SOLVING SYSTEMS OF NONLINEAR EQUATIONS WITHOUT USING PARTIAL DERIVATIVES

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หัวข้อวิทยานิพนธ์	การหาผลเฉลยของระบบสมการไม่เชิงเส้นโดยไม่ใช้อนุพันธ์ย่อย
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บทคัดย่อ

เนื้อหาของวิทยานิพนธ์ฉบับนี้ เป็นการนำเสนอวิธีการทำซ้ำเพื่อหาผลเฉลยเชิงตัวเลขของระบบสมการไม่เชิงเส้น ซึ่งวิธีการทำซ้ำแบบใหม่นี้จัดอยู่ในประเภทเดียวกันกับการปรับปรุงวิธีนิวตัน โดยการแทนเมตริกซ์จาโคเบียนในวิธีของนิวตันด้วยเมตริกซ์ใหม่ แล้วแทนอนุพันธ์ย่อยในเมตริกซ์จาโคเบียนด้วยรูปแบบอื่น เพื่อเป็นทางเลือกในการหาผลเฉลยเชิงตัวเลขของระบบสมการไม่เชิงเส้น

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ABSTRACT

In this thesis, we will present the iterative methods for finding numerical solution of systems of nonlinear equations. The new iterative methods are the same type as Modified Newton method by replacing the Jacobian matrix in Newton's method by new matrices, and use the other formulas instead of partial derivatives. These new methods will be the new ways for solving systems of nonlinear equations.

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CHAPTER 1

INTRODUCTION

Many problems in science and technology are involved in system of nonlinear equations, in specifically applied science, as for example, the first problem, Figure 1.1 shows a two-link robot arm. One end of arm is fixed at the origin of the coordinate system, and the other end is at coordinate (x, y). There are positioning actuators control the link angles θ and ϕ, as show. In term of θ and ϕ, the coordinates of the joint between links are (a, b), where

$$a = 0.5\cos\theta$$

$$b = 0.5\sin\theta.$$

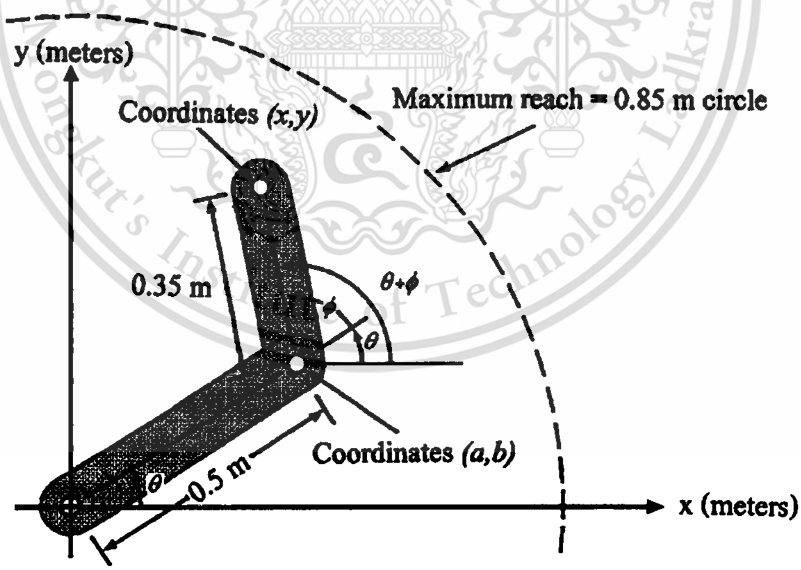


Figure 1.1 Position a robot arm in geometry.

The coordinates (x, y) are given by

$$x = 0.5\cos\theta + 0.35\cos(\theta + \phi) = 0$$
$$y = 0.5\sin\theta + 0.35\sin(\theta + \phi) = 0. \tag{1.1}$$

The angles θ and ϕ that will position the end of the arm at coordinates (0.2,0.4) are given by simultaneous solution of

$$0.5\cos\theta + 0.35\cos(\theta + \phi) - 0.2 = 0$$
$$0.5\sin\theta + 0.35\sin(\theta + \phi) - 0.4 = 0. \tag{1.2}$$

We want to find set of θ and ϕ are satisfied each of equation.

Problem (1.2) is system of two equations two unknowns, we can write in the model of two unknowns two equations as:

$$f(x, y) = 0$$
$$g(x, y) = 0. \tag{1.3}$$

There are several iterative methods for solving above system as for example, the method are base on Newton method i.e. Newton method, Newton-Raphson method, Newton-Like method and Modified Newton method. In these methods we have to find partial derivative and use only one initial vector for solving the system.

Consider the second applied problem, the turbulent flow of fluids in an interconnected network (see Fig. 1.2), the flow rate V from one node to another is about proportional to the square root of the difference in pressures at the nodes (thus fluid flow differs from flow of electrical current in a network in that nonlinear equations result).

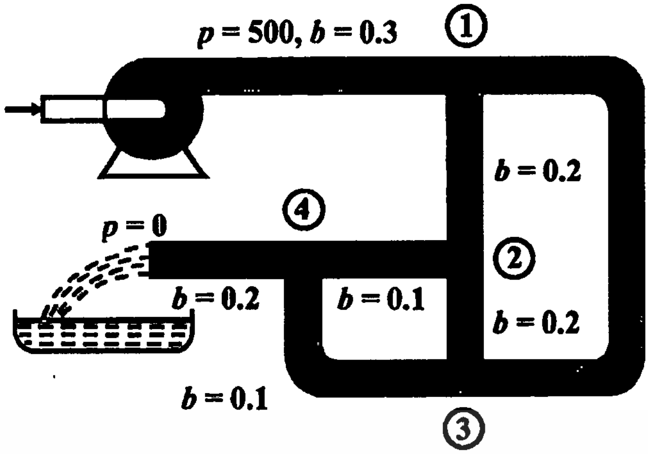


Figure 1.2 The turbulent flow of fluids in an interconnected network.

For the conduits in Figure 1.2 the pressure at each node. The values of b represent conductance factors in the relation $v_{ij} = b_{ij}(p_i - p_j)^{1/2}$.

These equations can be set up for the pressure at each node follow as :

at node 1 : $0.3\sqrt{500 - p_1} = 0.2\sqrt{p_1 - p_2} + 0.1\sqrt{p_1 - p_3}$

node 2 : $0.2\sqrt{p_1 - p_2} = 0.1\sqrt{p_2 - p_4} + 0.2\sqrt{p_2 - p_3}$

node 3 : $0.1\sqrt{p_1 - p_3} + 0.2\sqrt{p_2 - p_3} = 0.1\sqrt{p_3 - p_4}$

node 4 : $0.1\sqrt{p_2 - p_4} + 0.1\sqrt{p_3 - p_4} = 0.2\sqrt{p_4 - 0}$.

In order to find the solutions of two applied problems, we need to solve the systems of nonlinear equations and their numerical solution that will be shown in chapter 4.

The general form for n equations n unknowns of these problems we usually write in the vector

$$f(x) = 0 \tag{1.4}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $f(x) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$.

By using Newton method, approximate solutions of system (1.4) could be written as

$$x^{k+1} = x^k - [J(x)]^{-1}\,[f(x)]; \quad k = 0,1,2,\ldots \tag{1.5}$$

where $J(x)$ denotes Jacobian matrix,

$$J(x) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

and f_{ij} denotes the partial derivative of $f_i(x)$ with respect to the j^{th} variable.

If using the idea of Modified Newton method we can find the solution of (1.4) follow as

$$x^{k+1} = x^k - \left[H(x_1^k, x_2^k, \ldots, x_n^k)\right]^{-1} \begin{bmatrix} f_1(x_1^k, x_2^k, \ldots, x_n^k) \\ f_2(x_1^k, x_2^k, \ldots, x_n^k) \\ \vdots \\ f_n(x_1^k, x_2^k, \ldots, x_n^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{1.6}$$

where $H(x)$ is the diagonal matrix,

$$H(x) = \begin{bmatrix} f_{11} & & \cdots & 0 \\ \vdots & f_{22} & & \vdots \\ & \vdots & \ddots & \\ 0 & \cdots & & f_{nn} \end{bmatrix}$$

and f_{ii} denotes the partial derivative of $f_i(x)$ with respect to the i^{th} variable.

Above methods, we use partial derivative and use only one initial vector for solving the system.

The purpose of this thesis, we will present the methods for finding the approximate solution of the system (1.4). The process of new iterative methods, we will replaced Jacobian matrix in Newton's method by new matrices, and then uses the other formulas instead of partial derivatives. The new iterative methods will use two or three initial vectors for solving the system.

In chapter 2, we will discuss the important theorems concern with the Modified Newton method. Then we will introduce the new methods for solving the systems of nonlinear equations in chapter 3. In chapter 4, we will find the approximate solution of examples including application problems. Finally, we will make a conclusion and suggestions in chapter 5.

CHAPTER 2

LITERATURE REVIEWS

This chapter we shall review definitions, lemmas and fundamental theorems in numerical mathematics. The following theorems will be used to support our results.

First, we shall give the symbols which we use in this chapter,

$D \subset X$	denotes the domain of f ,
D_0	denotes closed convex subset of D,
$N(x,t)$	denotes the open neighborhood of radius t around x,
	i.e. $N(x,t) = \{y \in X : \|x-y\| < t\}$,
$L(X,Y)$	denotes a real Banach space,
$f \in Lip_k(D)$	if $\|f(x) - f(y)\| \le k\|x-y\|$ for some constant k ,
$\{x_k\}$	denotes sequence of vectors in R^n,

$B(x) = f'(x) - A(x)$,

and

$M(x) = f'(x) - H(x)$.

Consider the system of nonlinear equations

$$f(x) = 0 \tag{2.1}$$

where $f : X \to X$ and $X \equiv R^n$ denotes n - dimensional real vector space, f is assumed to be continuously differentiable.

In the analysis of Newton's method, it will be necessary to assume the Jacobian matrix,

$$J(x^k) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \tag{2.2}$$

where f_{ij} denotes the partial derivative of $f_i(x^k)$ with respect to the j^{th} variable and evaluated at x, is at least continuous at the solution x^*; that is

$$\left\|f'(x^* + h) - f'(x^*)\right\| \to 0 \quad \text{as} \quad h \to 0.$$

On occasion, it will also be useful to assume, instead of continuity of f', that the derivative satisfies the following property.

Definition 2.1 The mapping $f : R^n \to R^n$ is (Totally or Frechet) differentiable at x if the Jacobian matrix (2.2) exists at x and

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - h\, f'(x)\|}{\|h\|} = 0. \tag{2.3}$$

Not that if $n = 1$, Definition 2.1 reduces to the usual definition of differentiability. Note also that if f is differentiable at x, then f is continuous at x; this follows form the inequality

$$\|f(x+h) - f(x)\| \le \|f(x+h) - f(x) - h\, f'(x)\| + \|h\, f'(x)\|. \tag{2.4}$$

Finally, we note that it is possible to say that if the Jacobian matrix is continuous at x then f is differentiable at x.

One of the basic tools of nonlinear analysis is the mean value theorem. If f is a differentiable function from R^1 to R^1, which states that

$$f(x) - f(y) = f'(z)(x - y) \tag{2.5}$$

for some point z between x and y.

Unfortunately, this result does not extend verbatim to mappings from R^n to R^n. However, we are able to prove some results which are often just as useful. For the first, we define the integral $\int_a^b G(t)dt$ of a mapping $G:[a,b] \subset R^1 \to R^n$ as the vector with components $\int_a^b g_i(t)dt, i = 1,2,\ldots,n$ where $g_1, g_2, \ldots, g_n$ are the components of G. Thus, for example, if $f : R^n \to R^n$ the relation

$$f(y) - f(x) = \int_0^1 f'(x + t(y - x))(y - x)dt \tag{2.6}$$

is equivalent to

$$f_i(y) - f_i(x) = \int_0^1 \sum_{j=1}^n f_{ij}(x + t(y - x))(y_j - x_j)dt \tag{2.7}$$

for $i = 1,2,\ldots,n$.

For $n = 1$ (2.6) is simply the fundamental theorem of the integral calculus. Hence the next result is a natural extension of the theorem to n - dimensions.

Lemma 2.1 Assume that $f : R^n \to R^n$ is continuously differentiable on a convex set $D \subset R^n$. Then for any $x, y \in D$, (2.6) holds.

Proof: For fixed $x, y \in D$ define the function $g_i : [0,1] \subset R^1 \to R^1$ by

$$g_i(t) = f_i(x + t(y - x)), \quad t \in [0,1], i = 1,2,\ldots,n. \tag{2.8}$$

By the convexity of D, it follows that g_i is continuously differentiable on $[0,1]$ and thus the fundamental theorem of the integral calculus implies that

$$g_i(1) - g_i(0) = \int_0^1 g_i'(t)dt . \tag{2.9}$$

But a simple calculation show that

$$g_i'(t) = \sum_{j=1}^{n} f_{ij}(x + t(y - x))(y_i - x_i) \tag{2.10}$$

so that (2.9) is equivalent to (2.7).

For the next result, we first need a lemma on integration.

Lemma 2.2 Assume that $G : [a,b] \subset R^1 \to R^n$ is continuous. Then

$$\left\| \int_a^b G(t)\dot{d}t \right\| \leq \int_a^b \|G(t)\| dt . \tag{2.11}$$

Proof : Since any norm is a continuous function on R^n, both integrals of (2.11) exist and therefore any $\varepsilon > 0$ there is a partition $a < t_0 < \ldots < t_p < b$ of $[a,b]$ such that

$$\left\| \int_a^b G(t)dt - \sum_{i=1}^{p} G(t_i)(t_i - t_{i-1}) \right\| < \varepsilon \tag{2.12}$$

and

$$\left| \int_a^b \|G(t)\| dt - \sum_{i=1}^{p} \|G(t_i)\|(t_i - t_{i-1}) \right| < \varepsilon .$$

Hence

$$\left\|\int_a^b G(t)dt\right\| \leq \left\|\sum_{i=1}^{p} G(t_i)(t_i - t_{i-1})\right\| + \varepsilon$$

$$\leq \sum_{i=1}^{p} \|G(t_i)\|(t_i - t_{i-1}) + \varepsilon$$

$$\leq \int_a^b \|G(t)\|dt + 2\varepsilon$$

and, since ε was arbitrary, (2.11) must be valid.

By mean of lemma 2.1 and 2.2 we can prove the following useful alternative of the mean value theorem.

Theorem 2.1 Assume that $f : R^n \to R^n$ is continuously differentiable on the convex set $D \subset R^n$. Then for any $x, y \in D$,

$$\|f(x) - f(y)\| \leq \sup_{0 \leq t \leq 1} \|f'(x + t(y - x))\| \, \|y - x\|. \tag{2.13}$$

Proof: By lemma 2.1 and 2.2 we have

$$\|f(x) - f(y)\| = \left\|\int_0^1 f'(x + t(y - x))(y - x)dt\right\|$$

$$\leq \int_0^1 \|f'(x + t(y - x))\| \, \|y - x\| dt$$

$$\leq \sup_{0 \leq t \leq 1} \|f'(x + t(y - x))\| \int_0^1 \|y - x\| dt$$

which is (2.13).

Lemma 2.3 Let $f' \in Lip_k(D_0)$ and let $x, y \in D_0$, then

$$\|f(y) - f(x) - f'(x)(y-x)\| \le \frac{1}{2} k\|x-y\|^2. \tag{2.14}$$

Proof: From $f : R^n \to R^n$ is differentiable and satisfied $f' \in Lip_k(D_0)$, that means

$$\|f'(x) - f'(y)\| \le k\|x-y\|$$

for all x, y in closed convex subset D_0 of D.

Since $f'(x)(y-x)$ is constant with respect to the integration, we have

$$\begin{aligned} f(y) - f(x) - f'(x)(y-x) &= \int_0^1 [f'(x+t(y-x))(y-x) - f'(x)(y-x)]dt \\ &= \int_0^1 [f'(x+t(y-x)) - f'(x)](y-x)dt. \end{aligned}$$

Hence, the result follow by taking norm of both side,

$$\begin{aligned} \|f(y) - f(x) - f'(x)(y-x)\| &= \left\|\int_0^1 [f'(x+t(y-x)) - f'(x)](y-x)dt\right\| \\ &\le \int_0^1 \|f'(x+t(y-x)) - f'(x)\|\|y-x\|dt \\ &\le \int_0^1 k\|x+t(y-x)-x\|\,\|y-x\|dt \\ &\le \int_0^1 k\|t(y-x)\|\,\|y-x\|dt \\ &\le \int_0^1 k\|t\|\,\|y-x\|^2 dt \end{aligned}$$

$$= k\|y - x\|^2 \int_0^1 t\,dt$$

$$= \frac{1}{2} k\|x - y\|^2 .$$

Definition 2.2 Let t_0 and t' be non-negative real numbers, g be a continuously differentiable real function on $[t_0, t_0 + t']$ and G be a continuously differentiable operator on $\overline{N}(x_0, t) \subset X$ into X. Then the equation $t = g(t)$ will be said to majorize the equation $x = G(x)$, or g majorizes G, on $N(x_0, t')$ if

$$\|G(x_0) - x_0\| \le g(t_0) - t_0 \tag{2.15}$$

and

$$\|G'(x)\| \le g'(t) \text{ where } \|x - x_0\| \le t - t_0 < t' . \tag{2.16}$$

Definition 2.3 Let t_0 and t' be non-negative real numbers, g be a real function on $[t_0, t_0 + t']$ and G be an operator sending $N(x_0, t')$ into X. Then the equation $t = g(t)$ will be said to weakly majorize the equation $x = G(x)$, or g weakly majorizes G, if (2.15) hold and in addition

$$\|G(G(x)) - G(x)\| \le g(g(t)) - g(t) \tag{2.17}$$

where $\qquad \|x - x_0\| \le t - t_0 < t'$

and $\qquad \|G(x) - x\| \le g(t) - t .$

We remark that Lemma 2.3 and Lemma 2.4 are know results in mathematical analysis and Lemma 2.5 is given by Ortega [6].

Lemma 2.4 (The Banach Lemma) Let $M \in L(X,X)$ and $\|I-J\| \le \delta < 1$, then J^{-1} exists in $L(X,X)$ and $J^{-1} \le (1-\delta)^{-1}$.

Lemma 2.5 Let $\{x_k\}$ be a sequence in X and $\{t_k\}$ be a sequence of non-negative real numbers such that

$$\|x_{k+1}-x_k\| \le t_{k+1}-t_k, \quad k=0,1,2,\dots, \tag{2.18}$$

and $t_k \to t^* < \infty$. By these conditions, there exists a point $s \in X$ such that $x_k \to s$ and

$$\|s-x_k\| \le t^*-t_k, \quad k=0,1,2,\dots, \tag{2.19}$$

Proof: The proof is immediate from

$$\begin{aligned}\|x_{k+p}-x_k\| &\le \sum_{i=1}^{p}\|x_{k+i}-x_{k+i-1}\| \\ &\le t_{k+p}-t_k \\ &\le t^*-t_k,\end{aligned}$$

which show that $\{x_k\}$ is a Cauchy Sequence.

We shall say that $\{t_k\}$ majorize $\{x_k\}$ if $\|x_{k+1}-x_k\| \le t_{k+1}-t_k, \quad k=0,1,2,\dots$ holds.

The following theorem is The Kantorovich Theorem. It is one of the fundamental theorems in numerical mathematics. The idea for proof this theorem in [4]

Theorem 2.2 (The Kantorovich Theorem) Let x_0 be in D_0 and let $\Gamma_0 = [f'(x_0)]^{-1}$ exist with $\Gamma_0 \in Lip_k(D_0)$,

$$\|\Gamma_0 f(x_0)\| \le \eta \text{ and } h = k\eta \le \frac{1}{2}. \tag{2.20}$$

Define

$$r_0(h) = \frac{1}{h}\left(1 - \sqrt{1-2h}\right)\eta \tag{2.21}$$

$$r_1(h) = \frac{1}{h}\left(1 + \sqrt{1-2h}\right)\eta. \tag{2.22}$$

Then if $N(x_0, r_0(h)) \subset D_0$, the sequence of iterates defined by Newton's method exists, remains in $N(x_0, r_0(h))$ and converges to s in $N(x_0, r_0(h))$ such that $f(s) = 0$. If $h < \frac{1}{2}$, s is the only solution in $N(x_0, r_1(h)) \cap D_0$, and if $h = \frac{1}{2}$, s is unique in $N(x_0, r_1(h)) \cap D_0$. Furthermore, the sequence of the iterates satisfy the error bounds

$$\|s - x_m\| \le \frac{1}{h}\frac{1}{2^m}\left(1 - \sqrt{1-2h}\right)^{2m}\eta. \tag{2.23}$$

This theorem together with Lemma 2.5 and Definition 2.2 give the convergence of the sequence $\{x_k\}$ in X when the sequence of $\{t_k\}$ converges.

Theorem 2.3 If g majorizes G on $\overline{N}(x_0, t')$ and g has a fixed point in $[t_0, t_0 + t']$, then G has a fixed point s in $\overline{N}(x_0, t')$. Furthermore, $x_{k+1} = G(x_k)$ and $t_{k+1} = g(t_k)$, $k = 0,1,2,\ldots$, converges to s and t^* respectively with the real sequence majorizing the vector sequence.

Next lemma uses the weakly majorizing property and it is given by Dennis [3].

Lemma 2.6 If $g(t) \in [t, t_0 + t']$ when $t \in [t_0, t_0 + t']$, and g weakly majorizes G on $N(x_0, t')$, then there are elements $t^* \in [t_0, t_0 + t']$, $s \in \overline{N}(x_0, t')$ such that

$$x_{k+1} = G(x_k) \tag{2.24}$$

and

$$t_{k+1} = g(t_k),\ k = 0,1,2,\ldots \tag{2.25}$$

converge to s and t^* respectively with t sequence majorizing the x sequence.

Theorem 2.4 Let A be a function on $N(x_0, r)$ such that $A(x) \in L(X,Y)$ for each x and $A(x)$ is invertible for each x in $N(x_0, r)$ and that there is a real, nonvanishing, nonincreasing function $a(t)$ on $[0, r)$ such that

$$\left\|[A(x)]^{-1}\right\| \le a(\|x - x_0\|)^{-1}. \tag{2.26}$$

If $f' \in Lip_k(\overline{N}(x_0, r))$ then if $\sigma \ge 1$ and $\delta > 0$ are real numbers such that

$$a(t) + \sigma k t \tag{2.27}$$

is isotonic on $(0, r)$, and

$$\|B(x)\| \le a(\|x - x_0\|) + \sigma k\|x - x_0\|\delta, \tag{2.28}$$

for every $x \in N(x_0, r)$, then

$$g(t) = t + (a(t))^{-1}\left(\frac{1}{2}\sigma k t^2 - \delta t + a(0)\left\|[A(x_0)]^{-1} - f(x_0)\right\|\right) \tag{2.29}$$

weakly majorizes $G(x) = x - [A(x)]^{-1} f(x)$ on $N(x_0, r)$.

Theorem 2.5 Let $f' \in Lip_k(\overline{N}(x_0, r))$ and $[A(x_0)]^{-1}$ exist and be bounded in the norm by $[a(0)]^{-1}$. If $\|B(x_0)\| < a(0)$,

$$h' = \frac{k}{a(0)} \left\| [A(x_0)]^{-1} f(x_0) \right\| a(0) - \left(\|B(x_0)\|\right)^2 \le \frac{1}{2} \tag{2.30}$$

and

$$r_0' = \frac{1}{k}\left(1 - \sqrt{1 - 2h'}\right)\left(a(0) - \|B(x_0)\|\right) \le r. \tag{2.31}$$

Then if f has a unique zero $s \in \overline{N}(x_0, r_0')$, and

$$x_{m+1}' = x_m' - \left[A(x_0)\right]^{-1} f(x_m'), \quad m = 0,1,2,\ldots$$

converges to s from any $x_0' \in \overline{N}(x_0, r)$ such that

$$\|x_0' - x_0\| < r_1' = \frac{1}{k}\left(1 - \sqrt{1 - 2h}\right)\left(a(0) - \|B(x_0)\|\right).$$

If, in addition σ, δ and a satisfy the condition of Theorem 2.4 and

$$h = \frac{1}{\delta^2} \sigma k \left\| \left[A(x_0)\right]^{-1} f(x_0) \right\| a(0) \le \frac{1}{2} \tag{2.32}$$

and

$$r_0 = \frac{1}{\sigma k}\left(1 - \sqrt{1 - 2h}\right)\delta < r, \tag{2.33}$$

then

$$x_{m+1}' = x_m' - \left[A(x_m)\right]^{-1} f(x_m'), \quad m = 0,1,2,\ldots \tag{2.34}$$

converges to s.

In the following theorem, we impose one more condition on $A(x)$ and one condition on $B(x)$ instead of $B(x_0)$.

Theorem 2.6 Let $f' \in Lip_k(\overline{D})$ where $x_0 \in D$ and D be an open convex subset of X. Assume that

$$\left\|[A(x_0)]^{-1} f(x_0)\right\| \leq \alpha \tag{2.35}$$

$$\left\|[A(x_0)]^{-1}\right\| \leq \beta \tag{2.36}$$

$$\|A(x) - A(x_0)\| \leq \eta_0 + \eta_1 \|x - x_0\|, \ \ \forall x \in D \tag{2.37}$$

$$\|B(x)\| \leq \delta_0 + \delta_1 \|x - x_0\|, \ \ \forall x \in D. \tag{2.38}$$

then

$$\beta\delta_0 < 1, \ \ h' = \frac{\beta\alpha k}{(1-\beta\delta_0)^2} \leq \frac{1}{2}$$

and $N(x_0, r_0') \subset D$ where

$$r_0' = \frac{1-\sqrt{1-2h'}}{\beta k}(1-\beta\delta_0)$$

implied that f has a solution $r \in \overline{N}(x_0, r_0')$ which is unique in $D \cap N(x_0, r_1')$ where

$$r_1' = \frac{1-\sqrt{1+2h'}}{\beta k}(1-\beta\delta_0).$$

Furthermore $\qquad x_{m+1}' = x_m' - [A(x_0)]^{-1} f(x_m')$

converges to s from any $\qquad x_0' \in \overline{D} \cap N(x_0, r_1')$.

If, in addition $\qquad \beta(\delta_0 + \eta_0) < 1$

and $$h = \frac{\sigma\beta\alpha k}{(1-\beta\eta_0-\beta\delta_0)^2} \le \frac{1}{2}$$

where $$\sigma = \max\left(1, \frac{\delta_1+\eta_1}{k}\right)$$

and $$N(x_0, r_0) \subset D,$$

$$r_0' = \frac{1-\sqrt{1-2h}}{\sigma\beta k}(1-\beta\eta_0-\beta\delta_0),$$

then $$x_{m+1} = x_m - [A(x_m)]^{-1} f(x_m)$$

converges to s.

The next theorems will ensure that the convergence of the Modified Newton method for solving systems of nonlinear equations which is a special kind of the Newton-Like method.

