots, n \\
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n) & \\
&= g_{2,i,j}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \\
&\qquad i \neq j, i, j = 1, 2, 3, \cdots, n \\
&\vdots \\
u(x_1, 0, 0, \cdots, 0) &= g_{n-1,2,3,\cdots,n}(x_1) \\
u(0, x_2, 0, \cdots, 0) &= g_{n-1,1,3,\cdots,n}(x_2) \\
&\vdots \\
u(0, 0, \cdots, 0, x_n) &= g_{n-1,1,2,\cdots,n-1}(x_n) \\
u(0, 0, 0, \cdots, 0) &= c = [c_1, c_2, \cdots, c_n]^T
\end{aligned} \tag{1.17}$$

where $u^k(x)$ denotes the k^{th} iteration of the (unknown) function $u(x)$ and satisfies equation (1.16) – (1.17) and find the best numerical method for solving it when we can not find the exact solution.

In this thesis, we study the numerical methods for solving an iterative partial differential equation, especially, in two dimensions.

CHAPTER 2

DEFINITIONS AND THEOREMS

In this chapter we will discuss the uniqueness and the existence of the solutions of the iterative partial differential equations of the form

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} = f\left(x, u(x), u^2(x), \cdots, u^m(x)\right), \qquad x \in Z \tag{2.1}$$

where $Z \in [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ with the initial conditions

$$\begin{aligned}
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) &= g_{1,i}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \\
&\qquad\qquad i = 1,2,\cdots,n \\
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n)& \\
&= g_{2,i,j}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \\
&\qquad\qquad i \neq j, i, j = 1,2,3,\cdots,n \\
&\vdots \\
u(x_1, 0, 0, \cdots, 0) &= g_{n-1,2,3,\cdots,n}(x_1) \\
u(0, x_2, 0, \cdots, 0) &= g_{n-1,1,3,\cdots,n}(x_2) \\
&\vdots \\
u(0, 0, \cdots, 0, x_n) &= g_{n-1,1,2,\cdots,n-1}(x_n) \\
u(0, 0, 0, \cdots, 0) &= c = [c_1, c_2, \cdots, c_n]^T
\end{aligned} \tag{2.2}$$

or the compatibility initial condition

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix} \tag{2.3}$$

$$= (-1)^2\left(g_{1,1}(x_2, x_3, \cdots, x_n) + \cdots + g_{1,n}(x_1, x_2, \cdots, x_{n-1})\right)$$
$$+(-1)^3\left(g_{2,1,2}(x_3, x_4, \cdots x_n) + \cdots + g_{2,n-1,n}(x_1, x_2, \cdots, x_{n-2})\right)$$
$$\vdots$$
$$+ (-1)^n\left(g_{n-1,2,3,\cdots n}(x_1) + \cdots + g_{n-1,1,2,\cdots n-1}(x_n)\right)$$
$$+ (-1)^{n+1} c$$

where m is a positive integer greater than 1 and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix},$$

$$f(x) = \begin{bmatrix} f_1(x, u(x), u^2(x), \cdots, u^m(x)) \\ f_2(x, u(x), u^2(x), \cdots, u^m(x)) \\ \vdots \\ f_n(x, u(x), u^2(x), \cdots, u^m(x)) \end{bmatrix}$$

and

$$u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x), u_2(x), \cdots, u_n(x)) \\ u_2(u_1(x), u_2(x), \cdots, u_n(x)) \\ \vdots \\ u_n(u_1(x), u_2(x), \cdots, u_n(x)) \end{bmatrix}$$

$$u^3(x) = u(u^2(x))$$
$$\vdots$$
$$u^m(x) = u(u^{m-1}(x))$$

and

$$Z = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$$
$$u_i : Z \to R, \qquad f_i : Z \times R^{mn} \to R, \quad i = 1,2,\cdots,n$$
$$u : Z \to R^n, \qquad f : Z \times R^{mn} \to R^n.$$

When f and g are continuous then the problem (2.1)-(2.2) is equivalent to the problem of the continuous solution of the integral equation

$$u(x) = g(x) + \int_0^x f(t, u(t), u^2(t), u^3(t), \cdots u^m(t))dt . \tag{2.4}$$

2.1 Uniqueness.

Let $f(x, u(x), u^2(x), \cdots, u^m(x))$ be defined and continuous in the set $Z \times R^{mn}$, say D, and g be defined and continuous in Z and

let

$$\| f(x, z_1, z_2, \cdots, z_m) \| \le K \tag{2.5}$$

$$\| f(x, z_1, z_2, \cdots, z_m) - f(x, \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_m) \| \le M_1 \| z_1 - \bar{z}_2 \| + M_2 \| z_2 - \bar{z}_2 \| + \cdots + M_m \| z_m - \bar{z}_m \| \tag{2.6}$$

$$\| g(x) \| \le L \le K \tag{2.7}$$

for all $(x, z_1, z_2, \cdots, z_m)$, $(x, \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_m)$ in D and for x in Z and K, L, M_1, $M_2, \cdots, M_m$ are in R^+. The norm $\| \cdot \|$ is the Euclidean norm.

We are looking for the solution $u(x)$ of the problem (2.1)-(2.2) or (2.4) where

$u(x)$ belong to Z for all x in Z (2.8)

$\| u(x) - u(y) \| \le N \| x - y \|$ for all x and y in Z and N in R^+. (2.9)

Let $S_1 = M_1 + M_2 N + M_3 N^2 + \cdots M_m N^{m-1}$

$S_2 = M_2 + M_3 N + M_4 N^2 + \cdots M_m N^{m-2}$

$\vdots$

$S_{m-1} = M_{m-1} + M_m N$

$S_m = M_m$

$$d = (S_1)^{1/m} \quad , \ d > 0$$

$$a = a_1 a_2 a_3 \cdots a_n$$

$$b = S_2 + S_3 + \cdots + S_m$$

$$h = 2L + 2K$$

$$A = h$$

$$B = ab$$

$$C = a_1 + a_2 + \cdots + a_n$$

thus we have the following theorem.

Theorem 2.1 If $Be^{dC} < 1$ and f and g satisfy the above conditions then there exist at most one solution to the problem (2.1)-(2.2) .

Proof. Suppose that $u(x)$ and $v(x)$ are two solutions of the problem (2.1)-(2.2) and let

$$p(x) = \| u(x) - v(x) \| .$$

Thus

$$p(x) = \left\| \int_0^x f(t, u(t), u^2(t), \cdots, u^m(t)) - f(t, v(t), v^2(t), \cdots, v^m(t)) dt \right\|$$

$$\leq \int_0^x \left\| f\left(t, u(t), u^2(t), \cdots u^m(t)\right) - f\left(t, v(t), v^2(t), \cdots, v^m(t)\right) \right\| dt$$

$$\leq M_1 \int_0^x \| u(t) - v(t) \| dt + M_2 \int_0^x \| u^2(t) - v^2(t) \| dt + \cdots$$

$$+ M_m \int_0^x \| u^m(t) - v^m(t) \| dt$$

but

$$\| u^2(t) - v^2(t) \| = \| u(u(t)) - v(v(t)) \|$$

$$= \| u(u(t)) - u(v(t)) + u(v(t)) - v(v(t)) \|$$

(From triangular inequality)

$$\leq \| u(u(t)) - u(v(t)) \| + \| u(v(t)) - v(v(t)) \|$$

$$\leq N \| u(t) - v(t) \| + p(v(t))$$

$$= Np(t) + p(v(t))$$

and

$$\| u^3(t) - v^3(t) \| = \| u^2(u(t)) - u^2(v(t)) + u^2(v(t))$$

$$- u(v^2(t)) + u(v^2(t)) - v(v^2(t)) \|$$

$$\leq \| u^2(u(t)) - u^2(v(t)) \| + \| u^2(v(t)) - u(v^2(t)) \|$$

$$+ \| u(v^2(t)) - v(v^2(t)) \|$$

$$= \| u^2(u(t)) - u^2(v(t)) \| + \| u(u(v(t))) - u(v(v(t))) \|$$

$$+ \| u(v^2(t)) - v(v^2(t)) \|$$

$$\leq N^2 \| u(t) - v(t) \| + N \| u(v(t)) - v(v(t)) \|$$

$$+ \| u(v^2(t)) - v(v^2(t)) \|$$

$$= N^2 p(t) + Np(v(t)) + p(v^2(t)).$$

In the same way, we get

$$\| u^4(t) - v^4(t) \| \leq N^3 p(t) + N^2 p(v(t)) + Np(v^2(t)) + p(v^3(t))$$

$$\| u^5(t) - v^5(t) \| \leq N^4 p(t) + N^3 p(v(t)) + N^2 p(v^2(t)) + Np(v^3(t)) + p(v^4(t))$$

$$\vdots$$

$$\| u^{m-1}(t) - v^{m-1}(t) \| \leq N^{m-2} p(t) + N^{m-3} p(v(t)) + N^{m-4} p(v^2(t)) + \cdots + Np(v^{m-3}(t)) + p(v^{m-2}(t))$$

$$\| u^m(t) - v^m(t) \| \leq N^{m-1} p(t) + N^{m-2} p(v(t)) + N^{m-3} p(v^2(t)) + N^{m-4} p(v^3(t)) + \cdots + Np(v^{m-2}(t)) + p(v^{m-1}(t)).$$

Thus we have

$$p(x) \leq \int_0^x [M_1 p(t) + M_2 (Np(t) + p(v(t))) + M_3 (N^2 p(t) + Np(v(t)) + p(v^2(t))) + \cdots + M_{m-1} (N^{m-2} p(t) + N^{m-3} p(v(t)) + N^{m-4} p(v^2(t)) + \cdots + p(v^{m-2}(t))) + M_m (N^{m-1} p(t) + N^{m-2} p(v(t)) + N^{m-3} p(v^2(t)) + N^{m-4} p(v^3(t)) + \cdots + Np(v^{m-2}(t)) + p(v^{m-1}(t)))] \, dt$$

$$= \int_0^x \left[M_1 + M_2 N + M_3 N^2 + \cdots + M_{m-1} N^{m-2} + M_m N^{m-1} \right] p(t)dt$$

$$+ \int_0^x \left[M_2 + M_3 N + M_4 N^2 + \cdots + M_{m-1} N^{m-3} + M_m N^{m-2} \right] p(v(t))dt$$

$$+ \int_0^x \left[M_3 + M_4 N + M_5 N^2 + \cdots + M_{m-1} N^{m-4} + M_m N^{m-3} \right] p(v^2(t))dt$$

$$\vdots$$

$$+ \int_0^x \left[M_{m-1} + M_m N \right] p(v^{m-2}(t))dt$$

$$+ \int_0^x M_m p(v^{m-1}(t))dt .$$

Thus we have

$$p(x) \le S_1 \int_0^x p(t)dt + S_2 \int_0^x p(v(t))dt + S_3 \int_0^x p(v^2(t)) + \cdots + S_m \int_0^x p(v^{m-1}(t))dt . \qquad (2.10)$$

But for $j = 0,1,2,\cdots, m-1$, we have

$$p(v^j(t)) = \| u(v^j(t)) - v(v^j(t)) \|$$

$$\le 2L + 2K$$

thus

$$p(x) \le S_1 \int_0^x p(t)dt + (2L+2K)S_2 \int_0^x dt + (2L+2K)S_3 \int_0^x dt + \cdots + (2L+2K)S_m \int_0^x dt$$

$$= S_1 \int_0^x p(t)dt + (a_1 a_2 a_3 \cdots a_n)(2L+2K)S_2 + (a_1 a_2 a_3 \cdots a_n)(2L+2K)S_3 + \cdots$$

$$+ (a_1 a_2 a_3 \cdots a_n)(2L+2K)S_m$$

$$= S_1 \int_0^x p(t)dt + a_1 a_2 a_3 \cdots a_n (2L+2K)(S_2 + S_3 + \cdots S_m)$$

then

$$p(x) \le d^m \int_0^x p(t)dt + abh$$

so

$$p(x) \le abh\left(e^{dx_1} e^{dx_2} e^{dx_3} \cdots e^{dx_n}\right)$$

or $\qquad p(x) \le abhe^{d(x_1 + x_2 + \cdots + x_n)} \qquad (2.11)$

and

$$p(x) \le abhe^{d(a_1+a_2+\cdots+a_n)} \tag{2.12}$$

and

$$p(x) \le ABe^{dC}. \tag{2.13}$$

From (2.10)-(2.13), we have

$$p(x) \le A[Be^{dC}]^2 \tag{2.14}$$

and by induction we have

$$p(x) \le A[Be^{dC}]^k \quad , \qquad k = 1,2,3,\cdots$$

thus by the hypothesis that $Be^{dC} < 1$ then $p(x)$ tends to zero as k tends to infinity. Therefore $u(x) = v(x)$. This ends the proof of the theorem 2.1.

2.2 Existence.

Let us suppose that

$$\| g(x) \| + aK \le a \quad \text{for all } x \text{ in } Z \quad \text{and} \tag{2.15}$$

$$a[M_1 + (N+1)M_2 + (N^2+N+1)M_3 + \cdots + (N^{m-1} + N^{m-2} + \cdots + N + 1)M_m] < 1 \tag{2.16}$$

and let consider the following sequences

$$u_{1,k+1}(x) = g(x) + \int_0^x f(t, u_{1,k}(t), u_{1,k}^2(t), \cdots u_{1,k}^m(t))dt \tag{2.17.1}$$

$$u_{2,k+1}(x) = g(x) + \int_0^x f(t, u_{2,k}(t), u_{2,k}^2(t), \cdots u_{2,k}^{m-1}(t), u_{2,k}^{m-1}(u_{2,k+1}(t))dt \tag{2.17.2}$$

$$u_{3,k+1}(x) = g(x) + \int_0^x f(t, u_{3,k}(t), u_{3,k}^2(t), \cdots u_{3,k}^{m-2}(t), u_{3,k}^{m-2}(u_{3,k+1}(t) u_{3,k}^{m-1}(u_{3,k+1}(t)))dt \tag{2.17.3}$$

⋮

$$u_{m+1,k+1}(x) = g(x) + \int_0^x f(t, u_{m+1,k+1}(t), u_{m+1,k}(u_{m+1,k+1}(t)), u_{m+1,k}^2(u_{m+1,k+1}(t)) , u_{m+1,k}^{m-1}(u_{m+1,k+1}(t)))dt \tag{2.17.m+1}$$

where $u_{1,0}(x), u_{2,0}(x), \cdots, u_{m+1,0}(x)$ are fixed functions of the class C^1 map Z to Z such that

$$\left\| \frac{\partial^n u_{i,0}(x)}{\partial x_1, \partial x_2, \cdots, \partial x_n} \right\| \le k \qquad , \ i = 1,2,3,\cdots, m+1$$

Hence we have the following theorem.

Theorem 2.2 Let the condition of theorem 2.1 holds and the conditions (2.15) and (2.16) be satisfied then the sequences (2.17.1) ,$\cdots$, (2.17.$m+1$) converge uniformly to the (unique) solution $u = u(x)$ of the problem (2.1)-(2.2).

Proof. We put

$$U_{1,k} = \max_{x \in Z} \| u_{1,k}(x) - u_{1,k-1}(x) \|$$
$$U_{2,k} = \max_{x \in Z} \| u_{2,k}(x) - u_{2,k-1}(x) \|$$
$$\vdots$$
$$U_{m+1,k} = \max_{x \in Z} \| u_{m+1,k}(x) - u_{m+1,k-1}(x) \|$$

where $k = 0,1,2,\cdots$.

It can be shown by induction k on that

$$U_{1,k} \le (W_1)^k U_{1,0}$$
$$U_{2,k} \le (W_2)^k U_{2,0}$$
$$\vdots$$
$$U_{m+1,k} \le (W_{m+1})^k U_{m+1,0}$$

where

$$W_1 = a\left[M_1 + (N+1)M_2 + (N^2 + N + 1)M_3 + \cdots + (N^{m-1} + N^{m-2} + \cdots + N + 1)M_m \right]$$
$$W_2 = \frac{A_2}{B_2},$$
$$W_3 = \frac{A_3}{B_3},$$
$$\vdots$$

$$W_{m+1} = \frac{A_{m+1}}{B_{m+1}},$$

$$A_2 = a[\, \mathrm{M}_1 + (N+1)M_2 + \cdots + (N^{m-3} + \cdots + N + 1)M_{m-2} + (N^{m-2} + \cdots + N + 1)(M_{m-1} + M_m)$$

$$A_3 = a[\, \mathrm{M}_1 + (N+1)M_2 + \cdots + (N^{m-4} + \cdots + N + 1)M_3 + (N^{m-3} + \cdots + N + 1)(M_{m-2} + M_{m-1}) + (N^{m-2} + \cdots + N + 1)M_m]$$

$$\vdots$$

$$A_{m+1} = a[\, (\mathrm{M}_1 + \mathrm{M}_2) + (\mathrm{N}+1)\mathrm{M}_3 + (N^2 + N + 1)M_4 + \cdots + (N^{m-2} + \cdots + N + 1)M_m$$

$$B_2 = 1 - aN^{m-1}M_m$$

$$B_3 = 1 - aN^{m-2}M_{m-1} - aN^{m-1}M_m$$

$$\vdots$$

$$B_{m+1} = 1 - aNM_2 - aN^2M_3 - \cdots - aN^{m-1}M_m .$$

Since $W_1 < 1$ then $W_i < 1$ for $i = 1,2,3,\cdots,m+1$.
Thus for $i = 1,2,3,\cdots,m+1$, W_i tends to zero as k tends to infinity then $U_{i,k}$ tends to zero as k tends to infinity. This mean that if $\{ u_{\mathrm{i},\ell_j}(\cdot) \}$ is subsequence of $\{ u_{\mathrm{i},k}(\cdot) \}$ tending uniformly to some $\overline{u}_i(\cdot)$ then $\overline{u}_i(\cdot)$ is a solution of the problem (2.1)-(2.2). Since the family $\{ u_{\mathrm{i},k} \}$ is an Arzela – Ascoli family , thus for every subsequence $\{ u_{\mathrm{i},\ell_j} \}$ of $\{ u_{\mathrm{i},k} \}$ there exist a subsequence $\{ u_{\mathrm{i},k_j} \}$ uniformly convergent and the limit needs to be a solution of the problem (2.1)-(2.2) as it was mentioned above. Thus the sequence $\{ u_{\mathrm{i},k} \}$ tends uniformly to the (unique) solution of the problem (2.1)-(2.2). This ends the proof of the theorem 2.2.

CHAPTER 3

NUMERICAL METHODS FOR SOLVING ITERATIVE PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we shall look for a numerical method for solving the iterative partial differential equation of the form

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} = f\big(x, u(x), u^2(x), \cdots, u^m(x)\big), \qquad x \in Z \tag{3.1}$$

where $Z \in [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ with the initial conditions

$$\begin{aligned}
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) &= g_{1,i}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \\
&\qquad\qquad i = 1,2,\cdots,n \\
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n) & \\
&= g_{2,i,j}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \\
&\qquad\qquad i \neq j, i, j = 1,2,3,\cdots,n \\
&\ \ \vdots \\
u(x_1, 0, 0, \cdots, 0) &= g_{n-1,2,3,\cdots,n}(x_1) \\
u(0, x_2, 0, \cdots, 0) &= g_{n-1,1,3,\cdots,n}(x_2) \\
&\ \ \vdots \\
u(0, 0, \cdots, 0, x_n) &= g_{n-1,1,2,\cdots,n-1}(x_n) \\
u(0, 0, 0, \cdots, 0) &= c = [c_1, c_2, \cdots, c_n]^T
\end{aligned} \tag{3.2}$$

by using the integral equation

$$u_{k+1}(x) = g(x) + \int_0^x f(t, u_k(t), u_k^2(t), u_k^3(t), \cdots u_k^m(t))dt. \tag{3.3}$$

The formula (3.3) already proved to be converged by the theorem 2.2.1.

In general, The existing numerical methods for solving the partial differential equations can not solve the iterative partial differential equations. The reason is that to find the value of $u(x+h)$, we need to use the values of $u^2(x), u^3(x), \ldots, u^m(x)$ which may involve in using the unknown values of $u(r)$ where r is not equal to x_i as show in Figure 1 and Figure 2. We must use their approximating values instead. The author has devised the three methods to find the approximating values. Then we will combine them with the Backward Finite Difference Approximation to solve our problems. The three methods use the same idea which include the iteration technique, the combination of Backward Finite Difference Approximation for solving partial differential equation and interpolation. For each iteration, we shall solve the value of $u(x)$ for the hold interval $[0, a_i]$. After that we partition the closed interval $[0, a_i]$ in to k_i partitions at $x_{i0}, = 0, x_{i1} = h_i, \ldots, x_{ij} = jh, \ldots, x_{ik} = k_i h_i = a_i$ where $h_i = 1/k_i$ for $i = 1,2,\ldots,n$.