Theorem 2.7 Let D be an open subset of the space X and $f' \in Lip_k(\overline{D})$. Assuming that $f(x)$ and $H(x)$ satisfy all the conditions of the previous theorems, then there exists a unique zero s in D, so that for any point $x_0 \in D$ the sequence $\{x_m\}$ where

$$x_{m+1} = x_m - [H(x_m)]^{-1} f(x_m)$$

converges to s.

Proof: By the results of the previous theorems, the existence any sequence of s in D is ensured and the sequence $\{x_m\}$ of points in D, where

$$x_{m+1} = x_m - \left[H(x_m)\right]^{-1} f(x_m)$$

for $x_0 \in D$, converges to this unique point s.

The last theorem of this section. Theorem 2.8 show that the order of the convergence of the Modified Newton method for solving a system of nonlinear equations which is of second order. The proof of this theorem see [9]

Theorem 2.8 Let the conditions of theorem 2.6 be satisfied and

$$\|B(x)\| \le \rho \|x - x_0\|, \quad \forall x \in D. \tag{2.39}$$

Then the order of the convergence of the method is equal to 2.

CHAPTER 3

SOLVING SYSTEMS OF NONLINEAR EQUATIONS WITHOUT USING PARTIAL DERIVATIVES

In this chapter, we shall introduce new methods for solving system of nonlinear equations. First section, we will give the definitions of matrices that will replace the Jacobian matrix. Next, we will state about new iterative methods.

3.1 Definition

Definition 3.1.1 We define the matrix $D(x)$ of $f(x)$ as

$$D(x) = \begin{bmatrix} f_{11} & & \cdots & 0 \\ \vdots & f_{22} & & \vdots \\ & \vdots & \ddots & \\ 0 & \cdots & & f_{nn} \end{bmatrix} \tag{3.1}$$

where $f_{ii}(x^k) \approx \dfrac{f_i(x^k) - f_i(x^{k-1})}{x_i^k - x_i^{k-1}}$ for using initial vector two points backward,

$f_{ii}(x^k) \approx \dfrac{f_i(x^{k+1}) - f_i(x^k)}{x_i^{k+1} - x_i^k}$ for using initial vector two points forward,

$f_{ii}(x^k) \approx \dfrac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_i^{k+1} - x_i^{k-1}}$ for using initial vector three points central

and $\quad i = 1,2,\ldots,n$,

$\quad k = 1,2,\ldots$.

Definition 3.1.2 We define the matrix $L_2(x)$, $L_3(x)$, ..., $L_n(x)$ of $f(x)$ as

$$L_2(x) = \begin{bmatrix} f_{11} & & & & \cdots & 0 \\ f_{21} & f_{22} & & & & \\ & f_{32} & f_{33} & & & \vdots \\ & & \ddots & \ddots & & \\ \vdots & & & & & \\ & & & f_{(n-1)(n-2)} & f_{(n-1)(n-1)} & \\ 0 & \cdots & & & f_{(n)(n-1)} & f_{nn} \end{bmatrix}, \qquad (3.2)$$

$$L_3(x) = \begin{bmatrix} f_{11} & & & & \cdots & 0 \\ f_{21} & f_{22} & & & & \\ f_{31} & f_{32} & f_{33} & & & \vdots \\ & \ddots & \ddots & \ddots & & \\ \vdots & & & & & \\ & & & f_{(n-1)(n-2)} & f_{(n-1)(n-1)} & \\ 0 & \cdots & & f_{(n)(n-2)} & f_{(n)(n-1)} & f_{nn} \end{bmatrix}, \qquad (3.3)$$

$$\cdots,$$

$$L_n(x) = \begin{bmatrix} f_{11} & & & & \cdots & 0 \\ f_{21} & f_{22} & & & & \\ f_{31} & f_{32} & f_{33} & & & \vdots \\ & \ddots & \ddots & \ddots & & \\ \vdots & & & & & \\ & & & f_{(n-1)(n-2)} & f_{(n-1)(n-1)} & \\ f_{n1} & \cdots & & f_{(n)(n-2)} & f_{(n)(n-1)} & f_{nn} \end{bmatrix}, \qquad (3.4)$$

where $f_{ij}(x^k) \approx \dfrac{f_i(x^k) - f_i(x^{k-1})}{x_j^k - x_j^{k-1}}$ for using initial vector two points backward,

$$f_{ij}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^k)}{x_j^{k+1} - x_j^k}$$ for using initial vector two points forward,

$$f_{ij}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_j^{k+1} - x_j^{k-1}}$$ for using initial vector three points central

and $i, j = 1,2,\ldots,n$,

$k = 1,2,\ldots$.

Definition 3.1.3 We define the matrix $U_2(x)$, $U_3(x)$, ...,$U_n(x)$ of $f(x)$ as

$$U_2(x) = \begin{bmatrix} f_{11} & f_{12} & & & \cdots & 0 \\ & f_{22} & f_{23} & & & \vdots \\ & & \ddots & \ddots & & \\ \vdots & & & f_{(n-2)(n-2)} & f_{(n-2)(n-1)} & \\ & & & & f_{(n-1)(n-1)} & f_{(n-1)(n)} \\ 0 & \cdots & & & & f_{nn} \end{bmatrix}, \quad (3.5)$$

$$U_3(x) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & & \cdots & 0 \\ & f_{22} & f_{23} & f_{24} & & \vdots \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & f_{(n-2)(n-2)} & f_{(n-2)(n-1)} & f_{(n-2)(n)} \\ & & & & f_{(n-1)(n-1)} & f_{(n-1)(n)} \\ 0 & \cdots & & & & f_{nn} \end{bmatrix}, \quad (3.6)$$

$\cdots,$

$$U_n(x) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & & & & \cdots & f_{1n} \\ & f_{22} & f_{23} & f_{24} & & & & f_{2n} \\ & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & f_{(n-2)(n-2)} & f_{(n-2)(n-1)} & & f_{(n-2)(n)} \\ & & & & & f_{(n-1)(n-1)} & & f_{(n-1)(n)} \\ 0 & \cdots & & & & & & f_{nn} \end{bmatrix} \tag{3.7}$$

where $f_{ij}(x^k) \approx \dfrac{f_i(x^k) - f_i(x^{k-1})}{x_j^k - x_j^{k-1}}$ for using initial vector two points backward,

$f_{ij}(x^k) \approx \dfrac{f_i(x^{k+1}) - f_i(x^k)}{x_j^{k+1} - x_j^k}$ for using initial vector two points forward,

$f_{ij}(x^k) \approx \dfrac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_j^{k+1} - x_j^{k-1}}$ for using initial vector three points central

and $\quad i, j = 1,2,\ldots,n, \quad k = 1,2,\ldots.$

For the system of nonlinear equation

$$f(x) = 0 \tag{3.8}$$

The methods which state in chapter 1, we use partial derivative and use only one initial vector for solving system. In this part, we shall introduce the iterative methods for find the solution of equation (3.8). The processing is replace Jacobian matrix in Newton's method by diagonal matrix $D(x)$, $L_2(x)$ matrix, $L_3(x)$ matrix, ..., $L_n(x)$ matrix, $U_2(x)$ matrix, $U_3(x)$ matrix, ..., and $U_n(x)$ matrix, then used the other formulas instead of partial derivatives, therefore we obtained the new iterative methods which use two initial vectors or three initial vectors for solving systems of nonlinear equations.

3.2 The New Iterative Methods

By the process above, for the system of n equations n unknowns and choosing initial vector, we to obtained $(2n-1)$ new iterative methods for solving problems, the first method follow as :

D method

The approximate solution is in the form

$$x^{k+1} = x^k - \left[D(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ \vdots \\ f_n(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{3.9}$$

where $D(x^k) = \begin{bmatrix} f_{11} & & \cdots & 0 \\ \vdots & f_{22} & & \vdots \\ & \vdots & \ddots & \\ 0 & \cdots & & f_{nn} \end{bmatrix}$,

$f_{ii}(x^k) \approx \dfrac{f_i(x^k) - f_i(x^{k-1})}{x_i^k - x_i^{k-1}}$ for using initial vector two points backward,

$f_{ii}(x^k) \approx \dfrac{f_i(x^{k+1}) - f_i(x^k)}{x_i^{k+1} - x_i^k}$ for using initial vector two points forward,

$f_{ii}(x^k) \approx \dfrac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_i^{k+1} - x_i^{k-1}}$ for using initial vector three points central

and $\quad i = 1,2,\ldots,n$, $\;k = 1,2,\ldots$.

The next method, we explain diagonal matrix to lower triangular matrix and upper triangular matrix respectively.

L_2 method

The approximate solution is in the form

$$x^{k+1} = x^k - \left[L_2(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ \vdots \\ f_n(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{3.10}$$

where $L_2(x^k)$ is the matrix in Definition 3.1.2,

L_3 method

The approximate solution is in the form

$$x^{k+1} = x^k - \left[L_3(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ \vdots \\ f_n(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{3.11}$$

where $L_3(x^k)$ is the matrix in Definition 3.1.2,

$\ldots,$

L_n method

The approximate solution is in the form

$$x^{k+1} = x^k - \left[L_n(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ \vdots \\ f_n(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{3.12}$$

where $L_n(x^k)$ is the matrix in Definition 3.1.2.

U_2 method

The approximate solution is in the form

$$x^{k+1} = x^k - \left[U_2(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ \vdots \\ f_n(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{3.13}$$

where $U_2(x^k)$ is the matrix in Definition 3.1.3,

U_3 method

The approximate solution is in the form

$$x^{k+1} = x^k - \left[U_3(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ \vdots \\ f_n(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{3.14}$$

where $U_3(x^k)$ is the matrix in Definition 3.1.3,

$\ldots,$

and the $(2n-1)^{\text{th}}$ method

U_n method

The approximate solution is in the form

$$x^{k+1} = x^k - \left[U_n(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ \vdots \\ f_n(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots \tag{3.15}$$

where $U_n(x^k)$ is the matrix in Definition 3.1.3.

Theorem 3.1 If $D(x)$ matrix, $L_2(x)$ matrix, $L_3(x)$ matrix, ... , $L_n(x)$ matrix, $U_2(x)$ matrix, $U_3(x)$ matrix, ... , $U_n(x)$ matrix of D *method*, L_2 *method*, L_3 *method*, ... , L_n *method*, U_2 *method*, U_3 *method*, ... , U_n *method* respectively are satisfy the condition of theorem 2.7 and theorem 2.8 then there exists a unique zero in D so that for any point x^0 in D the sequence $\{x^k\}$ converges to this zero and the order of convergence is equal to 2.

CHAPTER 4

SOME EXAMPLES AND APPLICATION PROBLEMS OF SOLVING SYSTEMS OF NONLINEAR EQUATIONS

In this chapter, we shall use the new iterative methods are obtained from chapter 3 solving some examples and application problems of systems of nonlinear equations and comparison the number of iterations of those methods.

Example4.1 Find the solution of system

$$\begin{aligned} &\frac{1}{2}\sin(xy) - \frac{1}{4\pi}y - \frac{1}{2}x = 0 \\ &(1-\frac{1}{4\pi})(e^{2x}-e) + \frac{1}{\pi}ey - 2ex = 0. \end{aligned} \tag{4.1}$$

We solving this system by D *method*, L_2 *method* and U_2 *method*, the approximate solutions are form :

$$x^{k+1} = x^k - \left[D(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots, \qquad \text{where } D(x^k) = \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix},$$

$$x^{k+1} = x^k - \left[L_2(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots, \qquad \text{where } L_2(x^k) = \begin{bmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{bmatrix},$$

$$x^{k+1} = x^k - \left[U_2(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots, \qquad \text{where } U_2(x^k) = \begin{bmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{bmatrix}.$$

By using initial vectors two points backward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix and $U_2(x^k)$ matrix as below

$$f_{11}(x^k, y^k) = \frac{f_1(x^k, y^k) - f_1(x^{k-1}, y^k)}{x^k - x^{k-1}}, \qquad f_{12}(x^k, y^k) = \frac{f_1(x^k, y^k) - f_1(x^k, y^{k-1})}{y^k - y^{k-1}},$$

$$f_{21}(x^k, y^k) = \frac{f_2(x^k, y^k) - f_2(x^{k-1}, y^k)}{x^k - x^{k-1}}, \qquad f_{22}(x^k, y^k) = \frac{f_2(x^k, y^k) - f_2(x^k, y^{k-1})}{y^k - y^{k-1}}.$$

By using initial vectors two points forward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix and $U_2(x^k)$ matrix as below

$$f_{11}(x^k, y^k) = \frac{f_1(x^{k+1}, y^k) - f_1(x^k, y^k)}{x^{k+1} - x^k}, \qquad f_{12}(x^k, y^k) = \frac{f_1(x^k, y^{k+1}) - f_1(x^k, y^k)}{y^{k+1} - y^k},$$

$$f_{21}(x^k, y^k) = \frac{f_2(x^{k+1}, y^k) - f_2(x^k, y^k)}{x^{k+1} - x^k}, \qquad f_{22}(x^k, y^k) = \frac{f_2(x^k, y^{k+1}) - f_2(x^k, y^k)}{y^{k+1} - y^k}.$$

By using initial vectors three points central, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix and $U_2(x^k)$ matrix as below

$$f_{11}(x^k, y^k) = \frac{f_1(x^{k+1}, y^k) - f_1(x^{k-1}, y^k)}{x^{k+1} - x^{k-1}},$$

$$f_{12}(x^k, y^k) = \frac{f_1(x^k, y^{k+1}) - f_1(x^k, y^{k-1})}{y^{k+1} - y^{k-1}},$$

$$f_{21}(x^k, y^k) = \frac{f_2(x^{k+1}, y^k) - f_2(x^{k-1}, y^k)}{x^{k+1} - x^{k-1}},$$

$$f_{22}(x^k, y^k) = \frac{f_2(x^k, y^{k+1}) - f_2(x^k, y^{k-1})}{y^{k+1} - y^{k-1}}.$$

We repeat each of the methods until error less than 0.0000005. The results of example 4.1 showing in table 4.1

Table 4.1 The results of example 4.1

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward (0.6,3.5) and $\Delta x = 0.2,\ \Delta y = 1.0$	D *method*	13	x ≈ 0.49999990 y ≈ 3.14159244
	L_2 *method*	10	x ≈ 0.50000019 y ≈ 3.14159300
	U_2 *method*	9	x ≈ 0.50000000 y ≈ 3.14159254
Two Points Forward (1.0,3.5) and $\Delta x = 0.5,\ \Delta y = 0.5$	D *method*	9	x ≈ 0.50000029 y ≈ 3.14159254
	L_2 *method*	7	x ≈ 0.49999981 y ≈ 3.14159261
	U_2 *method*	5	x ≈ 0.50000002 y ≈ 3.14159278
Three Points Central (0.6,2.4) and $\Delta x = 0.5,\ \Delta y = 1.0$	D *method*	11	x ≈ 0.50000023 y ≈ 3.14159298
	L_2 *method*	8	x ≈ 0.49999981 y ≈ 3.14159221
	U_2 *method*	8	x ≈ 0.50000000 y ≈ 3.14159247

Showing the iterations of table 4.1 in appendix.

Example 4.2 Find the solution of system

$$\frac{x-50}{\sqrt{|z-50|}}+\frac{x-y}{\sqrt{|x-y|}}=0$$

$$\frac{y-x}{\sqrt{|y-x|}}+\frac{y-z}{\sqrt{|y-z|}}=0 \qquad (4.2)$$

$$\frac{0.9\sqrt{2}(z-50)}{\sqrt{|z-50|}}+\frac{0.9\sqrt{2}z}{\sqrt{|z|}}+\frac{3.2(z-y)}{\sqrt{|z-y|}}=0.$$

We solving this system by D *method*, L_2 *method*, L_3 *method*, U_2 *method* and U_3 *method*, the approximate solutions are form :

$$x^{k+1}=x^k-\left[D(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k)\\ f_2(x^k)\\ f_3(x^k)\end{bmatrix};\ k=0,1,2,\ldots,\ \text{where } D(x^k)=\begin{bmatrix} f_{11} & 0 & 0\\ 0 & f_{22} & 0\\ 0 & 0 & f_{33}\end{bmatrix},$$

$$x^{k+1}=x^k-\left[L_2(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k)\\ f_2(x^k)\\ f_3(x^k)\end{bmatrix};\ k=0,1,2,\ldots,\ \text{where } L_2(x^k)=\begin{bmatrix} f_{11} & 0 & 0\\ f_{21} & f_{22} & 0\\ 0 & f_{32} & f_{33}\end{bmatrix},$$

$$x^{k+1}=x^k-\left[L_3(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k)\\ f_2(x^k)\\ f_3(x^k)\end{bmatrix};\ k=0,1,2,\ldots,\ \text{where } L_3(x^k)=\begin{bmatrix} f_{11} & 0 & 0\\ f_{21} & f_{22} & 0\\ f_{31} & f_{32} & f_{33}\end{bmatrix},$$

$$x^{k+1}=x^k-\left[U_2(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k)\\ f_2(x^k)\\ f_3(x^k)\end{bmatrix};\ k=0,1,2,\ldots,\ \text{where } U_2(x^k)=\begin{bmatrix} f_{11} & f_{12} & 0\\ 0 & f_{22} & f_{23}\\ 0 & 0 & f_{33}\end{bmatrix},$$

$$x^{k+1} = x^k - \left[U_3(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \end{bmatrix};\ k = 0,1,2,\ldots,\ \text{where } U_3(x^k) = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ 0 & f_{22} & f_{23} \\ 0 & 0 & f_{33} \end{bmatrix}.$$

By using initial vectors two points backward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix, $L_3(x^k)$ matrix, $U_2(x^k)$ matrix and $U_3(x^k)$ matrix as below

$$f_{11}(x^k, y^k, z^k) = \frac{f_1(x^k, y^k, z^k) - f_1(x^{k-1}, y^k, z^k)}{x^k - x^{k-1}},$$

$$f_{12}(x^k, y^k, z^k) = \frac{f_1(x^k, y^k, z^k) - f_1(x^k, y^{k-1}, z^k)}{y^k - y^{k-1}},$$

$$f_{13}(x^k, y^k, z^k) = \frac{f_1(x^k, y^k, z^k) - f_1(x^k, y^k, z^{k-1})}{z^k - z^{k-1}},$$

$$f_{21}(x^k, y^k, z^k) = \frac{f_2(x^k, y^k, z^k) - f_2(x^{k-1}, y^k, z^k)}{x^k - x^{k-1}},$$

$$f_{22}(x^k, y^k, z^k) = \frac{f_2(x^k, y^k, z^k) - f_2(x^k, y^{k-1}, z^k)}{y^k - y^{k-1}},$$

$$f_{23}(x^k, y^k, z^k) = \frac{f_2(x^k, y^k, z^k) - f_2(x^k, y^k, z^{k-1})}{z^k - z^{k-1}},$$

$$f_{31}(x^k, y^k, z^k) = \frac{f_3(x^k, y^k, z^k) - f_3(x^{k-1}, y^k, z^k)}{x^k - x^{k-1}},$$

$$f_{32}(x^k, y^k, z^k) = \frac{f_3(x^k, y^k, z^k) - f_3(x^k, y^{k-1}, z^k)}{y^k - y^{k-1}},$$

$$f_{33}(x^k, y^k, z^k) = \frac{f_3(x^k, y^k, z^k) - f_3(x^k, y^k, z^{k-1})}{z^k - z^{k-1}},$$

By using initial vectors two points forward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix , $L_3(x^k)$ matrix, $U_2(x^k)$ matrix and $U_3(x^k)$ matrix as below

$$f_{11}(x^k, y^k, z^k) = \frac{f_1(x^{k+1}, y^k, z^k) - f_1(x^k, y^k, z^k)}{x^{k+1} - x^k},$$

$$f_{12}(x^k, y^k, z^k) = \frac{f_1(x^k, y^{k+1}, z^k) - f_1(x^k, y^k, z^k)}{y^{k+1} - y^k},$$

$$f_{13}(x^k, y^k, z^k) = \frac{f_1(x^k, y^k, z^{k+1}) - f_1(x^k, y^k, z^k)}{z^{k+1} - z^k},$$

$$f_{21}(x^k, y^k, z^k) = \frac{f_2(x^{k+1}, y^k, z^k) - f_2(x^k, y^k, z^k)}{x^{k+1} - x^k},$$

$$f_{22}(x^k, y^k, z^k) = \frac{f_2(x^k, y^{k+1}, z^k) - f_2(x^k, y^k, z^k)}{y^{k+1} - y^k},$$

$$f_{23}(x^k, y^k, z^k) = \frac{f_2(x^k, y^k, z^{k+1}) - f_2(x^k, y^k, z^k)}{z^{k+1} - z^k},$$

$$f_{31}(x^k, y^k, z^k) = \frac{f_3(x^{k+1}, y^k, z^k) - f_3(x^k, y^k, z^k)}{x^{k+1} - x^k},$$

$$f_{32}(x^k, y^k, z^k) = \frac{f_3(x^k, y^{k+1}, z^k) - f_3(x^k, y^k, z^k)}{y^{k+1} - y^k},$$

$$f_{33}(x^k, y^k, z^k) = \frac{f_3(x^k, y^k, z^{k+1}) - f_3(x^k, y^k, z^k)}{z^{k+1} - z^k}.$$

By using initial vectors three points central, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix , $L_3(x^k)$ matrix, $U_2(x^k)$ matrix and $U_3(x^k)$ matrix as below

$$f_{11}(x^k, y^k, z^k) = \frac{f_1(x^{k+1}, y^k, z^k) - f_1(x^{k-1}, y^k, z^k)}{x^{k+1} - x^{k-1}},$$

$$f_{12}(x^k, y^k, z^k) = \frac{f_1(x^k, y^{k+1}, z^k) - f_1(x^k, y^{k-1}, z^k)}{y^{k+1} - y^{k-1}},$$

$$f_{13}(x^k, y^k, z^k) = \frac{f_1(x^k, y^k, z^{k+1}) - f_1(x^k, y^k, z^{k-1})}{z^{k+1} - z^{k-1}},$$

$$f_{21}(x^k, y^k, z^k) = \frac{f_2(x^{k+1}, y^k, z^k) - f_2(x^{k-1}, y^k, z^k)}{x^{k+1} - x^{k-1}},$$

$$f_{22}(x^k, y^k, z^k) = \frac{f_2(x^k, y^{k+1}, z^k) - f_2(x^k, y^{k-1}, z^k)}{y^{k+1} - y^{k-1}},$$

$$f_{23}(x^k, y^k, z^k) = \frac{f_2(x^k, y^k, z^{k+1}) - f_2(x^k, y^k, z^{k-1})}{z^{k+1} - z^{k-1}}.$$

$$f_{31}(x^k, y^k, z^k) = \frac{f_3(x^{k+1}, y^k, z^k) - f_3(x^{k-1}, y^k, z^k)}{x^{k+1} - x^{k-1}},$$

$$f_{32}(x^k, y^k, z^k) = \frac{f_3(x^k, y^{k+1}, z^k) - f_3(x^k, y^{k-1}, z^k)}{y^{k+1} - y^{k-1}},$$

$$f_{33}(x^k, y^k, z^k) = \frac{f_3(x^k, y^k, z^{k+1}) - f_3(x^k, y^k, z^{k-1})}{z^{k+1} - z^{k-1}}.$$

Repeat each of the methods until error less than 0.0000005. The results of example 4.2 showing in the table 4.2.

Table 4.2 The results of example 4.2

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward (40.0,45.0,44.0) and $\Delta x = 2.5$, $\Delta y = 2.5$, $\Delta z = 2.5$	*D method*	63	x ≈45.89787591 y ≈43.84681370 z ≈41.79575223
	L_2 *method*	30	x ≈45.89787626 y ≈43.84681454 z ≈41.79575296
	L_3 *method*	30	x ≈45.89787626 y ≈43.84681454 z ≈41.79575296
	U_2 *method*	44	x ≈45.89787610 y ≈43.84681429 z ≈41.79575238
	U_3 *method*	41	x ≈45.89787744 y ≈43.84681613 z ≈41.79575398
Two Points Forward (47.5,45.5,43.0) and $\Delta x = 1.5$, $\Delta y = 1.5$, $\Delta z = 2.0$	*D method*	79	x ≈45.89787714 y ≈43.84681560 z ≈41.79575372
	L_2 *method*	29	x ≈45.89787633 y ≈43.84681460 z ≈41.79575296
	L_3 *method*	29	x ≈45.89787633 y ≈43.84681460 z ≈41.79575296

Table 4.2 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Forward	U_2 *method*	29	x $\approx$45.89787669 y $\approx$43.84681489 z $\approx$41.79575319
	U_3 *method*	29	x $\approx$45.89787646 y $\approx$43.84681474 z $\approx$41.79575269
Three Points Central (30.0,12.0,20.0) and $\Delta x = 3.0$, $\Delta y = 3.0$, $\Delta z = 3.0$	D *method*	Diverge	-
	L_2 *method*	47	x $\approx$45.89787633 y $\approx$43.84681460 z $\approx$41.79575294
	L_3 *method*	47	x $\approx$45.89787633 y $\approx$43.84681460 z $\approx$41.79575294
	U_2 *method*	39	x $\approx$45.89787604 y $\approx$43.84681423 z $\approx$41.79575230
	U_3 *method*	39	x $\approx$45.89787632 y $\approx$43.84681443 z $\approx$41.79575287

Showing the iterations of table 4.2 in appendix.

Example 4.3 Find the solution of system

$$xy - 1.5 = 0$$

$$x + (1-y)\lambda - 1.45 = 0$$

$$x^2 + z - 1.09 = 0 \tag{4.3}$$

$$0.25z + \lambda + 1.0875 = 0.$$

We solving this system by D *method*, L_2 *method*, L_3 *method*, L_4 *method*, U_2 *method*, U_3 *method* and U_4 *method*, the approximate solutions are form :

$$x^{k+1} = x^k - \left[D(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \end{bmatrix}; \quad k = 0,1,2,\dots,$$

where $D(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ 0 & f_{22} & 0 & 0 \\ 0 & 0 & f_{33} & 0 \\ 0 & 0 & 0 & f_{44} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[L_2(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \end{bmatrix}; \quad k = 0,1,2,\dots,$$

where $L_2(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ 0 & f_{32} & f_{33} & 0 \\ 0 & 0 & f_{43} & f_{44} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[L_3(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \end{bmatrix}; \quad k = 0,1,2,\dots,$$

where $L_3(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ f_{31} & f_{32} & f_{33} & 0 \\ 0 & f_{42} & f_{43} & f_{44} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[L_4(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \end{bmatrix}; \quad k = 0,1,2,\dots,$$

where $L_4(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ f_{31} & f_{32} & f_{33} & 0 \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[U_2(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \end{bmatrix}; \quad k = 0,1,2,\dots,$$

where $U_2(x^k) = \begin{bmatrix} f_{11} & f_{12} & 0 & 0 \\ 0 & f_{22} & f_{23} & 0 \\ 0 & 0 & f_{33} & f_{34} \\ 0 & 0 & 0 & f_{44} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[U_3(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \end{bmatrix}; \quad k = 0,1,2,\dots,$$

where $U_3(x^k) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & 0 \\ 0 & f_{22} & f_{23} & f_{24} \\ 0 & 0 & f_{33} & f_{34} \\ 0 & 0 & 0 & f_{44} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[U_4(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \end{bmatrix}; \quad k = 0,1,2,\dots,$$

where $U_4(x^k) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ 0 & f_{22} & f_{23} & f_{24} \\ 0 & 0 & f_{33} & f_{34} \\ 0 & 0 & 0 & f_{44} \end{bmatrix}$.

By using initial vectors two points backward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix , $L_3(x^k)$ matrix, $L_4(x^k)$ matrix, $U_2(x^k)$ matrix, $U_3(x^k)$ matrix and $U_4(x^k)$ matrix as below

$$f_{i1}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k) - f_i(x^{k-1}, y^k, z^k, \lambda^k)}{x^k - x^{k-1}},$$

$$f_{i2}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k) - f_i(x^k, y^{k-1}, z^k, \lambda^k)}{y^k - y^{k-1}},$$

$$f_{i3}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k) - f_i(x^k, y^k, z^{k-1}, \lambda^k)}{z^k - z^{k-1}},$$

$$f_{i4}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k) - f_i(x^k, y^k, z^k, \lambda^{k-1})}{\lambda^k - \lambda^{k-1}},$$

where $i = 1,2,3,4$.

By using initial vectors two points forward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix , $L_3(x^k)$ matrix, $L_4(x^k)$ matrix, $U_2(x^k)$ matrix, $U_3(x^k)$ matrix and $U_4(x^k)$ matrix as below

$$f_{i1}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^{k+1}, y^k, z^k, \lambda^k) - f_i(x^k, y^k, z^k, \lambda^k)}{x^{k+1} - x^k},$$

$$f_{i2}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^{k+1}, z^k, \lambda^k) - f_i(x^k, y^k, z^k, \lambda^k)}{y^{k+1} - y^k},$$

$$f_{i3}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^k, z^{k+1}, \lambda^k) - f_i(x^k, y^k, z^k, \lambda^k)}{z^{k+1} - z^k},$$

$$f_{i4}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^k, z^k, \lambda^{k+1}) - f_i(x^k, y^k, z^k, \lambda^k)}{\lambda^{k+1} - \lambda^k},$$

where $i = 1,2,3,4$.

By using initial vectors three points central, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix , $L_3(x^k)$ matrix, $L_4(x^k)$ matrix, $U_2(x^k)$ matrix, $U_3(x^k)$ matrix and $U_4(x^k)$ matrix as below

$$f_{i1}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^{k+1}, y^k, z^k, \lambda^k) - f_i(x^{k-1}, y^k, z^k, \lambda^k)}{x^{k+1} - x^{k-1}},$$

$$f_{i2}(x^k, y^k, z^k, \lambda^k) = \frac{f_i(x^k, y^{k+1}, z^k, \lambda^k) - f_i(x^k, y^{k-1}, z^k, \lambda^k)}{y^{k+1} - y^{k-1}},$$

$$f_{i3}(x^k,y^k,z^k,\lambda^k)=\frac{f_i(x^k,y^k,z^{k+1},\lambda^k)-f_i(x^k,y^k,z^{k-1},\lambda^k)}{z^{k+1}-z^{k-1}},$$

$$f_{i4}(x^k,y^k,z^k,\lambda^k)=\frac{f_i(x^k,y^k,z^k,\lambda^{k+1})-f_i(x^k,y^k,z^k,\lambda^{k-1})}{\lambda^{k+1}-\lambda^{k-1}},$$

where $i=1,2,3,4$.

Repeat each of the methods until error less than 0.0000005. The results of example 4.3 showing in the table 4.3.

Table 4.3 The results of example 4.3

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward (1.3,1.5,0.0,-0.5) and $\Delta x=1.0,\ \Delta y=0.5,$ $\Delta z=0.5,\ \Delta\lambda=0.5$	*D method*	95	x $\approx$ 1.19999951 y $\approx$ 1.25000036 z $\approx$ -0.34999893 $\lambda\approx$ -1.00000038
	L_2 method	38	x $\approx$ 1.19999984 y $\approx$ 1.25000010 z $\approx$ -0.34999936 $\lambda\approx$ -1.00000016
	L_3 method	43	x $\approx$ 1.19999867 y $\approx$ 1.25000107 z $\approx$ -0.34999680 $\lambda\approx$ -1.00000080

Table 4.3 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward	L_4 *method*	43	x ≈ 1.19999867 y ≈ 1.25000107 z ≈ -0.34999680 λ ≈ -1.00000080
	U_2 *method*	36	x ≈ 1.19999982 y ≈ 1.25000019 z ≈ -0.34999933 λ ≈ -1.00000025
	U_3 *method*	48	x ≈ 1.19999966 x ≈ 1.25000035 z ≈ -0.34999895 λ ≈ -1.00000034
	U_4 *method*	48	x ≈ 1.19999966 x ≈ 1.25000035 z ≈ -0.34999895 λ ≈ -1.00000034
Two Points Forward (0.8,1.6,0.8,-0.8) and $\Delta x = 0.5$, $\Delta y = 0.5$, $\Delta z = 0.5$, $\Delta\lambda = 0.5$	*D method*	102	x ≈ 1.19999974 y ≈ 1.25000015 z ≈ -0.34999954 λ ≈ -1.00000020
	L_2 *method*	41	x ≈ 1.19999983 y ≈ 1.25000010 z ≈ -0.34999930 λ ≈ -1.00000018

Table 4.3 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Forward	L_3 *method*	37	$x \approx 1.19999868$ $y \approx 1.25000107$ $z \approx -0.34999682$ $\lambda \approx -1.00000079$
	L_4 *method*	37	$x \approx 1.19999868$ $y \approx 1.25000107$ $z \approx -0.34999682$ $\lambda \approx -1.00000079$
	U_2 *method*	46	$x \approx 1.19999990$ $y \approx 1.25000010$ $z \approx -0.34999956$ $\lambda \approx -1.00000019$
	U_3 *method*	43	$x \approx 1.19999965$ $y \approx 1.25000036$ $z \approx -0.34999892$ $\lambda \approx -1.00000035$
	U_4 *method*	43	$x \approx 1.19999965$ $y \approx 1.25000036$ $z \approx -0.34999892$ $\lambda \approx -1.00000035$
Three Points Central	D *method*	104	$x \approx 1.19999988$ $y \approx 1.25000003$ $z \approx -0.34999992$ $\lambda \approx -1.00000010$

Table 4.3 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Three Points Central (1.3,1.3,0.0,-0.5) and $\Delta x = 0.5$, $\Delta y = 1.0$, $\Delta z = 0.5$, $\Delta\lambda = 0.5$	L_2 *method*	31	$x \approx 1.19999899$ $y \approx 1.25000087$ $z \approx -0.34999735$ $\lambda \approx -1.00000066$
	L_3 *method*	43	$x \approx 1.19999856$ $y \approx 1.25000116$ $z \approx -0.34999655$ $\lambda \approx -1.00000086$
	L_4 *method*	43	$x \approx 1.19999856$ $y \approx 1.25000116$ $z \approx -0.34999655$ $\lambda \approx -1.00000086$
	U_2 *method*	32	$x \approx 1.20000005$ $y \approx 1.24999995$ $z \approx -0.35000032$ $\lambda \approx -0.99999983$
	U_3 *method*	45	$x \approx 1.20000037$ $y \approx 1.24999962$ $z \approx -0.35000114$ $\lambda \approx -0.99999963$
	U_4 *method*	45	$x \approx 1.20000037$ $y \approx 1.24999962$ $z \approx -0.35000114$ $\lambda \approx -0.99999963$

Showing the iterations of table 4.3 in appendix.

Example 4.4 Find the solution of the system

$$0.25\mu + x^2 - 0.0275 = 0$$

$$y^2 + z^2 - 9.49 = 0$$

$$x\lambda + y + z + 2.0 = 0 \tag{4.4}$$

$$z\lambda\mu + 0.0525 = 0$$

$$0.5(x - \mu) - 0.125 = 0.$$

We solving this system by D *method*, L_2 *method*, L_3 *method*, L_4 *method*, L_5 *method*, U_2 *method*, U_3 *method*, U_4 *method* and U_5 *method*, the approximate solutions are form :

$$x^{k+1} = x^k - \left[D(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots,$$

where $D(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 & 0 \\ 0 & f_{22} & 0 & 0 & 0 \\ 0 & 0 & f_{33} & 0 & 0 \\ 0 & 0 & 0 & f_{44} & 0 \\ 0 & 0 & 0 & 0 & f_{55} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[L_2(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots$$

$$\text{where} \quad L_2(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 & 0 \\ 0 & f_{32} & f_{33} & 0 & 0 \\ 0 & 0 & f_{43} & f_{44} & 0 \\ 0 & 0 & 0 & f_{54} & f_{55} \end{bmatrix},$$

$$x^{k+1} = x^k - \left[L_3(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots$$

$$\text{where} \quad L_3(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 & 0 \\ f_{22} & f_{22} & 0 & 0 & 0 \\ f_{31} & f_{32} & f_{33} & 0 & 0 \\ 0 & f_{42} & f_{43} & f_{44} & 0 \\ 0 & 0 & f_{53} & f_{54} & f_{55} \end{bmatrix},$$

$$x^{k+1} = x^k - \left[L_4(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots$$

$$\text{where} \quad L_4(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 & 0 \\ f_{22} & f_{22} & 0 & 0 & 0 \\ f_{31} & f_{32} & f_{33} & 0 & 0 \\ f_{41} & f_{42} & f_{43} & f_{44} & 0 \\ 0 & f_{52} & f_{53} & f_{54} & f_{55} \end{bmatrix},$$

$$x^{k+1} = x^k - \left[L_5(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots$$

where $L_5(x^k) = \begin{bmatrix} f_{11} & 0 & 0 & 0 & 0 \\ f_{22} & f_{22} & 0 & 0 & 0 \\ f_{31} & f_{32} & f_{33} & 0 & 0 \\ f_{41} & f_{42} & f_{43} & f_{44} & 0 \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[L_5(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots,$$

where $U_2(x^k) = \begin{bmatrix} f_{11} & f_{12} & 0 & 0 & 0 \\ 0 & f_{22} & f_{23} & 0 & 0 \\ 0 & 0 & f_{33} & f_{34} & 0 \\ 0 & 0 & 0 & f_{44} & f_{45} \\ 0 & 0 & 0 & 0 & f_{55} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[U_3(x^k)\right]^{-1}\begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots$$

where $U_3(x^k) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & 0 & 0 \\ 0 & f_{22} & f_{23} & f_{24} & 0 \\ 0 & 0 & f_{33} & f_{34} & f_{35} \\ 0 & 0 & 0 & f_{44} & f_{45} \\ 0 & 0 & 0 & 0 & f_{55} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[U_4(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots$$

where $U_4(x^k) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & 0 \\ 0 & f_{22} & f_{23} & f_{24} & f_{25} \\ 0 & 0 & f_{33} & f_{34} & f_{35} \\ 0 & 0 & 0 & f_{44} & f_{45} \\ 0 & 0 & 0 & 0 & f_{55} \end{bmatrix}$,

$$x^{k+1} = x^k - \left[U_5(x^k)\right]^{-1} \begin{bmatrix} f_1(x^k) \\ f_2(x^k) \\ f_3(x^k) \\ f_4(x^k) \\ f_5(x^k) \end{bmatrix}; \quad k = 0,1,2,\ldots$$

where $U_5(x^k) = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ 0 & f_{22} & f_{23} & f_{24} & f_{25} \\ 0 & 0 & f_{33} & f_{34} & f_{35} \\ 0 & 0 & 0 & f_{44} & f_{45} \\ 0 & 0 & 0 & 0 & f_{55} \end{bmatrix}$,

By using initial vectors two points backward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix, $L_3(x^k)$ matrix, $L_4(x^k)$ matrix, $L_5(x^k)$ matrix, $U_2(x^k)$ matrix, $U_3(x^k)$ matrix, $U_4(x^k)$ matrix and $U_5(x^k)$ matrix as below

$$f_{i1}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k, \mu^k) - f_i(x^{k-1}, y^k, z^k, \lambda^k, \mu^k)}{x^k - x^{k-1}},$$

$$f_{i2}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k, \mu^k) - f_i(x^k, y^{k-1}, z^k, \lambda^k, \mu^k)}{y^k - y^{k-1}},$$

$$f_{i3}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k, \mu^k) - f_i(x^k, y^k, z^{k-1}, \lambda^k, \mu^k)}{z^k - z^{k-1}},$$

$$f_{i4}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k, \mu^k) - f_i(x^k, y^k, z^k, \lambda^{k-1}, \mu^k)}{\lambda^k - \lambda^{k-1}},$$

$$f_{i5}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k, \mu^k) - f_i(x^k, y^k, z^k, \lambda^k, \mu^{k-1})}{\mu^k - \mu^{k-1}}$$

where $i = 1,2,3,4,5$.

By using initial vectors two points forward, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix, $L_3(x^k)$ matrix, $L_4(x^k)$ matrix, $L_5(x^k)$ matrix, $U_2(x^k)$ matrix, $U_3(x^k)$ matrix, $U_4(x^k)$ matrix and $U_5(x^k)$ matrix as below

$$f_{i1}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^{k+1}, y^k, z^k, \lambda^k, \mu^k) - f_i(x^k, y^k, z^k, \lambda^k, \mu^k)}{x^{k+1} - x^k},$$

$$f_{i2}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^{k+1}, z^k, \lambda^k, \mu^k) - f_i(x^k, y^k, z^k, \lambda^k, \mu^k)}{y^{k+1} - y^k},$$

$$f_{i3}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^{k+1}, \lambda^k, \mu^k) - f_i(x^k, y^k, z^k, \lambda^k, \mu^k)}{z^{k+1} - z^k},$$

$$f_{i4}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^{k+1}, \mu^k) - f_i(x^k, y^k, z^k, \lambda^k, \mu^k)}{\lambda^{k+1} - \lambda^k},$$

$$f_{i5}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k, \mu^{k+1}) - f_i(x^k, y^k, z^k, \lambda^k, \mu^k)}{\mu^{k+1} - \mu^k}$$

where $i = 1,2,3,4,5$.

By using initial vectors three points central, we can find $D(x^k)$ matrix, $L_2(x^k)$ matrix, $L_3(x^k)$ matrix, $L_4(x^k)$ matrix, $L_5(x^k)$ matrix, $U_2(x^k)$ matrix, $U_3(x^k)$ matrix, $U_4(x^k)$ matrix and $U_5(x^k)$ matrix as below

$$f_{i1}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^{k+1}, y^k, z^k, \lambda^k, \mu^k) - f_i(x^{k-1}, y^k, z^k, \lambda^k, \mu^k)}{x^{k+1} - x^{k-1}},$$

$$f_{i2}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^{k+1}, z^k, \lambda^k, \mu^k) - f_i(x^k, y^{k-1}, z^k, \lambda^k, \mu^k)}{y^{k+1} - y^{k-1}},$$

$$f_{i3}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^{k+1}, \lambda^k, \mu^k) - f_i(x^k, y^k, z^{k-1}, \lambda^k, \mu^k)}{z^{k+1} - z^{k-1}},$$

$$f_{i4}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^{k+1}, \mu^k) - f_i(x^k, y^k, z^k, \lambda^{k-1}, \mu^k)}{\lambda^{k+1} - \lambda^{k-1}},$$

$$f_{i5}(x^k, y^k, z^k, \lambda^k, \mu^k) = \frac{f_i(x^k, y^k, z^k, \lambda^k, \mu^{k+1}) - f_i(x^k, y^k, z^k, \lambda^k, \mu^{k-1})}{\mu^{k+1} - \mu^{k-1}}$$

where $i = 1,2,3,4,5$.

Repeat each of the methods until error less than 0.0000005. The results of example 4.4 showing in the table 4.4.

Table 4.4 The results of example 4.4

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward (-0.5,-1.5,2.5,1.5,-0.5) and $\Delta x = 0.5$, $\Delta y = 0.5$, $\Delta z = 2.0$, $\Delta\lambda = 1.0$, $\Delta\mu = 1.0$	*D method*	35	x ≈ -0.45000000 y ≈ -2.92662028 z ≈ 0.96171399 $\lambda \approx$ 0.07798577 $\mu \approx$ -0.70000000
	L_2 method	21	x ≈ -0.44999996 y ≈ -2.92662029 z ≈ 0.96171387 $\lambda \approx$ 0.07798579 $\mu \approx$ -0.70000012
	L_3 method	21	x ≈ -0.44999996 y ≈ -2.92662027 z ≈ 0.96171384 $\lambda \approx$ 0.07798579 $\mu \approx$ -0.70000012
	L_4 method	21	x ≈ -0.44999996 y ≈ -2.92662027 z ≈ 0.96171384 $\lambda \approx$ 0.07798579 $\mu \approx$ -0.70000012
	L_5 method	17	x ≈ -0.45000000 y ≈ -2.92662034 z ≈ 0.96171391 $\lambda \approx$ 0.07798578 $\mu \approx$ -0.70000006

Table 4.4 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward	U_2 *method*	20	x $\approx$-0.45000012 y $\approx$-2.92662031 z $\approx$0.96171388 $\lambda \approx$ 0.07798574 $\mu \approx$-0.69999984
	U_3 *method*	20	x $\approx$-0.45000012 y $\approx$-2.92662031 z $\approx$0.96171388 $\lambda \approx$ 0.07798574 $\mu \approx$-0.69999984
	U_4 *method*	20	x $\approx$-0.45000012 y $\approx$-2.92662031 z $\approx$0.96171388 $\lambda \approx$ 0.07798574 $\mu \approx$-0.69999984
	U_5 *method*	16	x $\approx$-0.45000022 y $\approx$-2.92662024 z $\approx$0.96171408 $\lambda \approx$ 0.07798574 $\mu \approx$-0.70000032
Two Points Forward	D *method*	31	x $\approx$-0.45000000 y $\approx$-2.92662035 z $\approx$0.96171380 $\lambda \approx$ 0.07798578 $\mu \approx$-0.70000001

Table 4.4 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Forward (-0.2,-3.5,-1.0,-0.5,-0.5) and $\Delta x = 0.3$, $\Delta y = 0.5$, $\Delta z = 2.0$, $\Delta\lambda = 1.0$, $\Delta\mu = 0.4$	L_2 *method*	23	x ≈ -0.45000042 y ≈ -2.92662032 z ≈ 0.96171383 $\lambda \approx 0.07798561$ $\mu \approx -0.70000089$
	L_3 *method*	24	x ≈ -0.45000025 y ≈ -2.92662033 z ≈ 0.96171388 $\lambda \approx 0.07798568$ $\mu \approx -0.70000042$
	L_4 *method*	24	x ≈ -0.45000025 y ≈ -2.92662033 z ≈ 0.96171388 $\lambda \approx 0.07798568$ $\mu \approx -0.70000042$
	L_5 *method*	22	x ≈ -0.45000021 y ≈ -2.92662032 z ≈ 0.96171386 $\lambda \approx 0.07798569$ $\mu \approx -0.70000037$
	U_2 *method*	23	x ≈ -0.45000042 y ≈ -2.92662031 z ≈ 0.96171388 $\lambda \approx 0.07798563$ $\mu \approx -0.70000089$

Table 4.4 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Forward	U_3 *method*	23	x $\approx$-0.45000042 y $\approx$-2.92662031 z $\approx$0.96171388 $\lambda \approx$ 0.07798563 $\mu \approx$-0.70000089
	U_4 *method*	23	x $\approx$-0.45000042 y $\approx$-2.92662031 z $\approx$0.96171388 $\lambda \approx$ 0.07798563 $\mu \approx$-0.70000089
	U_5 *method*	21	x $\approx$-0.45000041 y $\approx$-2.92662031 z $\approx$0.96171387 $\lambda \approx$ 0.07798561 $\mu \approx$-0.70000084
Three Points Central (-1.0,-3.0,-0.5,-0.5,-1.0) and $\Delta x = 0.5$, $\Delta y = 1.5$, $\Delta z = 1.0$, $\Delta\lambda = 0.5$, $\Delta\mu = 0.5$	D *method*	31	x $\approx$-0.45000000 y $\approx$-2.92662030 z $\approx$0.96171400 $\lambda \approx$ 0.07798577 $\mu \approx$-0.70000000
	L_2 *method*	22	x $\approx$-0.45000036 y $\approx$-2.92662033 z $\approx$0.96171387 $\lambda \approx$ 0.07798563 $\mu \approx$-0.70000056

Table 4.4 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Three Points Central	L_3 *method*	23	$x \approx -0.45000016$ $y \approx -2.92662031$ $z \approx 0.96171385$ $\lambda \approx 0.07798571$ $\mu \approx -0.70000036$
	L_4 *method*	23	$x \approx -0.45000016$ $y \approx -2.92662031$ $z \approx 0.96171385$ $\lambda \approx 0.07798571$ $\mu \approx -0.70000036$
	L_5 *method*	20	$x \approx -0.45000032$ $y \approx -2.92662034$ $z \approx 0.96171385$ $\lambda \approx 0.07798565$ $\mu \approx -0.70000056$
	U_2 *method*	22	$x \approx -0.45000036$ $y \approx -2.92662031$ $z \approx 0.96171387$ $\lambda \approx 0.07798565$ $\mu \approx -0.70000056$
	U_3 *method*	22	$x \approx -0.45000036$ $y \approx -2.92662031$ $z \approx 0.96171387$ $\lambda \approx 0.07798565$ $\mu \approx -0.70000056$

Table 4.4 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Three Points Central	U_4 *method*	22	$x \approx -0.45000036$ $y \approx -2.92662031$ $z \approx 0.96171387$ $\lambda \approx 0.07798565$ $\mu \approx -0.70000056$
	U_5 *method*	20	$x \approx -0.45000032$ $y \approx -2.92662031$ $z \approx 0.96171387$ $\lambda \approx 0.07798565$ $\mu \approx -0.70000065$

Showing the iterations of table 4.4 in appendix.

Example 4.5 The Robot

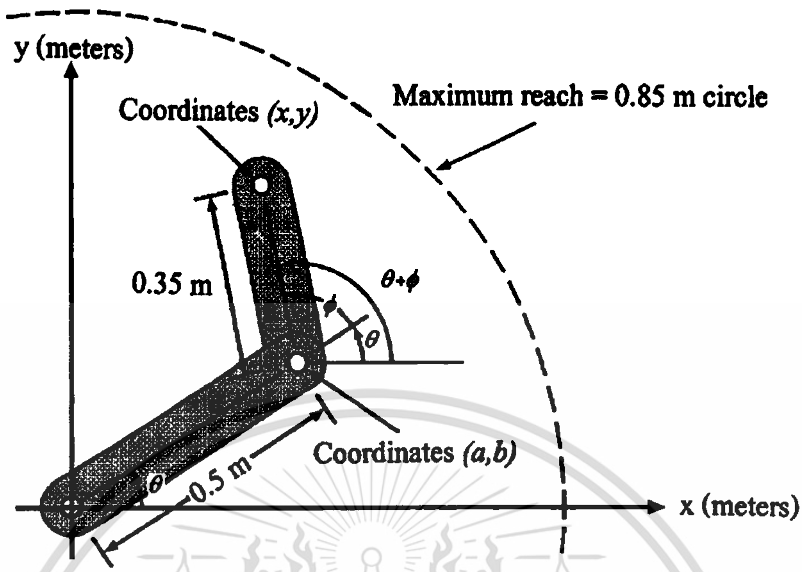


Figure 4.1 Position a robot arm in geometry.

Figure 4.1 shows a two-link robot arm. One end of arm is fixed at the origin of the coordinate system, and the other end is at coordinate (x, y). There are positioning actuators control the link angles θ and ϕ, as show. In term of θ and ϕ, the coordinates of the joint between links are (a, b), where

$$a = 0.5\cos\theta \;\text{ and }\; b = 0.5\sin\theta\,.$$

The coordinates (x, y) are given by

$$x - a = 0.35\cos(\theta + \phi)$$

$$y - b = 0.35\sin(\theta + \phi)$$

or

$$x = 0.5\cos\theta + 0.35\cos(\theta + \phi) = 0$$

$$y = 0.5\sin\theta + 0.35\sin(\theta + \phi) = 0$$

The angles θ and ϕ that will position the end of the arm at coordinates (0.2,0.4) are given by simultaneous solution of

$$f(\theta+\phi)=0.5\cos\theta+0.35\cos(\theta+\phi)-0.2=0$$
$$g(\theta+\phi)=0.5\sin\theta+0.35\sin(\theta+\phi)-0.4=0 \tag{4.5}$$

We solving this system by D *method*, L_2 *method* and U_2 *method*, the approximate solutions are form :

$$\begin{bmatrix}\theta^{k+1}\\ \phi^{k+1}\end{bmatrix}=\begin{bmatrix}\theta^{k}\\ \phi^{k}\end{bmatrix}-\left[D(\theta^k,\phi^k)\right]^{-1}\begin{bmatrix}f(\theta^k,\phi^k)\\ g(\theta^k,\phi^k)\end{bmatrix};\quad k=0,1,2,\ldots,$$

where $D(\theta^k,\phi^k)=\begin{bmatrix}f_\theta & 0\\ 0 & g_\phi\end{bmatrix}$,

$$\begin{bmatrix}\theta^{k+1}\\ \phi^{k+1}\end{bmatrix}=\begin{bmatrix}\theta^{k}\\ \phi^{k}\end{bmatrix}-\left[L_2(\theta^k,\phi^k)\right]^{-1}\begin{bmatrix}f(\theta^k,\phi^k)\\ g(\theta^k,\phi^k)\end{bmatrix};\quad k=0,1,2,\ldots,$$

where $L_2(\theta^k,\phi^k)=\begin{bmatrix}f_\theta & 0\\ g_\theta & g_\phi\end{bmatrix}$,

$$\begin{bmatrix}\theta^{k+1}\\ \phi^{k+1}\end{bmatrix}=\begin{bmatrix}\theta^{k}\\ \phi^{k}\end{bmatrix}-\left[U_2(\theta^k,\phi^k)\right]^{-1}\begin{bmatrix}f(\theta^k,\phi^k)\\ g(\theta^k,\phi^k)\end{bmatrix};\quad k=0,1,2,\ldots,$$

where $U_2(\theta^k,\phi^k)=\begin{bmatrix}f_\theta & f_\phi\\ 0 & g_\phi\end{bmatrix}$.