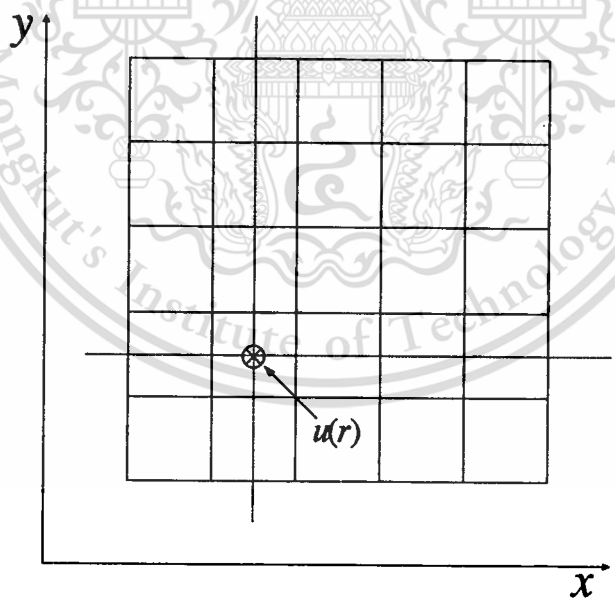

Figure 3.1 Show the unknown value of $u(r)$ in two dimensions.

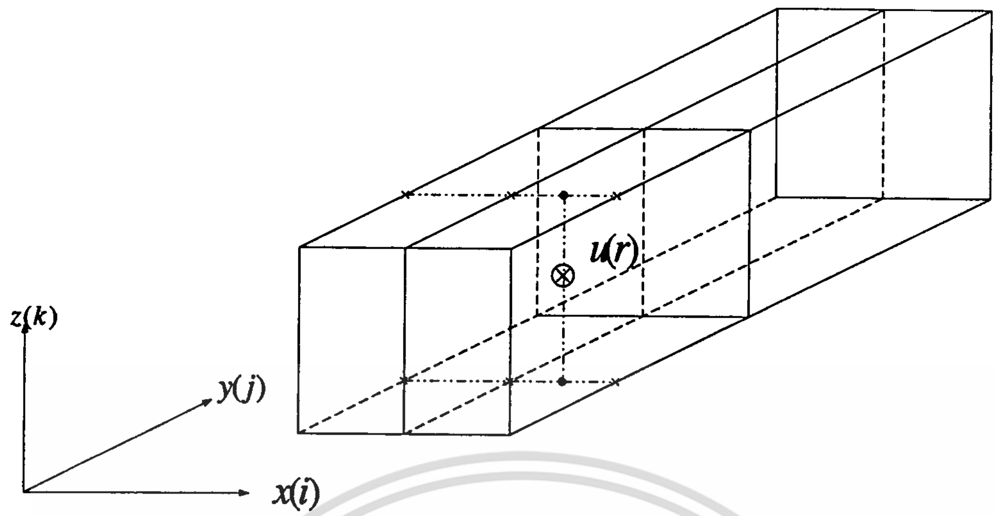

Figure 3.2 Show the unknown value of $u(r)$ in three dimensions.

3.1 First Method.

In this method, we solve the problem (3.1)-(3.2) by using the Backward Finite Difference Approximation and the Two Points Newton's Divided Difference Method, especially, in two dimensions.

We consider the Backward Finite Difference Approximation to find the value of $u(x,y)$.

Since $$\frac{\partial u(x,y)}{\partial x} = \frac{u_1(x,y) - u_1(x-h,y)}{h} \tag{3.4}$$

and then $$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\dfrac{\partial u_1(x,y)}{\partial y} - \dfrac{\partial u_1(x-h,y)}{\partial y}}{h}$$

$$= \frac{1}{h}\left(\frac{u_1(x,y) - u_1(x,y-h)}{h} - \frac{u_1(x-h,y) - u_1(x-h,y-h)}{h}\right)$$

$$= \frac{u_1(x,y) - u_1(x,y-h) - u_1(x-h,y) + u_1(x-h,y-h)}{h^2}. \tag{3.5}$$

Form equation (3.1) we write $\frac{\partial^n u(x)}{\partial x_1, \partial x_2, \ldots, \partial x_n}$ in form of two dimensions as follow

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} \frac{\partial^2 u_1(x,y)}{\partial x \partial y} \\ \frac{\partial^2 u_2(x,y)}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} f_1(x,y,u_1(x,y),u_1^2(x,y),\ldots,u_1^m(x,y)) \\ f_2(x,y,u_1(x,y),u_2^2(x,y),\ldots,u_2^m(x,y)) \end{bmatrix} \tag{3.6}$$

for $m = 1,2,3,\ldots.$

From equation (3.5) and (3.6), we get

$$f_1(x,y,u_1(x,y),u_1^2(x,y),\ldots,u_1^m(x,y)) = \frac{u_1(x,y) - u_1(x,y-h) - u_1(x-h,y) + u_1(x-h,y-h)}{h^2}. \tag{3.7}$$

Therefore

$$u_1(x,y) = h^2 f_1(x,y,u_1(x,y),u_1^2(x,y),\ldots,u_1^m(x,y)) + u_1(x,y-h) + u_1(x-h,y) - u_1(x-h,y-h). \tag{3.8}$$

In the same way, we get

$$u_2(x,y) = h^2 f_2(x,y,u_2(x,y),u_2^2(x,y),\ldots,u_2^m(x,y)) + u_2(x,y-h) + u_2(x-h,y) - u_2(x-h,y-h). \tag{3.9}$$

From (3.8) and (3.9), we get the solution of $u(x,y)$. We have to know the value of function $f(x,y,u(x,y),u^2(x,y),\ldots,u^m(x,y))$.

In order to find the value of f ,we have to know the values of $u(x,y), u^2(x,y),\ldots,$ $u^m(x,y)$ which we use the Two Points Newton's Divided Difference Method to find them.

First, from initial function $u_0(x,y) = g(x,y)$ that satisfies the conditions of theorem 2.2.2 and use the values of h, $f(x,y,u(x,y),u^2(x,y),\ldots,u^m(x,y))$, $u(x,y-h)$, $u(x-h,y)$ and $u(x-h,y-h)$ to find $u^{(1)}(x,y)$ at each point as follow .

Since

$$u^2(x,y) = \begin{bmatrix} u_1(u_1(x,y), u_2(x,y)) \\ u_2(u_1(x,y), u_2(x,y)) \end{bmatrix} \quad (3.10)$$

and letting $u_1(x,y) = x^*,\ u_2(x,y) = y^*$. (3.11)

So $$u^2(x,y) = u(u(x^*,y^*)) = \begin{bmatrix} u_1(x^*,y^*) \\ u_2(x^*,y^*) \end{bmatrix} = \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix} \quad (3.12)$$

where

$$u(x^*,y_j) = u(x_i,y_j) + \frac{(x^*-x_i)}{(x_{i+1}-x_i)}\left[u(x_{i+1},y_j) - u(x_i,y_j)\right]$$

$$u(x^*,y_{j+1}) = u(x_i,y_{j+1}) + \frac{(x^*-x_i)}{(x_{i+1}-x_i)}\left[u(x_{i+1},y_{j+1}) - u(x_i,y_{j+1})\right] \quad (3.13)$$

$$u(x^*,y^*) = u(x^*,y_j) + \frac{(y^*-y_i)}{(y_{i+1}-y_i)}\left[u(x^*,y_{j+1}) - u(x^*,y_j)\right].$$

In the same way, we use the values of $u^2(x,y)$ to find $u^3(x,y)\ldots$ and use $u^{m-1}(x,y)$ to find $u^m(x,y)$. Then we use $u^{(1)}(x,y)$ to find $u^{(2)}(x,y)$ and continue this process until

$$\sum_{\ell=0}^{\ell=k} \| u_1^{(\ell+1)}(x,y) - u_1^{(\ell)}(x,y) \| < \varepsilon \text{ and } \sum_{\ell=0}^{\ell=k} \| u_2^{(\ell+1)}(x,y) - u_2^{(\ell)}(x,y) \| < \varepsilon$$

for a sufficiently small ε.

3.2 Second Method.

This method uses the Backward Finite Difference Approximation and the k Points Newton's Divided Difference Method to solve the problem (3.1) –(3.2).

First, we use the Backward Finite Difference Approximation to find the values of $u(x,y)$. The same way as the first method but instead of two points, we will used k points.

Second, we use the k Points Newton's Divided Difference Method to find the value of

$$u^2(x,y) \quad = \quad \begin{bmatrix} u_1(u_1(x,y),u_2(x,y)) \\ u_2(u_1(x,y),u_2(x,y)) \end{bmatrix}.$$

Let $u_1(x,y)=x^*$, $u_2(x,y)=y^*$.

So $u^2(x,y) \quad = u(u(x^*,y^*)) = \begin{bmatrix} u_1(x^*,y^*) \\ u_2(x^*,y^*) \end{bmatrix} = \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix}.$

From the Newton's Interpolation Polynomials of degree k ,$P_n^j(x)$, for $j=0,1,2,\ldots,k$, that

$$P_k^j(x^*) = a_0 + a_1(x^* - x_0) + \cdots + a_n(x^* - x_0)(x^* - x_1)\cdots(x^* - x_{k-1}) \quad (3.14)$$

where

$$a_0 = u(x_0^j, y_0^j)$$

$$a_1 = u[x_1^j, x_0^j] = \frac{u(x_1^j, y_1^j) - u(x_0^j, y_0^j)}{x_1 - x_0}$$

$$a_2 = u[x_2^j, x_1^j, x_0^j] = \frac{u[x_2^j, x_1^j] - u[x_1^j, x_0^j]}{x_2 - x_0}$$

$$= \frac{\dfrac{u(x_2^j, y_2^j) - u(x_1^j, y_1^j)}{x_2 - x_1} - \dfrac{u(x_1^j, y_1^j) - u(x_0^j, y_0^j)}{x_1 - x_0}}{x_2 - x_0} \quad (3.15)$$

$\vdots$

$$a_n = u[x_k^j, x_{k-1}^j, \ldots, x_1^j, x_0^j] = \frac{u[x_k^j, x_{k-1}^j, \ldots, x_1^j] - u[x_{k-1}^j, x_{k-2}^j, \ldots, x_0^j]}{x_k - x_0}$$

and then

$$P_k(y^*) = b_0 + b_1(y^* - y_0) + \cdots + b_n(y^* - y_0)(y^* - y_1)\cdots(y^* - y_{k-1}) \quad (3.16)$$

where

$$b_0 = u(x^*, y_0)$$

$$b_1 = u[y_1, y_0] = \frac{u(x_1, y_0) - u(x_0, y_0)}{y_1 - y_0}$$

$$b_2 = u[y_2, y_1, y_0] = \frac{u(y_2, y_1] - u[y_1, y_0]}{y_2 - y_0} \tag{3.17}$$

$$\vdots$$

$$b_k = u[y_k, y_{k-1}, \ldots, y_1, y_0] = \frac{u[y_k, y_{k-1}, \ldots, y_1] - u[y_{k-1}, y_{k-2}, \ldots, y_0]}{y_k - y_0}.$$

In the same way, we use the values of $u^2(x, y)$ to find $u^3(x, y)\ldots$ and use $u^{m-1}(x, y)$ to find $u^m(x, y)$. Then we use $u^{(1)}(x, y)$ to find $u^{(2)}(x, y)$ and continue this process until

$$\sum_{\ell=0}^{\ell=k} \| u_1^{(\ell+1)}(x, y) - u_1^{(\ell)}(x, y) \| < \varepsilon \text{ and } \sum_{\ell=0}^{\ell=k} \| u_2^{(\ell+1)}(x, y) - u_2^{(\ell)}(x, y) \| < \varepsilon$$

for a sufficiently small ε.

3.3 Third Method.

This method uses the Backward Finite Difference Approximation and Lagrange Interpolation Method to solve the problem (3.1) –(3.2).

First, we use the Backward Finite Difference Approximation to find the value of $u(x, y)$. The same way as the first method. Then we will use the Lagrange Interpolation Method used to find the values of $u(x, y), u^2(x, y), \ldots, u^m(x, y)$.

Second, we use the Lagrange Interpolation Method to find

the value of $u^2(x, y) \quad = \quad \begin{bmatrix} u_1(u_1(x, y), u_2(x, y)) \\ u_2(u_1(x, y), u_2(x, y)) \end{bmatrix}.$

Let $u_1(x,y) = x^*$, $u_2(x,y) = y^*$.

So $u^2(x,y) \quad = u(u(x^*,y^*)) = \begin{bmatrix} u_1(x^*,y^*) \\ u_2(x^*,y^*) \end{bmatrix} = \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix}$.

From the Lagrange's Interpolation Polynomials of degree k, $P_k^j(x^*)$, for i, j and $q = 0,1,2,\ldots,k$ that

$$P_k^j(x^*) = f_0^j L_0^j(x^*) + f_1^j L_1^j(x^*) + \cdots + f_k^j L_k^j(x^*) = \sum_{q=0}^{k} f_q^j L_q^j(x^*) \qquad (3.18)$$

where

$$L_q^j(x^*) = \frac{(x^* - x_0^j)(x^* - x_1^j)\cdots(x^* - x_{q-1}^j)(x^* - x_{q+1}^j)\cdots(x^* - x_q^j)}{(x_q^j - x_0^j)(x_q^j - x_1^j)\cdots(x_q^j - x_{q-1}^j)(x_q^j - x_{q+1}^j)\cdots(x_q^j - x_q^j)} \qquad (3.19)$$

and

$$x_q^j = u(x_q^j, y_q^j) \qquad (3.20)$$

and then $$P_k(y^*) = f_0 L_0(y^*) + f_1 L_1(y^*) + \cdots + f_k L_k(y^*) = \sum_{q=0}^{k} f_q L_q(x^*) \qquad (3.21)$$

where

$$L_q(y^*) = \frac{(y^* - y_0)(y^* - y_1)\cdots(y^* - y_{q-1})(y^* - y_{q+1})\cdots(y^* - y_q)}{(y_q - y_0)(y_q - y_1)\cdots(y_q - y_{q-1})(y_q - y_{q+1})\cdots(y_q - y_q)} \qquad (3.22)$$

and

$$y_q = u(x^*, y_q). \qquad (3.23)$$

As before, we use the values of $u^2(x,y)$ to find $u^3(x,y)\ldots$ and use $u^{m-1}(x,y)$ to find $u^m(x,y)$. Then we use $u^{(1)}(x,y)$ to find $u^{(2)}(x,y)$ and continue this process until

$$\sum_{\ell=0}^{\ell=k} \| u_1^{(\ell+1)}(x,y) - u_1^{(\ell)}(x,y) \| < \varepsilon \text{ and } \sum_{\ell=0}^{\ell=k} \| u_2^{(\ell+1)}(x,y) - u_2^{(\ell)}(x,y) \| < \varepsilon$$

for a sufficiently small ε.

CHAPTER 4

EXAMPLES OF THE ITERATIVE PARTIAL DIFFERENTIAL EQUATIONS

In this chapter, we shall find the analytic solutions and numerical solutions by using the three methods obtained from chapter 3 to solve some iterative partial differential equations.

4.1 Analytical Solutions.

From chapter 2, we can find the analytical solutions of iterative partial differential equations

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} = f\left(x, u(x), u^2(x), \cdots, u^m(x)\right), \qquad x \in Z \tag{4.1}$$

where $Z \in [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ with the initial conditions

$$\begin{aligned}
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) &= g_{1,i}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \\
&\qquad i = 1,2,\cdots,n \\
u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n) & \\
&= g_{2,i,j}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \\
&\qquad i \neq j, i, j = 1,2,3,\cdots,n \\
&\ \vdots \\
u(x_1, 0, 0, \cdots, 0) &= g_{n-1,2,3,\cdots,n}(x_1) \\
u(0, x_2, 0, \cdots, 0) &= g_{n-1,1,3,\cdots,n}(x_2) \\
&\ \vdots \\
u(0, 0, \cdots, 0, x_n) &= g_{n-1,1,2,\cdots,n-1}(x_n) \\
u(0,0,0,\cdots,0) &= c = [c_1, c_2, \cdots, c_n]^T
\end{aligned} \tag{4.2}$$

by letting

$$u(x) = \lim_{n\to\infty} u_{k+1}(x) \qquad (4.3)$$

$$u_0(x) = g(x)$$

where

$$u_{k+1}(x) = g(x) + \int_0^x f\left(t, u_k(t), u_k^2(t), \cdots, u_k^m(t)\right)dt \quad ,k = 0,1,2,\cdots. \qquad (4.4)$$

Example 4.1 Find the solution, in $Z = [0,1]\times[0,1]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} 0 \\ \frac{1}{2} - \frac{xy}{8} \end{bmatrix} - \frac{xy}{4}u(x,y) + u^2(x,y)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}^T.$$

We have $g(x,y) = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$.

Solution. Let $u_0(x,y) = g(x,y)$.

Then by the equation (4.4), we get

$$u_1(x,y) = g(x,y) + \int_0^y\int_0^x f\left(x, y, u_0(x,y), u_0^2(x,y)\right)dxdy$$

$$= g(x,y) + \int_0^y\int_0^x \left(\begin{bmatrix} 0 \\ \frac{1}{2} - \frac{xy}{8} \end{bmatrix} - \frac{xy}{4}u_0(x,y) + u_0(u_0(x,y))\right)dxdy$$

$$= \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \int_0^y\int_0^x \left(\begin{bmatrix} 0 \\ \frac{1}{2} - \frac{xy}{8} \end{bmatrix} - \frac{xy}{4}\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right)dxdy$$

$$=\begin{bmatrix}\frac{1}{2}+\frac{xy}{2}-\frac{x^2y^2}{32}\\ \frac{xy}{2}-\frac{x^2y^2}{16}\end{bmatrix}.$$

Hence $u_2(x,y)=g(x,y)+\int_0^y\int_0^x f\left(x,y,u_1(x,y),u_1^2(x,y)\right)dxdy$.

Since $u_1^2(x,y)=u_1(u_1(x,y))$

$$=\begin{bmatrix}\frac{1}{2}+\frac{\left(\frac{1}{2}+\frac{xy}{2}-\frac{x^2y^2}{32}\right)\left(\frac{xy}{2}-\frac{x^2y^2}{16}\right)}{2}-\frac{\left(\frac{1}{2}+\frac{xy}{2}-\frac{x^2y^2}{32}\right)^2\left(\frac{xy}{2}-\frac{x^2y^2}{16}\right)^2}{32}\\ \frac{\left(\frac{1}{2}+\frac{xy}{2}-\frac{x^2y^2}{32}\right)\left(\frac{xy}{2}-\frac{x^2y^2}{16}\right)}{2}-\frac{\left(\frac{1}{2}+\frac{xy}{2}-\frac{x^2y^2}{32}\right)^2\left(\frac{xy}{2}-\frac{x^2y^2}{16}\right)^2}{16}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2}+\frac{1}{8}xy+\frac{55}{512}x^2y^2-\frac{55}{2048}x^3y^3+\frac{7}{32768}x^4y^4+\frac{5}{8192}x^5y^5-\frac{25}{262144}x^6y^6\\ +\frac{3}{524288}x^7y^7-\frac{1}{8388608}x^8y^8\\ \frac{1}{8}xy+\frac{27}{256}x^2y^2-\frac{31}{1024}x^3y^3-\frac{9}{16384}x^4y^4+\frac{5}{4096}x^5y^5-\frac{25}{131072}x^6y^6\\ +\frac{3}{262144}x^7y^7-\frac{1}{4194304}x^8y^8\end{bmatrix}$$

or

$$=\begin{bmatrix}\frac{1}{2}+\frac{1}{8}xy+\frac{55}{512}x^2y^2+\text{higher power terms}\\ \frac{1}{8}xy+\frac{27}{256}x^2y^2+\text{higher power terms}\end{bmatrix}.$$

Thus

$$u_2(x,y)=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}+\int_0^y\int_0^x\left(\begin{bmatrix}0\\ \frac{1}{2}-\frac{xy}{8}\end{bmatrix}-\begin{bmatrix}\frac{1}{8}xy+\frac{1}{8}x^2y^2-\frac{1}{128}x^3y^3\\ \frac{1}{8}x^2y^2-\frac{1}{64}x^3y^3\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}\frac{1}{2}+\frac{1}{8}xy+\frac{55}{512}x^2y^2+\text{higher power terms}\\ \frac{1}{8}xy+\frac{27}{256}x^2y^2+\text{higher power terms}\end{bmatrix}\right)dxdy$$

$$=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}+\int_0^y\int_0^x\left(\begin{bmatrix}\frac{1}{2}+\frac{119}{512}x^2y^2+\text{ higher power terms}\\ \frac{1}{2}-\frac{5}{256}x^2y^2+\text{higher power terms}\end{bmatrix}\right)dxdy$$

$$=\begin{bmatrix}\frac{1}{2}+\frac{xy}{2}+\frac{119}{4608}x^3y^3+\text{higher power terms}\\ \frac{xy}{2}-\frac{5}{2304}x^3y^3+\text{higher power terms}\end{bmatrix}.$$

Thus we obtain $u_k(x,y)$ tends to $\begin{bmatrix}\frac{1}{2}+\frac{xy}{2}\\ \frac{xy}{2}\end{bmatrix}$ as k tends to infinity.