By using initial vectors two points backward, we can find $D(\theta^k,\phi^k)$ matrix, $L_2(\theta^k,\phi^k)$ matrix and $U_2(\theta^k,\phi^k)$ matrix as below

$$f_\theta(\theta^k,\phi^k)=\frac{f(\theta^k,\phi^k)-f(\theta^{k-1},\phi^k)}{\theta^k-\theta^{k-1}},\quad f_\phi(\theta^k,\phi^k)=\frac{f(\theta^k,\phi^k)-f(\theta^k,\phi^{k-1})}{\phi^k-\phi^{k-1}},$$

$$g_\theta(\theta^k,\phi^k)=\frac{g(\theta^k,\phi^k)-g(\theta^{k-1},\phi^k)}{\theta^k-\theta^{k-1}},\quad g_\phi(\theta^k,\phi^k)=\frac{g(\theta^k,\phi^k)-g(\theta^k,\phi^{k-1})}{\phi^k-\phi^{k-1}}.$$

By using initial vectors two points forward, we can find $D(\theta^k,\phi^k)$ matrix, $L_2(\theta^k,\phi^k)$ matrix and $U_2(\theta^k,\phi^k)$ matrix as below

$$f_\theta(\theta^k,\phi^k)=\frac{f(\theta^{k+1},\phi^k)-f(\theta^k,\phi^k)}{\theta^{k+1}-\theta^k},\quad f_\phi(\theta^k,\phi^k)=\frac{f(\theta^k,\phi^{k+1})-f(\theta^k,\phi^k)}{\phi^{k+1}-\phi^k},$$

$$g_\theta(\theta^k,\phi^k)=\frac{g(\theta^{k+1},\phi^k)-g(\theta^k,\phi^k)}{\theta^{k+1}-\theta^k},\quad g_\phi(\theta^k,\phi^k)=\frac{g(\theta^k,\phi^{k+1})-g(\theta^k,\phi^k)}{\phi^{k+1}-\phi^k}.$$

By using initial vectors three points central, we can find $D(\theta^k,\phi^k)$ matrix, $L_2(\theta^k,\phi^k)$ matrix and $U_2(\theta^k,\phi^k)$ matrix as below

$$f_\theta(\theta^k,\phi^k)=\frac{f(\theta^{k+1},\phi^k)-f(\theta^{k-1},\phi^k)}{\theta^{k+1}-\theta^{k-1}},$$

$$f_\phi(\theta^k,\phi^k)=\frac{f(\theta^k,\phi^{k+1})-f(\theta^k,\phi^{k-1})}{\phi^{k+1}-\phi^{k-1}},$$

$$g_\theta(\theta^k,\phi^k)=\frac{g(\theta^{k+1},\phi^k)-g(\theta^{k-1},\phi^k)}{\theta^{k+1}-\theta^{k-1}},$$

$$g_\phi(\theta^k,\phi^k)=\frac{g(\theta^k,\phi^{k+1})-g(\theta^k,\phi^{k-1})}{\phi^{k+1}-\phi^{k-1}}.$$

Repeat each of the methods until error less than 0.0000005. The results of D *method*, L_2 *method* and U_2 *method* in the table 4.5.

Table 4.5 The results of example 4.5

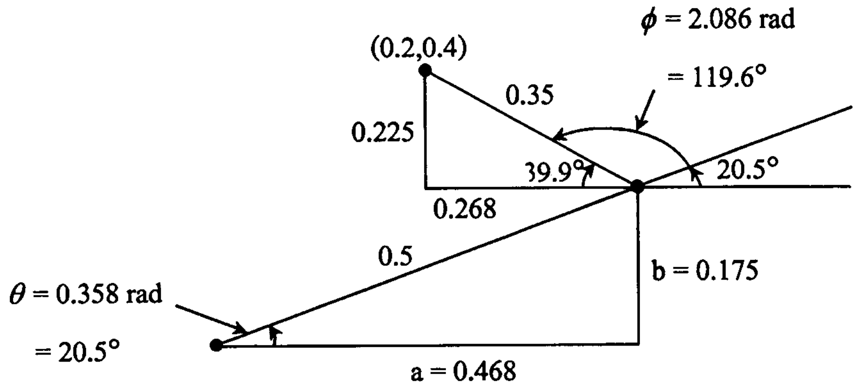
Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward (1.5,2.5) and $\Delta\theta = 0.5,\ \Delta\phi = 1.0$	D *method*	32	$\theta \approx 0.35806360$ $\phi \approx 2.08616734$
	L_2 *method*	29	$\theta \approx 0.35806315$ $\phi \approx 2.08616612$
	U_2 *method*	29	$\theta \approx 0.35806319$ $\phi \approx 2.08616594$
Two Points Backward (1.5,-1.0) and $\Delta\theta = 0.5,\ \Delta\phi = 1.0$	D *method*	13	$\theta \approx 1.85623355$ $\phi \approx -2.08616632$
	L_2 *method*	12	$\theta \approx 1.85623331$ $\phi \approx -2.08616693$
	U_2 *method*	12	$\theta \approx 1.85623324$ $\phi \approx -2.08616704$
Two Points Forward (0.4,2.1) and $\Delta\theta = 1.0,\ \Delta\phi = 1.0$	D *method*	25	$\theta \approx 0.35806358$ $\phi \approx 2.08616749$
	L_2 *method*	23	$\theta \approx 0.35806424$ $\phi \approx 2.08616605$
	U_2 *method*	23	$\theta \approx 0.35806420$ $\phi \approx 2.08616605$
Two Points Forward (1.3,-2.5) and $\Delta\theta = 0.5,\ \Delta\phi = 1.0$	D *method*	13	$\theta \approx 1.85623334$ $\phi \approx -2.08616678$
	L_2 *method*	12	$\theta \approx 1.85623364$ $\phi \approx -2.08616768$
	U_2 *method*	11	$\theta \approx 1.85623460$ $\phi \approx -2.08616710$

Table 4.5 (cont.)

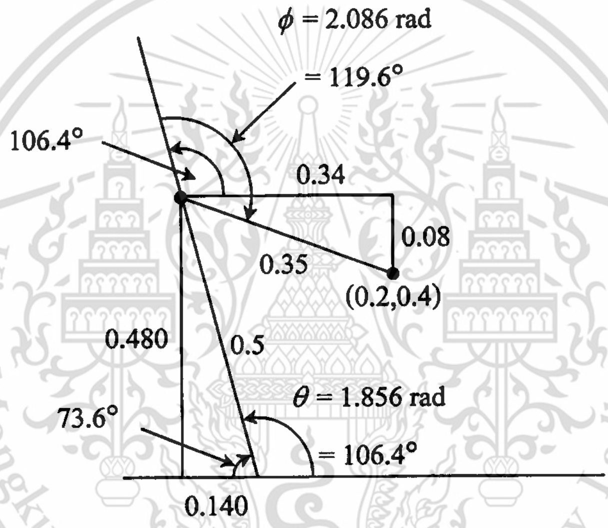
Initial Vectors	Method	Number of Iterations	Solution
Three Points Central (0.8,2.5) and $\Delta\theta = 0.5,\ \Delta\phi = 1.0$	*D method*	31	$\theta \approx 0.35806433$ $\phi \approx 2.08616681$
	L_2 method	28	$\theta \approx 0.35806311$ $\phi \approx 2.08616735$
	U_2 method	28	$\theta \approx 0.35806302$ $\phi \approx 2.08616737$
Three Points Central (2.0,-1.5) and $\Delta\theta = 0.5,\ \Delta\phi = 0.5$	*D method*	13	$\theta \approx 1.85623354$ $\phi \approx -2.08616736$
	L_2 method	12	$\theta \approx 1.85623424$ $\phi \approx -2.08616736$
	U_2 method	12	$\theta \approx 1.85623421$ $\phi \approx -2.08616724$

Showing the iterations of table 4.5 in appendix.

This problem, there are two sets of link angles that will position the arm as desired, $\theta \approx 0.358$, $\phi \approx 2.086$ and $\theta \approx 1.856$, $\phi \approx -2.086$. The geometry of these solutions is shown in Figure 4.2 (a) and (b) respectively. The existence of multiple solutions, as in this case, is one of the complications of robotics.



(a) First solution



(b) Second solution

Figure 4.2 The solutions of robot arm

Example 4.6 The fluid flow

The turbulent flow of fluids in an interconnected network, the flow rate V from one node to another is about proportional to the square root of the difference in pressures at the nodes (thus fluid flow differs from flow of electrical current in a network in that nonlinear equations result).

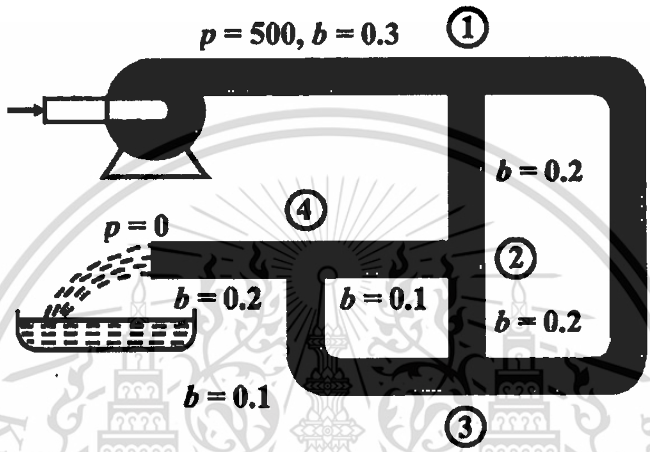


Figure 4.3 The turbulent flow of fluids in an interconnected network.

For the conduits in Figure 1.1 the pressure at each node. The values of b represent conductance factors in the relation $v_{ij} = b_{ij}(p_i - p_j)^{1/2}$.

These equations can be set up for the pressure at each node :

at node 1 : $0.3\sqrt{500-p_1} = 0.2\sqrt{p_1-p_2} + 0.1\sqrt{p_1-p_3}$

node 2 : $0.2\sqrt{p_1-p_2} = 0.1\sqrt{p_2-p_4} + 0.2\sqrt{p_2-p_3}$

node 3 : $0.1\sqrt{p_1-p_3} + 0.2\sqrt{p_2-p_3} = 0.1\sqrt{p_3-p_4}$

node 4 : $0.1\sqrt{p_2-p_4} + 0.1\sqrt{p_3-p_4} = 0.2\sqrt{p_4-0}$.

We want to find the value each of unknowns such satisfied each of equation. Repeat each of the method until error less than 0.0000005. The results of *D method*, *L_2 method*, *L_3 method*, *L_4 method*, *U_2 method*, *U_3 method* and *U_4 method* in the table 4.6.

Table 4.6 The results of example 4.6

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward (425.0,350.0,345.0,170.0) and $\Delta p_1 = 5.0$, $\Delta p_2 = 5.0$, $\Delta p_3 = 5.0$, $\Delta p_4 = 5.0$	*D method*	114	$p_1 \approx 423.18977547$ $p_2 \approx 347.79369978$ $p_3 \approx 343.51182508$ $p_4 \approx 172.82306773$
	L_2 method	114	$p_1 \approx 423.18977516$ $p_2 \approx 347.79369922$ $p_3 \approx 343.51182450$ $p_4 \approx 172.82306742$
	L_3 method	114	$p_1 \approx 423.18977514$ $p_2 \approx 347.79369919$ $p_3 \approx 343.51182446$ $p_4 \approx 172.82306740$
	L_4 method	114	$p_1 \approx 423.18977514$ $p_2 \approx 347.79369919$ $p_3 \approx 343.51182446$ $p_4 \approx 172.82306740$
	U_2 method	114	$p_1 \approx 423.18977516$ $p_2 \approx 347.79369924$ $p_3 \approx 343.51182453$ $p_4 \approx 172.82306743$

Table 4.6 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Backward	U_3 *method*	114	$p_1 \approx 423.18977515$ $p_2 \approx 347.79369921$ $p_3 \approx 343.51182450$ $p_4 \approx 172.82306741$
	U_4 *method*	114	$p_1 \approx 423.18977515$ $p_2 \approx 347.79369921$ $p_3 \approx 343.51182450$ $p_4 \approx 172.82306741$
Two Points Forward (430.0,345.0,340.0,170.0) and $\Delta p_1 = 5.0,\ \Delta p_2 = 5.0,$ $\Delta p_3 = 5.0,\ \Delta p_4 = 5.0$	D *method*	116	$p_1 \approx 423.18973771$ $p_2 \approx 347.79363163$ $p_3 \approx 343.51175552$ $p_4 \approx 172.82302983$
	L_2 *method*	115	$p_1 \approx 423.18973616$ $p_2 \approx 347.79362885$ $p_3 \approx 343.51175269$ $p_4 \approx 172.82302829$
	L_3 *method*	115	$p_1 \approx 423.18973617$ $p_2 \approx 347.79362885$ $p_3 \approx 343.51175269$ $p_4 \approx 172.82302829$
	L_4 *method*	115	$p_1 \approx 423.18973617$ $p_2 \approx 347.79362885$ $p_3 \approx 343.51175269$ $p_4 \approx 172.82302829$

Table 4.6 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Two Points Forward	U_2 *method*	115	$p_1 \approx 423.18973622$ $p_2 \approx 347.79362895$ $p_3 \approx 343.51175277$ $p_4 \approx 172.82302834$
	U_3 *method*	115	$p_1 \approx 423.18973624$ $p_2 \approx 347.79362898$ $p_3 \approx 343.51175281$ $p_4 \approx 172.82302836$
	U_4 *method*	115	$p_1 \approx 423.18973624$ $p_2 \approx 347.79362898$ $p_3 \approx 343.51175281$ $p_4 \approx 172.82302836$
Three Points Central (420.0,350.0,345.0,175.0) and $\Delta p_1 = 5.0$, $\Delta p_2 = 5.0$, $\Delta p_3 = 5.0$, $\Delta p_4 = 5.0$	D *method*	111	$p_1 \approx 423.18977560$ $p_2 \approx 347.79370002$ $p_3 \approx 343.51182532$ $p_4 \approx 172.82306786$
	L_2 *method*	111	$p_1 \approx 423.18977531$ $p_2 \approx 347.79369949$ $p_3 \approx 343.51182478$ $p_4 \approx 172.82306757$
	L_3 *method*	111	$p_1 \approx 423.18977530$ $p_2 \approx 347.79369948$ $p_3 \approx 343.51182477$ $p_4 \approx 172.82306756$

Table 4.6 (cont.)

Initial Vectors	Method	Number of Iterations	Solution
Three Points Central	L_4 *method*	111	$p_1 \approx 423.18977530$ $p_2 \approx 347.79369948$ $p_3 \approx 343.51182477$ $p_4 \approx 172.82306756$
	U_2 *method*	111	$p_1 \approx 423.18977530$ $p_2 \approx 347.79369948$ $p_3 \approx 343.51182477$ $p_4 \approx 172.82306756$
	U_3 *method*	111	$p_1 \approx 423.18977528$ $p_2 \approx 347.79369944$ $p_3 \approx 343.51182474$ $p_4 \approx 172.82306754$
	U_4 *method*	111	$p_1 \approx 423.18977528$ $p_2 \approx 347.79369944$ $p_3 \approx 343.51182474$ $p_4 \approx 172.82306754$

Showing the iterations of table 4.6 in appendix.

CHAPTER 5

CONCLUSION AND SUGGESTIONS

In this chapter, we will conclude and give suggestion about the new iterative methods are obtained in chapter 3 and the results from examples in chapter 4.

5.1 Conclusion

In this thesis, we found the new iterative methods for solving system of nonlinear equations, these methods are same type as Modified Newton method. The processing is replace Jacobian matrix in Newton's method by diagonal matrix $D(x)$, $L_2(x)$ matrix, $L_3(x)$ matrix, ..., $L_n(x)$ matrix, $U_2(x)$ matrix, $U_3(x)$ matrix, ..., and $U_n(x)$ matrix, then instead of partial derivative in Jacobian matrix by

$$f_{ij}(x^k) \approx \frac{f_i(x^k) - f_i(x^{k-1})}{x_j^k - x_j^{k-1}}$$ for using initial vector two points backward,

$$f_{ij}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^k)}{x_j^{k+1} - x_j^k}$$ for using initial vector two points forward,

$$f_{ij}(x^k) \approx \frac{f_i(x^{k+1}) - f_i(x^{k-1})}{x_j^{k+1} - x_j^{k-1}}$$ for using initial vector three points central

where $i, j = 1,2,\ldots,n$, and $k = 1,2,\ldots$.

We stop the iterative when $\sum_{i,j=1}^{n} \left|f_i(x_j) - f_i(x_{j-1})\right| < \varepsilon$, where ε denotes absolute error, in this thesis, we set $\varepsilon = 5 \times 10^{-7}$ all of examples. For the system of n equations n unknowns and choosing initial vector, we to obtained $(2n-1)$ methods. However, the number of iteration of the new iterative methods in chapter 3 may be more than the number of other method, because we without using partial derivative.

The system may have many solutions or unique solution or have no solution. For solving problems in chapter 4, we choose initial vectors by guessing, then sometimes initial vector can give the solution and sometimes cannot give the solution, though in actually the system has solution, which all obtained solution does not satisfy real world problem. In this case, scientists who to set up the problem need to improve and resolve.

5.2 Suggestions

In this thesis, the new iterative methods are obtained from chapter 3, we solved the problems by using initial two points backward difference, two points forward difference and three points central difference. For the person who interested in solving systems of nonlinear equations without using partial derivatives, may use more than or equal to three points and then compare their results.

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APPENDIX

The iterations of table 4.1

Two points backward

The result from *D method*

k	x^k	y^k	ER
1	0.41763120	3.12970460	0.08235074
2	0.46420316	3.06324025	0.06465247
3	0.57642463	3.11645731	0.07292035
4	0.49194411	3.14423818	0.00977060
5	0.49742493	3.13719141	0.00427240
10	0.50000525	3.14158460	0.00001122
11	0.50000128	3.14159528	0.00000257
12	0.49999958	3.14159329	0.00000089
13	0.49999990	3.14159244	0.00000020

Estimates values are $x \approx 0.49999990$ and $y \approx 3.14159244$ after 13 iterations.

The result from L_2 *method*

k	x^k	y^k	ER
1	0.41763120	3.06671230	0.02926339
2	0.47185743	3.07803599	0.05455599
3	0.55333903	3.19028252	0.07378276
4	0.48700306	3.13403063	0.00668721
5	0.49814287	3.13627331	0.00511578
6	0.50104695	3.14191640	0.00071955
7	0.49993865	3.14169103	0.00013452
8	0.49998410	3.14157878	0.00001418
9	0.50000221	3.14159149	0.00000298
10	0.50000019	3.14159300	0.00000034

Estimates values are $x \approx 0.50000019$ and $y \approx 3.14159300$ after 10 iterations.

The result from U_2 *method*

k	x^k	y^k	ER
1	0.46812148	3.12970460	0.02257815
2	0.49044244	3.11989923	0.02027303
3	0.50412081	3.13628895	0.00795136
4	0.49975578	3.14355458	0.00183755
5	0.49993094	3.14147020	0.00012031
6	0.50000910	3.14155810	0.00003564
7	0.50000084	3.14159721	0.00000436
8	0.49999977	3.14159307	0.00000055
9	0.50000000	3.14159254	0.00000011

Estimates values are $x \approx 0.50000000$ and $y \approx 3.14159254$ after 9 iterations.

Two points forward

The result from D *method*

k	x^k	y^k	ER
1	0.50738371	3.14159265	0.00674661
2	0.50048492	3.14496766	0.00322524
3	0.49943242	3.14183375	0.00071969
4	0.49996580	3.14130700	0.00027218
5	0.50004529	3.14157555	0.00005567
6	0.50000275	3.14161528	0.00002157
7	0.49999640	3.14159403	0.00000444
8	0.49999978	3.14159085	0.00000172
9	0.50000029	3.14159254	0.00000035

Estimates values are $x \approx 0.50000029$ and $y \approx 3.14159254$ after 9 iterations.

The result from L_2 *method*

k	x^k	y^k	ER
1	0.50738371	3.12139378	0.02252383
2	0.50343269	3.14387198	0.00247954
3	0.49963437	3.14186690	0.00055688
4	0.49996305	3.14153853	0.00005362
5	0.50000855	3.14159115	0.00000915
6	0.50000024	3.14159382	0.00000112
7	0.49999981	3.14159261	0.00000014

Estimates values are $x \approx 0.49999981$ and $y \approx 3.14159261$ after 7 iterations.

The result from U_2 *method*

k	x^k	y^k	ER
1	0.50010864	3.14159265	0.00010129
2	0.49999294	3.14164691	0.00005079
3	0.49999825	3.14158912	0.00000345
4	0.50000025	3.14159178	0.00000091
5	0.50000002	3.14159278	0.00000012

Estimates values are $x \approx 0.50000002$ and $y \approx 3.14159278$ after 5 iterations.

Three points central

The result from D *method*

k	x^k	y^k	ER
1	0.61253367	3.12970460	0.09442700
2	0.53008888	3.11880418	0.04304204
3	0.50627848	3.15129472	0.00993964
4	0.49873727	3.14450297	0.00347062
5	0.49953999	3.14095208	0.00063439
6	0.50010352	3.14136143	0.00027817
7	0.50003682	3.14164435	0.00005134
8	0.49999178	3.14161106	0.00002212
9	0.49999707	3.14158854	0.00000408

The result from D *method* (cont.)

k	x^k	y^k	ER
10	0.50000065	3.14159119	0.00000176
11	0.50000023	3.14159298	0.00000032

Estimates values are $x \approx 0.50000023$ and $y \approx 3.14159298$ after 11 iterations.

The result from L_2 *method*

k	x^k	y^k	ER
1	0.61253367	3.11076891	0.08850458
2	0.53184704	3.17682409	0.04414160
3	0.49942049	3.14926942	0.00721692
4	0.49878023	3.14106002	0.00072235
5	0.50009328	3.14147954	0.00017583
6	0.50001803	3.14161114	0.00001868
7	0.49999706	3.14159382	0.00000366
8	0.49999981	3.14159221	0.00000043

Estimates values are $x \approx 0.49999981$ and $y \approx 3.14159221$ after 8 iterations.

The result from U_2 *method*

k	x^k	y^k	ER
1	0.55081206	3.12970460	0.04920761
2	0.50705710	3.15154836	0.01031753
3	0.49955611	3.14483182	0.00303157
4	0.49989704	3.14136957	0.00021761
5	0.50001516	3.14154111	0.00005463
6	0.50000116	3.14160023	0.00000724
7	0.49999963	3.14159323	0.00000080
8	0.50000000	3.14159247	0.00000018

Estimates values are $x \approx 0.50000000$ and $y \approx 3.14159247$ after 8 iterations.

The iterations of table 4.2

Two points backward

The result from D *method*

k	x^k	y^k	z^k	ER
1	50.37075540	42.19059522	42.53368044	13.13899526
2	46.55189846	45.77765402	39.18440931	6.29160141
3	46.81739868	42.81328135	43.43503980	11.19890030
4	45.63979140	45.31879133	40.17163089	5.70949395
5	46.57919615	42.89295733	42.82271335	6.36905635
10	45.74811183	44.17280858	41.44246354	1.41285906
15	45.93650784	43.76014398	41.87629002	0.36892210
20	45.88646043	43.86584903	41.77278695	0.09195518
25	45.89989973	43.84090713	41.80040858	0.02283440
30	45.89702510	43.84782333	41.79415083	0.00579694
35	45.89794894	43.84637216	41.79597632	0.00141364
40	45.89780135	43.84684818	41.79562515	0.00036825
45	45.89787211	43.84677497	41.79575564	0.00008613
50	45.89786812	43.84681185	41.79574029	0.00002396
55	45.89787467	43.84681015	41.79575114	0.00000501
60	45.89787529	43.84681362	41.79575126	0.00000165
61	45.89787572	43.84681327	41.79575203	0.00000086
62	45.89787564	43.84681387	41.79575176	0.00000098
63	45.89787591	43.84681370	41.79575223	0.00000047

Estimates values are $x \approx 45.89787591$, $y \approx 43.84681370$ and $z \approx 41.79575223$ after 63 iterations.

The result from L_2 *method* and L_3 *method*

k	x^k	y^k	z^k	ER
1	50.37075540	44.59190669	40.84948844	4.90168331
2	47.08535839	44.93671769	42.52904172	0.62917087
3	46.45048385	44.69638295	42.33465860	0.41493732
4	46.39076217	44.38205967	42.00503210	0.54867443

The result from L_2 *method* and L_3 *method* (cont.)

k	x^k	y^k	z^k	ER
5	46.19068580	44.15176526	41.88251139	0.37913156
10	45.89937223	43.84397759	41.78617945	0.01429514
15	45.89712738	43.84621658	41.79602539	0.00164944
20	45.89793586	43.84688079	41.79577360	0.00007883
25	45.89787470	43.84681159	41.79574968	0.00000167
28	45.89787556	43.84681380	41.79575279	0.00000147
29	45.89787600	43.84681428	41.79575295	0.00000090
30	45.89787626	43.84681454	41.79575296	0.00000044

Estimates values are $x \approx 45.89787626$, $y \approx 43.84681454$ and $z \approx 41.79575296$ after 30 iterations.

The result from U_2 *method*

k	x^k	y^k	z^k	ER
1	48.23667709	41.25926884	42.53368044	14.20203266
2	45.61699073	40.46863753	36.87844912	4.37768306
3	44.35169392	49.14534264	38.22501235	16.40253424
4	48.60024674	42.29590420	47.60072482	20.60144635
5	43.08104472	35.10962856	39.84178124	16.58962472
10	41.32857616	39.51725627	26.32942821	14.00439954
15	44.96577319	42.42469184	40.43997122	0.56893436
20	45.75973669	43.69199564	41.60489093	0.13327250
25	45.88667305	43.83522640	41.78032965	0.01285885
30	45.89715610	43.84608381	41.79476280	0.00085520
35	45.89783184	43.84676945	41.79569157	0.00005301
40	45.89787364	43.84681180	41.79574901	0.00000327
42	45.89787549	43.84681367	41.79575154	0.00000107
43	45.89787587	43.84681406	41.79575207	0.00000061
44	45.89787610	43.84681429	41.79575238	0.00000035

Estimates values are $x \approx 45.89787610$, $y \approx 43.84681429$ and $z \approx 41.79575238$ after 44 iterations.

The result from U_3 *method*

k	x^k	y^k	z^k	ER
1	46.91734729	41.25926884	42.53368044	13.19358247
2	43.89671858	40.23490850	36.87844912	3.05398217
3	45.72326467	47.02396366	38.12792105	12.57550769
4	47.86049287	42.51137227	45.68961372	17.04353647
5	41.22330912	34.51692487	39.76603916	16.32065669
10	37.32399098	37.60841799	25.22703472	18.33425187
15	44.37704648	41.66048057	39.57743992	1.01878775
20	45.75978311	43.67323876	41.55618752	0.17484705
25	45.90004148	43.85076505	41.79587917	0.00630905
30	45.89820931	43.84718766	41.79649943	0.00081023
35	45.89784115	43.84676746	41.79569749	0.00003147
39	45.89787611	43.84681492	41.79575013	0.00000554
40	45.89787728	43.84681613	41.79575304	0.00000209
41	45.89787744	43.84681613	41.79575398	0.00000038

Estimates values are $x \approx 45.89787744$, $y \approx 43.84681613$ and $z \approx 41.79575398$ after 41 iterations.