Therefor the solution of the given equation is

$$u(x,y)=\begin{bmatrix}\frac{1}{2}+\frac{xy}{2}\\ \frac{xy}{2}\end{bmatrix}.$$

Example 4.2 Find the solution, in $Z = [0,1]\times[0,1]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \begin{bmatrix} -\frac{7}{60} - \frac{x^2y^2}{45}\ \frac{1}{10}u_1(x,y) + u_1(u_1,u_2) \\ \frac{1}{5} - \frac{x^2y^2}{75} - \frac{1}{10}u_2(x,y) + u_2(u_1,u_2) \end{bmatrix}$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}^T.$$

We have $$g(x,y) = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

Solution. Let $$u_0(x,y) = \begin{bmatrix} u_{1,0}(x,y) \\ u_{2,0}(x,y) \end{bmatrix} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix} = g(x,y).$$

Then by the equation (4.4), we get

$$u_1(x,y) = g(x,y) + \int_0^y\int_0^x f\left(x,y,u_0(x,y),u_0^2(x,y)\right)dxdy$$

$$= g(x,y) + \int_0^y\int_0^x \left(\begin{bmatrix} -\frac{7}{60} - \frac{x^2y^2}{45} - \frac{u_{1,0}(x,y)}{10} + u_{1,0}(u_{1,0}(x,y),u_{2,0}(x,y)) \\ \frac{1}{5} - \frac{x^2y^2}{75} - \frac{u_{2,0}(x,y)}{10} + u_{2,0}(u_{1,0}(x,y),u_{2,0}(x,y)) \end{bmatrix}\right) dxdy$$

$$= \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \int_0^y\int_0^x \left(\begin{bmatrix} \frac{1}{3} - \frac{x^2y^2}{45} \\ \frac{1}{5} - \frac{x^2y^2}{75} \end{bmatrix}\right) dxdy$$

$$=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}+\begin{bmatrix}\frac{xy}{3}-\frac{x^3y^3}{405}\\\frac{xy}{5}-\frac{x^3y^3}{675}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}+\frac{xy}{3}-\frac{x^3y^3}{405}\\\frac{xy}{5}-\frac{x^3y^3}{675}\end{bmatrix}.$$

Hence $u_2(x,y)=g(x,y)+\int_0^y\int_0^x f\left(x,y,u_1(x,y),u_1^2(x,y)\right)dxdy$

$$=\begin{bmatrix}\frac{1}{2}\\2\\0\end{bmatrix}+\int_0^y\int_0^x\left(\begin{bmatrix}-\frac{7}{60}-\frac{x^2y^2}{45}-\frac{u_{1,1}(x,y)}{10}+u_{1,1}(u_{1,1}(x,y),u_{2,1}(x,y))\\\frac{1}{5}-\frac{x^2y^2}{75}-\frac{u_{2,1}(x,y)}{10}+u_{2,1}(u_{1,1}(x,y),u_{2,1}(x,y))\end{bmatrix}\right)dxdy.$$

Since $$u_1^2(x,y)=u_1(u_1(x,y))=\begin{bmatrix}u_{1,1}(u_{1,1}(x,y),u_{2,1}(x,y))\\u_{2,1}(u_{1,1}(x,y),u_{2,1}(x,y))\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2}+\frac{xy}{3}+\frac{x^2y^2}{45}-\frac{8201}{32886000}x^3y^3+\text{higher power terms}\\\frac{xy}{50}+\frac{x^2y^2}{75}-\frac{101}{675000}x^3y^3\quad+\quad\text{higher power terms}\end{bmatrix}.$$

Thus, we have

$$u_2(x,y)=\begin{bmatrix}\frac{1}{2}\\2\\0\end{bmatrix}+\int_0^y\int_0^x\left(\begin{bmatrix}-\frac{7}{60}-\frac{x^2y^2}{45}\\\frac{1}{5}-\frac{x^2y^2}{75}\end{bmatrix}-\frac{1}{10}\begin{bmatrix}\frac{1}{2}+\frac{xy}{3}-\frac{x^3y^3}{405}\\\frac{xy}{5}-\frac{x^3y^3}{675}\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}\frac{1}{2}+\frac{xy}{3}+\frac{x^2y^2}{45}-\frac{8201}{32886000}x^3y^3+\text{higher power terms}\\\frac{xy}{50}+\frac{x^2y^2}{75}-\frac{101}{675000}x^3y^3\quad+\quad\text{higher power terms}\end{bmatrix}\right)dxdy$$

$$= \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \int_0^y \int_0^x \left(\begin{bmatrix} \frac{1}{3} - \frac{3}{10}xy + \frac{8111}{36540000}x^3y^3 + \text{higher power terms} \\ \frac{1}{5} - \frac{x^3y^3}{675000} + \text{higher power terms} \end{bmatrix} \right.$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{xy}{3} - \frac{3}{40}x^2y^2 + \frac{8111}{584640000} + \text{higher power terms} \\ \frac{xy}{5} - \frac{x^3y^3}{10800000} + \text{higher power terms} \end{bmatrix}.$$

Thus we obtain $u_k(x, y)$ tends to $\begin{bmatrix} \frac{1}{2} + \frac{xy}{3} \\ \frac{xy}{5} \end{bmatrix}$ as k tends to infinity.

Therefor the solution of the given equation is

$$u(x, y) = \begin{bmatrix} \frac{1}{2} + \frac{xy}{3} \\ \frac{xy}{5} \end{bmatrix}.$$

Example 4.3 Find the solution, in $Z = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$, of the equation

$$\frac{\partial^2 u(x, y)}{\partial x \partial y} = u^3(x, y)$$

with the initial conditions

$$u(x,0) = u(0, y) = u(0,0) = \begin{bmatrix} \frac{1}{8} & \frac{1}{16} \end{bmatrix}^T.$$

Solution. Let $u_0(x, y) = g(x, y)$.

By the equation (4.4), we get

$$u_1(x, y) = g(x, y) + \int_0^y \int_0^x u^3(x, y)\, dx dy$$

$$= g(x,y) + \int_0^y \int_0^x u_0(u_0(u_0(x,y)))\, dxdy$$

$$= \begin{bmatrix} \frac{1}{8} \\ \frac{1}{16} \end{bmatrix} + \int_0^y \int_0^x \begin{bmatrix} \frac{1}{8} \\ \frac{1}{16} \end{bmatrix} dxdy$$

$$= \begin{bmatrix} \frac{1}{8} + \frac{xy}{8} \\ \frac{1}{16} + \frac{xy}{16} \end{bmatrix}.$$

Hence $\quad u_2(x,y) = g(x,y) + \int_0^y \int_0^x u_1(u_1(u_1(x,y)))\, dxdy$.

Since

$$u_1(u_1(x,y)) = \begin{bmatrix} \frac{129}{1024} + \frac{xy}{512} + \frac{x^2y^2}{1024} \\ \frac{129}{2048} + \frac{xy}{1024} + \frac{x^2y^2}{2048} \end{bmatrix}$$

and

$$u_1(u_1(u_1(x,y))) = \begin{bmatrix} \frac{56741874268161}{4503599627370496} + \frac{272679297}{562949953421312} xy + \frac{276973447}{1125899906842624} x^2y^2 + \frac{2147591}{562949953421312} x^3y^3 \\ + \frac{2150179}{22517998136885248} x^4y^4 + \frac{391}{562949953421312} x^5y^5 + \frac{135}{1125899906842624} x^6y^6 \\ + \frac{1}{562949953421312} x^7y^7 + \frac{1}{4503599627370496} x^8y^8 \\ \frac{567418074268161}{9007199254740992} + \frac{272679297}{1125899906842624} xy + \frac{276973447}{2251799813685248} x^2y^2 + \frac{2147591}{1125899906842624} x^3y^3 \\ + \frac{2150179}{450599627370496} x^4y^4 + \frac{391}{1125899906842624} x^5y^5 + \frac{135}{2251799813685248} x^6y^6 \\ + \frac{1}{1125899906842624} x^7y^7 + \frac{1}{9007199254740992} x^8y^8 \end{bmatrix}$$

so ,we find the value of $\int_0^y \int_0^x u_1(u_1(u_1(x,y)))\, dxdy$, we get

$$\begin{bmatrix} \frac{567418874268161}{4503599627370496} xy + \frac{272679297}{2251799813685248} x^2y^2 + \frac{276973447}{10133099161583616} x^3y^3 + \frac{2147591}{9007199254740992} x^4y^4 \\ + \frac{2150179}{562949953421312 00} x^5y^5 + \frac{391}{20266198323167232} x^6y^6 + \frac{135}{5516909543528 8576} x^7y^7 \\ + \frac{1}{36028797018963968} x^8y^8 + \frac{1}{364791569817010176} x^9y^9 \\ \frac{567418074268161}{9007199254740992} xy + \frac{272679297}{4503599627384 96} x^2y^2 + \frac{276973447}{20266198323167232} x^3y^3 + \frac{2147591}{18014398509481984} x^4y^4 \\ + \frac{2150179}{1125899906842624 00} x^5y^5 + \frac{391}{40532396646334464} x^6y^6 + \frac{135}{110602790870577152} x^7y^7 \\ + \frac{1}{72057594037927936} x^8y^8 + \frac{1}{729583139634743 52} x^9y^9 \end{bmatrix}$$

and then

$$u_2(x,y) = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{16} \end{bmatrix} + \int_0^y \int_0^x u_1(u_1(u_1(x,y)))\, dxdy.$$

$$= \begin{bmatrix} \frac{1}{8} + \frac{567418874268161}{4503599627370496} xy + \frac{272679297}{2251799813685248} x^2y^2 + \frac{276973447}{10133099161583616} x^3y^3 + \frac{2147591}{9007199254740992} x^4y^4 \\ + \frac{2150179}{56294995342131200} x^5y^5 + \frac{391}{20266198323167232} x^6y^6 + \frac{135}{55169095435288576} x^7y^7 \\ + \frac{1}{36028797018963968} x^8y^8 + \frac{1}{364791569817010176} x^9y^9 \\ \frac{1}{16} + \frac{567418074268161}{9007199254740992} xy + \frac{272679297}{450359962738496} x^2y^2 + \frac{276973447}{20266198323167232} x^3y^3 + \frac{2147591}{18014398509481984} x^4y^4 \\ + \frac{2150179}{112649906842624 00} x^5y^5 + \frac{391}{40532396646334464} x^6y^6 + \frac{135}{110602790870577152} x^7y^7 \\ + \frac{1}{72057594037927936} x^8y^8 + \frac{1}{72958313963474352} x^9y^9 \end{bmatrix}.$$

Thus, we use $u_2(x,y)$ as the approximating solution for the problem.

Example 4.4 Find the solution, in $Z = [0,1] \times [0,1]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} yu_1(u_1(x,y), u_2(x,y)) - xu_2(u_1(x,y), u_2(x,y)) + \frac{x}{2} \\ yu_1(x,y) - xu_2(x,y) + \frac{y}{2} \end{bmatrix}$$

with the initial condition

$$u(0,y) = \begin{bmatrix} 0 \\ \frac{y}{4} \end{bmatrix}, \; u(x,0) = \begin{bmatrix} \frac{x}{4} \\ 0 \end{bmatrix} \text{ and } \; u(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we have $$g(x,y)=\begin{bmatrix}\frac{x}{4}\\ \frac{y}{4}\end{bmatrix}.$$

Solution.

Let $u_0(x,y)=g(x,y)$

then by (4.4), we get

$$u_1(x,y)=\begin{bmatrix}\frac{x}{4}+\frac{x^2y}{4}\\ \frac{y}{4}+\frac{xy^2}{4}\end{bmatrix}$$

$$u_2(x,y)=\begin{bmatrix}\frac{x}{4}+\frac{x^2y}{4}\\ \frac{y}{4}+\frac{xy^2}{4}\end{bmatrix}$$

and

$$u_k(x,y)=\begin{bmatrix}\frac{x}{4}+\frac{x^2y}{4}\\ \frac{y}{4}+\frac{xy^2}{4}\end{bmatrix} \text{ for } k=0,1,2,3\dots.$$

Thus $u_k(x,y)$ tends to $\begin{bmatrix}\frac{x}{4}+\frac{x^2y}{4}\\ \frac{y}{4}+\frac{xy^2}{4}\end{bmatrix}$ as k tends to infinity.

Hence the solution of the given equation is

$$u(x,y)=\begin{bmatrix}\frac{x}{4}+\frac{x^2y}{4}\\ \frac{y}{4}+\frac{xy^2}{4}\end{bmatrix}.$$

4.2 Numerical Solutions.

We shall find the solution of the iterative partial differential equation using the three numerical methods in chapter 3. Then we shall compare them with the exact solution.

Example 4.5 Find the solution, in $Z = [0,1]\times[0,1]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} 0 \\ \frac{1}{2}-\frac{xy}{8} \end{bmatrix} - \frac{xy}{4}u(x,y) + u^2(x,y)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}^T.$$

The exact solution is $u(x,y) = \begin{bmatrix} \frac{1}{2}+\frac{xy}{2} \\ \frac{xy}{2} \end{bmatrix}$. We shall divided the interval [0,1], for y-axis and x-axis, into 4, 8, and 16 equally subintervals and repeat the methods until the error, ε, less than 0.000005.

We obtain the results as follows in table 4.1-4.9.

Table 4.1 The mean of error, $u(x, y)$, for k = h = 4.

	First Method	
	$u_1(x, y)$	$u_2(x, y)$
Iteration	Mean of Error	Mean of Error
1	0.0766906738	0.0766906738
2	0.0048587225	0.0048587225
3	0.0004749587	0.0004749587
4	0.0000375245	0.0000375245
5	0.0000025612	0.0000025612
6	0.0000001594	0.0000001594

Table 4.2 The mean of error, $u(x, y)$, for k = h = 4.

	Second Method	
	$u_1(x, y)$	$u_2(x, y)$
Iteration	Mean of Error	Mean of Error
1	0.0122070312	0.0122070312
2	0.0012833007	0.0012833007
3	0.0001021547	0.0001021547
4	0.0000065928	0.0000065928
5	0.0000003645	0.0000003645
6	0.0000000177	0.0000000177

Table 4.3 The mean of error, $u(x, y)$, for k = h = 4.

	Third Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0766906738	0.0766906738
2	0.0048033703	0.0048033703
3	0.0004296399	0.0004296399
4	0.0000281045	0.0000281045
5	0.0000014371	0.0000014371
6	0.0000000600	0.0000000600

Table 4.4 The mean of error, $u(x, y)$, for k = h = 8.

	First Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0707731247	0.0707731247
2	0.0030105359	0.0030105359
3	0.0001706412	0.0001706412
4	0.0000067423	0.0000067423
5	0.0000002084	0.0000002084
6	0.0000000055	0.0000000055

Table 4.5 The mean of error, $u(x, y)$, for k = h = 8.

	Second Method	
Iteration	$u_1(x, y)$ Mean of Error	$u_2(x, y)$ Mean of Error
1	0.0068664551	0.0068664551
2	0.0004136432	0.0004136432
3	0.0000166168	0.0000166168
4	0.0000005021	0.0000005021
5	0.0000000124	0.0000000124
6	0.0000000003	0.0000000003

Table 4.6 The mean of error, $u(x, y)$, for k = h = 8.

	Third Method	
Iteration	$u_1(x, y)$ Mean of Error	$u_2(x, y)$ Mean of Error
1	0.0707731247	0.0707731247
2	0.0030034875	0.0030034875
3	0.0001647392	0.0001647392
4	0.0000060939	0.0000060939
5	0.0000001712	0.0000001712
6	0.0000000040	0.0000000040

Table 4.7 The mean of error, $u(x, y)$, for k = h = 16.

	First Method	
Iteration	$u_1(x, y)$ Mean of Error	$u_2(x, y)$ Mean of Error
1	0.0675933063	0.0675933063
2	0.0023390272	0.0023390272
3	0.0000952787	0.0000952787
4	0.0000024173	0.0000024173
5	0.0000000443	0.0000000443

Table 4.8 The mean of error, $u(x, y)$, for k = h = 16.

	Second Method	
Iteration	$u_1(x, y)$ Mean of Error	$u_2(x, y)$ Mean of Error
1	0.0049610138	0.0049610138
2	0.0002112379	0.0002112379
3	0.0000054585	0.0000054585
4	0.0000000996	0.0000000996
5	0.0000000014	0.0000000014

Table 4.9 The mean of error, $u(x, y)$, for k = h = 16.

	Third Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0675933063	0.0675933063
2	0.0023378300	0.0023378300
3	0.0000941629	0.0000941629
4	0.0000023242	0.0000023242
5	0.0000000407	0.0000000407

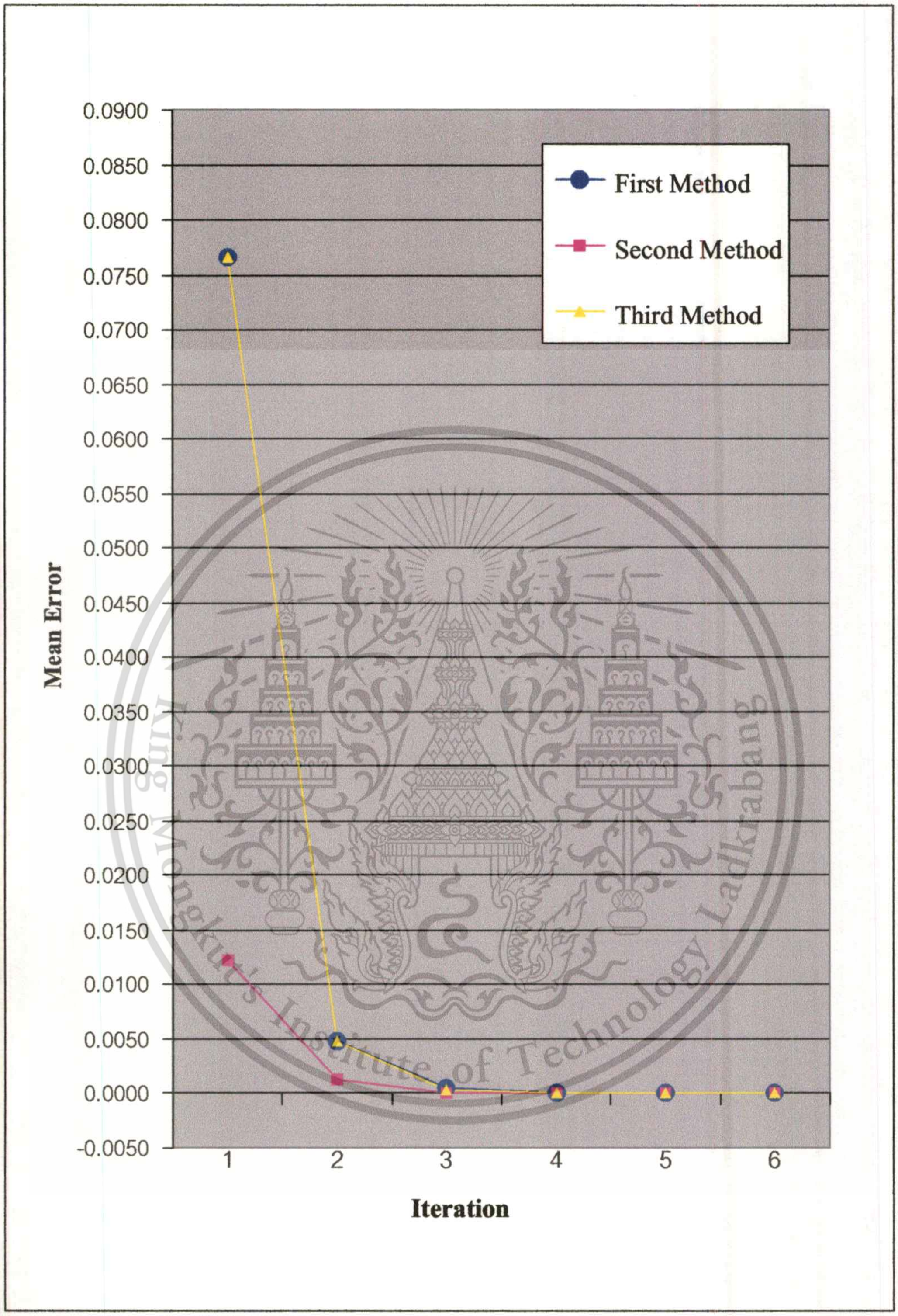

Figure 4.1 Graph of the mean error, $u_1(x, y)$, from the three numerical methods of example 4.5 for k = h =4.

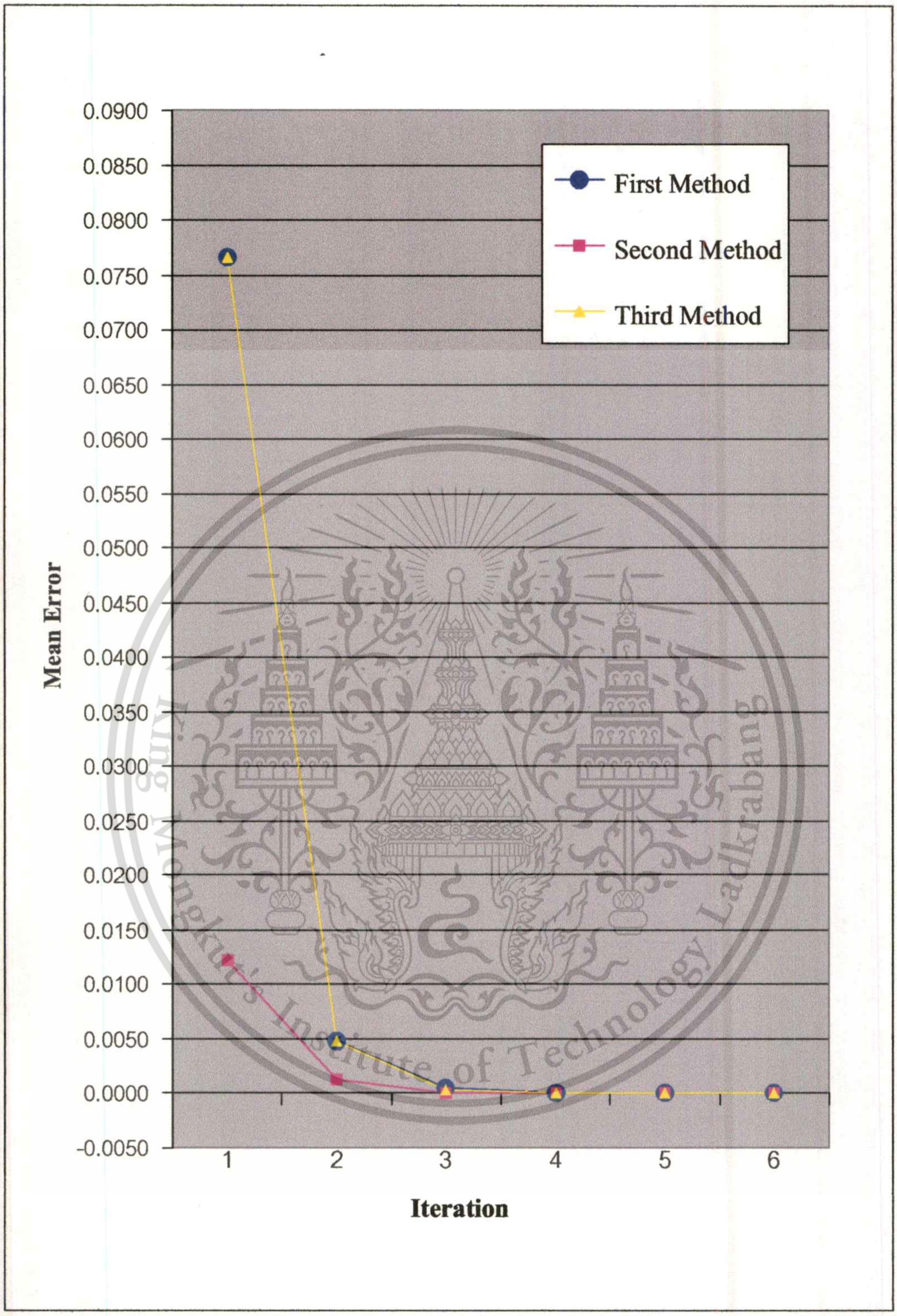

Figure 4.2 Graph of the mean error, $u_2(x, y)$, from the three numerical methods of example 4.5 for k = h =4.

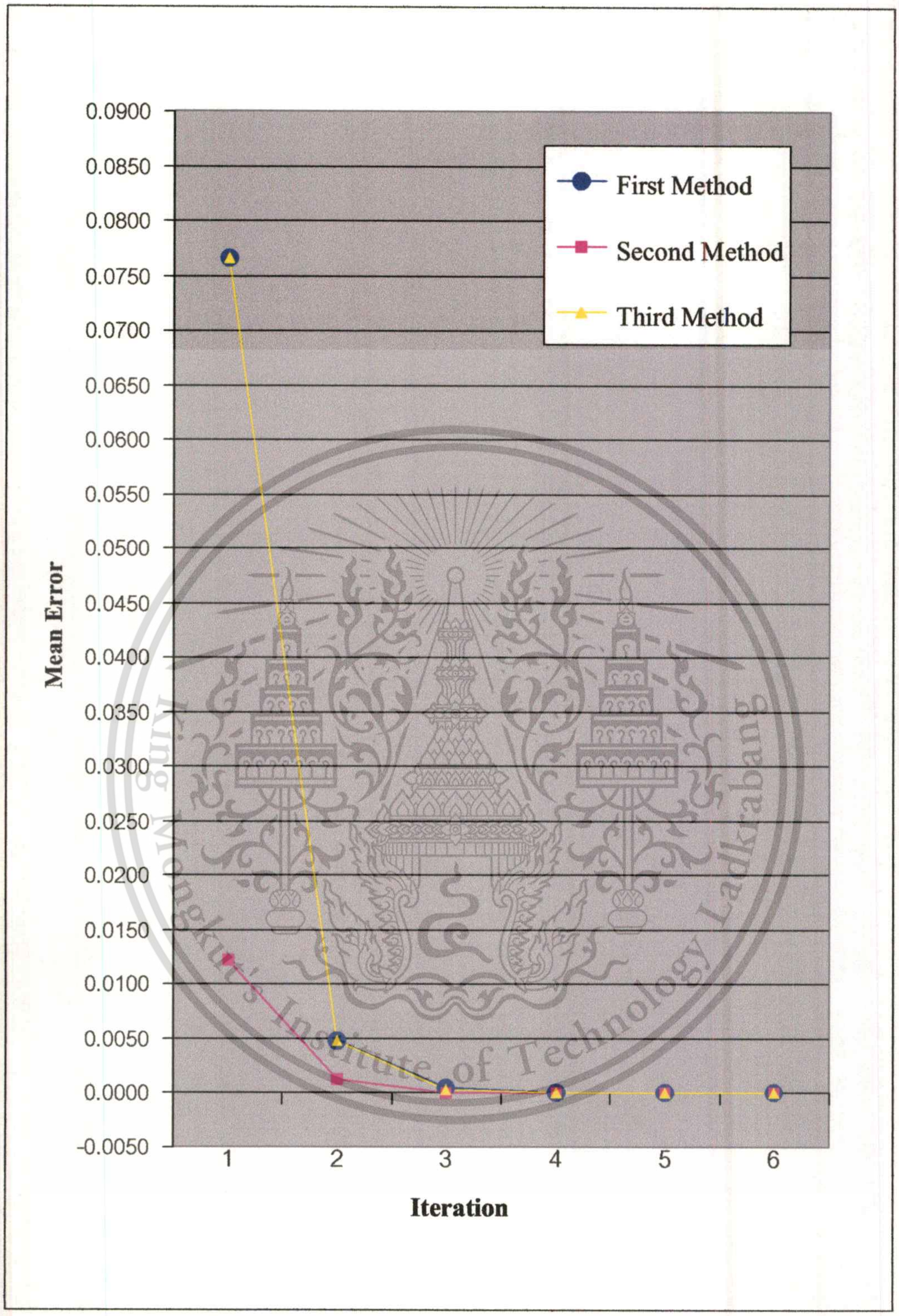

Figure 4.3 Graph of the mean error, $u_1(x, y)$, from the three numerical methods of example 4.5 for k = h =8.

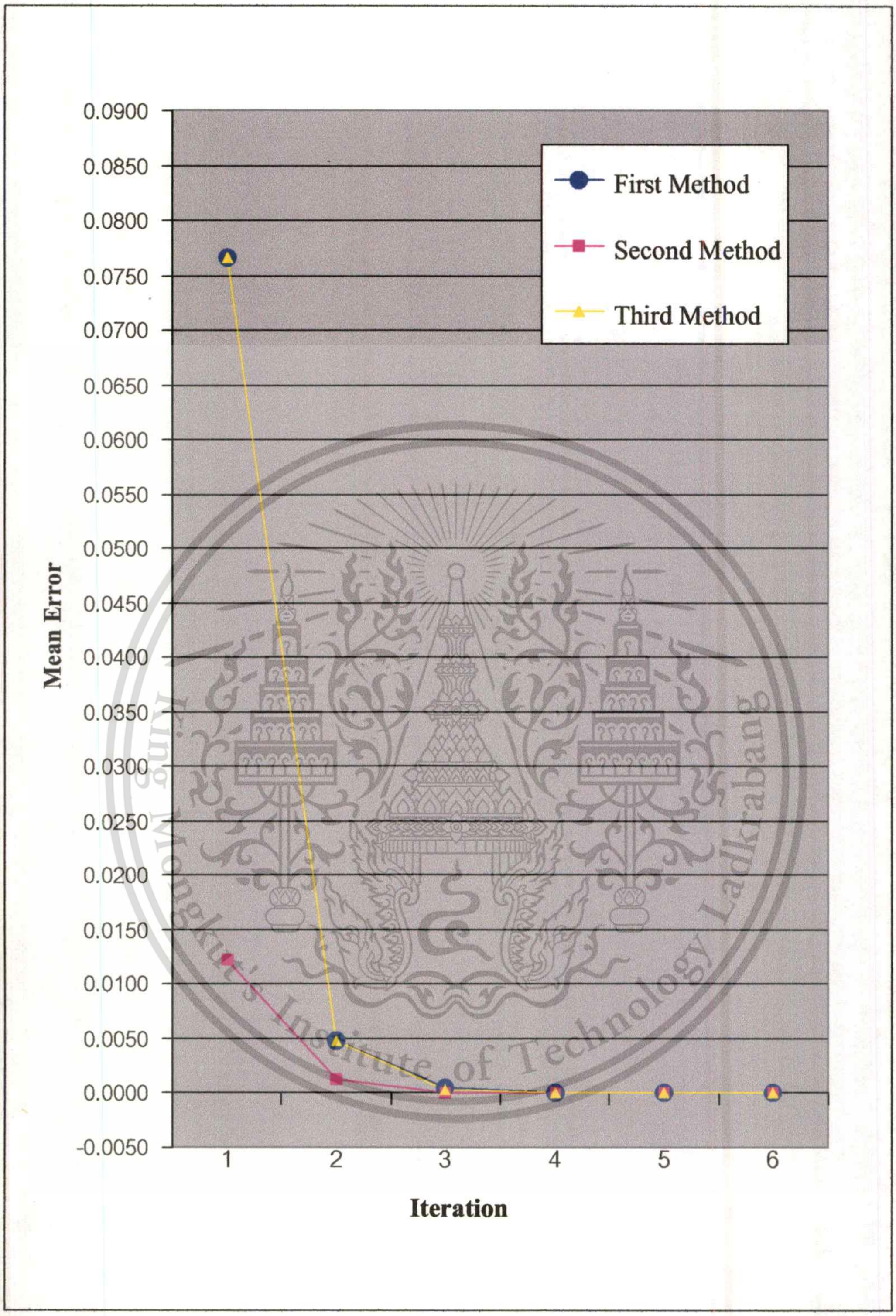

Figure 4.4 Graph of the mean error, $u_2(x, y)$, from the three numerical methods of example 4.5 for k = h =8.

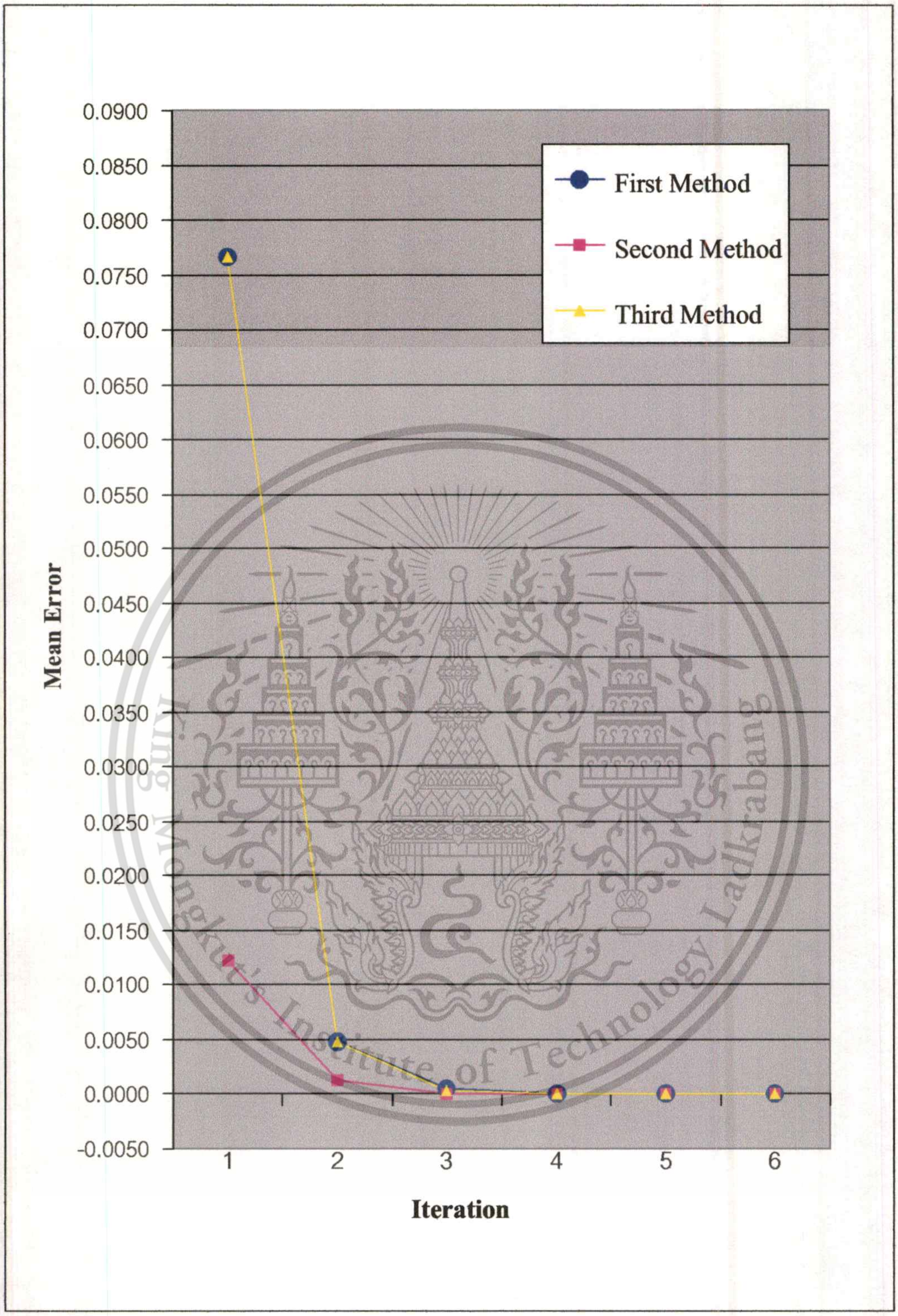

Figure 4.5 Graph of the mean error, $u_1(x, y)$, from the three numerical methods of example 4.5 for k = h =16.

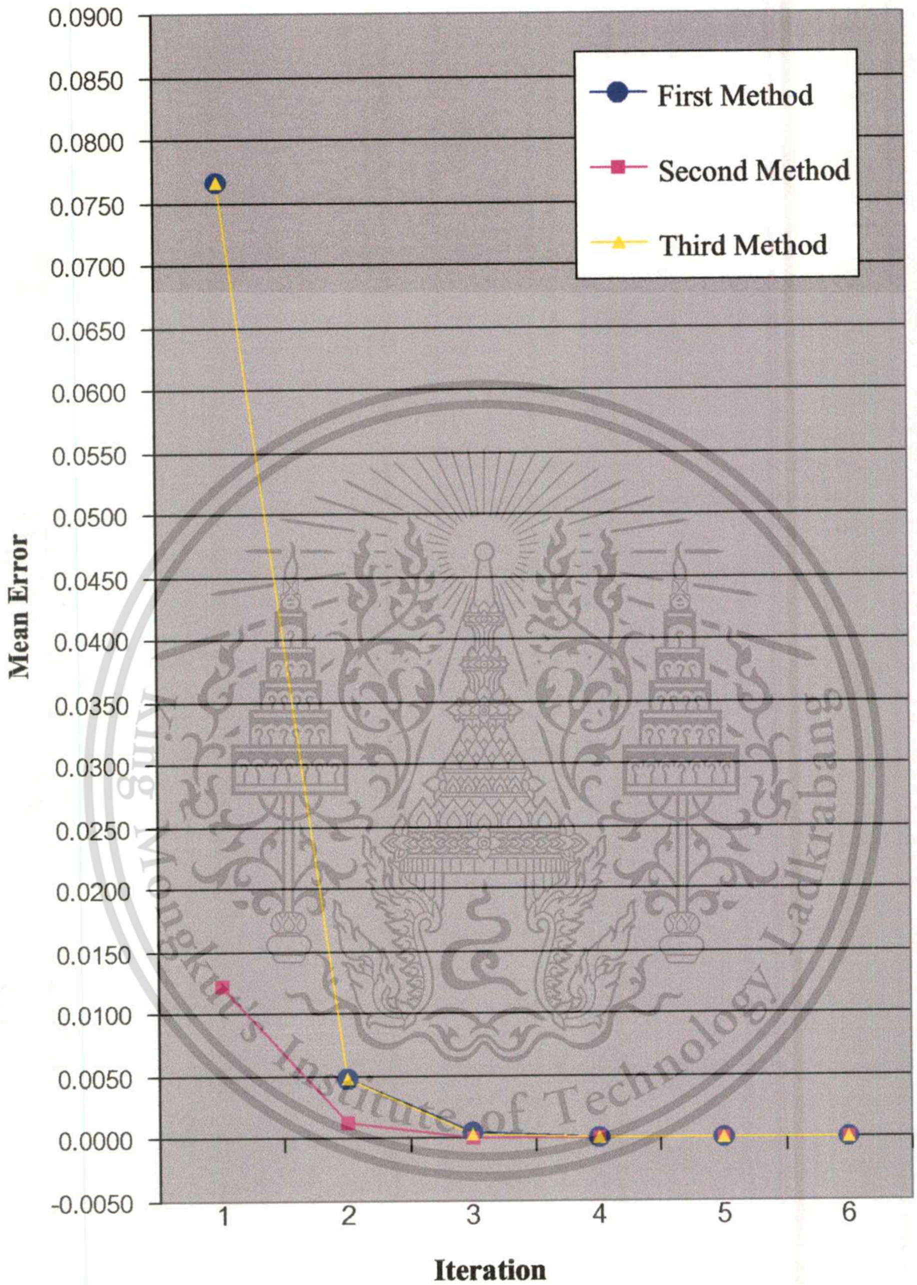

Figure 4.6 Graph of the mean error, $u_2(x, y)$, from the three numerical methods of example 4.5 for k = h =16.

Example 4.6 Find the solution, in $Z = [0,1]\times[0,1]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} -\frac{7}{60} - \frac{x^2y^2}{45}\ \frac{1}{10}u_1(x,y) + u_1(u_1,u_2) \\ \frac{1}{5} - \frac{x^2y^2}{75} - \frac{1}{10}u_2(x,y) + u_2(u_1,u_2) \end{bmatrix}$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}^T.$$

The exact solution is $u(x,y) = \begin{bmatrix} \frac{1}{2} + \frac{xy}{3} \\ \frac{xy}{5} \end{bmatrix}$. We shall divided the interval [0,1], for y-axis and x-axis, into 4, 8 and16 equally subintervals and repeat the methods until the error, ε, less than 0.000005. We obtain the result as follows in table 4.10 – 4.18.

Table 4.10 The mean of error, $u(x,y)$, for k = h = 4.

	First Method	
Iteration	$u_1(x,y)$ Mean of Error	$u_2(x,y)$ Mean of Error
1	0.0461869982	0.0277121989
2	0.0002443936	0.0001466362
3	0.0000034689	0.0000020814
4	0.0000000446	0.0000000268

Table 4.11 The mean of error, $u(x,y)$, for k = h = 4.

	Second Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0008477105	0.0005086263
2	0.0000115604	0.0000069362
3	0.0000001272	0.0000000763
4	0.0000000012	0.0000000007

Table 4.12 The mean of error, $u(x,y)$, for k = h = 4.

	Third Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0461869982	0.0277121989
2	0.0002417071	0.0001450243
3	0.0000021471	0.0000012883
4	0.0000000137	0.0000000082

Table 4.13 The mean of error, $u(x,y)$, for k = h = 8.

	First Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0446754853	0.0268052912
2	0.000146329	0.0000876797
3	0.0000009132	0.0000005479
4	0.0000000046	0.0000000028

Table 4.14 The mean of error, $u(x,y)$, for k = h = 8.

	Second Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0003862381	0.0002317429
2	0.0000025131	0.0000015079
3	0.0000000125	0.0000000075
4	0.0000000001	0.0000000000

Table 4.15 The mean of error, $u(x,y)$, for k = h = 8.

	Third Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0446754853	0.0268052912
2	0.0001460278	0.0000876167
3	0.0000008407	0.0000005044
4	0.0000000037	0.0000000022

Table 4.16 The mean of error, $u(x,y)$, for k = h = 16.

	First Method	
	$u_1(x,y)$	$u_2(x,y)$
Iteration	Mean of Error	Mean of Error
1	0.0433743992	0.026024395
2	0.0001083710	0.0000650226
3	0.0000004308	0.0000002585
4	0.0000000012	0.0000000007

Table 4.17 The mean of error, $u(x, y)$, for k = h = 16.

	Second Method	
	$u_1(x, y)$	$u_2(x, y)$
Iteration	Mean of Error	Mean of Error
1	0.0002489120	0.0001493472
2	0.0000010321	0.0000006193
3	0.0000000030	0.0000000018
4	0.0000000000	0.0000000000

Table 4.18 The mean of error, $u(x, y)$, for k = h = 16.