Two points forward

The result from D *method*

k	x^k	y^k	z^k	ER
1	46.81225106	45.27762431	43.04608193	0.49906072
2	46.77668617	44.92990932	42.89483605	0.59567099
3	46.57569469	44.83588826	42.63139595	0.32745461
4	46.49989514	44.60393004	42.55575084	0.39896916
5	46.37157000	44.52785348	42.38145378	0.23518686
10	46.09654175	44.10571892	42.04743128	0.12350875
15	45.97618808	43.95521359	41.89378403	0.04240899
20	45.93002516	43.88974714	41.83639992	0.01897849
25	45.91077860	43.86443353	41.81197816	0.00722428
30	45.90313092	43.85389303	41.80238523	0.00304021
35	45.89999782	43.84969662	41.79842477	0.00120290

The result from D *method* (cont.)

k	x^k	y^k	z^k	ER
40	45.89873760	43.84797851	41.79683900	0.00049446
45	45.89822484	43.84728703	41.79619190	0.00019852
50	45.89801767	43.84700576	41.79593092	0.00008088
55	45.89793360	43.84689209	41.79582489	0.00003265
60	45.89789957	43.84684597	41.79578201	0.00001326
65	45.89788578	43.84682731	41.79576462	0.00000536
70	45.89788020	43.84681974	41.79575758	0.00000217
75	45.89787793	43.84681667	41.79575472	0.00000088
77	45.89787746	43.84681604	41.79575414	0.00000061
78	45.89787729	43.84681580	41.79575391	0.00000051
79	45.89787714	43.84681560	41.79575372	0.00000043

Estimates values are $x \approx 45.89787714$, $y \approx 43.84681560$ and $z \approx 41.79575372$ after 79 iterations.

The result from L_2 *method* and L_3 *method*

k	x^k	y^k	z^k	ER
1	46.81225106	45.12047092	42.45085803	0.90644695
2	46.63626452	44.58023084	42.10298140	0.73151997
3	46.30370695	44.29091722	41.96705562	0.46042117
4	46.14483186	44.09560656	41.85675521	0.33093521
5	46.03086757	43.97305452	41.80695183	0.20827831
10	45.89425430	43.84123077	41.79097111	0.00176381
15	45.89773226	43.84681405	41.79606460	0.00063554
20	45.89790624	43.84684104	41.79574790	0.00005836
25	45.89787445	43.84681227	41.79575175	0.00000214
26	45.89787510	43.84681326	41.79575257	0.00000200
27	45.89787570	43.84681398	41.79575291	0.00000137
28	45.89787611	43.84681441	41.79575299	0.00000076
29	45.89787633	43.84681460	41.79575296	0.00000033

Estimates values are $x \approx 45.89787633$, $y \approx 43.84681460$ and $z \approx 41.79575296$ after 29 iterations.

The result from U_2 *method*

k	x^k	y^k	z^k	ER
1	26.81112927	7.57281555	7.08212228	11.72483374
2	34.09492116	13.91764860	12.13246385	12.73758928
3	29.22941593	42.54228315	15.77976212	34.96523770
4	65.31445807	112.51168961	22.93857970	51.07141528
5	74.62041773	12.93962365	74.83206956	71.04317313
10	39.02325776	43.35044630	38.89349023	12.62495418
15	46.60584859	44.78941056	42.79297173	0.53640676
20	45.97211843	43.92272528	41.89780494	0.08716539
25	45.90251230	43.85151146	41.80212162	0.00551473
30	45.89816243	43.84710436	41.79614585	0.00034044
35	45.89789403	43.84683246	41.79577702	0.00002100
40	45.89787748	43.84681569	41.79575428	0.00000129
41	45.89787701	43.84681522	41.79575364	0.00000074
42	45.89787675	43.84681495	41.79575327	0.00000042

Estimates values are $x \approx 45.89787675$, $y \approx 43.84681495$ and $z \approx 41.79575327$ after 42 iterations.

The result from U_3 *method*

k	x^k	y^k	z^k	ER
1	46.75785864	45.36026734	43.04608193	0.51267283
2	46.67251850	44.82815950	42.95555389	0.69953055
3	46.42231652	44.38897631	42.56349069	0.69819800
4	46.20384156	44.13376903	42.21337585	0.41474988
5	46.06512085	44.01127298	42.01519602	0.19341778
10	45.90949821	43.85860859	41.81173686	0.01381758
15	45.89859723	43.84754525	41.79674347	0.00085726
20	45.89792088	43.84685966	41.79581392	0.00005295
25	45.89787913	43.84681737	41.79575655	0.00000327
26	45.89787796	43.84681618	41.79575494	0.00000187
27	45.89787729	43.84681550	41.79575402	0.00000107
28	45.89787691	43.84681511	41.79575349	0.00000061
29	45.89787669	43.84681489	41.79575319	0.00000035

Estimates values are $x \approx 45.89787669$, $y \approx 43.84681489$ and $z \approx 41.79575319$ after 29 iterations.

Three points central

The result from L_2 *method* and L_3 *method*

k	x^k	y^k	z^k	ER
1	28.03497606	35.63059160	22.07220307	25.88009272
2	46.60605753	20.85731283	33.76323284	26.66179208
3	34.69412585	46.89220233	31.93943805	25.04351526
4	51.62699083	28.54758548	40.14996273	28.50304896
5	39.31210837	52.50995204	36.48007626	23.98065874
10	51.34784006	41.87518013	40.00618100	5.56275473
15	50.71648673	51.97832062	47.06164195	4.59178353
20	46.53423726	44.55418915	42.11343160	0.64213473
25	45.92257205	43.86358463	41.78136935	0.05998733
30	45.89572070	43.84454751	41.79530111	0.00323090
35	45.89796012	43.84694011	41.79585525	0.00003543
40	45.89787920	43.84681424	41.79574591	0.00001336
45	45.89787575	43.84681403	41.79575290	0.00000127
46	45.89787613	43.84681442	41.79575298	0.00000070
47	45.89787633	43.84681460	41.79575294	0.00000030

Estimates values are $x \approx 45.89787633$, $y \approx 43.84681460$ and $z \approx 41.79575294$ after 47 iterations.

The result from U_2 *method*

k	x^k	y^k	z^k	ER
1	28.72721001	31.78127353	10.69635676	29.99082156
2	39.22126943	17.99112549	37.71265509	28.14677168
3	35.79448503	44.16638519	18.52363054	31.24996365
4	49.84259752	24.97214737	44.15820736	33.68732845
5	40.71324557	47.29477487	23.22429392	27.98381579
10	43.45662089	41.97133310	38.89470008	3.19670383
15	45.71093218	43.64462040	41.53900427	0.19313107

The result from U_2 *method* (cont.)

k	x^k	y^k	z^k	ER
20	45.88416134	43.83278195	41.77688403	0.01605160
25	45.89701393	43.84594029	41.79456748	0.00102542
30	45.89782314	43.84676064	41.79567960	0.00006338
35	45.89787311	43.84681126	41.79574827	0.00000391
36	45.89787451	43.84681268	41.79575020	0.00000224
37	45.89787531	43.84681349	41.79575130	0.00000128
38	45.89787577	43.84681396	41.79575193	0.00000073
39	45.89787604	43.84681423	41.79575230	0.00000042

Estimates values are $x \approx 45.89787604$, $y \approx 43.84681423$ and $z \approx 41.79575230$ after 39 iterations.

The result from U_3 *method*

k	x^k	y^k	z^k	ER
1	27.40564330	31.78127353	10.69635676	30.89004816
2	41.70760361	16.99674280	37.71265509	30.04727961
3	34.80395594	46.69131034	17.98585277	33.88757453
4	54.32337208	23.58978313	45.80747521	39.00430516
5	41.60999531	50.36827190	21.91659390	30.69134030
10	47.65941535	38.18442564	40.98556280	16.73306920
15	45.82347687	43.76465581	41.59440276	0.23960642
20	45.90739260	43.85925938	41.81157516	0.00992614
25	45.89752249	43.84625671	41.79546019	0.00039322
30	45.89785389	43.84679215	41.79569370	0.00007542
35	45.89787980	43.84681901	41.79575853	0.00000377
36	45.89787818	43.84681677	41.79575622	0.00000298
37	45.89787713	43.84681540	41.79575448	0.00000190
38	45.89787657	43.84681470	41.79575342	0.00000100
39	45.89787632	43.84681443	41.79575287	0.00000042

Estimates values are $x \approx 45.89787632$, $y \approx 43.84681443$ and $z \approx 41.79575287$ after 39 iterations.

The iterations of table 4.3

Two points backward

The result from *D method*

k	x^k	y^k	z^k	λ^k	ER
1	1.00000000	1.30000000	-0.60000000	-1.08750000	1.16375000
2	1.15384615	1.41379310	0.09000000	-0.93750000	0.72693787
3	1.06097561	1.31589744	-0.24136095	-1.11000000	0.43077509
4	1.13990647	1.35047242	-0.03566924	-1.02715976	0.31445033
5	1.11072242	1.30189416	-0.20938676	-1.07858269	0.17672858
10	1.17430813	1.28091843	-0.25419054	-1.02078054	0.05323798
15	1.18428681	1.26121498	-0.31620497	-1.01258535	0.01537681
20	1.19383247	1.25711523	-0.32865677	-1.00483903	0.01005286
25	1.19602295	1.25290451	-0.34132747	-1.00310687	0.00357957
30	1.19834907	1.25188971	-0.34436352	-1.00128391	0.00256522
35	1.19892153	1.25079190	-0.34764065	-1.00083779	0.00095089
40	1.19954608	1.25051851	-0.34845561	-1.00035221	0.00069647
45	1.19970246	1.25021879	-0.34934852	-1.00023080	0.00026088
50	1.19987430	1.25014350	-0.34957273	-1.00009747	0.00019220
55	1.19991753	1.25006067	-0.34981938	-1.00006395	0.00007220
60	1.19996512	1.25003981	-0.34988148	-1.00002704	0.00005328
65	1.19997711	1.25001684	-0.34994987	-1.00001775	0.00002003
70	1.19999032	1.25001105	-0.34996710	-1.00000751	0.00001479
75	1.19999365	1.25000468	-0.34998608	-1.00000493	0.00000556
80	1.19999731	1.25000307	-0.34999086	-1.00000208	0.00000411
85	1.19999824	1.25000130	-0.34999614	-1.00000137	0.00000154
90	1.19999925	1.25000085	-0.34999746	-1.00000058	0.00000114
93	1.19999937	1.25000047	-0.34999861	-1.00000049	0.00000055
94	1.19999955	1.25000051	-0.34999848	-1.00000035	0.00000068
95	1.19999951	1.25000036	-0.34999893	-1.00000038	0.00000043

Estimates values are $x \approx 1.19999951$, $y \approx 1.25000036$, $z \approx$ -0.34999893 and $\lambda \approx$ -1.00000038 after 95 iterations.

The result from L_2 *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.00000000	1.45000000	-0.60000000	-0.93750000	0.76812500
2	1.03448276	1.44767241	0.09000000	-1.11000000	0.15396156
3	1.03614601	1.37249364	0.01984542	-1.09246136	0.08825924
4	1.09290124	1.31682423	0.01640144	-1.09160036	0.19292904
5	1.13910419	1.27669808	-0.10443311	-1.06139172	0.16604387
10	1.20926127	1.24150951	-0.37138254	-0.99465437	0.00275988
15	1.20270451	1.24805534	-0.35911845	-0.99772039	0.00385648
20	1.20031799	1.24980467	-0.35124564	-0.99968859	0.00069025
25	1.20001002	1.24999830	-0.35006299	-0.99998425	0.00005381
30	1.19999681	1.25000266	-0.34999122	-1.00000220	0.00000193
35	1.19999935	1.25000044	-0.34999769	-1.00000058	0.00000110
36	1.19999958	1.25000028	-0.34999844	-1.00000039	0.00000078
37	1.19999973	1.25000017	-0.34999899	-1.00000025	0.00000054
38	1.19999984	1.25000010	-0.34999936	-1.00000016	0.00000036

Estimates values are $x \approx 1.19999984$, $y \approx 1.25000010$, $z \approx -0.34999936$ and $\lambda \approx -1.00000016$ after 38 iterations.

The result from L_3 *method* and L_4 *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.00000000	1.45000000	-0.12000000	-1.05750000	0.28587500
2	1.03448276	1.38906640	0.01068966	-1.09017241	0.08082274
3	1.07986199	1.33167701	-0.07247784	-1.06938054	0.08104577
4	1.12639926	1.29635770	-0.17449776	-1.04387556	0.05830136
5	1.15708805	1.27796407	-0.24648277	-1.02587931	0.03140739
10	1.19318127	1.25540382	-0.33366892	-1.00408277	0.00246045
15	1.19822244	1.25142761	-0.34573635	-1.00106591	0.00059397
20	1.19951406	1.25039136	-0.34883393	-1.00029152	0.00015962
25	1.19986564	1.25010829	-0.34967755	-1.00008061	0.00004393
30	1.19996273	1.25003004	-0.34991057	-1.00002236	0.00001217
35	1.19998966	1.25000834	-0.34997517	-1.00000621	0.00000338
40	1.19999713	1.25000232	-0.34999311	-1.00000172	0.00000094

The result from L_3 *method* and L_4 *method* (cont.)

k	x^k	y^k	z^k	λ^k	ER
41	1.19999778	1.25000179	-0.34999467	-1.00000133	0.00000073
42	1.19999828	1.25000139	-0.34999587	-1.00000103	0.00000056
43	1.19999867	1.25000107	-0.34999680	-1.00000080	0.00000043

Estimates values are $x \approx 1.19999867$, $y \approx 1.25000107$, $z \approx -0.34999680$ and $\lambda \approx -1.00000080$ after 43 iterations.

The result from U_2 *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.04333333	1.30000000	-0.60000000	-1.08750000	0.97553889
2	1.08365942	1.37394636	0.00145556	-0.93750000	0.26301314
3	1.08008799	1.39076329	-0.08431773	-1.08786389	0.08650297
4	1.12516779	1.34003519	-0.07659007	-1.06642057	0.14689717
5	1.15321022	1.30460047	-0.17600257	-1.06835248	0.12185386
10	1.20235192	1.24757169	-0.35225805	-1.00152918	0.00580821
15	1.20111176	1.24884405	-0.35363498	-0.99881976	0.00157617
20	1.20014285	1.24985126	-0.35055053	-0.99978600	0.00034344
25	1.20000576	1.24999400	-0.35003218	-0.99998390	0.00003068
30	1.19999876	1.25000129	-0.34999678	-1.00000069	0.00000058
34	1.19999959	1.25000043	-0.34999861	-1.00000047	0.00000067
35	1.19999972	1.25000029	-0.34999902	-1.00000035	0.00000051
36	1.19999982	1.25000019	-0.34999933	-1.00000025	0.00000038

Estimates values are $x \approx 1.19999982$, $y \approx 1.25000019$, $z \approx -0.34999933$ and $\lambda \approx -1.00000025$ after 36 iterations.

The result from U_3 *method* and U_4 *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.17062500	0.71250000	-0.60000000	-1.08750000	1.72759805
2	1.12450853	1.21724138	-0.28036289	-0.93750000	0.43878211
3	1.14181705	1.32867400	-0.17451943	-1.01740928	0.10900325
4	1.15946787	1.29436134	-0.21374617	-1.04387014	0.06793950

The result from U_3 *method* and U_4 *method* (cont.)

k	x^k	y^k	z^k	λ^k	ER
5	1.17182934	1.28108749	-0.25436573	-1.03406346	0.05268071
10	1.19387748	1.25644348	-0.33064880	-1.00644001	0.00830888
15	1.19838427	1.25168728	-0.34496666	-1.00163707	0.00202286
20	1.19955724	1.25046152	-0.34862530	-1.00044487	0.00054382
25	1.19987750	1.25012763	-0.34962000	-1.00012281	0.00014969
30	1.19996602	1.25003540	-0.34989461	-1.00003405	0.00004146
35	1.19999057	1.25000983	-0.34997075	-1.00000945	0.00001151
40	1.19999738	1.25000273	-0.34999188	-1.00000262	0.00000319
45	1.19999927	1.25000076	-0.34999774	-1.00000073	0.00000089
46	1.19999944	1.25000059	-0.34999825	-1.00000056	0.00000069
47	1.19999956	1.25000045	-0.34999865	-1.00000044	0.00000053
48	1.19999966	1.25000035	-0.34999895	-1.00000034	0.00000041

Estimates values are $x \approx 1.19999966$, $y \approx 1.25000035$, $z \approx -0.34999895$ and $\lambda \approx -1.00000034$ after 48 iterations.

Two points forward

The result from D *method*

k	x^k	y^k	z^k	λ^k	ER
1	0.93750000	1.81250000	0.45000000	-1.28750000	1.05921875
2	0.82758621	1.39805825	0.21109375	-1.20000000	0.74146405
3	1.07291667	1.51867816	0.40510107	-1.14027344	0.85851978
4	0.98770104	1.33069553	-0.06115017	-1.18877527	0.54700656
5	1.12723006	1.38888676	0.11444665	-1.07221246	0.49878764
10	1.15141154	1.27507097	-0.27261968	-1.04272496	0.10388470
15	1.18697787	1.26908052	-0.29185095	-1.01037035	0.04645179
20	1.18983474	1.25596194	-0.33213930	-1.00804254	0.01776755
25	1.19665224	1.25470947	-0.33591091	-1.00261205	0.01045383
30	1.19734230	1.25159477	-0.34524468	-1.00207086	0.00445267
35	1.19908878	1.25126960	-0.34621554	-1.00070764	0.00275639
40	1.19927364	1.25043839	-0.34869433	-1.00056389	0.00120350

The result from *D method* (cont.)

k	x^k	y^k	z^k	λ^k	ER
45	1.19974835	1.25034972	-0.34895852	-1.00019518	0.00075475
50	1.19979918	1.25012139	-0.34963857	-1.00015574	0.00033172
55	1.19993023	1.25009689	-0.34971152	-1.00005410	0.00020877
60	1.19994431	1.25003368	-0.34989973	-1.00004318	0.00009192
65	1.19998063	1.25002689	-0.34991995	-1.00001501	0.00005791
70	1.19998454	1.25000935	-0.34997216	-1.00001199	0.00002551
75	1.19999462	1.25000746	-0.34997778	-1.00000417	0.00001607
80	1.19999571	1.25000260	-0.34999227	-1.00000333	0.00000708
85	1.19999851	1.25000207	-0.34999383	-1.00000116	0.00000446
90	1.19999881	1.25000072	-0.34999785	-1.00000092	0.00000197
95	1.19999959	1.25000058	-0.34999829	-1.00000032	0.00000124
100	1.19999967	1.25000020	-0.34999940	-1.00000026	0.00000055
101	1.19999981	1.25000027	-0.34999921	-1.00000015	0.00000057
102	1.19999974	1.25000015	-0.34999954	-1.00000020	0.00000042

Estimates values are $x \approx 1.19999974$, $y \approx 1.25000015$, $z \approx$ -0.34999954 and $\lambda \approx$ -1.00000020 after 102 iterations.

The result from L_2 *method*

k	x^k	y^k	z^k	λ^k	ER
1	0.93750000	1.70250000	0.45000000	-1.20000000	0.66550000
2	0.88105727	1.49481461	0.21109375	-1.14027344	0.29034583
3	1.00346892	1.35937014	0.31373809	-1.16593452	0.39413130
4	1.10345222	1.26640729	0.08305013	-1.10826253	0.36453557
5	1.18445307	1.22292445	-0.12760680	-1.05559830	0.26705391
10	1.23510717	1.22059982	-0.44989178	-0.97502705	0.02217161
15	1.20994754	1.24259688	-0.38258821	-0.99185295	0.01266016
20	1.20148458	1.24902520	-0.35549293	-0.99862677	0.00278007
25	1.20009847	1.24995192	-0.35045184	-0.99988704	0.00030305
30	1.19999331	1.25000705	-0.34998946	-1.00000263	0.00000663

The result from L_2 *method* (cont.)

k	x^k	y^k	z^k	λ^k	ER
35	1.19999748	1.25000182	-0.34999159	-1.00000210	0.00000350
39	1.19999952	1.25000030	-0.34999817	-1.00000046	0.00000097
40	1.19999971	1.25000018	-0.34999885	-1.00000029	0.00000065
41	1.19999983	1.25000010	-0.34999930	-1.00000018	0.00000041

Estimates values are $x \approx 1.19999983$, $y \approx 1.25000010$, $z \approx -0.34999930$ and $\lambda \approx -1.00000018$ after 41 iterations.

The result from L_3 *method* and L_4 *method*

k	x^k	y^k	z^k	λ^k	ER
1	0.93750000	1.70250000	0.16125000	-1.12781250	0.42572578
2	0.88105727	1.51807632	0.32468475	-1.16867119	0.20995262
3	0.98809261	1.36173963	0.13117092	-1.12029273	0.22862618
4	1.10153216	1.28522404	-0.12264662	-1.05683835	0.13204300
5	1.16711169	1.26041976	-0.27528833	-1.01867792	0.04969246
10	1.19863254	1.25109475	-0.34672008	-1.00081998	0.00046405
15	1.19962678	1.25030066	-0.34910441	-1.00022390	0.00012238
20	1.19989670	1.25008326	-0.34975209	-1.00006198	0.00003376
25	1.19997134	1.25002310	-0.34993122	-1.00001719	0.00000936
30	1.19999204	1.25000641	-0.34998091	-1.00000477	0.00000260
35	1.19999779	1.25000178	-0.34999470	-1.00000133	0.00000072
36	1.19999829	1.25000138	-0.34999590	-1.00000103	0.00000056
37	1.19999868	1.25000107	-0.34999682	-1.00000079	0.00000043

Estimates values are $x \approx 1.19999868$, $y \approx 1.25000107$, $z \approx -0.34999682$ and $\lambda \approx -1.00000079$after 37 iterations.

The result from U_2 *method*

k	x^k	y^k	z^k	λ^k	ER
1	0.86950000	1.81250000	0.45000000	-1.28750000	0.74509275
2	1.11515778	1.45087379	0.33396975	-1.20000000	0.84071369
3	1.22405072	1.27903518	-0.15357687	-1.17099244	0.54301269
4	1.28571943	1.19295537	-0.40830016	-1.04910578	0.29041105
5	1.30051712	1.15659104	-0.56307446	-0.98542496	0.08595618
10	1.27521737	1.17714954	-0.55786979	-0.94301978	0.03555187
15	1.23271162	1.21729945	-0.44851364	-0.97019696	0.03115462
20	1.20793766	1.24185410	-0.37669238	-0.99090640	0.01248941
25	1.20102021	1.24894071	-0.35390382	-0.99848977	0.00240661
30	1.20004690	1.24995117	-0.35024721	-0.99987993	0.00022489
35	1.19999208	1.25000825	-0.34998046	-1.00000363	0.00000302
40	1.19999806	1.25000202	-0.34999325	-1.00000236	0.00000345
44	1.19999968	1.25000033	-0.34999874	-1.00000050	0.00000082
45	1.19999982	1.25000019	-0.34999924	-1.00000031	0.00000053
46	1.19999990	1.25000010	-0.34999956	-1.00000019	0.00000033

Estimates values are $x \approx 1.19999990$, $y \approx 1.25000010$, $z \approx$-0.34999956 and $\lambda \approx$-1.00000019 after 46 iterations.

The result from U_3 *method* and U_4 *method*

k	x^k	y^k	z^k	λ^k	ER
1	0.98650000	1.44687500	0.45000000	-1.28750000	0.60519163
2	1.10057962	1.39037015	0.11681775	-1.20000000	0.47062566
3	1.16102200	1.31828040	-0.12127550	-1.11670444	0.29321958
4	1.18456381	1.27574271	-0.25797208	-1.05718112	0.12666593
5	1.19213022	1.25999278	-0.31319143	-1.02300698	0.04196797
10	1.19832178	1.25175276	-0.34476715	-1.00170751	0.00211423
15	1.19954046	1.25047902	-0.34857315	-1.00046178	0.00056458
20	1.19987288	1.25013245	-0.34960565	-1.00012745	0.00015535
25	1.19996474	1.25003673	-0.34989064	-1.00003533	0.00004303

The result from U_3 *method* and U_4 *method* (cont.)

k	x^k	y^k	z^k	λ^k	ER
30	1.19999021	1.25001020	-0.34996964	-1.00000981	0.00001194
35	1.19999728	1.25000283	-0.34999157	-1.00000272	0.00000331
40	1.19999925	1.25000079	-0.34999766	-1.00000076	0.00000092
41	1.19999942	1.25000061	-0.34999819	-1.00000059	0.00000071
42	1.19999955	1.25000047	-0.34999860	-1.00000045	0.00000055
43	1.19999965	1.25000036	-0.34999892	-1.00000035	0.00000043

Estimates values are $x \approx 1.19999965$, $y \approx 1.25000036$, $z \approx$ -0.34999892 and $\lambda \approx$ -1.00000035 after 43 iterations.

Three points central

The result from D *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.15384615	1.30000000	-0.60000000	-1.08750000	0.53873521
2	1.15384615	1.26990385	-0.24136095	-0.93750000	0.16750509
3	1.18162722	1.31589744	-0.24136095	-1.02715976	0.17588652
4	1.13990647	1.26127657	-0.30624288	-1.02715976	0.21706004
5	1.18927128	1.30189416	-0.20938676	-1.01093928	0.23196671
10	1.17430813	1.25449244	-0.33649423	-1.02078054	0.10764902
15	1.19746656	1.26121498	-0.31620497	-1.00197419	0.05365572
20	1.19383247	1.25121074	-0.34639121	-1.00483903	0.02509722
25	1.19930715	1.25290451	-0.34132747	-1.00053784	0.01360550
30	1.19834907	1.25033363	-0.34900646	-1.00128391	0.00666125
35	1.19980841	1.25079190	-0.34764065	-1.00014858	0.00368903
40	1.19954608	1.25009244	-0.34972477	-1.00035221	0.00182727
45	1.19994686	1.25021879	-0.34934852	-1.00004120	0.00101770
50	1.19987430	1.25002565	-0.34992363	-1.00009747	0.00050569
55	1.19998525	1.25006067	-0.34981938	-1.00001144	0.00028208
60	1.19996512	1.25000712	-0.34997880	-1.00002704	0.00014028
65	1.19999590	1.25001684	-0.34994987	-1.00000318	0.00007829
70	1.19999032	1.25000198	-0.34999411	-1.00000751	0.00003894

The result from D *method* (cont.)

k	x^k	y^k	z^k	λ^k	ER
75	1.19999886	1.25000468	-0.34998608	-1.00000088	0.00002173
80	1.19999731	1.25000055	-0.34999837	-1.00000208	0.00001081
85	1.19999968	1.25000130	-0.34999614	-1.00000024	0.00000603
90	1.19999925	1.25000015	-0.34999955	-1.00000058	0.00000300
95	1.19999991	1.25000036	-0.34999893	-1.00000007	0.00000168
100	1.19999979	1.25000004	-0.34999987	-1.00000016	0.00000083
101	1.19999996	1.25000017	-0.34999950	-1.00000003	0.00000078
102	1.19999984	1.25000003	-0.34999990	-1.00000012	0.00000065
103	1.19999997	1.25000013	-0.34999961	-1.00000002	0.00000060
104	1.19999988	1.25000003	-0.34999992	-1.00000010	0.00000050

Estimates values are $x \approx 1.19999988$, $y \approx 1.25000003$, $z \approx -0.34999992$ and $\lambda \approx -1.00000010$ after 104 iterations.