	Third Method	
	$u_1(x, y)$	$u_2(x, y)$
Iteration	Mean of Error	Mean of Error
1	0.0433743992	0.0260246395
2	0.0001083604	0.0000650162
3	0.0000004221	0.0000002533
4	0.0000000012	0.0000000007

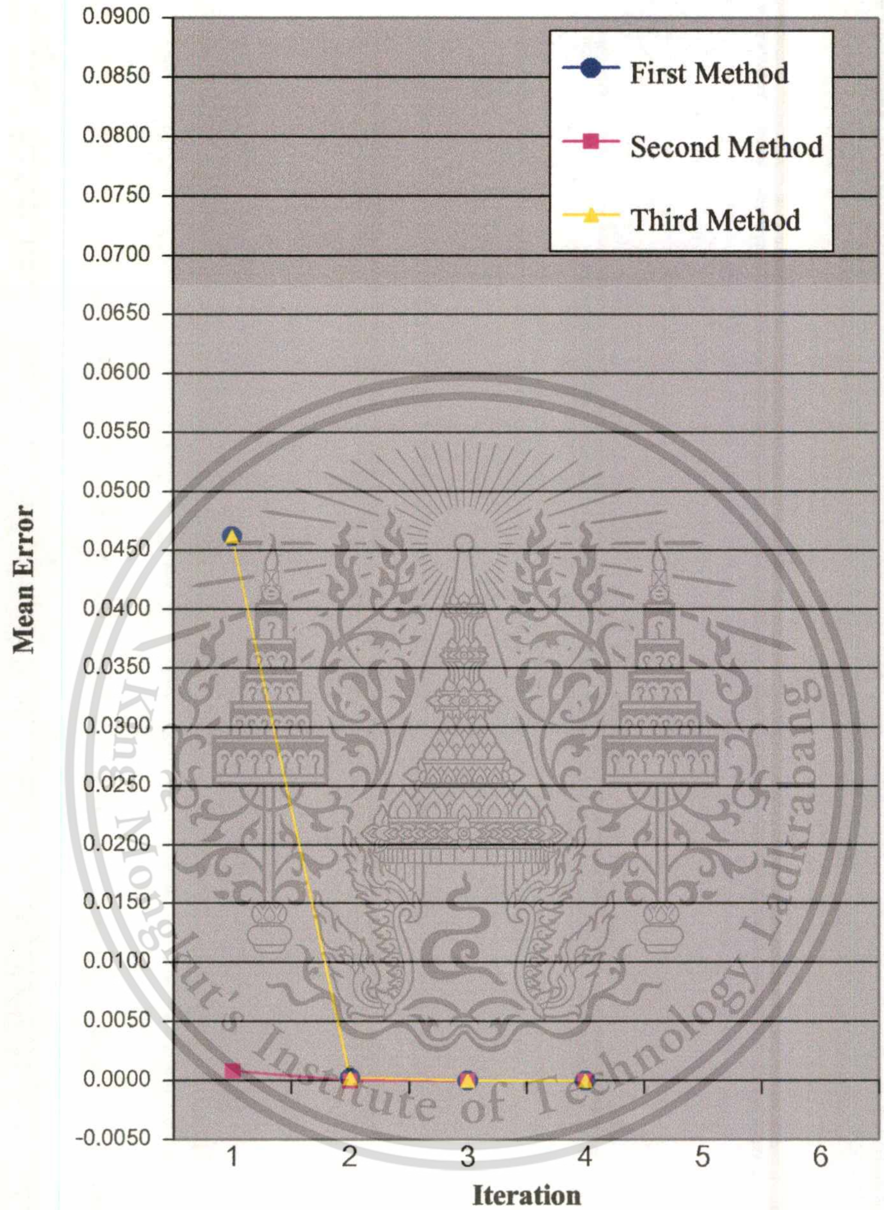

Figure 4.7 Graph of the mean error, $u_1(x, y)$, from the three numerical methods of example 4.6 for k = h =4.

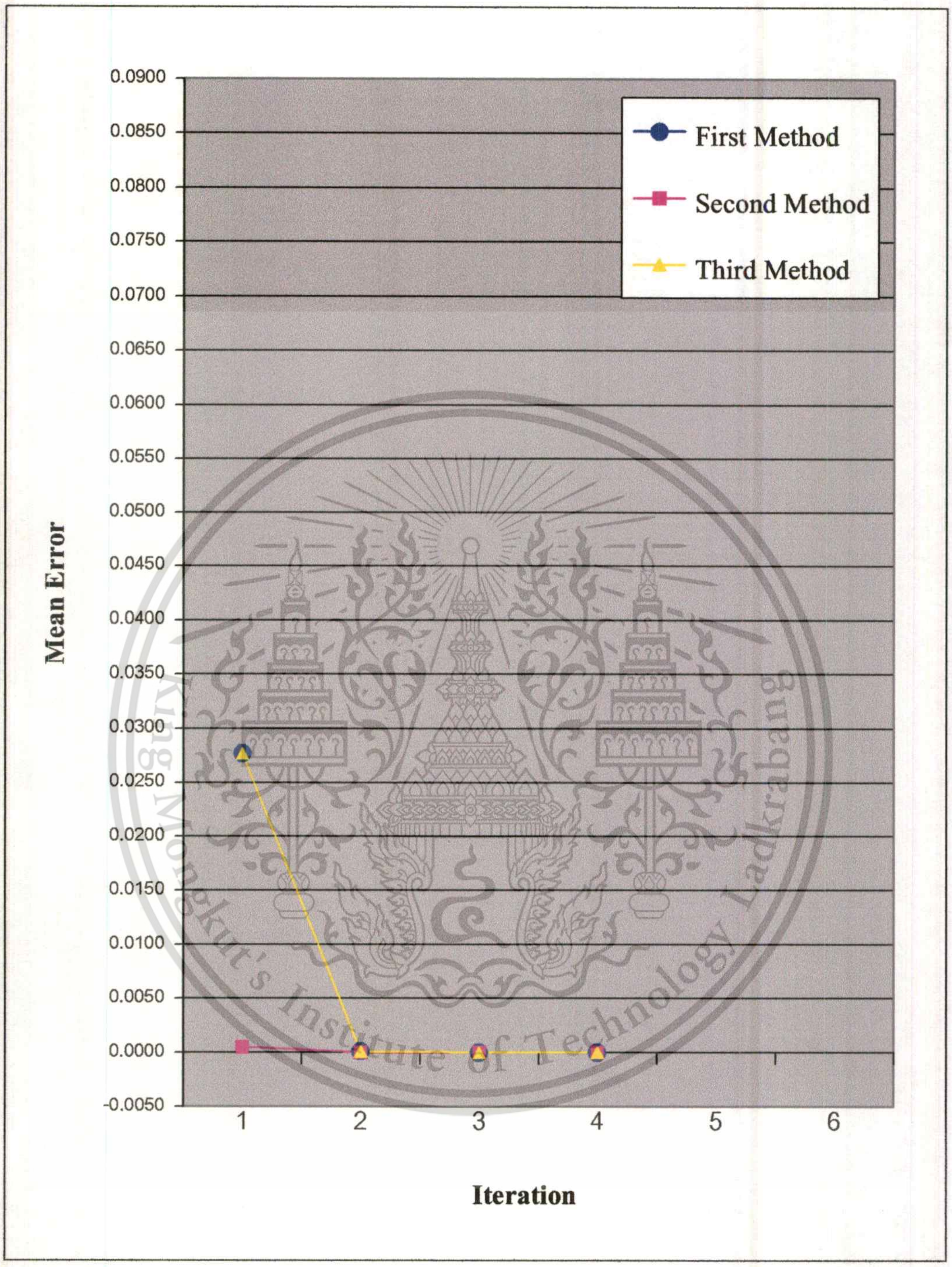

Figure 4.8 Graph of the mean error, $u_2(x, y)$, from the three numerical methods of example 4.6 for k = h =4.

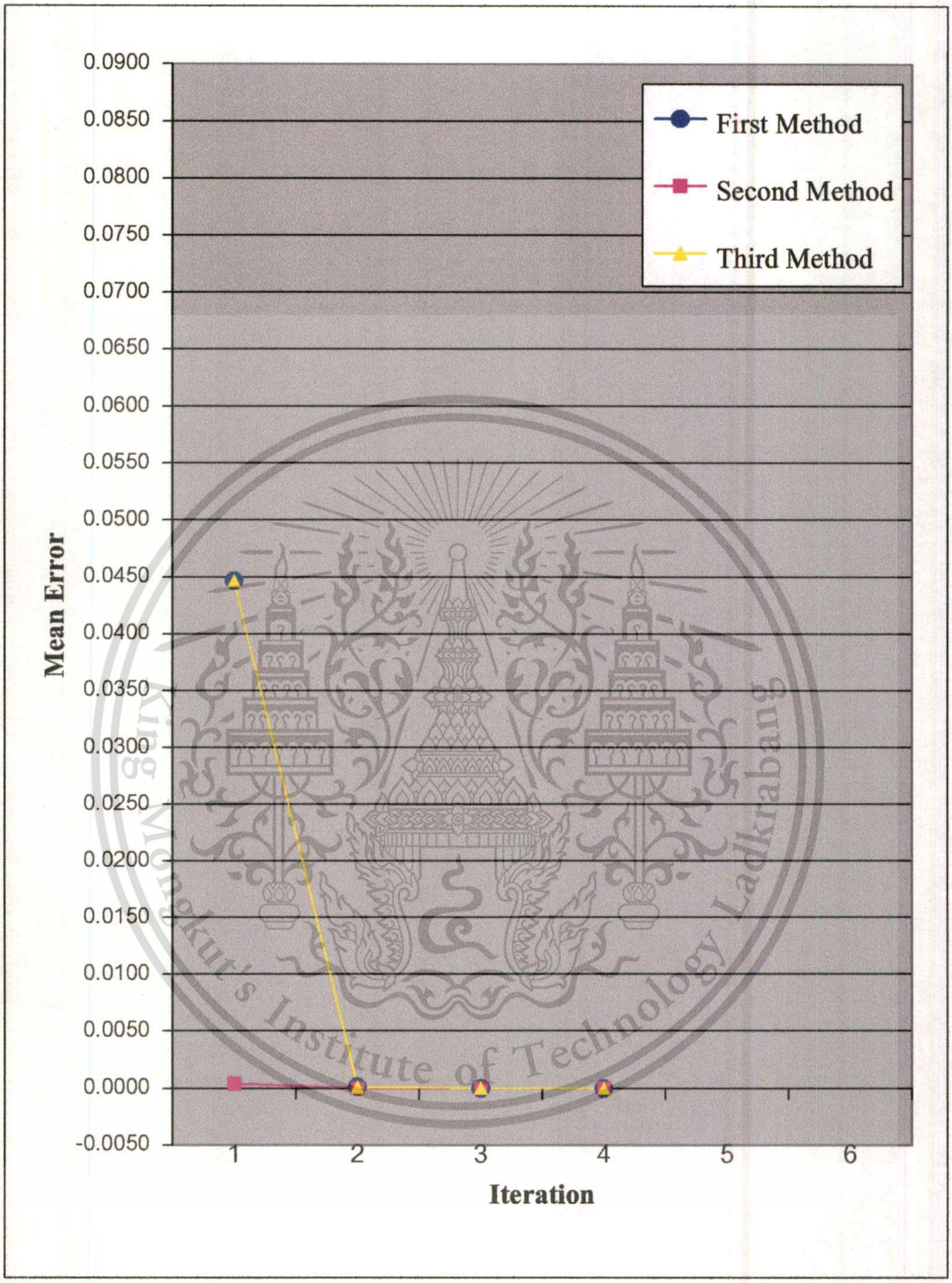

Figure 4.9 Graph of the mean error, $u_1(x, y)$, from the three numerical methods of example 4.6 for k = h =8.

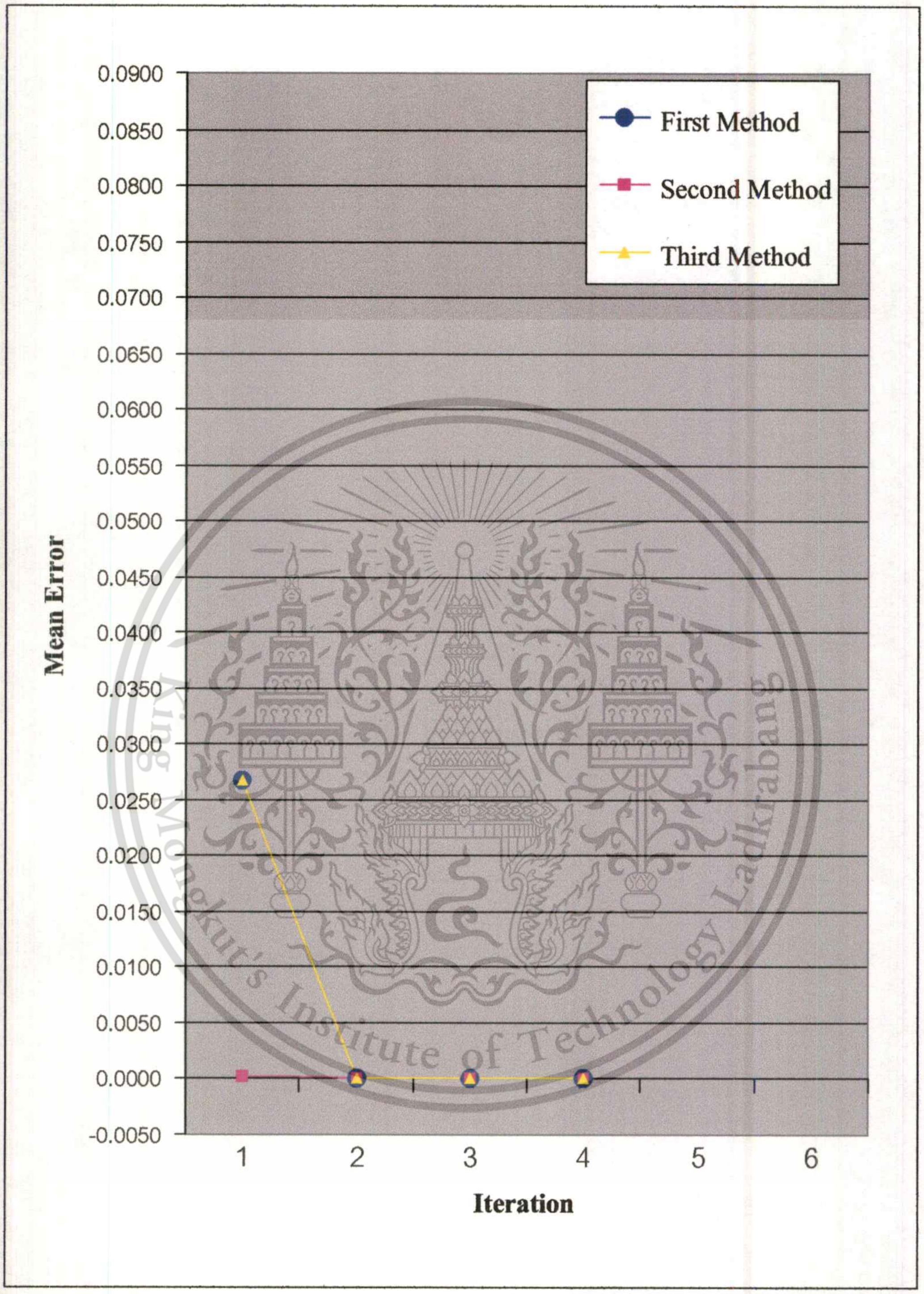

Figure 4.10 Graph of the mean error, $u_2(x, y)$, from the three numerical methods of example 4.6 for k = h =8.

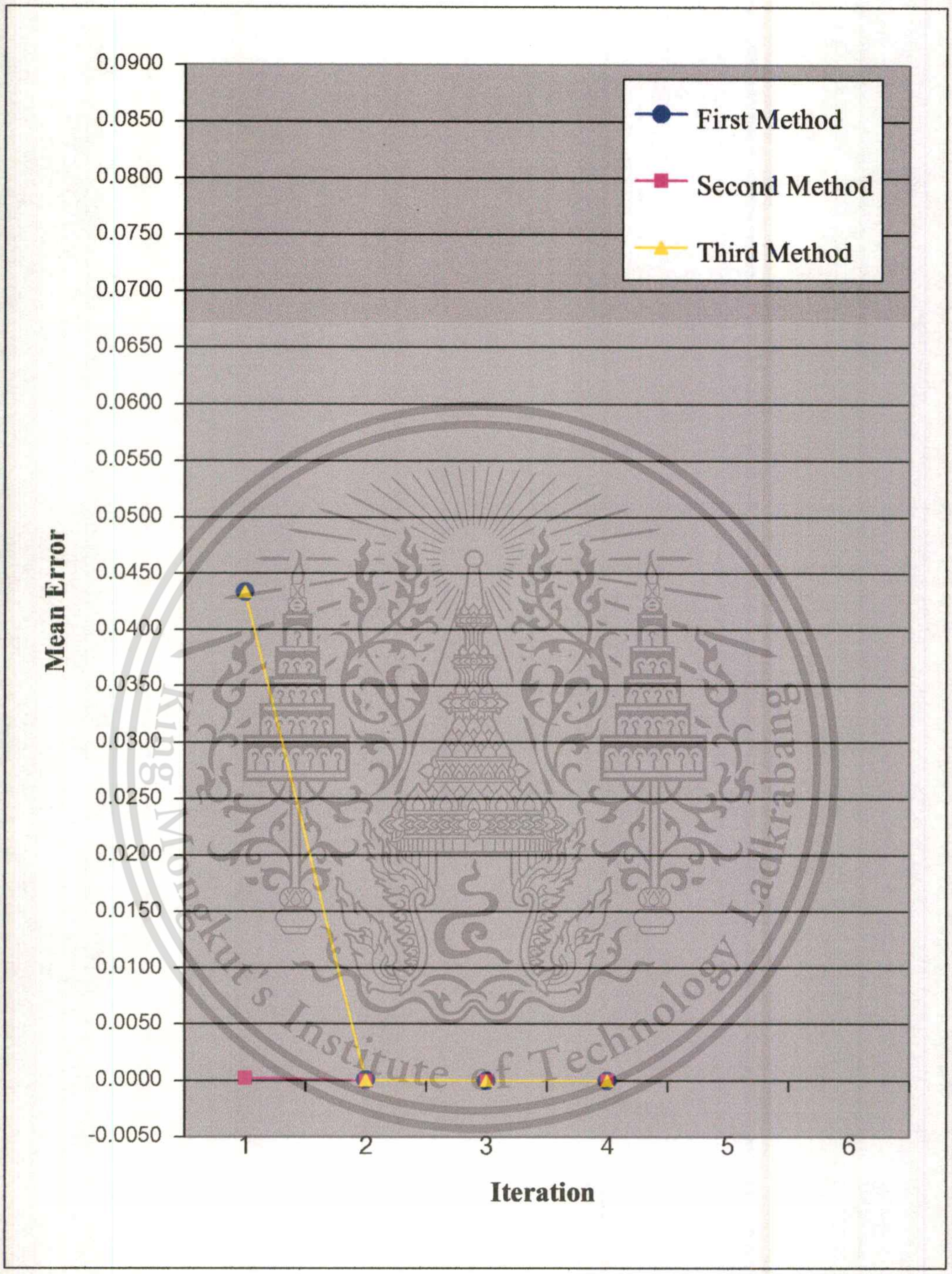

Figure 4.11 Graph of the mean error, $u_1(x, y)$, from the three numerical methods of example 4.6 for k = h =16.

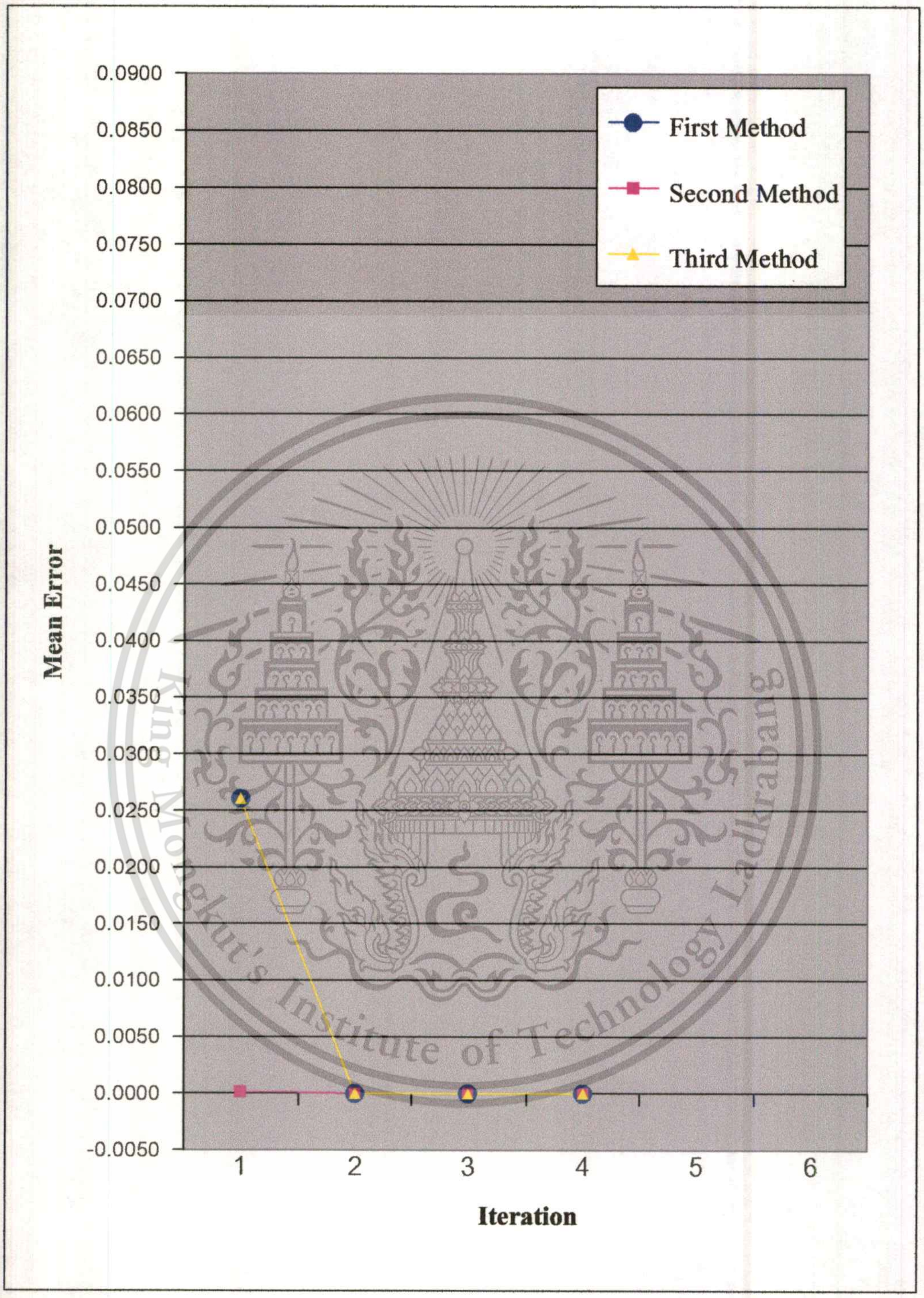

Figure 4.12 Graph of the mean error, $u_2(x, y)$, from the three numerical methods of example 4.6 for k = h =16.

Example 4.7 Find the solution, in $Z = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = u^3(x,y) \qquad (4.7.1)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \left[\frac{1}{8} \ \frac{1}{16}\right]^T. \qquad (4.7.2)$$

We do not know the analytical solution of the problem (4.7.1)-(4.7.2). We shall divided the interval [0,1], for y-axis and x-axis, into 4, 8 and 16 equally subintervals and repeat the methods until the error, ε, less than 0.000005. We obtain the numerical result as follows in figure 4.13 – 4.18.

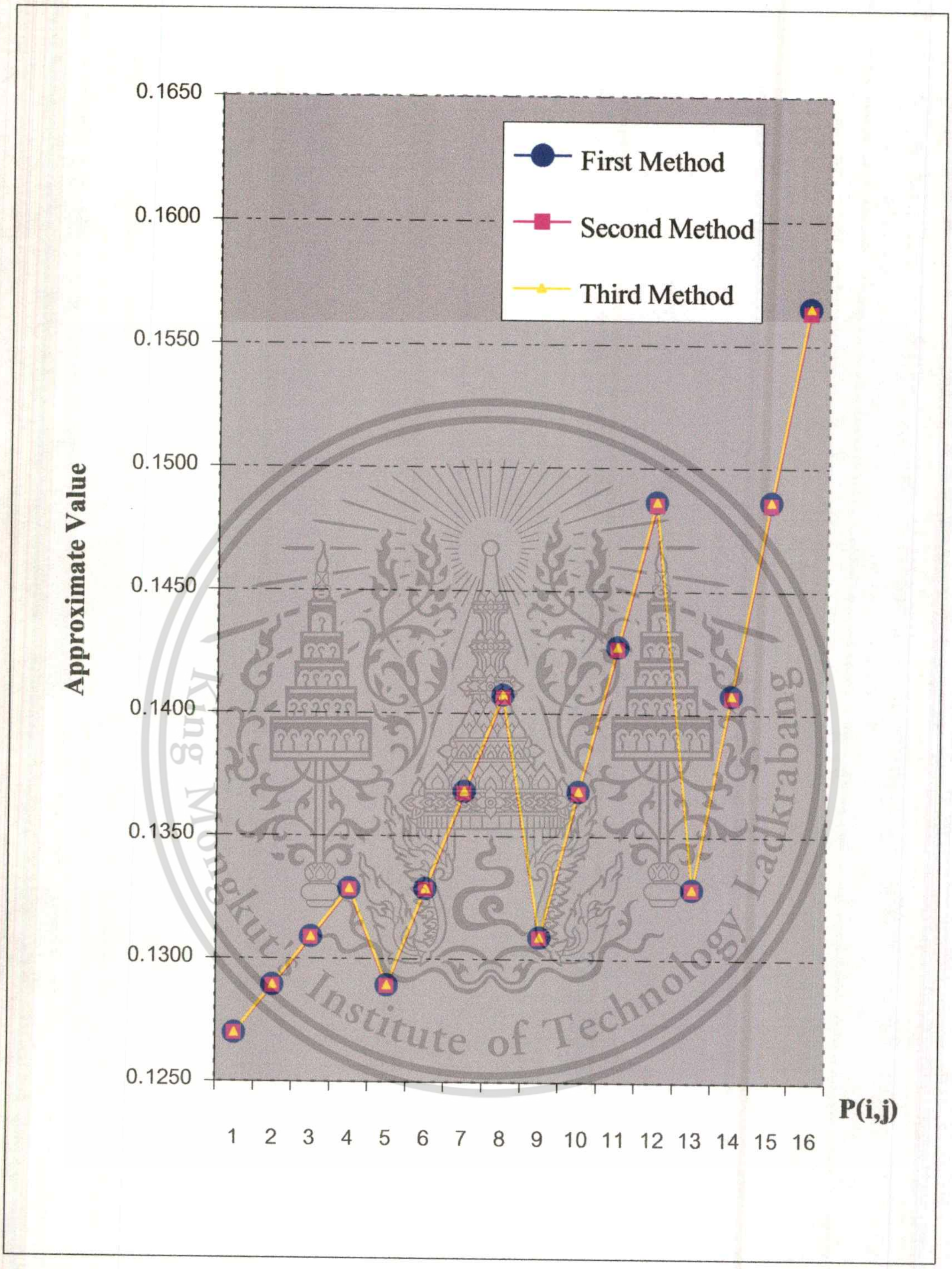

Figure 4.13 Graph of the result, $u_1(x, y)$, from the three numerical methods of example 4.7 for k = h =4.

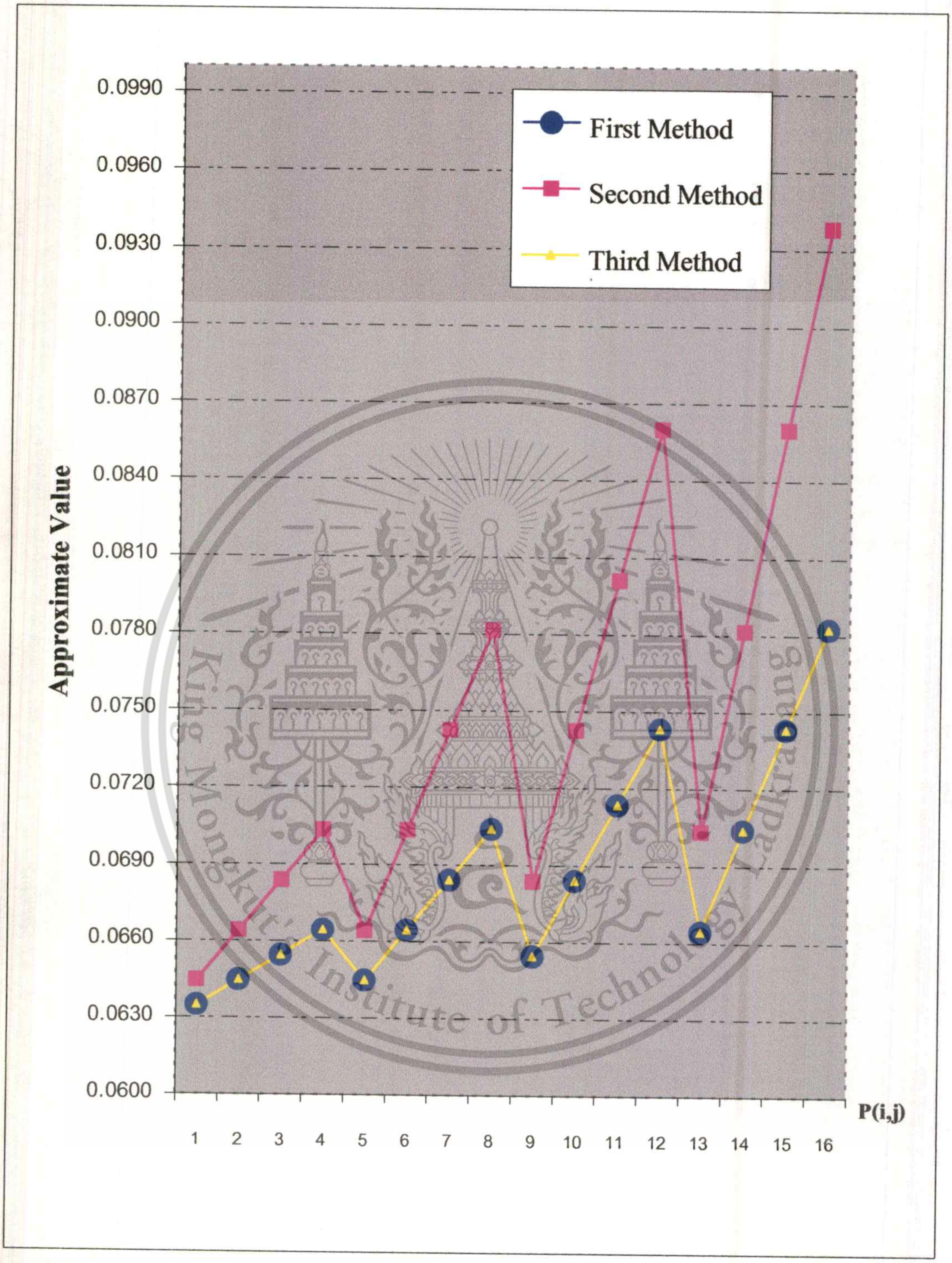

Figure 4.14 Graph of the result, $u_2(x, y)$, from the three numerical methods of example 4.7 for k = h =4.

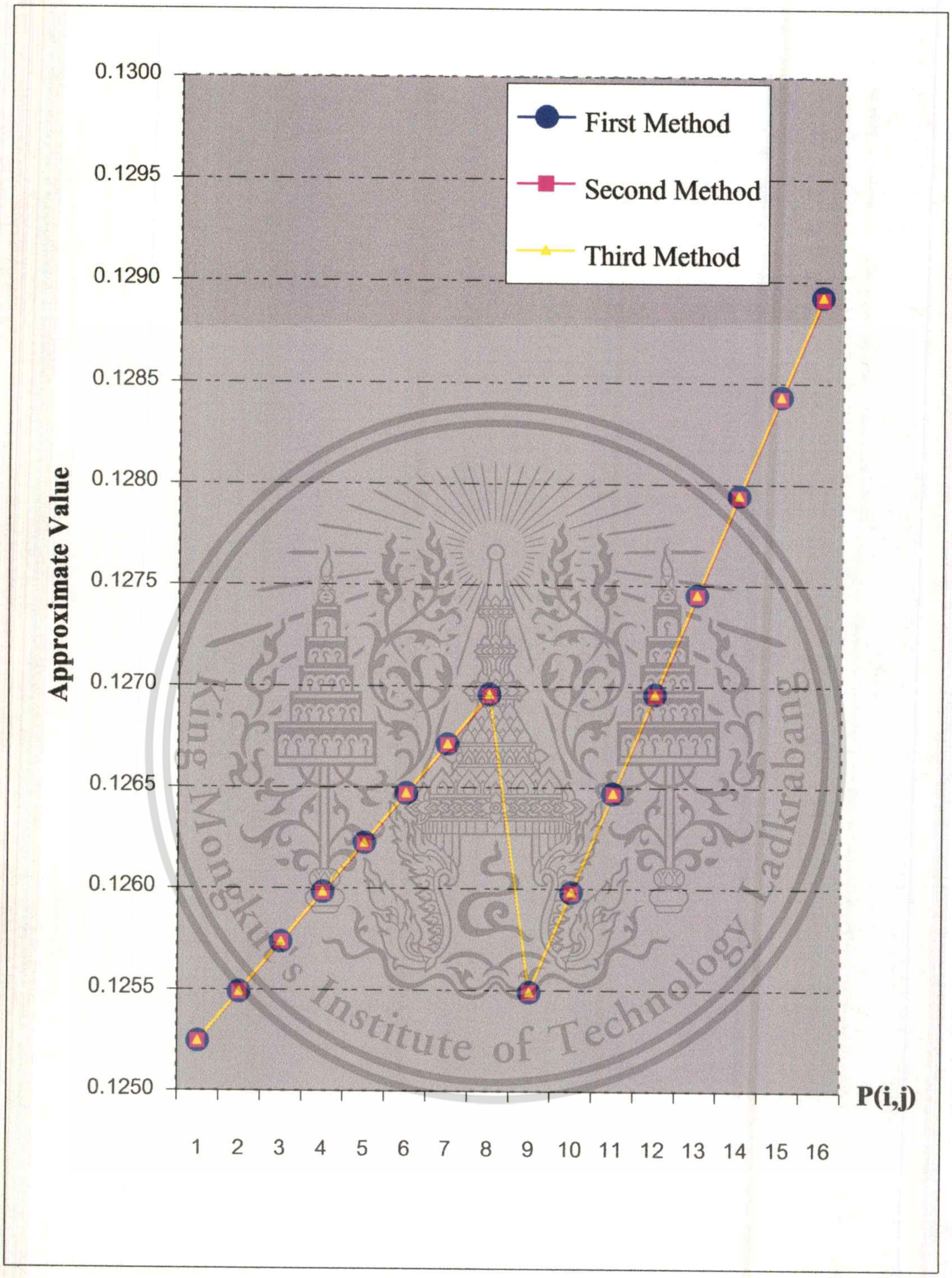

Figure 4.15 Graph of the result, $u_1(x, y)$, from the three numerical methods of example 4.7 for k = h =8.

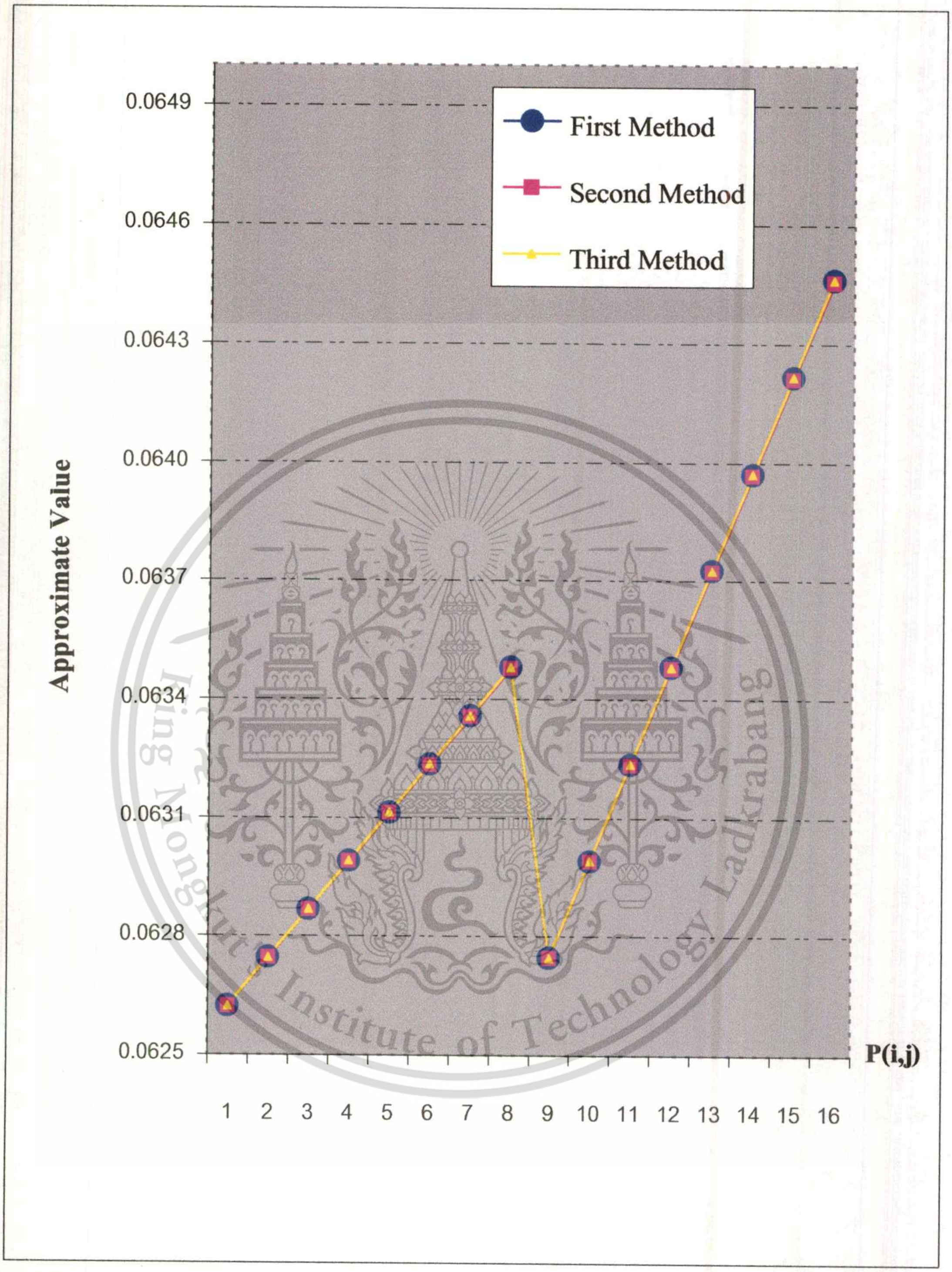

Figure 4.16 Graph of the result, $u_2(x, y)$, from the three numerical methods of example 4.7 for k = h =8.

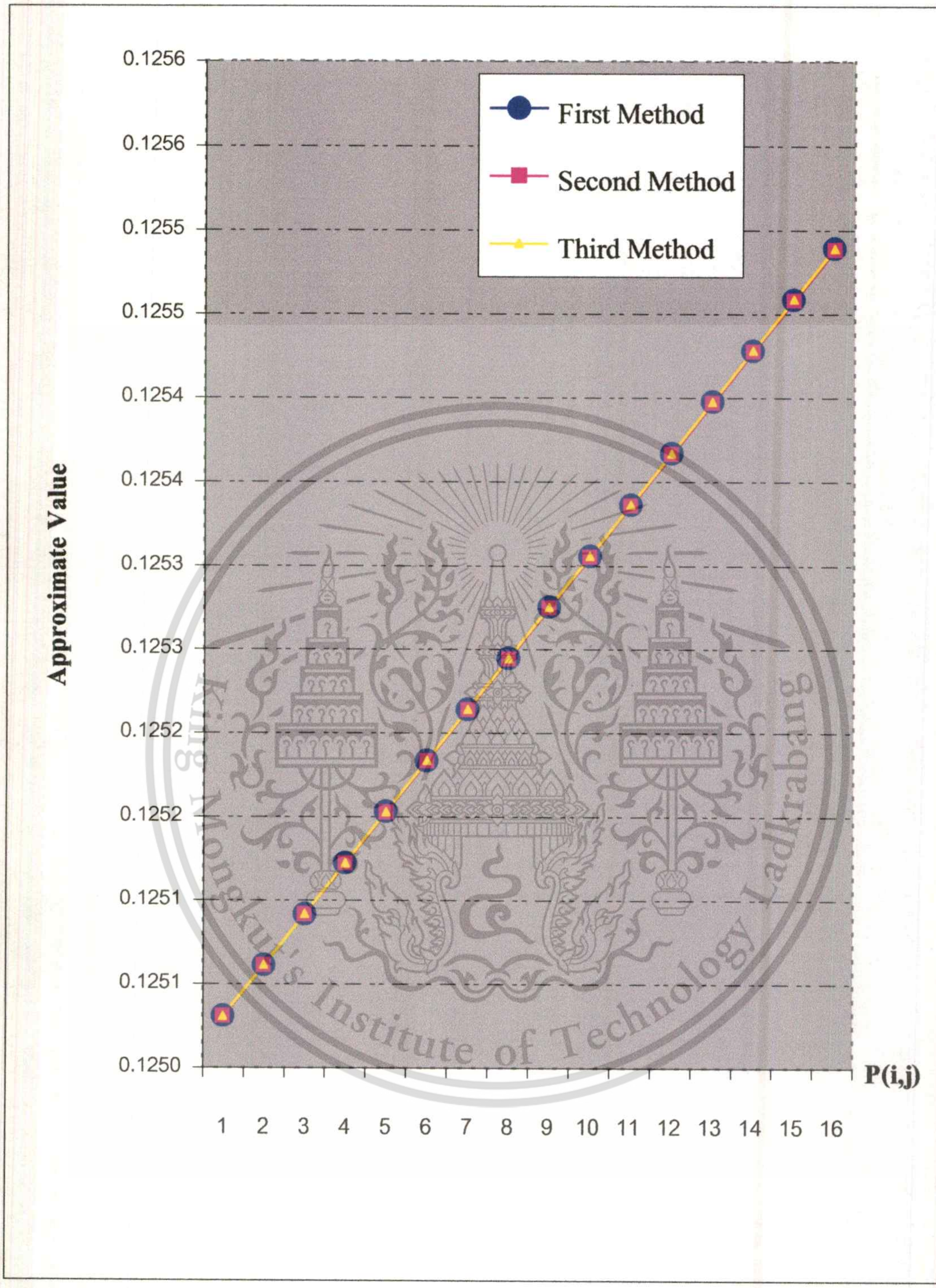

Figure 4.17 Graph of the result, $u_1(x, y)$, from the three numerical methods of example 4.7 for k = h = 16.