The result from L_2 *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.15384615	1.37307692	-0.60000000	-0.93750000	0.49656435
2	1.09243697	1.37346854	-0.24136095	-1.02715976	0.16441905
3	1.09212549	1.34842843	-0.10341854	-1.06164536	0.04006032
4	1.11240609	1.31556341	-0.10273808	-1.06181548	0.08379228
5	1.14019590	1.28843263	-0.14744732	-1.05063817	0.10029954
10	1.20080989	1.24798841	-0.34510295	-1.00122426	0.00914271
15	1.20123218	1.24905943	-0.35385002	-0.99903749	0.00135355
20	1.20018544	1.24987986	-0.35069292	-0.99982677	0.00035746
25	1.20001072	1.24999524	-0.35005170	-0.99998708	0.00003639
29	1.19999904	1.25000115	-0.34999925	-1.00000019	0.00000197
30	1.19999890	1.25000106	-0.34999770	-1.00000057	0.00000056
31	1.19999899	1.25000087	-0.34999735	-1.00000066	0.00000046

Estimates values are $x \approx 1.19999899$, $y \approx 1.25000087$, $z \approx -0.34999735$ and $\lambda \approx -1.00000066$ after 31 iterations.

The result from L_3 *method* and L_4 *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.15384615	1.37307692	-0.22000000	-1.03250000	0.19472855
2	1.09243697	1.35023679	-0.09964746	-1.06258814	0.04331693
3	1.11091626	1.31686613	-0.14379345	-1.05155164	0.04329614
4	1.13906795	1.29285742	-0.20668328	-1.03582918	0.03572185
5	1.16022074	1.27826628	-0.25566473	-1.02358382	0.02232679
10	1.19260427	1.25585833	-0.33229887	-1.00442528	0.00266922
15	1.19807905	1.25154236	-0.34539308	-1.00115173	0.00064254
20	1.19947545	1.25042242	-0.34874133	-1.00031467	0.00017235
25	1.19985501	1.25011686	-0.34965204	-1.00008699	0.00004741
30	1.19995979	1.25003241	-0.34990350	-1.00002413	0.00001313
35	1.19998884	1.25000900	-0.34997321	-1.00000670	0.00000364
40	1.19999690	1.25000250	-0.34999256	-1.00000186	0.00000101
41	1.19999760	1.25000193	-0.34999424	-1.00000144	0.00000078
42	1.19999814	1.25000150	-0.34999555	-1.00000111	0.00000061
43	1.19999856	1.25000116	-0.34999655	-1.00000086	0.00000047

Estimates values are $x \approx 1.19999856$, $y \approx 1.25000116$, $z \approx -0.34999655$ and $\lambda \approx -1.00000086$ after 43 iterations.

The result from U_2 *method*

k	x^k	y^k	z^k	λ^k	ER
1	1.15384615	1.30000000	-0.60000000	-1.08750000	0.53873521
2	1.18278476	1.26990385	-0.24136095	-0.93750000	0.17348191
3	1.16880970	1.28502959	-0.30897980	-1.02715976	0.06330406
4	1.17810763	1.27375517	-0.27611611	-1.01025505	0.03532828
5	1.18198413	1.26913240	-0.29793760	-1.01847097	0.02078627
10	1.19802948	1.25206898	-0.34199442	-1.00322153	0.00542640
15	1.19998550	1.25001516	-0.34975455	-1.00016384	0.00035482
20	1.20002739	1.24997147	-0.35008257	-0.99997621	0.00002705
25	1.20000449	1.24999532	-0.35001656	-0.99999383	0.00000954
30	1.20000029	1.24999970	-0.35000135	-0.99999939	0.00000108
31	1.20000013	1.24999986	-0.35000070	-0.99999966	0.00000063

The result from U_2 *method* (cont.)

k	x^k	y^k	z^k	λ^k	*ER*
32	1.20000005	1.24999995	-0.35000032	-0.99999983	0.00000035

Estimates values are $x \approx 1.20000005$, $y \approx 1.24999995$, $z \approx -0.35000032$ and $\lambda \approx -0.99999983$ after 32 iterations.

The result from U_3 *method* and U_4 *method*

k	x^k	y^k	z^k	λ^k	*ER*
1	1.24197115	0.94750000	-0.60000000	-1.08750000	0.88586258
2	1.21641179	1.18404951	-0.45249235	-0.93750000	0.22046161
3	1.21458502	1.24192110	-0.38965765	-0.97437691	0.02887567
4	1.21172046	1.23770547	-0.38521676	-0.99008559	0.01123824
5	1.20933499	1.24039905	-0.37826648	-0.99119581	0.00995338
10	1.20279361	1.24710188	-0.35859932	-0.99725220	0.00327448
15	1.20079342	1.24917450	-0.35245512	-0.99920947	0.00095661
20	1.20022173	1.24976910	-0.35068719	-0.99977822	0.00026955
25	1.20006168	1.24993576	-0.35019124	-0.99993824	0.00007516
30	1.20001714	1.24998215	-0.35005314	-0.99998284	0.00002089
35	1.20000476	1.24999504	-0.35001476	-0.99999523	0.00000580
40	1.20000132	1.24999862	-0.35000410	-0.99999868	0.00000161
43	1.20000061	1.24999936	-0.35000190	-0.99999939	0.00000075
44	1.20000047	1.24999951	-0.35000147	-0.99999953	0.00000058
45	1.20000037	1.24999962	-0.35000114	-0.99999963	0.00000045

Estimates values are $x \approx 1.20000037$, $y \approx 1.24999962$, $z \approx -0.35000114$ and $\lambda \approx -0.99999963$ after 45 iterations.

The iterations of table 4.4

Two points backward

The result from *D method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.43500000	-1.78285714	0.25000000	0.04200000	-0.75000000	6.80069327
2	-0.46256684	-3.68635770	-0.19887286	0.28000000	-0.68500000	6.27317812
3	-0.44561202	-2.92961590	1.81587642	-0.38538359	-0.71256684	4.01475827
4	-0.45339863	-2.56836045	0.75788433	0.04057396	-0.69561202	2.52943391
5	-0.44876319	-2.99017673	0.58675663	0.09958394	-0.70339863	0.66834780
10	-0.45007212	-2.94161604	0.97127202	0.08193485	-0.69990407	0.11706687
15	-0.44999794	-2.92620373	0.96395856	0.07788385	-0.70000556	0.00465285
20	-0.45000012	-2.92652191	0.96165222	0.07796146	-0.69999984	0.00076219
25	-0.45000000	-2.92662300	0.96169933	0.07798644	-0.70000001	0.00003014
30	-0.45000000	-2.92662094	0.96171430	0.07798593	-0.70000000	0.00000494
33	-0.45000000	-2.92662035	0.96171364	0.07798578	-0.70000000	0.00000054
34	-0.45000000	-2.92662039	0.96171395	0.07798579	-0.70000000	0.00000066
35	-0.45000000	-2.92662028	0.96171399	0.07798577	-0.70000000	0.00000020

Estimates values are $x \approx -0.45000000$, $y \approx -2.92662028$, $z \approx 0.96171399$, $\lambda \approx 0.07798577$ and $\mu \approx -0.70000000$ after 35 iterations.

The result from *L_2 method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.43500000	-1.78285714	0.53285714	1.88619643	-0.75000000	6.85755905
2	-0.46256684	-3.61890526	2.43940071	-0.94650416	-0.68500000	12.47856797
3	-0.44561202	-1.84963990	-0.58818154	-3.24864887	-0.71256684	8.05730868
4	-0.45339863	-2.89614762	-0.55148938	-0.16086193	-0.69561202	2.19016175
5	-0.44876319	-3.06433645	0.99140187	-0.20308402	-0.70339863	1.09963734
10	-0.45007212	-2.92606545	0.96127820	0.07789943	-0.69990407	0.00449958
15	-0.44999794	-2.92662043	0.96171337	0.07798663	-0.70000556	0.00000917
19	-0.44999984	-2.92662025	0.96171381	0.07798583	-0.70000043	0.00000113
20	-0.45000012	-2.92662033	0.96171394	0.07798573	-0.69999984	0.00000061
21	-0.44999996	-2.92662029	0.96171387	0.07798579	-0.70000012	0.00000032

Estimates values are $x \approx$-0.44999996, $y \approx$-2.92662029, $z \approx$0.96171387, $\lambda \approx$0.07798579 and $\mu \approx$-0.70000012 after 21 iterations.

The result from L_3 *method* and L_4 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.43500000	-1.78285714	0.43535714	1.97760268	-0.75000000	6.98113932
2	-0.46256684	-3.64766097	2.56243440	-0.86933885	-0.68500000	13.30582517
3	-0.44561202	-1.73596481	-0.65142303	-3.32939624	-0.71256684	8.65680173
4	-0.45339863	-2.86012805	-0.64941566	-0.11531267	-0.69561202	2.35365705
5	-0.44876319	-3.05331972	1.00157164	-0.17604125	-0.70339863	1.04396056
10	-0.45007212	-2.92609548	0.96131741	0.07790191	-0.69990407	0.00425272
15	-0.44999794	-2.92661959	0.96171199	0.07798657	-0.70000556	0.00001714
19	-0.44999984	-2.92662019	0.96171370	0.07798583	-0.70000043	0.00000174
20	-0.45000012	-2.92662036	0.96171400	0.07798573	-0.69999984	0.00000093
21	-0.44999996	-2.92662027	0.96171384	0.07798579	-0.70000012	0.00000049

Estimates values are $x \approx$-0.44999996, $y \approx$-2.92662027, $z \approx$0.96171384, $\lambda \approx$0.07798579 and $\mu \approx$-0.70000012 after 21 iterations.

The result from L_5 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.43500000	-1.78285714	0.43535714	1.97760268	-0.73375000	6.95496116
2	-0.45822193	-3.64766097	2.55384188	-0.81763982	-0.69080548	13.13184116
3	-0.44728851	-1.74406012	-0.62166078	-3.13455921	-0.70548857	8.42826673
4	-0.45149071	-2.86833789	-0.54688646	-0.19222675	-0.69833906	2.31623735
5	-0.44953350	-3.07724288	0.99083051	-0.21630134	-0.70100141	1.17613230
10	-0.45001001	-2.92605602	0.96126320	0.07792356	-0.69999302	0.00439626
15	-0.44999997	-2.92662100	0.96171439	0.07798586	-0.70000028	0.00000564
16	-0.45000008	-2.92662014	0.96171378	0.07798572	-0.70000000	0.00000139
17	-0.45000000	-2.92662034	0.96171391	0.07798578	-0.70000006	0.00000034

Estimates values are $x \approx$-0.45000000, $y \approx$-2.92662034, $z \approx$0.96171391, $\lambda \approx$0.07798578 and $\mu \approx$-0.70000006 after 17 iterations.

The result from U_2 *method*, U_3 *method* and U_4 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.43500000	-4.81332589	-1.03554688	-0.14550000	-0.75000000	18.65482280
2	-0.46256684	-1.59426317	2.76964732	-0.07074963	-0.68500000	4.14646026
3	-0.44561202	-2.30503669	-0.27340936	0.02838404	-0.71256684	4.76672640
4	-0.45339863	-2.98185299	0.31264994	-0.26930729	-0.69561202	1.16703265
5	-0.44876319	-3.07243628	0.87071780	0.24215177	-0.70339863	1.11848912
10	-0.45007212	-2.92646460	0.96218369	0.07810076	-0.69990407	0.00084443
15	-0.44999794	-2.92662039	0.96171363	0.07798610	-0.70000556	0.00000801
18	-0.45000043	-2.92662032	0.96171384	0.07798565	-0.69999943	0.00000122
19	-0.44999984	-2.92662029	0.96171393	0.07798581	-0.70000043	0.00000064
20	-0.45000012	-2.92662031	0.96171388	0.07798574	-0.69999984	0.00000034

Estimates values are $x \approx$ -0.45000012, $y \approx$ -2.92662031, $z \approx$ 0.96171388, $\lambda \approx$ 0.07798574 and $\mu \approx$ -0.69999984 after 20 iterations.

The result from U_5 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.44541667	-4.81332589	-1.03554688	-0.14550000	-0.75000000	18.63892784
2	-0.45937087	-1.59447431	2.76873712	-0.07024442	-0.69541667	4.12917956
3	-0.44965003	-2.30461146	-0.27113146	0.02761907	-0.70937087	4.75874561
4	-0.45190522	-2.98393502	0.31203223	-0.27286980	-0.69965003	1.15236374
5	-0.45005856	-3.07202680	0.87168506	0.24069990	-0.70190522	1.11197110
10	-0.45001894	-2.92645963	0.96219861	0.07811528	-0.70001432	0.00072401
14	-0.45000097	-2.92661943	0.96171655	0.07798620	-0.70000119	0.00000447
15	-0.45000032	-2.92662055	0.96171314	0.07798543	-0.70000097	0.00000145
16	-0.45000022	-2.92662024	0.96171408	0.07798574	-0.70000032	0.00000042

Estimates values are $x \approx$ -0.45000022, $y \approx$ -2.92662024, $z \approx$ 0.96171408, $\lambda \approx$ 0.07798574 and $\mu \approx$ -0.70000032 after 16 iterations.

Two points forward

The result from D *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-1.32500000	-2.92153846	1.40000000	-0.10500000	-0.45000000	3.91974852
2	-0.89701987	-2.73056782	0.78241346	0.08333333	-1.57500000	2.09231105
3	-0.61636254	-3.00035606	0.80531947	0.04260322	-1.14701987	0.60109472
4	-0.54740966	-2.97473196	1.02661509	0.05683556	-0.86636254	0.52573197
5	-0.49324403	-2.90501894	1.00584429	0.05902717	-0.79740966	0.15952894
10	-0.45194009	-2.92461734	0.95816482	0.07672734	-0.70326136	0.02205542
15	-0.45006977	-2.92677298	0.96139630	0.07799548	-0.70014949	0.00082871
20	-0.45000320	-2.92663409	0.96173573	0.07798789	-0.70000538	0.00013519
25	-0.45000012	-2.92661934	0.96171590	0.07798549	-0.70000025	0.00000502
29	-0.45000001	-2.92662017	0.96171417	0.07798574	-0.70000002	0.00000066
30	-0.45000001	-2.92662021	0.96171375	0.07798575	-0.70000001	0.00000086
31	-0.45000000	-2.92662035	0.96171380	0.07798578	-0.70000001	0.00000024

Estimates values are $x \approx$-0.45000000, $y \approx$-2.92662035, $z \approx$0.96171380, $\lambda \approx$0.07798578 and $\mu \approx$-0.70000001 after 31 iterations.

The result from L_2 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-1.32500000	-2.92153846	0.82153846	-0.33269231	-0.45000000	2.97412367
2	-0.89701987	-2.97466439	0.53384708	0.12608707	-1.57500000	1.56119886
3	-0.61636254	-3.03403661	1.14713922	-0.03996392	-1.14701987	1.48010207
4	-0.54740966	-2.86572723	0.84109497	0.02144085	-0.86636254	0.73345523
5	-0.49324403	-2.96821722	0.97995415	0.07016720	-0.79740966	0.34934672
10	-0.45194009	-2.92675977	0.96154417	0.07723700	-0.70326136	0.00247821
15	-0.45006977	-2.92662423	0.96170345	0.07795871	-0.70014949	0.0000834
20	-0.45000320	-2.92662057	0.96171364	0.07798453	-0.70000538	0.00000435
21	-0.45000150	-2.92662038	0.96171367	0.07798519	-0.70000320	0.00000178
22	-0.45000089	-2.92662037	0.96171383	0.07798543	-0.70000150	0.00000121
23	-0.45000042	-2.92662032	0.96171383	0.07798561	-0.70000089	0.0000005

Estimates values are $x \approx -0.45000042$, $y \approx -2.92662032$, $z \approx 0.96171383$, $\lambda \approx 0.07798561$ and $\mu \approx -0.70000089$ after 23 iterations.

The result from L_3 *method* and L_4 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-1.32500000	-2.92153846	0.25903846	-0.26237981	-0.45000000	3.46356876
2	-0.89701987	-3.09011908	0.85475918	0.45858260	-1.57500000	2.59841631
3	-0.61636254	-2.96577324	1.24842638	-0.34378482	-1.14701987	2.10969455
4	-0.54740966	-2.81692711	0.62873598	-0.31325521	-0.86636254	1.48950233
5	-0.49324403	-3.02834206	0.87383080	0.13261363	-0.79740966	0.74778345
10	-0.45194009	-2.92750612	0.96208772	0.07731598	-0.70326136	0.00804868
15	-0.45006977	-2.92662358	0.96169717	0.07795885	-0.70014949	0.00009945
20	-0.45000320	-2.92662071	0.96171362	0.07798455	-0.70000538	0.00000528
21	-0.45000150	-2.92662039	0.96171356	0.07798520	-0.70000320	0.00000198
22	-0.45000089	-2.92662041	0.96171382	0.07798543	-0.70000150	0.00000146
23	-0.45000042	-2.92662033	0.96171380	0.07798561	-0.70000089	0.00000055
24	-0.45000025	-2.92662033	0.96171388	0.07798568	-0.70000042	0.00000041

Estimates values are $x \approx -0.45000025$, $y \approx -2.92662033$, $z \approx 0.96171388$, $\lambda \approx 0.07798568$ and $\mu \approx -0.70000042$ after 24 iterations.

The result from L_5 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-1.32500000	-2.92153846	0.25903846	-0.26237981	-0.73125000	3.27174682
2	-0.91564570	-3.09011908	0.84987213	0.29863207	-1.47266142	2.2123163
3	-0.60428052	-2.96708539	1.14754294	-0.12189740	-1.08780440	1.27169888
4	-0.53104808	-2.85859535	0.79386197	-0.01648611	-0.83597241	0.88061634
5	-0.48501650	-2.98130038	0.97330435	0.08074960	-0.76954018	0.43323324
10	-0.45104992	-2.92678806	0.96160685	0.07759225	-0.70179890	0.0019626
15	-0.45002957	-2.92662313	0.96170859	0.07797449	-0.70005328	0.00004072
20	-0.45000087	-2.92662040	0.96171376	0.07798544	-0.70000153	0.00000131

The result from L_5 *method* (cont.)

k	x^k	y^k	z^k	λ^k	μ^k	ER
21	-0.45000042	-2.92662034	0.96171383	0.07798561	-0.70000076	0.00000061
22	-0.45000021	-2.92662032	0.96171386	0.07798569	-0.70000037	0.00000032

Estimates values are $x \approx$ -0.45000021, $y \approx$ -2.92662032, $z \approx$ 0.96171386, $\lambda \approx$ 0.07798569 and $\mu \approx$ -0.70000037 after 22 iterations.

The result from U_2 *method* , U_3 *method* and U_4 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-1.32500000	-2.92153846	1.41912500	-0.11750000	-0.45000000	4.01823802
2	-0.89701987	-3.22530744	0.91344876	0.15564778	-1.57500000	2.96728693
3	-0.61636254	-2.91778266	1.16332644	0.04706541	-1.14701987	0.80959708
4	-0.54740966	-2.95302366	0.94147861	0.04222379	-0.86636254	0.25949225
5	-0.49324403	-2.92134609	0.98450700	0.06520505	-0.79740966	0.08934642
10	-0.45194009	-2.92667231	0.96155574	0.07731267	-0.70326136	0.00187187
15	-0.45006977	-2.92662125	0.96171102	0.07796137	-0.70014949	0.00007237
20	-0.45000320	-2.92662036	0.96171370	0.07798468	-0.70000538	0.00000298
21	-0.45000150	-2.92662032	0.96171383	0.07798525	-0.70000320	0.00000155
22	-0.45000089	-2.92662032	0.96171384	0.07798547	-0.70000150	0.00000083
23	-0.45000042	-2.92662031	0.96171388	0.07798563	-0.70000089	0.00000043

Estimates values are $x \approx$ -0.45000042, $y \approx$ -2.92662031, $z \approx$ 0.96171388, $\lambda \approx$ 0.07798563 and $\mu \approx$ -0.70000089 after 23 iterations.

The result from U_5 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-1.29375000	-2.92153846	1.41912500	-0.11750000	-0.45000000	3.91710521
2	-0.89567063	-3.22619173	0.91256326	0.15360785	-1.54375000	2.95402141
3	-0.59876425	-2.91591391	1.17457076	0.04716909	-1.14567063	0.82661233
4	-0.52863759	-2.95646600	0.93862834	0.04207001	-0.84876425	0.26560573
5	-0.48391326	-2.92100860	0.98699246	0.06676794	-0.77863759	0.08568234

The result from U_5 *method* (cont.)

k	x^k	y^k	z^k	λ^k	μ^k	ER
10	-0.45101982	-2.92665876	0.96159694	0.07757500	-0.70198707	0.00109025
15	-0.45002870	-2.92662093	0.96171198	0.07797462	-0.70005942	0.00002973
19	-0.45000169	-2.92662034	0.96171378	0.07798511	-0.70000348	0.00000175
20	-0.45000084	-2.92662032	0.96171384	0.07798544	-0.70000169	0.00000086
21	-0.45000041	-2.92662031	0.96171387	0.07798561	-0.70000084	0.00000042

Estimates values are $x \approx$-0.45000041, $y \approx$-2.92662031, $z \approx$0.96171387, $\lambda \approx$0.07798561 and $\mu \approx$-0.70000084 after 21 iterations.

Three points central

The result from D *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.63875000	-3.04000000	0.50000000	-0.10500000	-1.25000000	0.84128281
2	-0.58551981	-3.03973684	0.97293125	0.08400000	-0.88875000	0.95247977
3	-0.50597857	-2.92515533	1.08892051	0.06071521	-0.83551981	0.44747073
4	-0.48657621	-2.88203256	0.95587593	0.05770406	-0.75597857	0.35672018
5	-0.46575546	-2.92890738	0.91010999	0.07265212	-0.73657621	0.15532787
10	-0.45077592	-2.92880849	0.96187538	0.07821915	-0.70122084	0.0161898
15	-0.45002618	-2.92661232	0.96202973	0.07797330	-0.70005986	0.00092703
20	-0.45000128	-2.92660642	0.96171242	0.07798183	-0.70000202	0.00010179
25	-0.45000004	-2.92662036	0.96171183	0.07798577	-0.70000010	0.0000059
29	-0.45000000	-2.92662031	0.96171362	0.07798577	-0.70000001	0.00000078
30	-0.45000000	-2.92662039	0.96171391	0.07798579	-0.70000000	0.00000066
31	-0.45000000	-2.92662030	0.96171400	0.07798577	-0.70000000	0.00000029

Estimates values are $x \approx$-0.45000000, $y \approx$-2.92662030, $z \approx$0.96171400, $\lambda \approx$0.07798577 and $\mu \approx$-0.70000000 after 31 iterations.

The result from L_2 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.63875000	-3.04000000	0.54000000	-0.36500000	-1.25000000	0.85755781
2	-0.58551981	-3.03289474	0.79975099	0.15777296	-0.88875000	0.85297778
3	-0.50597857	-2.97551521	1.06789441	0.04713790	-0.83551981	0.64246400
4	-0.48657621	-2.89080900	0.91465977	0.06422501	-0.75597857	0.34207415
5	-0.46575546	-2.94211312	0.97336348	0.07395513	-0.73657621	0.13287780
10	-0.45077592	-2.92666245	0.96164332	0.07768387	-0.70122084	0.00088021
15	-0.45002618	-2.92662158	0.96170943	0.07797561	-0.70005986	0.00003235
20	-0.45000128	-2.92662041	0.96171382	0.07798528	-0.70000202	0.00000179
21	-0.45000056	-2.92662033	0.96171380	0.07798555	-0.70000128	0.00000070
22	-0.45000036	-2.92662033	0.96171387	0.07798563	-0.70000056	0.00000050

Estimates values are $x \approx$-0.45000036, $y \approx$-2.92662033, $z \approx$0.96171387, $\lambda \approx$0.07798563 and $\mu \approx$-0.70000056 after 22 iterations.

The result from L_3 *method* and L_4 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER^{*}
1	-0.63875000	-3.04000000	0.72062500	-0.41015625	-1.25000000	0.99887571
2	-0.58551981	-2.99544401	0.75528940	0.07429167	-0.88875000	0.45919255
3	-0.50597857	-2.98657272	1.02416272	0.06629390	-0.83551981	0.54620581
4	-0.48657621	-2.90645984	0.93871688	0.06540261	-0.75597857	0.19778549
5	-0.46575546	-2.93420869	0.96467031	0.07306951	-0.73657621	0.07001503
10	-0.45077592	-2.92671567	0.96166128	0.07769033	-0.70122084	0.00125908
15	-0.45002618	-2.92662139	0.96170678	0.07797568	-0.70005986	0.00004115
20	-0.45000128	-2.92662048	0.96171383	0.07798528	-0.70000202	0.00000225
21	-0.45000056	-2.92662032	0.96171374	0.07798555	-0.70000128	0.00000088
22	-0.45000036	-2.92662035	0.96171388	0.07798564	-0.70000056	0.00000063
23	-0.45000016	-2.92662031	0.96171385	0.07798571	-0.70000036	0.00000025

Estimates values are $x \approx$-0.45000016, $y \approx$-2.92662031, $z \approx$0.96171385, $\lambda \approx$0.07798571 and $\mu \approx$-0.70000036 after 23 iterations.

The result from L_5 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.63875000	-3.04000000	0.72062500	-0.41015625	-1.15968750	0.94960402
2	-0.56784614	-2.99544401	0.76253837	0.07948230	-0.87102403	0.44626244
3	-0.49987594	-2.98473615	1.02446744	0.06699959	-0.80085359	0.52454432
4	-0.47770788	-2.90630653	0.93831276	0.06778216	-0.74433393	0.19141941
5	-0.46240424	-2.93434065	0.96568341	0.07420542	-0.72388197	0.06758310
10	-0.45038319	-2.92665750	0.96165511	0.07783954	-0.70065503	0.00053445
15	-0.45001079	-2.92662136	0.96171198	0.07798166	-0.70001945	0.00001509
18	-0.45000131	-2.92662045	0.96171370	0.07798527	-0.70000229	0.00000199
19	-0.45000064	-2.92662036	0.96171379	0.07798553	-0.70000114	0.00000091
20	-0.45000032	-2.92662034	0.96171385	0.07798565	-0.70000056	0.00000048

Estimates values are $x \approx$-0.45000032, $y \approx$-2.92662034, $z \approx$0.96171385, $\lambda \approx$0.07798565 and $\mu \approx$-0.70000056 after 20 iterations.