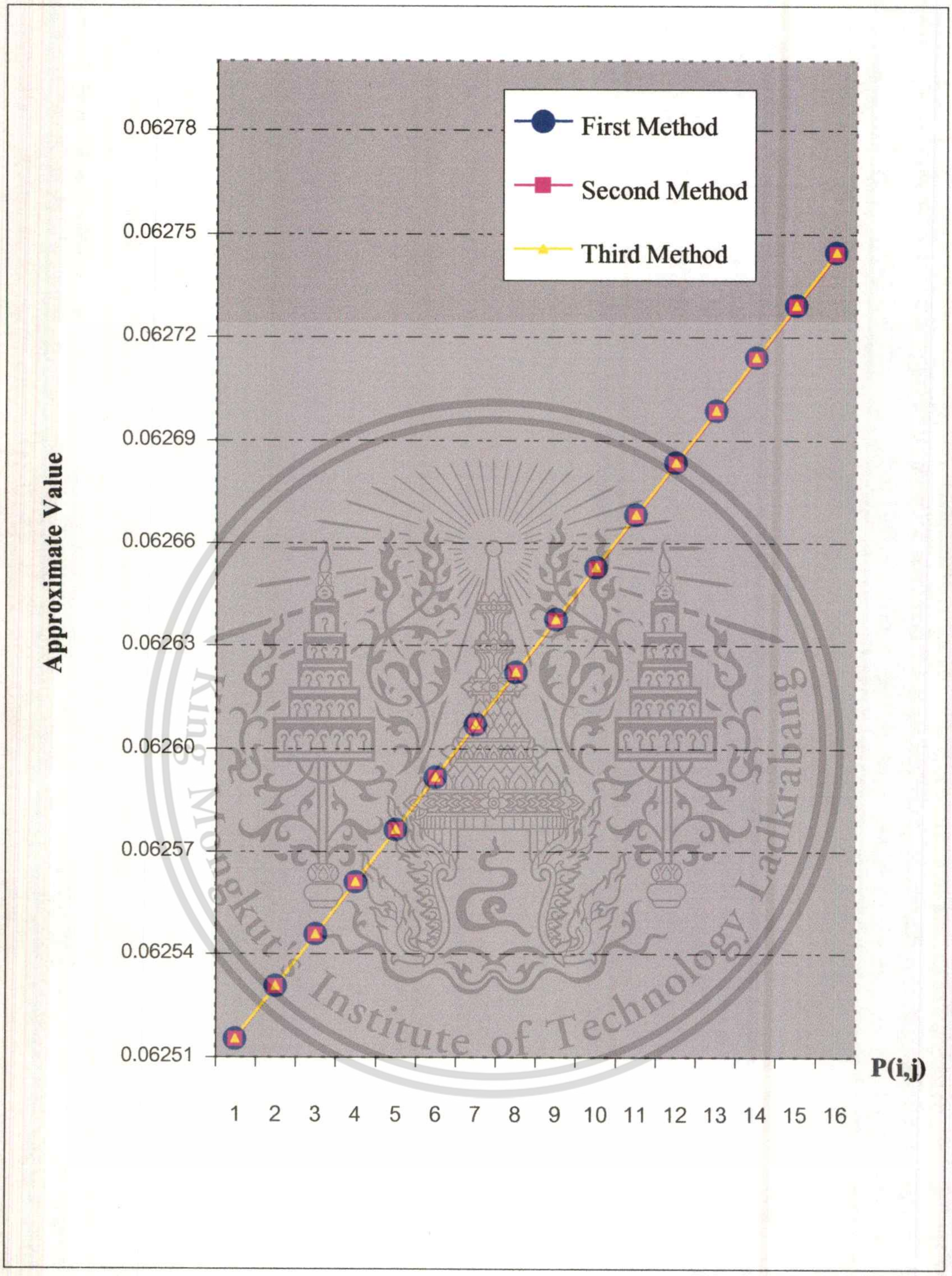

Figure 4.18 Graph of the result, $u_2(x, y)$, from the three numerical methods of example 4.7 for k = h =16.

CHAPTER 5

CONCLUSION AND SUGGESTIONS

In this chapter, we select the best of the three numerical methods by considering the mean of error from the examples in chapter 4.

5.1 Conclusion.

From the examples of chapter 4 . We summarize the mean of error for 4, 8 and 16 partitions which shown in table 5.1-5.3 and the ranking of these errors shown in table 5.4-5.6.

Table 5.1 The mean of errors from examples in chapter 4 for 4 partitions.

Example		Mean of Error		
		First Method	Second Method	Third Method
$u_1(x,y)$	Example 4.5	0.0000001594	0.0000000177	0.0000000603
	Example 4.6	0.0000000446	0.0000000012	0.0000000137
	Example 4.7	0.0000007858	0.0000000476	0.0000007857
$u_2(x,y)$	Example 4.5	0.0000001594	0.0000000177	0.0000000603
	Example 4.6	0.0000000268	0.0000000007	0.0000000082
	Example 4.7	0.0000003929	0.0000000476	0.0000003929

Table 5.2 The mean of errors from examples in chapter 4 for 8 partitions.

Example		Mean of Error		
		First Method	Second Method	Third Method
$u_1(x,y)$	Example 4.5	0.0000000055	0.0000000003	0.0000000040
	Example 4.6	0.0000000046	0.0000000001	0.0000000037
	Example 4.7	0.0000000773	0.0000000047	0.0000000773
$u_2(x,y)$	Example 4.5	0.0000000055	0.0000000003	0.0000000040
	Example 4.6	0.0000000028	0.0000000000	0.0000000022
	Example 4.7	0.0000000387	0.0000000024	0.0000000386

Table 5.3 The mean of errors from examples in chapter 4 for 16 partitions.

Example		Mean of Error		
		First Method	Second Method	Third Method
$u_1(x,y)$	Example 4.5	0.0000000443	0.0000000014	0.0000000407
	Example 4.6	0.0000000012	0.0000000000	0.0000000000
	Example 4.7	0.0000000085	0.0000010777	0.0000000085
$u_2(x,y)$	Example 4.5	0.0000000443	0.0000000014	0.0000000407
	Example 4.6	0.0000000007	0.0000000000	0.0000000007
	Example 4.7	0.0000000426	0.0000005388	0.0000000043

Table 5.4 The ranking of mean of errors from table 5.1 for 4 partitions.

Example		Mean of Error		
		First Method	Second Method	Third Method
$u_1(x,y)$	Example 4.5	3rd	1st	2nd
	Example 4.6	3rd	1st	2nd
	Example 4.7	3rd	1st	2nd
$u_2(x,y)$	Example 4.5	3rd	1st	2nd
	Example 4.6	3rd	1st	2nd
	Example 4.7	3rd	1st	2nd

Table 5.5 The ranking of mean of errors from table 5.2 for 8 partitions.

Example		Mean of Error		
		First Method	Second Method	Third Method
$u_1(x,y)$	Example 4.5	3rd	1st	2nd
	Example 4.6	3rd	1st	2nd
	Example 4.7	3rd	1st	2nd
$u_2(x,y)$	Example 4.5	3rd	1st	2nd
	Example 4.6	3rd	1st	2nd
	Example 4.7	3rd	1st	2nd

Table 5.6 The ranking of mean of errors from table 5.3 for 16 partitions.

Example		Mean of Error		
		First Method	Second Method	Third Method
$u_1(x,y)$	Example 4.5	3rd	1st	2nd
	Example 4.6	3rd	1st	2nd
	Example 4.7	3rd	1st	2nd
$u_2(x,y)$	Example 4.5	3rd	1st	2nd
	Example 4.6	3rd	1st	2nd
	Example 4.7	3rd	1st	2nd

Where, the first method is the Backward Finite Difference Approximation and the Two Points Newton's Divided Difference Method. The second methods is the Backward Finite Difference Approximation and the k points Newton's Divided Difference Method. The third methods is the Backward Finite Difference Approximation and the Lagrange Interpolation Method.

We can conclude that the above three numerical methods are accepted to solve the iterative partial differential equation and the more partitions, the more accuracy. However, the second method give minimum mean of error in all examples and the time used for solving the problem is more than the others because this method used operations in calculation more than the others. Moreover the first method used the time in solving the problem less than the others.

5.2 Suggestions.

We suggest to extend from two dimensions to three or more dimensions for further study.

Example, we show only three dimensions and solve this kind of problem by using only the Backward Finite Difference Approximation and the Newton's Divided Difference Method and the other two methods are kept for further study.

First, we consider the Backward Finite Difference Approximation to find the value of $u(x,y,z)$.

Since
$$\frac{\partial u(x,y,z)}{\partial x}=\frac{u_1(x,y,z)-u_1(x-h,y,z)}{h} \tag{5.1}$$

and
$$\frac{\partial^2 u(x,y,z)}{\partial x\partial y}=\frac{\dfrac{\partial u_1(x,y,z)}{\partial y}-\dfrac{\partial u_1(x-h,y,z)}{\partial y}}{h}$$

$$=\frac{1}{h}\left(\frac{u_1(x,y,z)-u_1(x,y-h,z)}{h}-\frac{u_1(x-h,y,z)-u_1(x-h,y-h,z)}{h}\right)$$

$$= \frac{u_1(x,y,z) - u_1(x,y-h,z) - u_1(x-h,y,z) + u_1(x-h,y-h,z)}{h^2} \tag{5.2}$$

and then
$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} = \frac{1}{h^2}\left(\frac{\partial u_1(x,y,z)}{\partial z} - \frac{\partial u_1(x,y-h,z)}{\partial z} \cdots - \frac{\partial u_1(x-h,y,z)}{\partial z} + \frac{\partial u_1(x-h,y-h,z)}{\partial z}\right)$$

$$= \frac{1}{h^2}\left(\frac{u_1(x,y,z) - u_1(x,y,z-h)}{h} - \frac{u_1(x,y-h,z) - u_1(x,y-h,z-h)}{h} \cdots - \frac{u_1(x-h,y,z) - u_1(x-h,y,z-h)}{h} + \frac{u_1(x-h,y-h,z) - u_1(x-h,y-h,z-h)}{h}\right). \tag{5.3}$$

Form equation (5.1) we write $\frac{\partial^n u(x)}{\partial x_1, \partial x_2, \ldots, \partial x_n}$ in form of three dimensions as follow

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} = \begin{bmatrix} \frac{\partial^3 u_1(x,y,z)}{\partial x \partial y \partial z} \\ \frac{\partial^3 u_2(x,y,z)}{\partial x \partial y \partial z} \\ \frac{\partial^3 u_3(x,y,z)}{\partial x \partial y \partial z} \end{bmatrix} = \begin{bmatrix} f_1(x,y,z,u_1(x,y,z),u_1^2(x,y,z),\ldots,u_1^m(x,y,z)) \\ f_2(x,y,z,u_1(x,y,z),u_2^2(x,y,z),\ldots,u_2^m(x,y,z)) \\ f_3(x,y,z,u_1(x,y,z),u_3^2(x,y,z),\ldots,u_3^m(x,y,z)) \end{bmatrix}$$

for $m = 1,2,3,\ldots$. (5.4)

From equation (5.3) and (5.4), we get

$$f_1(x,y,,z,u_1(x,y,z),u_1^2(x,y,z),\ldots,u_1^m(x,y,z)) = \frac{1}{h^3}[u_1(x,y,z) - u_1(x,y,z-h) - u_1(x,y-h,z) + u_1(x,y-h,z-h) - u_1(x-h,y,z) + u_1(x-h,y,z-h) + u_1(x-h,y-h,z) - u_1(x-h,y-h,z-h)]. \tag{5.5}$$

Therefore

$$u_1(x,y,z) = h^3 f_1(x,y,,z,u_1(x,y,z),u_1^2(x,y,z),\ldots,u_1^m(x,y,z)) - u_1(x,y-h,z-h) - u_1(x-h,y,z-h) - u_1(x-h,y-h,z) + u_1(x,y,z-h) + u_1(x,y-h,z) + u_1(x-h,y,z) + u_1(x-h,y-h,z-h). \tag{5.6}$$

In the same way, we get $u_2(x,y,z)$ and $u_3(x,y,z)$ with respectively as follows

$$u_2(x,y,z) = h^3 f_2(x,y,,z,u_2(x,y,z),u_2^2(x,y,z),\ldots,u_2^m(x,y,z)) - u_2(x,y-h,z-h)$$
$$- u_2(x-h,y,z-h) - u_2(x-h,y-h,z) + u_2(x,y,z-h) + u_2(x,y-h,z)$$
$$+ u_2(x-h,y,z) + u_2(x-h,y-h,z-h) \quad (5.7)$$

and

$$u_3(x,y,z) = h^3 f_3(x,y,,z,u_3(x,y,z),u_3^2(x,y,z),\ldots,u_3^m(x,y,z)) - u_3(x,y-h,z-h)$$
$$- u_3(x-h,y,z-h) - u_3(x-h,y-h,z) + u_3(x,y,z-h) + u_3(x,y-h,z)$$
$$+ u_3(x-h,y,z) + u_3(x-h,y-h,z-h). \quad (5.8)$$

From (5.6) - (5.8), we get the solution of $u(x,y,z)$. We have to know the value of function $f(x,y,z,u(x,y,z),u^2(x,y,z),\ldots,u^m(x,y,z))$.
In order to find the value of f, we have to know the values $u(x,y,z),u^2(x,y,z),\ldots,$ $u^m(x,y,z)$ which we use the Newton's Divided Difference Method to solve.

First, we use the Newton's Divided Difference Method to find the value of

$$u^2(x,y,z) = \begin{bmatrix} u_1(u_1(x,y,z),u_2(x,y,z),u_3(x,y,z)) \\ u_2(u_1(x,y,z),u_2(x,y,z),u_3(x,y,z)) \\ u_3(u_1(x,y,z),u_2(x,y,z),u_3(x,y,z)) \end{bmatrix}. \quad (5.9)$$

Let $u_1(x,y,z) = x^*, \quad u_2(x,y,z) = y^*, u_3(x,y,z) = z^*.$ (5.10)

So $$u^2(x,y,z) = u(u(x^*,y^*,z^*)) = \begin{bmatrix} u_1(x^*,y^*,z^*) \\ u_2(x^*,y^*,z^*) \\ u_3(x^*,y^*,z^*) \end{bmatrix} = \begin{bmatrix} u_1(r) \\ u_2(r) \\ u_3(r) \end{bmatrix}. \quad (5.11)$$

From the Newton's Interpolation Polynomials of degree n $,P_n^{kj}(x^*)$, for $k,j = 0,1,2,\ldots,n$ that

$$P_n^{kj}(x^*) = a_0 + a_1(x^* - x_0) + \cdots + a_n(x^* - x_0)(x^* - x_1)\cdots(x^* - x_{n-1}) \quad (5.12)$$

where

$a_0 = u(x_0^{kj}, y_0^{kj})$

$$a_1 = u[x_1^{kj}, x_0^{kj}] = \frac{u(x_1^{kj}, y_1^{kj}) - u(x_0^{kj}, y_0^{kj})}{x_1 - x_0}$$

$$a_2 = u[x_2^{kj}, x_1^{kj}, x_0^{kj}] = \frac{u[x_2^{kj}, x_1^{kj}] - u[x_1^{kj}, x_0^{kj}]}{x_2 - x_0} \tag{5.13}$$

$$\vdots$$

$$a_n = u[x_n^{kj}, x_{n-1}^{kj}, \ldots, x_1^{kj}, x_0^{kj}] = \frac{u[x_n^{kj}, x_{n-1}^{kj}, \ldots, x_1^{kj}] - u[x_{n-1}^{kj}, x_{n-2}^{kj}, \ldots, x_0^{kj}]}{x_n - x_0}$$

and

$$P_n^k(y^*) = b_0 + b_1(y^* - y_0) + \cdots + b_n(y^* - y_0)(y^* - y_1)\cdots(y^* - y_{n-1}) \tag{5.14}$$

where

$$b_0 = u(x^*, y_0^k)$$

$$b_1 = u[y_1^k, y_0^k] = \frac{u(x_1^k, y_1^k) - u(x_0^k, y_0^k)}{y_1 - y_0}$$

$$b_2 = u[y_2^k, y_1^k, y_0^k] = \frac{u[y_2^k, y_1^k] - u[y_1^k, y_0^k]}{y_2 - y_0} \tag{5.15}$$

$$\vdots$$

$$b_n = u[y_n^k, y_{n-1}^k, \ldots, y_1^k, y_0^k] = \frac{u[y_n^k, y_{n-1}^k, \ldots, y_1^k] - u[y_{n-1}^k, y_{n-2}^k, \ldots, y_0^k]}{y_n - y_0}$$

and then

$$P_n(z^*) = d_0 + d_1(z^* - z_0) + \cdots + d_n(z^* - z_0)(z^* - z_1)\cdots(z^* - z_{n-1}) \tag{5.16}$$

where

$$d_0 = u(x^*, y^*, z_0)$$

$$d_1 = u[z_1, z_0] = \frac{u(x_1, y_1, z_0) - u(x_0, y_0, z_0)}{z_1 - z_0}$$

$$d_2 = u[z_2, z_1, z_0] = \frac{u[z_2, z_1] - u[z_1, z_0]}{z_2 - z_0} \tag{5.17}$$

$$\vdots$$

$$d_n = u[z_n, z_{n-1}, \ldots, z_1, z_0] = \frac{u[z_n, z_{n-1}, \ldots, z_1] - u[z_{n-1}, z_{n-2}, \ldots, z_0]}{z_n - z_0}.$$

In the same way, we use the values of $u^2(x,y,z)$ to find $u^3(x,y,z)\ldots$ and use $u^{m-1}(x,y,z)$ to find $u^m(x,y,z)$. Then we use $u^{(1)}(x,y,z)$ to find $u^{(2)}(x,y,z)$ and continue this process until

$$\sum_{\ell=0}^{\ell=k}\left\| u_1^{(\ell+1)}(x,y,z)-u_1^{(\ell)}(x,y,z)\right\| < \varepsilon,$$
$$\sum_{\ell=0}^{\ell=k}\left\| u_2^{(\ell+1)}(x,y,z)-u_2^{(\ell)}(x,y,z)\right\| < \varepsilon \quad \text{and}$$
$$\sum_{\ell=0}^{\ell=k}\left\| u_3^{(\ell+1)}(x,y,z)-u_3^{(\ell)}(x,y,z)\right\| < \varepsilon \quad \text{for a sufficiently small } \varepsilon.$$

We will give you some examples in finding solution of problem for three dimensions.

5.2.1 Analytical Solution.

Example 5.1 Find the solution, in $Z=[0,1]\times[0,1]\times[0,1]$, of the equation

$$\frac{\partial^3 u(x,y,z)}{\partial x\partial y\partial z} = \begin{bmatrix} \frac{1}{4}+\frac{23}{96}xyz-\frac{1}{96}x^2y^2z^2+u_1(u_1,u_2,u_3)-u_1(x,y,z) \\ 0 \\ \frac{1}{2}-\frac{23}{48}xyz-\frac{1}{48}x^2y^2z^2+u_2(u_1,u_2,u_3)-u_2(x,y,z) \end{bmatrix}$$

with the initial conditions

$$u(0,y,z)=u(x,0,z)=u(x,y,0)=u(0,0,z)=u(0,y,0)=u(x,0,0)$$
$$=u(0,0,0)=\begin{bmatrix}\frac{1}{4} & \frac{1}{3} & 0\end{bmatrix}^T.$$

We have $$g(x,y,z)=\begin{bmatrix}\frac{1}{4}\\ \frac{1}{3}\\ 0\end{bmatrix}.$$

Solution. Let $u_0(x,y,z) = g(x,y,z) = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \\ 0 \end{bmatrix}$.