The result from U_2 *method* , U_3 *method* and U_4 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.63875000	-3.22442708	0.60656250	-0.07375000	-1.25000000	2.20264890
2	-0.58551981	-2.90986405	1.22786950	0.06391422	-0.88875000	0.90256336
3	-0.50597857	-2.94929711	0.93693387	0.04906614	-0.83551981	0.19688286
4	-0.48657621	-2.92098133	0.98006646	0.06823241	-0.75597857	0.06045598
5	-0.46575546	-2.92893606	0.95500008	0.07129668	-0.73657621	0.02587662
10	-0.45077592	-2.92663292	0.96167549	0.07772590	-0.70122084	0.00070757
15	-0.45002618	-2.92662062	0.96171292	0.07797654	-0.70005986	0.00002803
20	-0.45000128	-2.92662033	0.96171381	0.07798534	-0.70000202	0.00000118
21	-0.45000056	-2.92662031	0.96171388	0.07798557	-0.70000128	0.00000060
22	-0.45000036	-2.92662031	0.96171387	0.07798565	-0.70000056	0.00000033

Estimates values are $x \approx$-0.45000036, $y \approx$-2.92662031, $z \approx$0.96171387, $\lambda \approx$0.07798565 and $\mu \approx$-0.70000056 after 22 iterations.

The result from U_5 *method*

k	x^k	y^k	z^k	λ^k	μ^k	ER
1	-0.64656250	-3.22442708	0.60656250	-0.07375000	-1.25000000	2.20820798
2	-0.56912767	-2.90984807	1.22795444	0.06402946	-0.89656250	0.89572887
3	-0.50138772	-2.94971295	0.93596910	0.04906915	-0.81912767	0.19304863
4	-0.47811085	-2.92091459	0.98033442	0.06949171	-0.75138772	0.05518584
5	-0.46274009	-2.92902555	0.95474088	0.07181052	-0.72811085	0.02310155
10	-0.45039029	-2.92662568	0.96169752	0.07783694	-0.70076367	0.00040595
15	-0.45001103	-2.92662056	0.96171312	0.07798147	-0.70002278	0.00001137
18	-0.45000133	-2.92662033	0.96171380	0.07798525	-0.70000268	0.00000136
19	-0.45000065	-2.92662031	0.96171385	0.07798552	-0.70000133	0.00000067
20	-0.45000032	-2.92662031	0.96171387	0.07798565	-0.70000065	0.00000033

Estimates values are $x \approx$ -0.45000032, $y \approx$ -2.92662031, $z \approx$ 0.96171387, $\lambda \approx$ 0.07798565 and $\mu \approx$ -0.70000065 after 20 iterations.

The iterations of table 4.5

Two points backward

The result from D *method*

k	θ^k	ϕ^k	ER
1	0.05150029	1.97137220	0.20586665
2	0.38180293	1.70979531	0.17973988
3	0.58901146	2.13451901	0.12397823
4	0.33318881	2.20531776	0.05370964
5	0.29189879	2.07024035	0.03999799
10	0.35978610	2.07689623	0.00422665
15	0.35897444	2.08639329	0.00053638
20	0.35804140	2.08628570	0.00005421
25	0.35805198	2.08616377	0.00000691
30	0.35806401	2.08616516	0.00000070
31	0.35806458	2.08616691	0.00000051
32	0.35806360	2.08616734	0.00000029

Estimates values are $\theta \approx 0.35806360$ and $\phi \approx 2.08616734$ after 32 iterations.

The result from L_2 *method*

k	θ^k	ϕ^k	ER
1	0.05150029	2.02972715	0.19721772
2	0.35283453	1.73868226	0.17146255
3	0.57648425	2.11052634	0.12148380
4	0.34470661	2.19471473	0.05063846
5	0.29833573	2.07572030	0.03605277
10	0.35960331	2.07973470	0.00286317
15	0.35859035	2.08641474	0.00030508
20	0.35803803	2.08621795	0.00002013
25	0.35805963	2.08616333	0.00000248
27	0.35806525	2.08616808	0.00000099
28	0.35806294	2.08616771	0.00000051
29	0.35806315	2.08616612	0.00000040

Estimates values are $\theta \approx 0.35806315$ and $\phi \approx 2.08616612$ after 29 iterations.

The result from U_2 *method*

k	θ^k	ϕ^k	ER
1	0.02886956	1.97137220	0.22141167
2	0.38661675	1.66691020	0.19915945
3	0.60490804	2.14338053	0.13047757
4	0.32616268	2.20941422	0.05446974
5	0.29552387	2.06545552	0.03773525
10	0.36075315	2.07887505	0.00305544
15	0.35859526	2.08653447	0.00030276
20	0.35802772	2.08622276	0.00002404
25	0.35805982	2.08616216	0.00000302
26	0.35806620	2.08616379	0.00000162
27	0.35806516	2.08616855	0.00000120
28	0.35806271	2.08616777	0.00000065
29	0.35806319	2.08616594	0.00000048

Estimates values are $\theta \approx 0.35806319$ and $\phi \approx 2.08616594$ after 29 iterations.

The result from D *method*

k	θ^k	ϕ^k	ER
1	1.75667829	-1.79424350	0.13558848
2	1.88696324	-2.03945887	0.03157486
3	1.86364529	-2.10264595	0.00843274
4	1.85287100	-2.09054616	0.00315164
5	1.85534355	-2.08417111	0.00101700
10	1.85623915	-2.08615956	0.00000512
11	1.85623514	-2.08616988	0.00000162
12	1.85623308	-2.08616753	0.00000060
13	1.85623355	-2.08616632	0.00000019

Estimates values are $\theta \approx 1.85623355$ and $\phi \approx -2.08616632$ after 13 iterations.

The result from L_2 *method*

k	θ^k	ϕ^k	ER
1	1.75667829	-1.83659215	0.11992305
2	1.88936428	-2.04779437	0.03061038
3	1.86229472	-2.10200239	0.00789064
4	1.85300483	-2.08914885	0.00271084
5	1.85562923	-2.08443167	0.00085035
10	1.85623756	-2.08616412	0.00000298
11	1.85623423	-2.08616872	0.00000096
12	1.85623331	-2.08616693	0.00000031

Estimates values are $\theta \approx 1.85623331$ and $\phi \approx -2.08616693$ after 12 iterations.

The result from U_2 *method*

k	θ^k	ϕ^k	ER
1	1.75667829	-1.79424350	0.13558848
2	1.89376596	-2.03945887	0.03573631
3	1.86330029	-2.10631898	0.00991913
4	1.85242186	-2.09036884	0.00337493
5	1.85553830	-2.08390189	0.00109116
10	1.85623824	-2.08616301	0.00000368
11	1.85623430	-2.08616935	0.00000123
12	1.85623324	-2.08616704	0.00000038

Estimates values are $\theta \approx 1.85623324$ and $\phi \approx -2.08616704$ after 12 iterations.

Two points forward

The result from D *method*

k	θ^k	ϕ^k	ER
1	0.35301617	2.11256450	0.01200361
2	0.34304765	2.08288186	0.00891590
3	0.36002557	2.07452179	0.00534997
4	0.36457720	2.08774337	0.00382953
5	0.35719920	2.09094198	0.00218149
10	0.35759025	2.08605532	0.00027929
15	0.35807473	2.08610471	0.00002836
20	0.35806983	2.08616813	0.00000360
23	0.35806406	2.08616479	0.00000087
24	0.35806479	2.08616695	0.00000063
25	0.35806358	2.08616749	0.00000037

Estimates values are $\theta \approx 0.35806358$ and $\phi \approx 2.08616749$ after 25 iterations.

The result from L_2 *method*

k	θ^k	ϕ^k	ER
1	0.35301617	2.11309873	0.01226645
2	0.34274173	2.08232480	0.00908180
3	0.36034673	2.07523408	0.00493371
4	0.36417979	2.08816801	0.00357176
5	0.35695840	2.09027112	0.00180285
10	0.35772582	2.08599665	0.00019538
15	0.35808115	2.08613391	0.00001268
20	0.35806632	2.08616895	0.00000163
21	0.35806245	2.08616843	0.00000084
22	0.35806274	2.08616576	0.00000065
23	0.35806424	2.08616605	0.00000034

Estimates values are $\theta \approx 0.35806424$ and $\phi \approx 2.08616605$ after 23 iterations.

The result from U_2 *method*

k	θ^k	ϕ^k	ER
1	0.35286085	2.11256450	0.01197322
2	0.34430764	2.08276715	0.00815029
3	0.36036006	2.07552850	0.00478574
4	0.36352274	2.08797319	0.00318813
5	0.35697404	2.09017966	0.00176088
10	0.35776311	2.08600910	0.00017354
15	0.35808016	2.08613448	0.00001259
20	0.35806602	2.08616883	0.00000151
21	0.35806254	2.08616841	0.00000078
22	0.35806286	2.08616581	0.00000061
23	0.35806420	2.08616605	0.00000032

Estimates values are $\theta \approx 0.35806420$ and $\phi \approx 2.08616605$ after 23 iterations.

The result from D *method*

k	θ^k	ϕ^k	ER
1	1.58443891	-1.54770475	0.25575034
2	1.84228814	-1.93196150	0.06485159
3	1.87297146	-2.08262836	0.01096512
4	1.85710796	-2.09604515	0.00433997
5	1.85419451	-2.08665307	0.00135042
6	1.85613895	-2.08496860	0.00052266
7	1.85647164	-2.08611070	0.00015737
8	1.85624491	-2.08630631	0.00006096
9	1.85620586	-2.08617326	0.00001843
10	1.85623241	-2.08615035	0.00000714
11	1.85623698	-2.08616593	0.00000216
12	1.85623387	-2.08616861	0.00000084
13	1.85623334	-2.08616678	0.00000025

Estimates values are $\theta \approx 1.85623334$ and $\phi \approx -2.08616678$ after 13 iterations.

The result from L_2 *method*

k	θ^k	ϕ^k	ER
1	1.58443891	-1.56316559	0.25049830
2	1.84506962	-1.94463368	0.05936419
3	1.87244957	-2.08559313	0.00985646
4	1.85651346	-2.09467445	0.00364276
5	1.85448573	-2.08617373	0.00105092
6	1.85623397	-2.08525954	0.00038135
7	1.85641406	-2.08617891	0.00010501
8	1.85623130	-2.08626007	0.00003879
9	1.85621509	-2.08616417	0.00001052
10	1.85623422	-2.08615707	0.00000395
11	1.85623564	-2.08616709	0.00000105
12	1.85623364	-2.08616768	0.00000040

Estimates values are $\theta \approx 1.85623364$ and $\phi \approx -2.08616768$ after 12 iterations.

The result from U_2 *method*

k	θ^k	ϕ^k	ER
1	1.64813368	-1.54770475	0.24320284
2	1.85037349	-2.02068859	0.02822495
3	1.86410661	-2.08746110	0.00437570
4	1.85593414	-2.09077108	0.00187964
5	1.85541566	-2.08598620	0.00044384
6	1.85627689	-2.08568630	0.00019336
7	1.85631769	-2.08619198	0.00004378
8	1.85622812	-2.08621597	0.00001961
9	1.85622511	-2.08616341	0.00000431
10	1.85623441	-2.08616164	0.00000199
11	1.85623460	-2.08616710	0.00000042

Estimates values are $\theta \approx 1.85623460$ and $\phi \approx -2.08616710$ after 11 iterations.

Three points central

The result from *D method*

k	θ^k	ϕ^k	ER
1	0.12206763	2.16807484	0.14150176
2	0.34103125	1.83311728	0.12804596
3	0.50174983	2.11753557	0.08005210
4	0.34456888	2.16795500	0.03755826
5	0.31137511	2.07898733	0.02808679
10	0.35897435	2.07964043	0.00303610
15	0.35870410	2.08628785	0.00037887
20	0.35805180	2.08625038	0.00003888
25	0.35805547	2.08616514	0.00000488
29	0.35806227	2.08616642	0.00000086
30	0.35806387	2.08616561	0.00000050
31	0.35806433	2.08616681	0.00000036

Estimates values are $\theta \approx 0.35806433$ and $\phi \approx 2.08616681$ after 31 iterations.

The result from L_2 *method*

k	θ^k	ϕ^k	ER
1	0.12206763	2.16755771	0.14151872
2	0.34140467	1.84511315	0.12213141
3	0.49490869	2.12069871	0.07588287
4	0.34260608	2.15863360	0.03262963
5	0.31673365	2.07560500	0.02474528
10	0.35949395	2.08175983	0.00188633
15	0.35842078	2.08637572	0.00020508
20	0.35804285	2.08620089	0.00001401
25	0.35806102	2.08616406	0.00000184
26	0.35806520	2.08616491	0.00000097
27	0.35806472	2.08616777	0.00000073
28	0.35806311	2.08616735	0.00000039

Estimates values are $\theta \approx 0.35806311$ and $\phi \approx 2.08616735$ after 28 iterations.

The result from U_2 *method*

k	θ^k	ϕ^k	ER
1	0.11758520	2.16807484	0.14414621
2	0.35753162	1.82608956	0.12835934
3	0.49689402	2.13656091	0.07488616
4	0.33318926	2.16528118	0.03415434
5	0.31700783	2.07037784	0.02448977
10	0.36004347	2.08143195	0.00193745
15	0.35840584	2.08643180	0.00019917
20	0.35803832	2.08620253	0.00001680
25	0.35806126	2.08616353	0.00000205
26	0.35806544	2.08616486	0.00000112
27	0.35806462	2.08616798	0.00000082
28	0.35806302	2.08616737	0.00000045

Estimates values are $\theta \approx 0.35806302$ and $\phi \approx 2.08616737$ after 28 iterations.

The result from D *method*

k	θ^k	ϕ^k	ER
1	1.83090873	-2.25530200	0.07058178
2	1.80233093	-2.04479156	0.02765232
3	1.86449846	-2.05257637	0.01407048
4	1.86226780	-2.09082110	0.00317066
5	1.85530440	-2.08968264	0.00147143
6	1.85552506	-2.08561875	0.00037208
7	1.85634297	-2.08575046	0.00017426
8	1.85631662	-2.08623078	0.00004353
9	1.85622094	-2.08621534	0.00002037
10	1.85622401	-2.08615919	0.00000510
11	1.85623521	-2.08616100	0.00000238
12	1.85623485	-2.08616757	0.00000060
13	1.85623354	-2.08616736	0.00000028

Estimates values are $\theta \approx 1.85623354$ and $\phi \approx -2.08616736$ after 13 iterations.

The result from L_2 *method*

k	θ^k	ϕ^k	ER
1	1.83090873	-2.25057084	0.06839288
2	1.80445879	-2.04450663	0.02739358
3	1.86446899	-2.05852814	0.01240286
4	1.86129398	-2.09062900	0.00289308
5	1.85534054	-2.08871747	0.00119946
6	1.85572112	-2.08566678	0.00031272
7	1.85633335	-2.08590746	0.00012747
8	1.85628537	-2.08622188	0.00003354
9	1.85622271	-2.08619274	0.00001340
10	1.85622852	-2.08616063	0.00000359
11	1.85623493	-2.08616408	0.00000141
12	1.85623424	-2.08616736	0.00000038

Estimates values are $\theta \approx 1.85623424$ and $\phi \approx -2.08616736$ after 12 iterations.

The result from U_2 *method*

k	θ^k	ϕ^k	ER
1	1.84856998	-2.25530200	0.07329053
2	1.80884633	-2.05776753	0.02163420
3	1.86219862	-2.05680118	0.01149339
4	1.86094463	-2.08953085	0.00236099
5	1.85557732	-2.08891609	0.00110917
6	1.85575460	-2.08578002	0.00025840
7	1.85630838	-2.08588536	0.00011825
8	1.85628226	-2.08621049	0.00002813
9	1.85622534	-2.08619518	0.00001246
10	1.85622881	-2.08616178	0.00000305
11	1.85623465	-2.08616381	0.00000131
12	1.85623421	-2.08616724	0.00000033

Estimates values are $\theta \approx 1.85623421$ and $\phi \approx -2.08616724$ after 12 iterations.

The iterations of table 4.6

Two points backward

The result from *D method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.17598547	349.45671865	345.40476224	173.74381462	0.02522729
2	424.05898832	349.46694724	345.05831551	173.71241841	0.02167593
3	424.00573817	349.17608233	345.05429279	173.62781755	0.02011754
4	423.90722285	349.16185119	344.80376823	173.55453475	0.01786674
5	423.86144234	348.94696539	344.77447544	173.48798372	0.01657442
10	423.59719675	348.54897860	344.25403275	173.23509688	0.01011518
15	423.44513729	348.24779941	343.98530118	173.07827492	0.00631526
20	423.34693009	348.07964318	343.80033650	172.98114912	0.00389379
25	423.28743564	347.96917073	343.69208926	172.92095039	0.00241694
30	423.25012854	347.90289470	343.62288564	172.88367784	0.00149464
35	423.22717997	347.86111622	343.58077029	172.86059029	0.00092615
40	423.21291580	347.83549519	343.55443795	172.84629506	0.00057324
45	423.20410119	347.81954438	343.53821919	172.83744249	0.00035502
50	423.19863692	347.80969675	343.52814695	172.83196114	0.00021979
55	423.19525557	347.80358912	343.52192042	172.82856704	0.0001361
60	423.19316116	347.79981076	343.51806157	172.82646547	0.00008427
65	423.19186457	347.79747008	343.51567341	172.82516420	0.00005218
70	423.19106166	347.79602117	343.51419431	172.82435848	0.00003231
75	423.19056454	347.79512390	343.51327861	172.82385959	0.00002
80	423.19025672	347.79456837	343.51271158	172.82355069	0.00001239
85	423.19006613	347.79422438	343.51236050	172.82335942	0.00000767
90	423.18994812	347.79401139	343.51214312	172.82324100	0.00000475
95	423.18987505	347.79387952	343.51200852	172.82316767	0.00000294
100	423.18982981	347.79379786	343.51192518	172.82312226	0.00000182
105	423.18980180	347.79374730	343.51187358	172.82309415	0.00000113
110	423.18978445	347.79371600	343.51184163	172.82307674	0.0000007
111	423.18978187	347.79371134	343.51183688	172.82307416	0.00000063
112	423.18977953	347.79370711	343.51183256	172.82307181	0.00000058
113	423.18977740	347.79370327	343.51182864	172.82306967	0.00000052

The result from D *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
114	423.18977547	347.79369978	343.51182508	172.82306773	0.00000048

Estimates values are $p_1 \approx 423.18977547$, $p_2 \approx 347.79369978$, $p_3 \approx 343.51182508$ and $p_4 \approx 172.82306773$ after 114 iterations.

The result from L_2 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.17598547	349.45570665	345.40250525	173.74383788	0.025135
2	424.05827843	349.46512216	345.05751673	173.71157938	0.02165735
3	424.00499335	349.17527342	345.05185615	173.62716287	0.02009976
4	423.90655188	349.15973160	344.80291028	173.55370650	0.01784892
5	423.86058878	348.94612174	344.77200424	173.48723951	0.01655409
10	423.59651175	348.54708747	344.25294577	173.23430154	0.01009738
15	423.44444609	348.24680838	343.98375847	173.07762103	0.00629844
20	423.34643808	348.07859130	343.79943602	172.98062811	0.0038814
25	423.28704288	347.96850318	343.69129586	172.92056229	0.00240727
30	423.24985227	347.90236199	343.62236358	172.88339476	0.00148776
35	423.22697957	347.86075750	343.58038108	172.86038943	0.00092119
40	423.21277715	347.83523734	343.55417479	172.84615456	0.00056979
45	423.20400493	347.81936958	343.53803530	172.83734564	0.00035263
50	423.19857142	347.80957649	343.52802290	172.83189503	0.00021817
55	423.19521116	347.80350823	343.52183626	172.82852232	0.000135
60	423.19313135	347.79975631	343.51800530	172.82643543	0.00008353
65	423.19184466	347.79743381	343.51563586	172.82514415	0.00005169
70	423.19104844	347.79599708	343.51416943	172.82434517	0.00003198
75	423.19055580	347.79510799	343.51326218	172.82385079	0.00001979
80	423.19025097	347.79455790	343.51270078	172.82354490	0.00001224
85	423.19006236	347.79421751	343.51235342	172.82335563	0.00000758
90	423.18994565	347.79400691	343.51213850	172.82323851	0.00000469
95	423.18987344	347.79387659	343.51200551	172.82316605	0.0000029
100	423.18982876	347.79379596	343.51192322	172.82312121	0.00000179

The result from L_2 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
105	423.18980112	347.79374607	343.51187231	172.82309347	0.00000111
110	423.18978401	347.79371520	343.51184080	172.82307630	0.00000069
111	423.18978147	347.79371061	343.51183612	172.82307375	0.00000062
112	423.18977916	347.79370644	343.51183187	172.82307143	0.00000057
113	423.18977706	347.79370266	343.51182801	172.82306933	0.00000052
114	423.18977516	347.79369922	343.51182450	172.82306742	0.00000047

Estimates values are $p_1 \approx 423.18977516$, $p_2 \approx 347.79369922$, $p_3 \approx 343.51182450$ and $p_4 \approx 172.82306742$ after 114 iterations.

The result from L_3 *method* and L_4 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.17598547	349.45570665	345.40228825	173.74380686	0.02511297
2	424.05824289	349.46495064	345.05747650	173.71152536	0.02165633
3	424.00492905	349.17523428	345.05169072	173.62709355	0.02009812
4	423.90651163	349.15958849	344.80283230	173.55365428	0.01784783
5	423.86052787	348.94605246	344.77186312	173.48717210	0.01655254
10	423.59646399	348.54698969	344.25284724	173.23424924	0.01009616
15	423.44440501	348.24673220	343.98367681	173.07757613	0.0062974
20	423.34640631	348.07853334	343.79937263	172.98059464	0.00388061
25	423.28701941	347.96845945	343.69125035	172.92053739	0.00240667
30	423.24983524	347.90233101	343.62233046	172.88337701	0.00148733
35	423.22696757	347.86073541	343.58035806	172.86037691	0.00092089
40	423.21276878	347.83522210	343.55415874	172.84614589	0.00056958
45	423.20399919	347.81935908	343.53802436	172.83733970	0.00035249
50	423.19856751	347.80956936	343.52801546	172.83189100	0.00021807
55	423.19520853	347.80350342	343.52183126	172.82851961	0.00013494
60	423.19312958	347.79975310	343.51800196	172.82643362	0.00008349
65	423.19184348	347.79743167	343.51563363	172.82514295	0.00005166
70	423.19104766	347.79599566	343.51416795	172.82434437	0.00003196
75	423.19055528	347.79510705	343.51326121	172.82385027	0.00001977

The result from L_3 *method* and L_4 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
80	423.19025063	347.79455728	343.51270014	172.82354455	0.00001224
85	423.19006214	347.79421711	343.51235301	172.82335540	0.00000757
90	423.18994551	347.79400664	343.51213822	172.82323836	0.00000468
95	423.18987335	347.79387642	343.51200533	172.82316595	0.0000029
100	423.18982870	347.79379585	343.51192311	172.82312115	0.00000179
105	423.18980108	347.79374599	343.51187223	172.82309343	0.00000111
110	423.18978399	347.79371515	343.51184075	172.82307628	0.00000069
111	423.18978145	347.79371057	343.51183608	172.82307373	0.00000062
112	423.18977914	347.79370640	343.51183183	172.82307141	0.00000057
113	423.18977704	347.79370262	343.51182797	172.82306931	0.00000051
114	423.18977514	347.79369919	343.51182446	172.82306740	0.00000047

Estimates values are $p_1 \approx 423.18977514$, $p_2 \approx 347.79369919$, $p_3 \approx 343.51182446$ and $p_4 \approx 172.82306740$ after 114 iterations.

The result from U_2 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.17402899	349.45702697	345.40483209	173.74381462	0.02511517
2	424.05911522	349.46574932	345.05837884	173.71251260	0.02167883
3	424.00495863	349.17615804	345.05332881	173.62753590	0.02010199
4	423.90706878	349.16028219	344.80372405	173.55431113	0.0178624
5	423.86062026	348.94682503	344.77315277	173.48758281	0.01655724
10	423.59672558	348.54742869	344.25347585	173.23460434	0.01010335
15	423.44451275	348.24705470	343.98414551	173.07777859	0.00630078
20	423.34650190	348.07873257	343.79965680	172.98073611	0.00388332
25	423.28707497	347.96859489	343.69143897	172.92062582	0.0024083
30	423.24987470	347.90241746	343.62245009	172.88343527	0.00148845
35	423.22699261	347.86079233	343.58043534	172.86041413	0.0009216
40	423.21278551	347.83525875	343.55420813	172.84616998	0.00057005
45	423.20401001	347.81938289	343.53805602	172.83735514	0.00035279
50	423.19857460	347.80958470	343.52803569	172.83190093	0.00021827

The result from U_2 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
55	423.19521312	347.80351332	343.52184418	172.82852596	0.00013506
60	423.19313256	347.79975946	343.51801020	172.82643769	0.00008357
65	423.19184541	347.79743576	343.51563889	172.82514555	0.00005171
70	423.19104890	347.79599829	343.51417130	172.82434603	0.00003199
75	423.19055609	347.79510873	343.51326334	172.82385133	0.0000198
80	423.19025115	347.79455836	343.51270150	172.82354523	0.00001225
85	423.19006247	347.79421780	343.51235387	172.82335583	0.00000758
90	423.18994572	347.79400708	343.51213877	172.82323864	0.00000469
95	423.18987349	347.79387670	343.51200568	172.82316613	0.0000029
100	423.18982879	347.79379603	343.51192333	172.82312126	0.0000018
105	423.18980113	347.79374611	343.51187237	172.82309350	0.00000111
110	423.18978402	347.79371522	343.51184084	172.82307632	0.00000069
111	423.18978148	347.79371063	343.51183616	172.82307377	0.00000062
112	423.18977917	347.79370647	343.51183190	172.82307145	0.00000057
113	423.18977707	347.79370268	343.51182804	172.82306934	0.00000052
114	423.18977516	347.79369924	343.51182453	172.82306743	0.00000047

Estimates values are $p_1 \approx 423.18977516$, $p_2 \approx 347.79369924$, $p_3 \approx 343.51182453$ and $p_4 \approx 172.82306743$ after 114 iterations.

The result from U_3 *method* and U_4 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.17416809	349.45705751	345.40483209	173.74381462	0.02511681
2	424.05893544	349.46577363	345.05841748	173.71252019	0.0216758
3	424.00497029	349.17615377	345.05333248	173.62755166	0.02010231
4	423.90692719	349.16028710	344.80372326	173.55431099	0.01785996
5	423.86060487	348.94679758	344.77314309	173.48758383	0.01655699
10	423.59664563	348.54739819	344.25343191	173.23458531	0.01010183
15	423.44447227	348.24700690	343.98410672	173.07775743	0.00629993
20	423.34646380	348.07869668	343.79961700	172.98071636	0.00388252
25	423.28704987	347.96856335	343.69140908	172.92060976	0.00240775

The result from U_3 *method* and U_4 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
30	423.24985592	347.90239452	343.62242630	172.88342288	0.00148803
35	423.22697979	347.86077506	343.58041820	172.86040496	0.00092131
40	423.21277655	347.83524654	343.55419569	172.84616338	0.00056985
45	423.20400394	347.81937424	343.53804732	172.83735048	0.00035265
50	423.19857047	347.80957874	343.52802963	172.83189769	0.00021817
55	423.19521035	347.80350922	343.52184004	172.82852374	0.000135
60	423.19313071	347.79975669	343.51800739	172.82643618	0.00008352
65	423.19184418	347.79743389	343.51563700	172.82514453	0.00005168
70	423.19104809	347.79599704	343.51417003	172.82434535	0.00003198
75	423.19055555	347.79510790	343.51326249	172.82385087	0.00001978
80	423.19025080	347.79455781	343.51270094	172.82354493	0.00001224
85	423.19006224	347.79421744	343.51235350	172.82335563	0.00000757
90	423.18994557	347.79400684	343.51213853	172.82323851	0.00000469
95	423.18987339	347.79387654	343.51200552	172.82316604	0.0000029
100	423.18982873	347.79379593	343.51192322	172.82312120	0.00000179
105	423.18980109	347.79374604	343.51187231	172.82309346	0.00000111
110	423.18978399	347.79371518	343.51184080	172.82307630	0.00000069
111	423.18978146	347.79371059	343.51183612	172.82307375	0.00000062
112	423.18977915	347.79370643	343.51183187	172.82307143	0.00000057
113	423.18977705	347.79370265	343.51182801	172.82306933	0.00000051
114	423.18977515	347.79369921	343.51182450	172.82306741	0.00000047

Estimates values are $p_1 \approx 423.18977515$, $p_2 \approx 347.79369921$, $p_3 \approx 343.51182450$ and $p_4 \approx 172.82306741$ after 114 iterations.

Two points forward

The result from *D method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	421.72470885	346.42475085	340.93367740	171.24538670	0.15039099
2	422.30682214	345.40427420	342.13305059	171.83412338	0.12825325
3	422.16072720	346.39212928	341.45904897	171.88238560	0.07337647
4	422.38250148	345.97465227	342.22038544	171.95837211	0.06543406
5	422.36661789	346.58272678	341.87650945	172.04619934	0.04780701
10	422.72240519	346.85680356	342.69677380	172.33918052	0.01710508
15	422.88718113	347.27955073	342.93961919	172.52456771	0.00744499
20	423.00714177	347.45312427	343.18045744	172.63802440	0.00453143
25	423.07509794	347.59050398	343.29885408	172.70858454	0.00283251
30	423.11932184	347.66525898	343.38262710	172.75215795	0.00174452
35	423.14595801	347.71506032	343.43090913	172.77916522	0.00108334
40	423.16270319	347.74468899	343.46202379	172.79587543	0.00066968
45	423.17298375	347.76344568	343.48087010	172.80622475	0.00041503
50	423.17937881	347.77491816	343.49268162	172.81263144	0.00025685
55	423.18332822	347.78206974	343.49994606	172.81659861	0.00015908
60	423.18577705	347.78648133	343.50446065	172.81905483	0.00009848
65	423.18729212	347.78921850	343.50725026	172.82057570	0.00006098
70	423.18823062	347.79091137	343.50897947	172.82151737	0.00003776
75	423.18881158	347.79196022	343.51004949	172.82210044	0.00002338
80	423.18917134	347.79260941	343.51071225	172.82246146	0.00001448
85	423.18939408	347.79301146	343.51112253	172.82268499	0.00000896
90	423.18953200	347.79326037	343.51137660	172.82282340	0.00000555
95	423.18961740	347.79341450	343.51153391	172.82290910	0.00000344
100	423.18967028	347.79350993	343.51163131	172.82296217	0.00000213
105	423.18970302	347.79356902	343.51169162	172.82299502	0.00000132
110	423.18972329	347.79360561	343.51172896	172.82301536	0.00000082
114	423.18973379	347.79362456	343.51174830	172.82302590	0.00000056
115	423.18973584	347.79362826	343.51175208	172.82302796	0.00000051

The result from D *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
116	423.18973771	347.79363163	343.51175552	172.82302983	0.00000046

Estimates values are $p_1 \approx 423.18973771$, $p_2 \approx 347.79363163$, $p_3 \approx 343.51175552$ and $p_4 \approx 172.82302983$ after 116 iterations.

The result from L_2 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	421.72470885	346.42016672	340.93886499	171.24544133	0.14941549
2	422.30612248	345.40954870	342.12619021	171.83436319	0.12670822
3	422.16137007	346.38976890	341.46459703	171.88193607	0.07257308
4	422.38261193	345.97958942	342.21649990	171.95922729	0.0643725
5	422.36763953	346.58078314	341.88181757	172.04643066	0.04703445
10	422.72277122	346.86034355	342.69577563	172.34001190	0.01661836
15	422.88802025	347.27969170	342.94209772	172.52518222	0.00742598
20	423.00757388	347.45459228	343.18097196	172.63857049	0.00451989
25	423.07554108	347.59104318	343.29986268	172.70898771	0.00282186
30	423.11960315	347.66589331	343.38311635	172.75246108	0.0017374
35	423.14617923	347.71542161	343.43135722	172.77938139	0.00107791
40	423.16285204	347.74497992	343.46230048	172.79602854	0.00066597
45	423.17309005	347.76363365	343.48107608	172.80633090	0.0004124
50	423.17945076	347.77505234	343.49281720	172.81270440	0.00025506
55	423.18337755	347.78215894	343.50004004	172.81664819	0.00015786
60	423.18581019	347.78654217	343.50452316	172.81908828	0.00009766
65	423.18731438	347.78925897	343.50729234	172.82059810	0.00006043
70	423.18824543	347.79093841	343.50900735	172.82153229	0.00003739
75	423.18882140	347.79197809	343.51006798	172.82211033	0.00002314
80	423.18917782	347.79262121	343.51072442	172.82246798	0.00001432
85	423.18939834	347.79301921	343.51113053	172.82268928	0.00000886
90	423.18953479	347.79326545	343.51138183	172.82282621	0.00000548
95	423.18961922	347.79341781	343.51153732	172.82291094	0.00000339

The result from L_2 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
100	423.18967146	347.79351209	343.51163353	172.82296336	0.0000021
105	423.18970379	347.79357042	343.51169306	172.82299580	0.0000013
110	423.18972379	347.79360652	343.51172989	172.82301587	0.0000008
113	423.18973191	347.79362118	343.51174486	172.82302402	0.0000006
114	423.18973414	347.79362520	343.51174896	172.82302626	0.00000055
115	423.18973616	347.79362885	343.51175269	172.82302829	0.0000005

Estimates values are $p_1 \approx 423.18973616$, $p_2 \approx 347.79362885$, $p_3 \approx 343.51175269$ and $p_4 \approx 172.82302829$ after 115 iterations.

The result from L_3 *method* and L_4 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	421.72470885	346.42016672	340.93462290	171.24552255	0.14977657
2	422.30543316	345.40604256	342.12685268	171.83323612	0.1272433
3	422.16030205	346.38898728	341.46253252	171.88128695	0.07271738
4	422.38201168	345.97769732	342.21595027	171.95848983	0.06453733
5	422.36691360	346.58006050	341.88025961	172.04585683	0.0471272
10	422.72243650	346.85953091	342.69529175	172.33963986	0.0166566
15	422.88779564	347.27936972	342.94165103	172.52497664	0.00743138
20	423.00745425	347.45434702	343.18077222	172.63844673	0.00452288
25	423.07546712	347.59092240	343.29972375	172.70891708	0.00282367
30	423.11956193	347.66581519	343.38304446	172.75241977	0.00173842
35	423.14615484	347.71537959	343.43131268	172.77935774	0.00107851
40	423.16283827	347.74495470	343.46227612	172.79601494	0.00066631
45	423.17308211	347.76361968	343.48106180	172.80632318	0.00041259
50	423.17944629	347.77504428	343.49280929	172.81270003	0.00025517
55	423.18337503	347.78215447	343.50003556	172.81664574	0.00015792
60	423.18580880	347.78653967	343.50452070	172.81908692	0.00009769
65	423.18731361	347.78925761	343.50729099	172.82059736	0.00006045
70	423.18824501	347.79093767	343.50900663	172.82153190	0.0000374

The result from L_3 *method* and L_4 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
75	423.18882118	347.79197770	343.51006760	172.82211012	0.00002314
80	423.18917771	347.79262102	343.51072423	172.82246788	0.00001432
85	423.18939829	347.79301911	343.51113044	172.82268923	0.00000886
90	423.18953477	347.79326540	343.51138179	172.82282619	0.00000548
95	423.18961921	347.79341780	343.51153731	172.82291093	0.00000339
100	423.18967146	347.79351209	343.51163353	172.82296336	0.0000021
105	423.18970379	347.79357042	343.51169306	172.82299580	0.0000013
110	423.18972379	347.79360652	343.51172990	172.82301587	0.0000008
113	423.18973191	347.79362118	343.51174486	172.82302402	0.0000006
114	423.18973414	347.79362520	343.51174896	172.82302626	0.00000055
115	423.18973617	347.79362885	343.51175269	172.82302829	0.0000005

Estimates values are $p_1 \approx 423.18973617$, $p_2 \approx 347.79362885$, $p_3 \approx 343.51175269$ and $p_4 \approx 172.82302829$ after 115 iterations.

The result from U_2 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	421.72595557	346.44041939	340.93368880	171.24538670	0.15215603
2	422.31101726	345.40534112	342.14711093	171.83801218	0.12988249
3	422.16530364	346.39740869	341.46404549	171.88618289	0.07332255
4	422.38461688	345.98101157	342.22577201	171.96094065	0.06527459
5	422.37058676	346.58640892	341.88219466	172.04913452	0.04753144
10	422.72397337	346.86198664	342.69908868	172.34107079	0.01679433
15	422.88891256	347.28135147	342.94301606	172.52598903	0.00740459
20	423.00807631	347.45539646	343.18192080	172.63901713	0.0045079
25	423.07586846	347.59161992	343.30031979	172.70928207	0.00281402
30	423.11980065	347.66622364	343.38344385	172.75263751	0.00173268
35	423.14630326	347.71563529	343.43154496	172.77949261	0.00107494
40	423.16292823	347.74510907	343.46242177	172.79609673	0.00066415
45	423.17313740	347.76371463	343.48114943	172.80637333	0.00041127

The result from U_2 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
50	423.17948000	347.77510210	343.49286317	172.81273058	0.00025436
55	423.18339567	347.78218985	343.50006829	172.81666441	0.00015742
60	423.18582140	347.78656126	343.50454071	172.81909831	0.00009739
65	423.18732131	347.78927079	343.50730318	172.82060431	0.00006027
70	423.18824972	347.79094572	343.50901407	172.82153613	0.00003729
75	423.18882405	347.79198262	343.51007213	172.82211270	0.00002307
80	423.18917946	347.79262401	343.51072699	172.82246945	0.00001428
85	423.18939936	347.79302094	343.51113212	172.82269019	0.00000883
90	423.18953542	347.79326652	343.51138282	172.82282677	0.00000547
95	423.18961961	347.79341848	343.51153793	172.82291128	0.00000338
100	423.18967170	347.79351250	343.51163391	172.82296358	0.00000209
105	423.18970394	347.79357068	343.51169329	172.82299593	0.00000129
110	423.18972388	347.79360667	343.51173004	172.82301595	0.0000008
113	423.18973198	347.79362129	343.51174496	172.82302408	0.0000006
114	423.18973420	347.79362530	343.51174906	172.82302631	0.00000055
115	423.18973622	347.79362895	343.51175277	172.82302834	0.0000005

Estimates values are $p_1 \approx 423.18973622$, $p_2 \approx 347.79362895$, $p_3 \approx 343.51175277$ and $p_4 \approx 172.82302834$ after 115 iterations.

The result from U_3 *method* and U_4 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	421.72737635	346.44044077	340.93368880	171.24538670	0.15210166
2	422.31160708	345.40564112	342.14727739	171.83801749	0.12988554
3	422.16485991	346.39768350	341.46433549	171.88629935	0.07334554
4	422.38510500	345.98114824	342.22595459	171.96108188	0.06529492
5	422.37043178	346.58663294	341.88236386	172.04921440	0.04754833
10	422.72417693	346.86207025	342.69924260	172.34114138	0.01679729
15	422.88896069	347.28147105	342.94309735	172.52603932	0.00740339
20	423.00814580	347.45546000	343.18200363	172.63905682	0.00450641

The result from U_3 *method* and U_4 *method* (cont.)

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
25	423.07590459	347.59167807	343.30037026	172.70931076	0.00281319
30	423.11982986	347.66626023	343.38348424	172.75265822	0.00173203
35	423.14632165	347.71566288	343.43157151	172.77950713	0.00107452
40	423.16294119	347.74512738	343.46244084	172.79610681	0.00066385
45	423.17314591	347.76372743	343.48116220	172.80638023	0.00041107
50	423.17948574	347.77511066	343.49287193	172.81273526	0.00025423
55	423.18339945	347.78219564	343.50007415	172.81666757	0.00015733
60	423.18582390	347.78656511	343.50454464	172.81910042	0.00009733
65	423.18732296	347.78927336	343.50730578	172.82060572	0.00006023
70	423.18825080	347.79094742	343.50901579	172.82153706	0.00003726
75	423.18882476	347.79198373	343.51007327	172.82211332	0.00002306
80	423.18917992	347.79262475	343.51072773	172.82246985	0.00001427
85	423.18939966	347.79302142	343.51113261	172.82269045	0.00000883
90	423.18953562	347.79326683	343.51138314	172.82282695	0.00000546
95	423.18961974	347.79341868	343.51153814	172.82291140	0.00000338
100	423.18967179	347.79351263	343.51163404	172.82296365	0.00000209
105	423.18970399	347.79357076	343.51169338	172.82299598	0.00000129
110	423.18972391	347.79360673	343.51173009	172.82301598	0.0000008
113	423.18973201	347.79362134	343.51174501	172.82302411	0.0000006
114	423.18973423	347.79362534	343.51174910	172.82302633	0.00000055
115	423.18973624	347.79362898	343.51175281	172.82302836	0.0000005

Estimates values are $p_1 \approx 423.18973624$, $p_2 \approx 347.79362898$, $p_3 \approx 343.51175281$ and $p_4 \approx 172.82302836$ after 115 iterations.

Three points central

The result from *D method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.17020532	348.77484872	345.12899862	173.74565016	0.08732671
2	423.78473584	349.23859137	344.51109238	173.47356929	0.05370977
3	423.83944711	348.68782025	344.82444631	173.43339396	0.04949078
4	423.70540339	348.94196193	344.37464935	173.37537637	0.03472787
5	423.71728346	348.56989090	344.56007115	173.32539177	0.03178626
10	423.49074463	348.39584407	344.04096340	173.13444315	0.01032592
15	423.38497165	348.12593763	343.88027173	173.01578711	0.00480898
20	423.30769016	348.01335662	343.72600634	172.94248472	0.0029276
25	423.26379767	347.92496567	343.64922359	172.89698146	0.00182987
30	423.23526072	347.87658988	343.59525619	172.86884065	0.00112724
35	423.21805043	347.84445918	343.56403196	172.85139948	0.00069996
40	423.20723586	347.82530571	343.54394664	172.84060472	0.00043273
45	423.20059315	347.81319201	343.53176631	172.83391864	0.00026817
50	423.19646163	347.80577797	343.52413657	172.82977929	0.00016597
55	423.19390971	347.80115763	343.51944230	172.82721601	0.00010279
60	423.19232745	347.79830692	343.51652543	172.82562895	0.00006364
65	423.19134847	347.79653834	343.51472286	172.82464623	0.00003941
70	423.19074206	347.79544445	343.51360554	172.82403777	0.0000244
75	423.19036666	347.79476673	343.51291412	172.82366101	0.00001511
80	423.19013420	347.79434724	343.51248588	172.82342773	0.00000935
85	423.18999027	347.79408746	343.51222076	172.82328329	0.00000579
90	423.18990115	347.79392662	343.51205659	172.82319385	0.00000359
95	423.18984597	347.79382702	343.51195494	172.82313848	0.00000222
100	423.18981180	347.79376536	343.51189201	172.82310419	0.00000137
105	423.18979065	347.79372718	343.51185304	172.82308296	0.00000085
109	423.18977969	347.79370740	343.51183285	172.82307197	0.00000058
110	423.18977755	347.79370353	343.51182891	172.82306982	0.00000053
111	423.18977560	347.79370002	343.51182532	172.82306786	0.00000048

Estimates values are $p_1 \approx 423.18977560$, $p_2 \approx 347.79370002$, $p_3 \approx 343.51182532$ and $p_4 \approx 172.82306786$ after 111 iterations.

The result from L_2 *method*

k	$p_1^{\ k}$	$p_2^{\ k}$	$p_3^{\ k}$	$p_4^{\ k}$	*ER*
1	424.17020532	348.77878815	345.12337138	173.74565734	0.08620491
2	423.78513614	349.23455778	344.51523270	173.47309949	0.05280917
3	423.83876483	348.69134008	344.81952232	173.43345223	0.04849769
4	423.70577951	348.93819954	344.37788058	173.37498793	0.03394032
5	423.71654490	348.57258243	344.55588260	173.32528083	0.03098556
10	423.49070451	348.39349254	344.04198949	173.13402969	0.00997535
15	423.38447325	348.12606233	343.87871245	173.01544964	0.00479787
20	423.30745787	348.01245710	343.72576456	172.94217373	0.00292126
25	423.26353401	347.92466772	343.64861167	172.89674495	0.00182355
30	423.23509436	347.87620538	343.59496869	172.86865992	0.00112302
35	423.21791657	347.84424264	343.56375910	172.85126899	0.00069667
40	423.20714536	347.82512788	343.54377830	172.84051147	0.00043047
45	423.20052795	347.81307684	343.53163966	172.83385355	0.00026656
50	423.19641730	347.80569517	343.52405292	172.82973431	0.00016486
55	423.19387916	347.80110240	343.51938404	172.82718532	0.00010203
60	423.19230686	347.79826909	343.51648655	172.82560816	0.00006313
65	423.19133460	347.79651311	343.51469661	172.82463227	0.00003906
70	423.19073280	347.79542755	343.51358810	172.82402844	0.00002417
75	423.19036051	347.79475554	343.51290254	172.82365482	0.00001495
80	423.19013013	347.79433983	343.51247823	172.82342364	0.00000925
85	423.18998759	347.79408258	343.51221573	172.82328059	0.00000573
90	423.18989939	347.79392342	343.51205329	172.82319209	0.00000354
95	423.18984482	347.79382493	343.51195279	172.82313732	0.00000219
100	423.18981105	347.79376399	343.51189060	172.82310344	0.00000136
105	423.18979016	347.79372629	343.51185212	172.82308247	0.00000084
109	423.18977934	347.79370677	343.51183221	172.82307162	0.00000057
110	423.18977723	347.79370296	343.51182831	172.82306950	0.00000052
111	423.18977531	347.79369949	343.51182478	172.82306757	0.00000047

Estimates values are $p_1 \approx 423.18977531$, $p_2 \approx 347.79369949$, $p_3 \approx 343.51182478$ and $p_4 \approx 172.82306757$ after 111 iterations.

The result from L_3 *method* and L_4 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.17020532	348.77878815	345.12512653	173.74558739	0.08638747
2	423.78542395	349.23580652	344.51522645	173.47356671	0.05294918
3	423.83918433	348.69136825	344.82066297	173.43372961	0.048631
4	423.70597593	348.93913349	344.37797571	173.37529585	0.03403653
5	423.71687485	348.57268528	344.55671407	173.32551576	0.03107196
10	423.49082556	348.39388825	344.04212073	173.13418152	0.01000398
15	423.38456750	348.12616316	343.87891315	173.01552941	0.0048001
20	423.30750145	348.01255976	343.72582977	172.94222060	0.00292236
25	423.26356173	347.92470789	343.64866512	172.89677020	0.00182422
30	423.23510832	347.87623355	343.59499160	172.86867399	0.00112336
35	423.21792458	347.84425563	343.56377368	172.85127647	0.00069687
40	423.20714944	347.82513552	343.54378514	172.84051543	0.00043057
45	423.20053011	347.81308048	343.53164341	172.83385555	0.00026661
50	423.19641834	347.80569705	343.52405464	172.82973528	0.00016489
55	423.19387965	347.80110322	343.51938484	172.82718575	0.00010205
60	423.19230706	347.79826943	343.51648684	172.82560833	0.00006313
65	423.19133466	347.79651320	343.51469667	172.82463231	0.00003906
70	423.19073280	347.79542754	343.51358807	172.82402843	0.00002417
75	423.19036049	347.79475548	343.51290247	172.82365478	0.00001495
80	423.19013010	347.79433978	343.51247816	172.82342360	0.00000925
85	423.18998756	347.79408253	343.51221567	172.82328056	0.00000572
90	423.18989937	347.79392338	343.51205325	172.82319206	0.00000354
95	423.18984480	347.79382490	343.51195276	172.82313730	0.00000219
100	423.18981104	347.79376397	343.51189058	172.82310342	0.00000136
105	423.18979015	347.79372627	343.51185211	172.82308246	0.00000084
109	423.18977934	347.79370676	343.51183219	172.82307161	0.00000057
110	423.18977722	347.79370295	343.51182830	172.82306949	0.00000052
111	423.18977530	347.79369948	343.51182477	172.82306756	0.00000047

Estimates values are $p_1 \approx 423.18977530$, $p_2 \approx 347.79369948$, $p_3 \approx 343.51182477$ and $p_4 \approx 172.82306756$ after 111 iterations.

The result from U_3 *method* and U_4 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.16767256	348.77538313	345.12898341	173.74565016	0.08715083
2	423.78524892	349.23638170	344.51127220	173.47369837	0.05341865
3	423.83819360	348.68887521	344.82261090	173.43289056	0.04910634
4	423.70555985	348.93913666	344.37524170	173.37517723	0.03432876
5	423.71604455	348.57088846	344.55766867	173.32483916	0.03135796
10	423.49047783	348.39351646	344.04112937	173.13397452	0.01006906
15	423.38427817	348.12569387	343.87879101	173.01531696	0.00479352
20	423.30734726	348.01234655	343.72558985	172.94211675	0.00291893
25	423.26345788	347.92455095	343.64857160	172.89669672	0.00182188
30	423.23504704	347.87614491	343.59491551	172.86863172	0.00112199
35	423.21788569	347.84419803	343.56373277	172.85124928	0.00069599
40	423.20712572	347.82510083	343.54375778	172.84049897	0.00043004
45	423.20051533	347.81305871	343.53162719	172.83384526	0.00026628
50	423.19640923	347.80568368	343.52404437	172.82972895	0.00016469
55	423.19387400	347.80109492	343.51937856	172.82718182	0.00010192
60	423.19230357	347.79826431	343.51648293	172.82560590	0.00006305
65	423.19133250	347.79651003	343.51469427	172.82463081	0.00003901
70	423.19073147	347.79542558	343.51358657	172.82402750	0.00002414
75	423.19035966	347.79475427	343.51290155	172.82365421	0.00001494
80	423.19012959	347.79433902	343.51247759	172.82342325	0.00000924
85	423.18998725	347.79408206	343.51221532	172.82328034	0.00000572
90	423.18989917	347.79392309	343.51205303	172.82319192	0.00000354
95	423.18984468	347.79382472	343.51195262	172.82313722	0.00000219
100	423.18981096	347.79376386	343.51189049	172.82310337	0.00000135
105	423.18979010	347.79372620	343.51185205	172.82308243	0.00000084
109	423.18977931	347.79370672	343.51183216	172.82307159	0.00000057
110	423.18977719	347.79370290	343.51182827	172.82306947	0.00000052
111	423.18977528	347.79369944	343.51182474	172.82306754	0.00000047

Estimates values are $p_1 \approx 423.18977528$, $p_2 \approx 347.79369944$, $p_3 \approx 343.51182474$ and $p_4 \approx 172.82306754$ after 111 iterations.

The result from U_2 *method*

k	p_1^k	p_2^k	p_3^k	p_4^k	ER
1	424.16755800	348.77539695	345.12898341	173.74565016	0.08714435
2	423.78563965	349.23636683	344.51127254	173.47370181	0.05340119
3	423.83803188	348.68895032	344.82263798	173.43288695	0.04909259
4	423.70585617	348.93913138	344.37528746	173.37520271	0.03431195
5	423.71595587	348.57098161	344.55769479	173.32484936	0.031345
10	423.49059649	348.39354037	344.04120339	173.13400432	0.01006668
15	423.38430156	348.12575684	343.87882642	173.01534069	0.0047941
20	423.30738671	348.01237788	343.72563392	172.94213711	0.00291976
25	423.26347809	347.92458274	343.64859818	172.89671200	0.00182235
30	423.23506388	347.87616494	343.59493793	172.86864310	0.00112237
35	423.21789636	347.84421363	343.56374766	172.85125742	0.00069624
40	423.20713333	347.82511129	343.54376870	172.84050472	0.00043021
45	423.20052036	347.81306613	343.53163458	172.83384924	0.00026639
50	423.19641265	347.80568869	343.52404949	172.82973169	0.00016477
55	423.19387627	347.80109834	343.51938201	172.82718368	0.00010197
60	423.19230508	347.79826660	343.51648527	172.82560716	0.00006309
65	423.19133349	347.79651156	343.51469582	172.82463165	0.00003904
70	423.19073212	347.79542660	343.51358761	172.82402805	0.00002415
75	423.19036009	347.79475494	343.51290223	172.82365458	0.00001495
80	423.19012987	347.79433947	343.51247804	172.82342349	0.00000925
85	423.18998743	347.79408236	343.51221561	172.82328050	0.00000572
90	423.18989929	347.79392328	343.51205322	172.82319203	0.00000354
95	423.18984476	347.79382485	343.51195275	172.82313729	0.00000219
100	423.18981101	347.79376394	343.51189058	172.82310342	0.00000136
105	423.18979013	347.79372626	343.51185211	172.82308246	0.00000084
109	423.18977933	347.79370675	343.51183220	172.82307161	0.00000057
110	423.18977722	347.79370294	343.51182831	172.82306949	0.00000052
111	423.18977530	347.79369948	343.51182477	172.82306756	0.00000047

Estimates values are $p_1 \approx 423.18977530$, $p_2 \approx 347.79369948$, $p_3 \approx 343.51182477$ and $p_4 \approx 172.82306756$ after 111 iterations.

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