Then by the equation (4.4), we get

$$u_1(x,y,z) = g(x;y,z) + \int_0^z\int_0^y\int_0^x f\left(x,y,z,u_0(x,y,z),u_0^2(x,y,z)\right)dxdydz$$

$$= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \\ 0 \end{bmatrix} + \int_0^z\int_0^y\int_0^x \begin{bmatrix} \frac{1}{4}+\frac{23}{96}xyz-\frac{1}{96}x^2y^2z^2 \\ 0 \\ \frac{1}{2}-\frac{23}{48}xyz-\frac{1}{48}x^2y^2z^2 \end{bmatrix} dxdydz$$

$$= \begin{bmatrix} \frac{1}{4}+\frac{1}{4}xyz+\frac{23}{768}x^2y^2z^2-\frac{1}{768}x^3y^3z^3 \\ \frac{1}{3} \\ \frac{1}{2}xyz-\frac{23}{384}x^2y^2z^2-\frac{1}{384}x^3y^3z^3 \end{bmatrix}.$$

$$u_2(x,y,z) = g(x,y,z) + \int_0^z\int_0^y\int_0^x f\left(x,y,z,u_1(x,y,z),u_1^2(x,y,z)\right)dxdydz.$$

Since $u_1(u_1(u_1(x,y,z))) =$

$$\begin{bmatrix} \frac{1}{4}+\frac{1}{96}xyz+\frac{4079}{442368}x^2y^2z^2+\frac{1579}{42467328}x^3y^3z^3+\text{higher power terms} \\ \frac{1}{3} \\ \frac{1}{48}xyz-\frac{4033}{221184}x^2y^2z^2-\frac{2065}{7077888}x^3y^3z^3+\quad\text{higher power terms} \end{bmatrix}.$$

Thus

$$u_2(x,y,z)=\begin{bmatrix}\frac{1}{4}\\ \frac{1}{3}\\ 0\end{bmatrix}+\int_0^z\int_0^y\int_0^x\left(\begin{bmatrix}\frac{1}{4}+\frac{23}{96}xyz-\frac{1}{96}x^2y^2z^2\\ 0\\ \frac{1}{2}-\frac{23}{48}xyz-\frac{1}{48}x^2y^2z^2\end{bmatrix}\right.$$

$$+\begin{bmatrix}\frac{1}{4}+\frac{1}{96}xyz+\frac{4079}{442368}x^2y^2z^2+\frac{1579}{42467328}x^3y^3z^3+\text{higher power terms}\\ \frac{1}{3}\\ \frac{1}{48}xyz-\frac{4033}{221184}x^2y^2z^2-\frac{2065}{7077888}x^3y^3z^3+\text{higher power terms}\end{bmatrix}$$

$$\left.-\begin{bmatrix}\frac{1}{4}+\frac{1}{4}xyz+\frac{23}{768}x^2y^2z^2-\frac{1}{768}x^3y^3z^3\\ \frac{1}{3}\\ \frac{1}{2}xyz-\frac{23}{384}x^2y^2z^2-\frac{1}{384}x^3y^3z^3\end{bmatrix}\right)dxdydz.$$

We obtain

$$\begin{bmatrix}\frac{1}{4}+\frac{1}{4}xyz-\frac{13777}{11943936}x^3y^3z^3+\frac{56875}{2717908992}x^4y^4z^4+\text{higher power terms}\\ \frac{1}{3}\\ \frac{1}{2}xyz-\frac{23}{192}x^2y^2z^2+\frac{4607}{5971968}x^3y^3z^3+\frac{16367}{452984823}x^4y^4z^4+\text{higher power terms}\end{bmatrix}$$

.

Thus we obtain $u_k(x,y,z)$ tends to $\begin{bmatrix} \frac{1}{4}+\frac{xyz}{4} \\ \frac{1}{3} \\ \frac{xyz}{2} \end{bmatrix}$ as k tends to infinity.

Therefor the solution of the given equation is

$$u(x,y,z)=\begin{bmatrix} \frac{1}{4}+\frac{xyz}{4} \\ \frac{1}{3} \\ \frac{xyz}{2} \end{bmatrix}.$$

5.2 Numerical Solutions.

Example 5.2 Find the solution, in $Z=[0,1]\times[0,1]\times[0,1]$, of the equation

$$\frac{\partial^3 u(x,y,z)}{\partial x\partial y\partial z}=\begin{bmatrix} \frac{1}{4}+\frac{23}{96}xyz-\frac{1}{96}x^2y^2z^2+u_1(u_1,u_2,u_3)-u_1(x,y,z) \\ 0 \\ \frac{1}{2}-\frac{23}{48}xyz-\frac{1}{48}x^2y^2z^2+u_2(u_1,u_2,u_3)-u_2(x,y,z) \end{bmatrix}$$

with the initial conditions

$$u(0,y,z)=u(x,0,z)=u(x,y,0)=u(0,0,z)=u(0,y,0)=u(x,0,0)$$

$$=u(0,0,0)=\begin{bmatrix} \frac{1}{4} & \frac{1}{3} & 0 \end{bmatrix}^T.$$

The exact solution is $u(x,y,z)=\begin{bmatrix} \frac{1}{4}+\frac{xyz}{4} \\ \frac{1}{3} \\ \frac{xyz}{2} \end{bmatrix}$. We shall divided the interval [0,1], for x-axis, y-axis and z-axis, into 4 equally subintervals and repeat the methods until the error, ε, less than 0.000005.

We obtain the results as follows in table 5.7 and in table A1.1 - A1.3 of appendix A.

Table 5.7 The result, $u(x,y)$, of example 5.2.

	$u_1(x,y,z)$	$u_2(x,y,z)$	$u_3(x,y,z)$
The number of iterations	25	25	25
Mean of Error	0.0	0.0	0.00000000001

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APPENDIX A

NUMERICAL SOLUTIONS

Table A1.1 The result, $u_1(x, y)$, of example 5.2 .

x	y	z	Approximate value	Exact value	Absolute value
0.2500	0.2500	0.2500	0.2539063	0.2539063	0.0000000
0.2500	0.2500	0.5000	0.2578125	0.2578125	0.0000000
0.2500	0.2500	0.7500	0.2617188	0.2617188	0.0000000
0.2500	0.2500	1.0000	0.2656250	0.2656250	0.0000000
0.2500	0.5000	0.2500	0.2578125	0.2578125	0.0000000
0.2500	0.5000	0.5000	0.2656250	0.2656250	0.0000000
0.2500	0.5000	0.7500	0.2734375	0.2734375	0.0000000
0.2500	0.5000	1.0000	0.2812500	0.2812500	0.0000000
0.2500	0.7500	0.2500	0.2617188	0.2617188	0.0000000
0.2500	0.7500	0.5000	0.2734375	0.2734375	0.0000000
0.2500	0.7500	0.7500	0.2851563	0.2851563	0.0000000
0.2500	0.7500	1.0000	0.2968750	0.2968750	0.0000000
0.2500	1.0000	0.2500	0.2656250	0.2656250	0.0000000
0.2500	1.0000	0.5000	0.2812500	0.2812500	0.0000000
0.2500	1.0000	0.7500	0.2968750	0.2968750	0.0000000
0.2500	1.0000	1.0000	0.3125000	0.3125000	0.0000000
0.5000	0.2500	0.2500	0.2578125	0.2578125	0.0000000
0.5000	0.2500	0.5000	0.2656250	0.2656250	0.0000000
0.5000	0.2500	0.7500	0.2734375	0.2734375	0.0000000
0.5000	0.2500	1.0000	0.2812500	0.2812500	0.0000000
0.5000	0.5000	0.2500	0.2656250	0.2656250	0.0000000
0.5000	0.5000	0.5000	0.2812500	0.2812500	0.0000000
0.5000	0.5000	0.7500	0.2968750	0.2968750	0.0000000
0.5000	0.5000	1.0000	0.3125000	0.3125000	0.0000000
0.5000	0.7500	0.2500	0.2734375	0.2734375	0.0000000
0.5000	0.7500	0.5000	0.2968750	0.2968750	0.0000000

Table A1.1 (Cont.)

x	y	z	Approximate value	Exact value	Absolute value
0.5000	0.7500	0.7500	0.3203125	0.3203125	0.0000000
0.5000	0.7500	1.0000	0.3437500	0.3437500	0.0000000
0.5000	1.0000	0.2500	0.2812500	0.2812500	0.0000000
0.5000	1.0000	0.5000	0.3125000	0.3125000	0.0000000
0.5000	1.0000	0.7500	0.3437500	0.3437500	0.0000000
0.5000	1.0000	1.0000	0.3750000	0.3750000	0.0000000
0.7500	0.2500	0.2500	0.2617188	0.2617188	0.0000000
0.7500	0.2500	0.5000	0.2734375	0.2734375	0.0000000
0.7500	0.2500	0.7500	0.2851563	0.2851563	0.0000000
0.7500	0.2500	1.0000	0.2968750	0.2968750	0.0000000
0.7500	0.5000	0.2500	0.2734375	0.2734375	0.0000000
0.7500	0.5000	0.5000	0.2968750	0.2968750	0.0000000
0.7500	0.5000	0.7500	0.3203125	0.3203125	0.0000000
0.7500	0.5000	1.0000	0.3437500	0.3437500	0.0000000
0.7500	0.7500	0.2500	0.2851563	0.2851563	0.0000000
0.7500	0.7500	0.5000	0.3203125	0.3203125	0.0000000
0.7500	0.7500	0.7500	0.3554688	0.3554688	0.0000000
0.7500	0.7500	1.0000	0.3906250	0.3906250	0.0000000
0.7500	1.0000	0.2500	0.2968750	0.2968750	0.0000000
0.7500	1.0000	0.5000	0.3437500	0.3437500	0.0000000
0.7500	1.0000	0.7500	0.3906250	0.3906250	0.0000000
0.7500	1.0000	1.0000	0.4375000	0.4375000	0.0000000
1.0000	0.2500	0.2500	0.2656250	0.2656250	0.0000000
1.0000	0.2500	0.5000	0.2812500	0.2812500	0.0000000
1.0000	0.2500	0.7500	0.2968750	0.2968750	0.0000000
1.0000	0.2500	1.0000	0.3125000	0.3125000	0.0000000
1.0000	0.5000	0.2500	0.2812500	0.2812500	0.0000000
1.0000	0.5000	0.5000	0.3125000	0.3125000	0.0000000
1.0000	0.5000	0.7500	0.3437500	0.3437500	0.0000000
1.0000	0.5000	1.0000	0.3750000	0.3750000	0.0000000

Table A1.1 (Cont.)

x	y	z	Approximate value	Exact value	Absolute value
1.0000	0.7500	0.2500	0.2968750	0.2968750	0.0000000
1.0000	0.7500	0.5000	0.3437500	0.3437500	0.0000000
1.0000	0.7500	0.7500	0.3906250	0.3906250	0.0000000
1.0000	0.7500	1.0000	0.4375000	0.4375000	0.0000000
1.0000	1.0000	0.2500	0.3125000	0.3125000	0.0000000
1.0000	1.0000	0.5000	0.3750000	0.3750000	0.0000000
1.0000	1.0000	0.7500	0.4375000	0.4375000	0.0000000
1.0000	1.0000	1.0000	0.5000000	0.5000000	0.0000000

The 25^{th} iteration.

The mean of error is 0.0000000.

Table A1.2 The result, $u_2(x, y)$, of example 5.2 .

x	y	z	Approximate value	Exact value	Absolute value
0.2500	0.2500	0.2500	0.3333333	0.3333333	0.0000000
0.2500	0.2500	0.5000	0.3333333	0.3333333	0.0000000
0.2500	0.2500	0.7500	0.3333333	0.3333333	0.0000000
0.2500	0.2500	1.0000	0.3333333	0.3333333	0.0000000
0.2500	0.5000	0.2500	0.3333333	0.3333333	0.0000000
0.2500	0.5000	0.5000	0.3333333	0.3333333	0.0000000
0.2500	0.5000	0.7500	0.3333333	0.3333333	0.0000000
0.2500	0.5000	1.0000	0.3333333	0.3333333	0.0000000
0.2500	0.7500	0.2500	0.3333333	0.3333333	0.0000000
0.2500	0.7500	0.5000	0.3333333	0.3333333	0.0000000
0.2500	0.7500	0.7500	0.3333333	0.3333333	0.0000000
0.2500	0.7500	1.0000	0.3333333	0.3333333	0.0000000
0.2500	1.0000	0.2500	0.3333333	0.3333333	0.0000000
0.2500	1.0000	0.5000	0.3333333	0.3333333	0.0000000
0.2500	1.0000	0.7500	0.3333333	0.3333333	0.0000000
0.2500	1.0000	1.0000	0.3333333	0.3333333	0.0000000

Table A1.2 (Cont.)

x	y	z	Approximate value	Exact value	Absolute value
0.5000	0.2500	0.2500	0.3333333	0.3333333	0.0000000
0.5000	0.2500	0.5000	0.3333333	0.3333333	0.0000000
0.5000	0.2500	0.7500	0.3333333	0.3333333	0.0000000
0.5000	0.2500	1.0000	0.3333333	0.3333333	0.0000000
0.5000	0.5000	0.2500	0.3333333	0.3333333	0.0000000
0.5000	0.5000	0.5000	0.3333333	0.3333333	0.0000000
0.5000	0.5000	0.7500	0.3333333	0.3333333	0.0000000
0.5000	0.5000	1.0000	0.3333333	0.3333333	0.0000000
0.5000	0.7500	0.2500	0.3333333	0.3333333	0.0000000
0.5000	0.7500	0.5000	0.3333333	0.3333333	0.0000000
0.5000	0.7500	0.7500	0.3333333	0.3333333	0.0000000
0.5000	0.7500	1.0000	0.3333333	0.3333333	0.0000000
0.5000	1.0000	0.2500	0.3333333	0.3333333	0.0000000
0.5000	1.0000	0.5000	0.3333333	0.3333333	0.0000000
0.5000	1.0000	0.7500	0.3333333	0.3333333	0.0000000
0.5000	1.0000	1.0000	0.3333333	0.3333333	0.0000000
0.7500	0.2500	0.2500	0.3333333	0.3333333	0.0000000
0.7500	0.2500	0.5000	0.3333333	0.3333333	0.0000000
0.7500	0.2500	0.7500	0.3333333	0.3333333	0.0000000
0.7500	0.2500	1.0000	0.3333333	0.3333333	0.0000000
0.7500	0.5000	0.2500	0.3333333	0.3333333	0.0000000
0.7500	0.5000	0.5000	0.3333333	0.3333333	0.0000000
0.7500	0.5000	0.7500	0.3333333	0.3333333	0.0000000
0.7500	0.5000	1.0000	0.3333333	0.3333333	0.0000000
0.7500	0.7500	0.2500	0.3333333	0.3333333	0.0000000
0.7500	0.7500	0.5000	0.3333333	0.3333333	0.0000000
0.7500	0.7500	0.7500	0.3333333	0.3333333	0.0000000
0.7500	0.7500	1.0000	0.3333333	0.3333333	0.0000000
0.7500	1.0000	0.2500	0.3333333	0.3333333	0.0000000
0.7500	1.0000	0.5000	0.3333333	0.3333333	0.0000000

Table A1.2 (Cont.)

x	y	z	Approximate value	Exact value	Absolute value
0.7500	1.0000	0.7500	0.3333333	0.3333333	0.0000000
0.7500	1.0000	1.0000	0.3333333	0.3333333	0.0000000
1.0000	0.2500	0.2500	0.3333333	0.3333333	0.0000000
1.0000	0.2500	0.5000	0.3333333	0.3333333	0.0000000
1.0000	0.2500	0.7500	0.3333333	0.3333333	0.0000000
1.0000	0.2500	1.0000	0.3333333	0.3333333	0.0000000
1.0000	0.5000	0.2500	0.3333333	0.3333333	0.0000000
1.0000	0.5000	0.5000	0.3333333	0.3333333	0.0000000
1.0000	0.5000	0.7500	0.3333333	0.3333333	0.0000000
1.0000	0.5000	1.0000	0.3333333	0.3333333	0.0000000
1.0000	0.7500	0.2500	0.3333333	0.3333333	0.0000000
1.0000	0.7500	0.5000	0.3333333	0.3333333	0.0000000
1.0000	0.7500	0.7500	0.3333333	0.3333333	0.0000000
1.0000	0.7500	1.0000	0.3333333	0.3333333	0.0000000
1.0000	1.0000	0.2500	0.3333333	0.3333333	0.0000000
1.0000	1.0000	0.5000	0.3333333	0.3333333	0.0000000
1.0000	1.0000	0.7500	0.3333333	0.3333333	0.0000000
1.0000	1.0000	1.0000	0.3333333	0.3333333	0.0000000

The 25th iteration.

The mean of error is 0.0000000.

Table A1.3 The result, $u_3(x,y)$, of example 5.2 .

x	y	z	Approximate value	Exact value	Absolute value
0.2500	0.2500	0.2500	0.0078125	0.0078125	0.0000000
0.2500	0.2500	0.5000	0.0156250	0.0156250	0.0000000
0.2500	0.2500	0.7500	0.0234375	0.0234375	0.0000000
0.2500	0.2500	1.0000	0.0312500	0.0312500	0.0000000
0.2500	0.5000	0.2500	0.0156250	0.0156250	0.0000000
0.2500	0.5000	0.5000	0.0312500	0.0312500	0.0000000
0.2500	0.5000	0.7500	0.0468750	0.0468750	0.0000000

Table A1.3 (Cont.)

x	y	z	Approximate value	Exact value	Absolute value
0.2500	0.5000	1.0000	0.0625000	0.0625000	0.0000000
0.2500	0.7500	0.2500	0.0234375	0.0234375	0.0000000
0.2500	0.7500	0.5000	0.0468750	0.0468750	0.0000000
0.2500	0.7500	0.7500	0.0703125	0.0703125	0.0000000
0.2500	0.7500	1.0000	0.0937500	0.0937500	0.0000000
0.2500	1.0000	0.2500	0.0312500	0.0312500	0.0000000
0.2500	1.0000	0.5000	0.0625000	0.0625000	0.0000000
0.2500	1.0000	0.7500	0.0937500	0.0937500	0.0000000
0.2500	1.0000	1.0000	0.1250000	0.1250000	0.0000000
0.5000	0.2500	0.2500	0.0156250	0.0156250	0.0000000
0.5000	0.2500	0.5000	0.0312500	0.0312500	0.0000000
0.5000	0.2500	0.7500	0.0468750	0.0468750	0.0000000
0.5000	0.2500	1.0000	0.0625000	0.0625000	0.0000000
0.5000	0.5000	0.2500	0.0312500	0.0312500	0.0000000
0.5000	0.5000	0.5000	0.0625000	0.0625000	0.0000000
0.5000	0.5000	0.7500	0.0937500	0.0937500	0.0000000
0.5000	0.5000	1.0000	0.1250000	0.1250000	0.0000000
0.5000	0.7500	0.2500	0.0468750	0.0468750	0.0000000
0.5000	0.7500	0.5000	0.0937500	0.0937500	0.0000000
0.5000	0.7500	0.7500	0.1406250	0.1406250	0.0000000
0.5000	0.7500	1.0000	0.1875000	0.1875000	0.0000000
0.5000	1.0000	0.2500	0.0625000	0.0625000	0.0000000
0.5000	1.0000	0.5000	0.1250000	0.1250000	0.0000000
0.5000	1.0000	0.7500	0.1875000	0.1875000	0.0000000
0.5000	1.0000	1.0000	0.2500000	0.2500000	0.0000000
0.7500	0.2500	0.2500	0.0234375	0.0234375	0.0000000
0.7500	0.2500	0.5000	0.0468750	0.0468750	0.0000000
0.7500	0.2500	0.7500	0.0703125	0.0703125	0.0000000
0.7500	0.2500	1.0000	0.0937500	0.0937500	0.0000000
0.7500	0.5000	0.2500	0.0468750	0.0468750	0.0000000

Table A1.3 (Cont.)

x	y	z	Approximate value	Exact value	Absolute value
0.7500	0.5000	0.5000	0.0937500	0.0937500	0.0000000
0.7500	0.5000	0.7500	0.1406250	0.1406250	0.0000000
0.7500	0.5000	1.0000	0.1875000	0.1875000	0.0000000
0.7500	0.7500	0.2500	0.0703125	0.0703125	0.0000000
0.7500	0.7500	0.5000	0.1406250	0.1406250	0.0000000
0.7500	0.7500	0.7500	0.2109375	0.2109375	0.0000000
0.7500	0.7500	1.0000	0.2812500	0.2812500	0.0000000
0.7500	1.0000	0.2500	0.0937500	0.0937500	0.0000000
0.7500	1.0000	0.5000	0.1875000	0.1875000	0.0000000
0.7500	1.0000	0.7500	0.2812500	0.2812500	0.0000000
0.7500	1.0000	1.0000	0.3750000	0.3750000	0.0000000
1.0000	0.2500	0.2500	0.0312500	0.0312500	0.0000000
1.0000	0.2500	0.5000	0.0625000	0.0625000	0.0000000
1.0000	0.2500	0.7500	0.0937500	0.0937500	0.0000000
1.0000	0.2500	1.0000	0.1250000	0.1250000	0.0000000
1.0000	0.5000	0.2500	0.0625000	0.0625000	0.0000000
1.0000	0.5000	0.5000	0.1250000	0.1250000	0.0000000
1.0000	0.5000	0.7500	0.1875000	0.1875000	0.0000000
1.0000	0.5000	1.0000	0.2500000	0.2500000	0.0000000
1.0000	0.7500	0.2500	0.0937500	0.0937500	0.0000000
1.0000	0.7500	0.5000	0.1875000	0.1875000	0.0000000
1.0000	0.7500	0.7500	0.2812500	0.2812500	0.0000000
1.0000	0.7500	1.0000	0.3750000	0.3750000	0.0000000
1.0000	1.0000	0.2500	0.1250000	0.1250000	0.0000000
1.0000	1.0000	0.5000	0.2500000	0.2500000	0.0000000
1.0000	1.0000	0.7500	0.3750000	0.3750000	0.0000000
1.0000	1.0000	1.0000	0.5000001	0.5000000	0.0000001

The 25th iteration.

The mean of error is 0.0000000156.

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